METHODVS INVENIENDI FORMVLAS INTEGRALES,

QVAE

CERTIS CASIBVS DATAM INTER SE TENEANT RATIONEM,

VBI SIMVL METHODVS TRADITVR FRACTIONES CONTINVAS SVMMANDI.

§. 1.

Quemadmodum in seriebus recurrentibus quilibet terminus ex vno pluribusue praecedentibus secundum legem quandam constantem determinatur, ita hic eiusmodi series sum consideraturus, in quibus quilibet terminus ex vno pluribusue praecedentibus secundum quampiam legem variabilem determinatur. Quoniam autem in talibus seriebus formula generalis singulos terminos exprimens plerumque non est algebraica, sed transcendens, singulos terminos per formulas integrales exhiberi conueniet, quae vt valores determinatos praebeant, post integrationem quantitati variabili valorem determinatum tribui assumo, ita vt singuli termini prodeant quantitates determinatae; atque nunc quaestio principalis huc redit, quemadmodum istae formulae integrales debeant esse comparatae, vt quilibet terminus secundum datam legem ex vno pluribusue praecedentibus determinetur.

§. 2.

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us quilibet termibus secundum leita hic eiusmodi uilibet terminus ex quampiam legem in talibus seriebus nens plerumque non s terminos per forae vt valores dequantitati variabili a vt singuli termini nunc quaestio prinformulae integrales minus secundum daibus determinetur.
§. 2.

§. 2. Quod quo clarius perspiciatur, contemplemur seriem notissimam harum formularum integralium:

$$\int \frac{dx}{\sqrt{(1-xx)}},\ \int \frac{xx\,dx}{\sqrt{(1-xx)}},\ \int \frac{x^4\,dx}{\sqrt{(1-xx)}},\ \int \frac{x^6\,dx}{\sqrt{(1-xx)}},\ \text{etc.}$$

quae si singulae ita integrentur, vt euanescant posito $x=0$, tum vero variabili x tribuatur valor $=1$, quilibet terminus a praecedente ita pendet, vt sit

$$\int \frac{xx\,dx}{\sqrt{(1-xx)}} = \frac{1}{2}\int \frac{dx}{\sqrt{(1-xx)}},$$
$$\int \frac{x^4\,dx}{\sqrt{(1-xx)}} = \frac{3}{4}\int \frac{xx\,dx}{\sqrt{(1-xx)}},$$
$$\int \frac{x^6\,dx}{\sqrt{(1-xx)}} = \frac{5}{6}\int \frac{x^4\,dx}{\sqrt{(1-xx)}},$$

atque in genere

$$\int \frac{x^n\,dx}{\sqrt{(1-xx)}} = \frac{n-1}{n}\int \frac{x^{n-2}\,dx}{\sqrt{(1-xx)}}.$$

Vnde patet, hanc formulam generalem spectari posse tanquam terminum generalem illius seriei, atque quemlibet terminum ex praecedente oriri, si iste multiplicetur per $\frac{n-1}{n}$.

§. 3. Ad similitudinem igitur huius casus seriem formularum integralium ita in genere constituamus,

$$\int dv,\ \int x\,dv.\ \int xx\,dv,\ \int x^3\,dv,\ \int x^4\,dv,\ \text{etc.}$$

ita vt terminus indici n respondens sit $\int x^{n-1}\,dv$, quae singula integralia ita accipi sumamus, vt euanescant posito $x=0$; post integrationem autem quantitati variabili x tribuamus quempiam valorem constantem, veluti $x=1$, vel alio cuipiam numero. Quibus positis quaestio huc redit, qualis pro v assumi debeat functio ipsius x, vt quilibet terminus per vnum, vel duos pluresue praecedentes, secundum legem quandam datam vtcunque variabilem, siue a indice n pendentem, determinetur; vbi quidem imprimis eo erit respiciendum, ad quot dimensiones index n in scala relationis proposita ascendat: plerumque autem non vltra primam

mam dimenſionem aſſurgere erit opus. Hinc igitur ſequentia Problemata pertractemus.

Problema I.

Inuenire functionem v, vt iſta relatio inter binos terminos ſibi ſuccedentes locum habeat:

$$\int x^n\, dv = \frac{\alpha n + a}{\beta n + b} \int x^{n-1}\, dv.$$

Solutio.

§. 4. Requiritur igitur hic vt ſit

$$(\alpha n + a) \int x^{n-1}\, dv = (\beta n + b) \int x^n\, dv$$

ſi ſcilicet poſt integrationem variabili x certus valor tribuatur. Quoniam igitur iſta conditio tum demum locum habere debet, poſtquam variabili x iſte valor conſtans fuerit datus, ponamus in genere, dum x eſt variabilis, hanc aequationem locum habere:

$$(\alpha n + a) \int x^{n-1}\, dv = (\beta n + b) \int x^n\, dv + V,$$

quantitatem autem V ita eſſe comparatam, vt euaneſcat poſtquam variabili ille valor determinatus fuerit aſſignatus. Praeterea vero, quia ambo integralia ita capi aſſumimus, vt euaneſcant poſito $x = 0$, neceſſe eſt vt etiam iſta quantitas V eodem quoque caſu euaneſcat.

§. 5. Quoniam haec aequalitas ſubſiſtere debet pro omnibus indicibus n, quos quidem ſemper vt poſitiuos ſpectamus, facile intelligitur, quantitatem iſtam V factorem habere debere x^n; quo pacto iam iſti conditioni ſatisfit, vt poſito $x = 0$ etiam fiat $v = 0$. Quamobrem ſtatuamus $V = x^n Q$, vbi Q denotet functionem ipſius x propoſito accom-

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accommodatam, et quam ſimul ita comparatam eſſe deſideramus, vt euaneſcat ſi ipſi x certus quidam valor tribuatur.

§. 6. Cum igitur eſſe debeat

$$(\alpha n + a) \int x^{n-1}\, dv = (\beta n + b) \int x^n\, dv + x^n Q,$$

differentietur iſta aequatio, ac differentiali per x^{n-1} diuiſo peruenietur ad hanc aequationem differentialem:

$$(\alpha n + a)\, dv = (\beta n + b)\, x\, dv + n Q\, dx + x\, dQ,$$

quae cum ſubſiſtere debeat pro omnibus valoribus ipſius n, termini iſta littera affecti ſeorſim ſe tollere debent, vnde nanciſcimur has duas aequalitates:

$$\text{I. } (\alpha - \beta x)\, dv = Q\, dx \quad \text{et} \quad \text{II. } (a - bx)\, dv = x\, dQ.$$

Ex priore fit $dv = \frac{Q\, dx}{\alpha - \beta x}$, ex altera vero $dv = \frac{x\, dQ}{a - bx}$; qui duo valores inter ſe aequati ſuppeditant hanc aequationem: $\frac{dQ}{Q} = \frac{dx}{x} \cdot \frac{a - bx}{\alpha - \beta x}$, quae aequatio reſoluitur in has partes

$$\frac{dQ}{Q} = \frac{a}{\alpha} \cdot \frac{dx}{x} + \frac{a\beta - b\alpha}{\alpha} \cdot \frac{dx}{\alpha - \beta x},$$

cuius ergo integrale erit

$$lQ = \frac{a}{\alpha}\, lx - \frac{a\beta - b\alpha}{\alpha\beta}\, l(\alpha - \beta x)$$

vnde deducitur

$$Q = C x^{\frac{a}{\alpha}} \cdot (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

§. 7. Ex hoc valore pro Q inuento ſtatim patet eum euaneſcere caſu $x = \frac{\alpha}{\beta}$, ſi modo fuerit $\frac{b\alpha - a\beta}{\alpha\beta} > 0$; ſin autem ſecus eueniat, non patet quomodo haec quantitas vllo caſu euaneſcere queat. Inuento autem hoc valore Q, inde reperietur

$$dv = C x^{\frac{a}{\alpha}}\, dx\, (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1}$$

hinc

hincque nostrae seriei terminus indici n respondens erit

$$\int x^{n-1}\,dv = C\int x^{n+\frac{a}{\alpha}-1}\,dx(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}-1},$$

tum vero erit

$$V = Cx^{n+\frac{a}{\alpha}}(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}}.$$

Vbi res imprimis eo redit, vt ista quantitas praeter casum $x=0$ insuper alio casu euanescat.

Corollarium 1.

§. 8. Hic duo casus occurrunt, qui peculiarem euolutionem postulant; prior est, quo $\alpha=0$; tum autem inchoandum erit ab aequatione $\frac{dQ}{Q} = -\frac{(a-bx)dx}{\beta xx}$, vnde integrando elicitur $lQ = \frac{a}{\beta x} + \frac{b}{\beta}lx$, hinque sumendo e pro numero cuius logarithmus hyperbolicus $=1$, colligitur

$$Q = e^{\frac{a}{\beta x}} . x^{\frac{b}{\beta}}$$

quae formula in nihilum abire nequit, nisi fiat $\frac{a}{\beta x} = -\infty$, ideoque $x=0$, sicque non duo haberentur casus, quibus fieret $V=0$, cum tamen duo desiderentur. Interim autem hinc fiet

$$dv = \frac{e^{\frac{a}{\beta x}} x^{\frac{b}{\beta}}\,dx}{\alpha-\beta x}.$$

Corollarium 2.

§. 9. Alter casus peculiarem integrationem postulans erit $\beta=0$; tum autem erit $\frac{dQ}{Q} = \frac{dx(a-bx)}{\alpha x}$, vnde fit $lQ = \frac{a}{\alpha}lx - \frac{bx}{\alpha}$, ideoque $Q = x^{\frac{a}{\alpha}} . e^{\frac{-bx}{\alpha}}$, quae formula casu

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casu $x=\infty$ euanescit, si modo fuerit $\frac{b}{\alpha}$ numerus positiuus, sin autem $\frac{b}{\alpha}$ fuerit numerus negatiuus, tum Q euanescit casu $x=-\infty$. Porro vero hoc casu fiet

$$dv = \frac{x^{\frac{a}{\alpha}} . e^{\frac{-bx}{\alpha}}\,dx}{\alpha-\beta x}$$

Scholion.

§. 8. His in genere obseruatis aliquot casus speciales euoluamus, quibus litteris α, β et a, b certos valores tribuemus, qui ad casus iam satis cognitos perducant.

Exemplum I.

§. 9. *Quaerantur formulae integrales, vt fiat*

$$\int x^n\,dv = \frac{(2n-1)}{2n}\int x^{n-1}\,dv.$$

Cum igitur hic esse debeat

$$(2n-1)\int x^{n-1}\,dv = 2n\int x^n\,dv,$$

erit hoc casu $\alpha=2$ et $a=-1$, tum vero $\beta=2$ et $b=0$; hinc fit

$$\frac{dQ}{Q} = -\frac{dx}{2x(1-x)} = -\frac{dx}{2x} - \frac{dx}{2(1-x)},$$ inde integrando

$$lQ = -\tfrac{1}{2}lx + \tfrac{1}{2}l(1-x),$$ ideoque

$$Q = C\sqrt{\tfrac{1-x}{x}}, \text{ ergo } V = x^n\sqrt{\tfrac{1-x}{x}}.$$

Porro cum hic fit $dv = \frac{Q\,dx}{2(1-x)}$, erit

$$dv = \frac{dx\sqrt{\frac{1-x}{x}}}{2(1-x)} = \frac{C\,dx}{2\sqrt{(x-xx)}},$$

sumto ergo $C=2$ erit $dv = \frac{dx}{\sqrt{(x-xx)}}$, et formula nostra generalis:

$\int x^{n-}$

$$\int x^{n-1}\,dv = \int \frac{x^{n-1}\,dx}{\sqrt{(x-xx)}},$$

vnde cum fit $V = x^n \sqrt{\frac{1-x}{x}}$, haec quantitas manifesto euanefcit fumto $x = 1$, ita vt noftra formula, fi poft integrationem ftatuatur $x = 1$, quaefito fatisfaciat. Quod fi iam ponamus $x = yy$, ifta formula induet hanc formam: $2\int \frac{y^{2n-1}\,dy}{\sqrt{(1-yy)}}$, quae, pofito poft integrationem $y = 1$, praebet hanc relationem:

$$\int \frac{y^{2n}\,dy}{\sqrt{(1-yy)}} = \frac{2n-1}{2n}\int \frac{y^{2n-2}\,dy}{\sqrt{(1-yy)}},$$

quae continet relationes fupra § 2 commemoratas; hinc enim fiet

$$\int \frac{yy\,dy}{\sqrt{(1-yy)}} = \tfrac{1}{2}\int \frac{dy}{\sqrt{(1-yy)}},$$

$$\int \frac{y^4\,dy}{\sqrt{(1-yy)}} = \tfrac{3}{4}\int \frac{yy\,dy}{\sqrt{(1-yy)}} \text{ et}$$

$$\int \frac{y^6\,dy}{\sqrt{(1-yy)}} = \tfrac{5}{6}\int \frac{y^4\,dy}{\sqrt{(1-yy)}}.$$

Exemplum 2.

§. 10. *Quaerantur formulae integrales, vt fiat*

$$\int x^n\,dv = \frac{\alpha n - 1}{\alpha n}\int x^{n-1}\,dv.$$

Cum igitur hic effe debeat

$$(\alpha n - 1)\int x^{n-1}\,dv = \alpha n \int x^n\,dv,$$

erit hoc cafu $a = -1$, $\beta = \alpha$ et $b = 0$, vnde per formulas fupra datas colligitur

$$Q = C x^{\frac{-1}{\alpha}} (\alpha - \alpha x)^{\frac{-1}{\alpha}} = C x^{\frac{-1}{\alpha}} (1-x)^{\frac{+1}{\alpha}}$$

quae quantitas manifefto euanefcit pofito $x = 1$. Tum autem erit

is manifefto euanefcit
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Quod fi iam ponarmam: $2\int \frac{y^{2n-1}\,dy}{\sqrt{(1-yy)}}$,
praebet hanc re-
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ofito $x = 1$. Tum

$$dv = \frac{x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}\,dx}{(1-x)},$$

vnde formula noftra generalis erit

$$\int x^{n-1}\,dv = \int x^{n-\frac{1}{\alpha}-1}(1-x)^{+\frac{1}{\alpha}-1}\,dx = \int \frac{x^{n-\frac{1}{\alpha}-1}\,dx}{(1-x)^{1-\frac{1}{\alpha}}};$$

quae concinnior redditur, faciendo $x = y^\alpha$, tum enim ea induet hanc formam: $\int \frac{y^{\alpha n-1}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}$, vbi iterum poft integrationem ftatui debet $y = 1$. Erit hinc

$$\int \frac{y^{\alpha n+\alpha-1}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha n - 1}{\alpha n}\int \frac{y^{\alpha n-1}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}},$$

atque hinc oriuntur fequentes cafus fpeciales:

$$\int \frac{y^{2\alpha-1}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha-1}{\alpha}\int \frac{y^{\alpha-1}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} \text{ et}$$

$$\int \frac{y^{3\alpha-1}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{2\alpha-1}{2\alpha}\int \frac{y^{2\alpha-1}\,dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}.$$

§. 11. Hinc igitur fi fumatur $\alpha = 1$, vt fieri debeat

$$\int x^n\,dv = \frac{n-1}{n}\int x^{n-1}\,dv,$$

formula noftra generalis iam in y expreffa erit $\int y^{n-2}\,dy$, cuius ergo valor eft $\frac{1}{n-1}y^{n-1} = \frac{1}{n-1}$, vnde tota feries noftrarum formularum integralium abibit in hanc:

$$\tfrac{1}{0},\ \tfrac{1}{1},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{1}{6},\ \tfrac{1}{7},\ \text{etc.}$$

§. 12. Sumamus etiam $\alpha = \frac{1}{2}$, et iam non amplius opus erit ad y procedere. Hoc igitur casu erit

$$Q = \frac{(1-x)^2}{xx} \text{ et } dv = \frac{(1-x)dx}{xx}$$

vnde formula nostra generalis fit

$$\int x^{n-1} dv = \int x^{n-3}(1-x) dx,$$

cuius ergo valor algebraice expressus erit

$$\frac{1}{n-2} x^{n-2} - \frac{1}{n-1} x^{n-1} = \frac{1}{(n-1)(n-2)}$$

vnde series nostrarum formularum euadet

$$\frac{1}{0.-1}, \frac{1}{0.1}, \frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \frac{1}{4.5}, \text{ etc.}$$

Exemplum 3.

§. 13. *Quaerantur formulae integrales, vt sit*

$$\int x^n dv = n \int x^{n-1} dv.$$

Cum igitur esse debeat

$$n \int x^{n-1} dv = 1. \int x^n dv, \text{ erit}$$

$$\alpha = 1, \; a = 0, \; b = 1, \; \beta = 0.$$

Cum igitur sit $\beta = 0$, casus Coroll. 2. hic locum habet, indeque erit $Q = e^{-x}$ ideoque $V = e^{-x}.x^n$, quae quantitas his duobus casibus euanescit: $x = 0$ et $x = \infty$. Porro vero erit $dv = e^{-x} dx$, hincque formula nostra generalis fiet $\int x^{n-1} dx. e^{-x}$, vnde ipsi seriei termini ab initio sequenti modo se habebunt:

$$\int e^{-x} dx, \int e^{-x} x dx, \int e^{-x} xx dx, \int e^{-x} x^3 dx \text{ etc.}$$

quibus integratis ita vt euanescant posito $x = 0$, tum vero posito $x = \infty$, orietur sequens series satis simplex:

$$1, 1, 1.2, 1.2.3, 1.2.3.4, 1.2.3.4.5, \text{ etc.}$$

quae

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Porro vero
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$x^3 dx$ etc.
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4.5, etc.
quae

quae est series hypergeometrica *Wallisii*, cuius ergo terminus generalis est

$$\int x^{n-1} e^{-x} dx = 1.2.3.4 \ldots (n-1).$$

§. 14. Ope ergo huius termini generalis hanc seriem interpolare licebit. Ita si quaeratur terminus medius inter duos primos, poni debet $n = \frac{3}{2}$, ac valor huius termini erit $\int e^{-x} dx \sqrt{x}$, cuius autem valor nullo modo algebraice exprimi potest. Inueni autem singulari modo hunc ipsum terminum aequari $\frac{1}{2}\sqrt{\pi}$, denotante π peripheriam circuli cuius diameter $= 1$, vnde hic vicissim cognoscimus esse $\int e^{-x} dx \sqrt{x} = \frac{\sqrt{\pi}}{2}$, posito scilicet post integrationem $x = \infty$. Terminus autem hunc praecedens, indici $\frac{1}{2}$ respondens, erit $= \sqrt{\pi}$, cui ergo aequatur formula $\int \frac{e^{-x} dx}{\sqrt{x}}$. Quod si hic ponamus $e^x = y$, ita vt posito $x = 0$ fit $y = 1$, at posito $x = \infty$ fiat $y = \infty$, tum ergo ista formula $\int \frac{e^{-x} dx}{\sqrt{x}}$ abit in hanc $\int \frac{dy}{yy\sqrt{ly}}$, quae formula si ita integretur vt euanescat posito $y = 1$, tum vero fiat $y = \infty$, praebet valorem ipsius $\sqrt{\pi}$. Si porro fiat $y = \frac{1}{z}$, erunt termini integrationis $z = 1$, et $z = 0$, et formula integralis erit

$$-\int \frac{dz}{\sqrt{-lz}} \begin{bmatrix} \text{a} & z = 1 \\ \text{ad} & z = 0 \end{bmatrix} = \sqrt{\pi},$$

siue permutatis terminis integrationis erit

$$\int \frac{dz}{\sqrt{-lz}} \begin{bmatrix} \text{a} & z = 0 \\ \text{ad} & z = 1 \end{bmatrix} = \sqrt{\pi},$$

quemadmodum iam olim obseruaui

 Exem-

Exemplum 4.

§. 15. *Quaerantur formulae integrales, vt sit*
$\int x^n dv = \frac{1}{n}\int x^{n-1} dv$, *siue*
$\int x^{n-1} dv = n\int x^n dv$.

Hic est $\alpha = 0$ et $a = 1$, $\beta = 1$ et $b = 0$; qui ergo est casus in Coroll. 1. tractatus, vnde colligitur fore $Q = e^{\frac{1}{x}}$, ideoque $V = x^n e^{\frac{1}{x}}$, quae formula nequidem euanescit sumto $x = 0$, quandoquidem formula $e^{\frac{1}{0}}$ aequiualet infinito infinitesimae potestatis. Hic autem miro modo euenit, vt casus $x = -0$ reddat formulam $e^{\frac{-1}{0}}$ subito euanescentem. Scilicet, si ω denotet quantitatem infinite paruam, erit $e^{\frac{1}{\omega}} = \infty^{\infty}$, tum vero repente fiet $e^{\frac{-1}{\omega}} = \frac{1}{\infty^{\infty}} = 0$, quam ob causam formulam hinc exhibere non licet scopo nostro respondentem. Reperietur quidem $dv = -e^{\frac{1}{x}}\frac{dx}{x}$, ita vt formula nostra generalis futura sit $-\int x^{n-2} dx\, e^{\frac{1}{x}}$, quae autem nobis nullum vsum praestare potest.

§. 16. Quod si hic ponamus $\frac{1}{x} = y$, formula ista generalis transit in hanc: $+\int \frac{e^y dy}{y^n}$. At vero nunc erit $V = \frac{e^y}{y^n}$, quae formula euanescit posito $y = -\infty$. Quomodocunque autem hanc expressionem transformemus, semper idem incommodum occurret. Interim tamen etiam hunc casum sequenti modo resoluere licebit. Sit enim seriei, quam quaerimus

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rimus, primus terminus $= \omega$, ex quo per regulam praescriptam sequentes ordine ita procedent

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & & & n \\ \omega, & \frac{\omega}{1}, & \frac{\omega}{1\cdot 2}, & \frac{\omega}{1\cdot 2\cdot 3}, & \frac{\omega}{1\cdot 2\cdot 3\cdot 4}, & \ldots & \ldots & \frac{\omega}{1\cdot 2\cdot 3 \ldots (n-1)} \end{array}$$

Supra autem vidimus huius formulae $1 . 2 . 3 . 4 . 5 \ldots (n-1)$ valorem exprimi per hoc integrale: $\int x^{n-1} e^{-x} dx$, integratione ab $x = 0$ ad $x = \infty$ extensa; tantum igitur opus est vt hanc formulam integralem in denominatorem transferamus, et seriei quam quaerimus terminus generalis erit

$$\frac{1}{\int x^{n-1} e^{-x} dx},$$

vnde satis intelligitur, negotium non per simplicem formulam integralem expediri posse, quod idem quoque tenendum est de aliis casibus, quibus quantitas V non duobus casibus euanescere potest; tum enim tantum opus est fractionem $\frac{\alpha n + a}{\beta n + b}$ inuertere, atque formulam integralem in denominatorem transferre.

Scholion.

§. 17. Nisi sit vel $\alpha = 0$ vel $\beta = 0$, quos casus iam expediuimus, resolutio nostri problematis semper reduci potest ad casum, quo ambae litterae α et β sunt aequales vnitati. Cum enim esse debeat

$$\int x^n dv = \frac{\alpha n + a}{\beta n + b}\int x^{n-1} dv,$$

ponatur $x = \frac{\alpha y}{\beta}$, fietque

$$\frac{\alpha}{\beta}\int y^n dv = \frac{\alpha n + a}{\beta n + b}\int y^{n-1} dv,$$

quae aequatio reducitur ad hanc formam:

 $\int y^n$

$$\int y^n\,dv = \frac{n+\alpha a}{n+\beta b}\int y^{n-1}\,dv.$$

Quod ſi iam nunc loco $\frac{a}{\alpha}$ ſcribamus a, et b loco $\frac{b}{\beta}$, reſoluenda erit haec formula:

$$\int y^n\,dv = \frac{n+a}{n+b}\int y^{n-1}\,dv,$$

cuius reſolutio, ſi loco x ſcribamus y et loco litterarum α et β vnitatem, ex ſuperiori ſolutione praebet primo

$$Q = C y^a (1-y)^{b-a},$$

quod ergo euaneſcit poſito $y=1$, ſi modo fuerit $b>a$, tum autem erit ipſa formula

$$\int y^{n-1}\,dv = C\int y^{n+a-1}\,dy\,(1-y)^{b-a-1};$$

ſin autem fuerit $b<a$, haec ſolutio, vti vidimus, locum habere nequit; verum hoc caſu pro termino noſtrae ſeriei aſſumi debet haec forma: $\frac{1}{\int y^{n-1}\,dv}$, ita vt tum eſſe debeat

$$\frac{1}{\int y^n\,dv} = \frac{n+a}{n+b}\,\frac{1}{\int y^{n-1}\,dv},\ \text{ſiue}$$

$$\int y^n\,dv = \frac{n+b}{n+a}\int y^{n-1}\,dv,$$

cuius reſolutio permutatis litteris a et b praebet

$$Q = C y^b (1-y)^{a-b}$$

quae iam caſu $y=1$ euaneſcit, ſi fuerit $a>b$, atque tum erit formula generalis

$$\int y^{n-1}\,dv = C\int y^{n+b-1}\,dy\,(1-y)^{a-b-1}.$$

Siue igitur ſit $b>a$ ſiue $a>b$, ſolutio nulla amplius laborat difficultate.

§. 18. Sin autem fuerit vel $\alpha=0$ vel $\beta=0$, loco alterius etiam ſcribi poterit vnitas; vnde ſi eſſe debeat $\int x^n$

loco $\frac{b}{\beta}$, reſol-
co litterarum
bet primo
fuerit $b>a$,
$^{a-1}$;
us, locum ha-
noſtrae ſeriei
m eſſe debeat
bet
$>b$, atque tum
$^{b-1}$.
amplius labo-
vel $\beta=0$, lo-
ſi eſſe debeat
$\int x^n$

$$\int x^n\,dv = \frac{n+a}{b}\int x^{n-1}\,dv,$$

ob $\alpha=1$ et $\beta=0$ ſolutio noſtra generalis dat

$$\frac{dQ}{Q} = \frac{dx}{x}(a-bx)$$

vnde colligitur $Q = C x^a . e^{-bx}$, quae formula euaneſcit poſito $x=\infty$, ſi modo b fuerit numerus poſitiuus; tum autem ſit terminus generalis

$$\int x^{n-1}\,dv = C\int x^{n+a-1}\,dx . e^{-x}.$$

At vero numerus b negatiuus eſſe nequit, quia alioquin conditio praeſcripta eſſet incongrua.

§. 19. Conſideremus etiam alterum caſum, quo $\alpha=0$ et $\beta=1$, ideoque conditio praeſcripta

$$\int x^n\,dv = \frac{a}{n+b}\int x^{n-1}\,dv,$$

vnde ſit

$$\frac{dQ}{Q} = -\frac{dx}{xx}(a-bx).$$

Hinc autem pro Q orietur valor, qui praeter caſum $x=0$ euaneſcere non poſſet; quam ob cauſſam formula generalis ſtatui debet $\frac{1}{\int x^{n-1}\,dv}$, ita vt eſſe debeat

$$\int x^n\,dv = \frac{n+b}{a}\int x^{n-1}\,dv$$

vnde prodit

$$\frac{dQ}{Q} = \frac{dx}{x}(b-ax),\ \text{ideoque}\ Q = C e^{-ax} . x^b,$$

quae expreſſio euaneſcit poſito $x=\infty$, quoniam a neceſſario debet eſſe numerus poſitiuus; tum autem erit

$$dv = C e^{-ax} . x^b\,dx,$$

vnde

vnde formula generalis feriei erit

$$\frac{1}{C\int x^{a+b-1}\,dx.\,e^{-ax}}$$

Problema II.

§. 20. *Denotet* T *terminum indici* n *respondentem in ferie quam confiderandam fufcepimus, at vero* T′ *terminum fequentem, atque proponatur haec conditio adimplenda:*

$$T' = \frac{(\alpha n + a)(\alpha' n + a')}{(\beta n + b)(\beta' n + b')} T$$

Solutio.

Quoniam hic valores geminati occurrunt, huic conditioni commodiffime fatisfiet, fi terminus generalis T tanquam productum ex duobus factoribus fpectetur. Statuatur igitur $T = RS$, fitque terminus fequens $= R'S'$, et quaerantur formulae R et S, vt fiat

$$R' = \frac{\alpha n + a}{\beta n + b} R \text{ et } S' = \frac{\alpha' n + a'}{\beta' n + b'} S,$$

tum enim fumendo $T = RS$ conditioni praefcriptae manifefto fatisfiet Hoc igitur modo pro R et S vel huiusmodi formulae: $\int x^{n-1}dv$, vel inuerfae $\frac{1}{\int x^{n-1}\,dv}$ reperientur, id quod pro folutione generali fufficit, vnde rem exemplo illuftremus.

Exemplum.

§. 21. *Quaeratur formula generalis* T, *vt fiat*

$$T' = \frac{nn - cc}{nn} T.$$

Refoluamus igitur T in duos factores R et S, ac ftatuamus

$$R' = \frac{n - c}{n} R \text{ et } S' = \frac{n + c}{n} S.$$

Pro

n refpondentem
t vero T′ ter-
iditio adimplenda:
rrunt, huic con-
generalis T tan-
ectetur. Statuatur
= R′ S′, et quae-
raefcriptae mani-
S vel huiusmodi
reperientur, id
rem exemplo il-
is T, vt fiat
et S, ac ftatuamus
Pro

Pro priore forma fi ftatuamus $R = \int x^{n-1}\,dv$, ex folutione generali, vbi erit $\alpha = 1$, $a = -c$, $\beta = 1$ et $b = 0$, fiet

$$Q = C\,x^{-c}(1-x)^c;$$

quae forma manifefto euanefcit pofito $x=1$, hincque quia fit

$$V = C\,x^{n-c}(1-x)^c,$$

haec forma etiam cafu $x = 0$ euanefcit, fi modo n fuerit $> c$, id quod tuto affumi poteft, quia exponentem n fucceffiue in infinitum crefcere affumimus, ac plerumque pro c fractiones tantum accipi folent. Hinc ergo erit

$$R = C\int x^{n-c-1}(1-x)^{c-1}\,dx.$$

§. 22. Hinc iam alter valor litterae S deduci poffet, fcribendo tantum $-c$ loco c, tum autem non amplius fieret $Q = 0$ pofito $x = 1$, quamobrem pro S formulam inuerfam $\frac{1}{\int x^{n-1}\,dv}$ affumi oportet, vt effe debeat

$$\int x^n\,dv = \frac{n}{n+c}\int x^{n-1}\,dv,$$

vbi cum fit $\alpha = 1$, $a = 0$, $\beta = 1$ et $b = c$, reperitur $Q = C(1-x)^c$, quae forma manifefto fit $= 0$ pofito $x = 1$, hinc autem prodit

$$dv = C(1-x)^{c-1}\,dx,$$

ergo habebimus

$$S = \frac{1}{C\int x^{n-1}(1-x)^{c-1}\,dx},$$

confequenter formula noftra generalis quaefita erit

$$T = \frac{\int x^{n-c-1}(1-x)^{c-1}\,dx}{\int x^{n-1}(1-x)^{c-1}\,dx}.$$

§. 23.

§. 23. Quod si ergo nostrae seriei per factores procedentis primum terminum ponamus $= A$, ipsa series erit

$$\begin{array}{llll} \text{I.} & \text{II.} & \text{III.} & \text{IV.} \\ A, & \frac{1-cc}{1}\cdot A, & \frac{1-cc}{1}\cdot\frac{4-cc}{4}\cdot A, & \frac{1-cc}{1}\cdot\frac{4-cc}{4}\cdot\frac{9-cc}{9}A, \text{ etc.} \end{array}$$

vnde si sumamus $c = \frac{1}{2}$, erit haec series

$$A,\ \frac{1\cdot 3}{2\cdot 2}\cdot A,\ \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot A,\ \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot A, \text{ etc.}$$

cuius ergo terminus indici n respondens est

$$\frac{\int x^{n-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx}{\int x^{n-1}(1-x)^{-\frac{1}{2}}dx},$$

qui posito $x = yy$ transit in hanc formam:

$$\frac{\int y^{2n-2}(1-yy)^{-\frac{1}{2}}dy}{\int y^{2n-1}(1-yy)^{-\frac{1}{2}}dy},$$

vnde patet, terminum primum fore

$$A = \int\frac{dy}{\sqrt{(1-yy)}} : \int\frac{y\,dy}{\sqrt{(1-yy)}} = \frac{\pi}{2}$$

posito scilicet post integrationem $y = 1$.

Problema.

Denotet T *terminum seriei indici* n *respondentem, sintque* T′ *et* T″ *termini sequentes pro indicibus* $n+1$ *et* $n+2$, *si proponatur inter ternos terminos se insequentes talis relatio, vt sit*

$$(\alpha n+a)T = (\beta n+b)T' + (\gamma n+c)T'';$$

inuestigare formulam pro T, *qua terminus generalis huius seriei exprimatur.*

Solutio.

seriei per factores
= A, ipsa series erit
IV.
A, etc.
A, etc.
est
nam:
1.
ci n respondentem, sint-
indicibus n + 1 et
ninos se insequentes talis
-(γn+c)T″;
rminus generalis huius
Solutio.

Solutio.

§. 24. Assumatur pro T formula integralis $\int x^{n-1}dv$, huiusque integrale ita capiatur, vt euanescat posito $x = 0$, eruntque termini sequentes

$$T' = \int x^{n}dv \text{ et } T'' = \int x^{n+1}dv,$$

siquidem post integrationem variabili x certus valor determinatus tribuatur. Quamdiu autem haec quantitas x vt variabilis spectatur, ponamus esse

$$(\alpha n+a)T = (\beta n+b)T' + (\gamma n+c)T'' + x^{n}Q,$$

ac perspicuum est Q eiusmodi functionem esse debere ipsius x, quae euanescat, si loco x valor ille determinatus substituatur, quem autem a ciphra diuersum esse oportet, quoniam iam assumsimus, omnes istas formulas in nihilum abire posito $x = 0$. Quodsi vero, absoluto calculo, huic conditioni nullo modo satisfieri poterit, id erit indicio, problema nostrum hac ratione resolui non posse, vt scilicet eius terminus generalis T per talem formulam differentialem simplicem $\int x^{n-1}dv$ exhibeatur.

§. 25. Differentiemus nunc aequationem modo stabilitam, ac diuisione facta per x^{n-1} sequens prodibit aequatio:

$$(\alpha n+a)dv = (\beta n+b)x\,dv + (\gamma n+c)xx\,dv + nQ\,dx + x\,dQ,$$

quae, quia termini littera n affecti seorsim se destruere debent, discerpetur in binas sequentes aequationes:

$$1°\ \alpha\,dv = \beta\,x\,dv + \gamma\,xx\,dv + Q\,dx,$$
$$2°\ a\,dv = b\,x\,dv + c\,xx\,dv + x\,dQ,$$

ex

ex quarum priore fit

$$d v = \frac{Q\, d x}{\alpha - \beta x - \gamma x x},$$

ex altera vero fit

$$d v = \frac{x\, d Q}{a - b x - c x x},$$

quorum valorum posterior per priorem diuisus praebet

$$\frac{dQ}{Q} = \frac{d x (\alpha - b x - c x x)}{x (\alpha - \beta x - \gamma x x)},$$

ex cuius ergo integratione valor ipsius Q elici debet, quo facto facile patebit, vtrum is certo quodam casu praeter $x = 0$ euanescere possit. Imprimis autem hic notari conuenit, si hoc integrale inuoluat huiusmodi factorem $e^{\frac{\lambda}{x}}$, tum solutionem quoque successu esse carituram, quandoquidem posito $x = 0$ iste factor tantam inuoluet infiniti potestatem, vt, etiamsi per x^n multiplicetur, productum etiamnum infinitum maneat.

§. 26. Quod si igitur his conditionibus praescriptis satisfacere licuerit, tum inuento valore litterae Q, quem ponamus fieri $= 0$ posito $x = f$, habebitur

$$d v = \frac{Q\, d x}{\alpha - \beta x - \gamma x x},$$

et formula generalis naturam seriei complectens erit

$$T = \int x^{n-1}\, d v = \int \frac{x^{n-1}\, Q\, d x}{\alpha - \beta x - \gamma x x},$$

quippe cuius integrale, a termino $x = 0$ vsque ad terminum $x = f$ extensum, praebebit valorem termini T, indici cuicunque n respondentis.

Scholion.

te
fo ; praebet
re
ne
tu i debet, quo
m casu praeter
c notari con-
de torem $e^{\frac{\lambda}{x}}$, tum
quandoquidem
i potestatem,
Ex etiamnum in-
ded us praescriptis
Q, quem po-
Sim
ns erit
vnde
tes
conti que ad termi-
nini T, indici
nume
conti
Scholion.

Scholion.

§. 27. Inuenta autem tali relatione inter ternos terminos cuiuspiam seriei sibi inuicem succedentes, inde more solito formari poterit fractio continua, cuius valorem assignare licebit. Si enim characteres T′, T″, T‴, T^{IV}, etc. denotent ordine omnes terminos post T sequentes in infinitum, ex relationibus, quas inter se tenent, sequentes formulae deducentur. Ex relatione

$$(\alpha n + a) T = (\beta n + b) T' + (\gamma n + c) T''$$

deducitur

$$(\alpha n + a) \tfrac{T}{T'} = \beta n + b + \frac{(\gamma n + c)(\alpha n + \alpha + a)}{(\alpha n + \alpha + a) T' : T''}.$$

Ex relatione sequente

$$(\alpha n + \alpha + a) T' = (\beta n + \beta + b) T'' + (\gamma n + \gamma + c) T'''$$

deducitur

$$(\alpha n + \alpha + a) \tfrac{T'}{T''} = \beta n + \beta + b + \frac{(\gamma n + \gamma + c)(\alpha n + 2\alpha + a)}{(\alpha n + 2\alpha + a) T'' : T'''}.$$

Simili modo sequentes relationes suppeditabunt:

$$(\alpha n + 2\alpha + a) \tfrac{T''}{T'''} = \beta n + 2\beta + b + \frac{(\gamma n + 2\gamma + c)(\alpha n + 3\alpha + a)}{(\alpha n + 3\alpha + a) T''' : T''''};$$

$$(\alpha n + 3\alpha + a) \tfrac{T'''}{T''''} = (\beta n + 3\beta + b) + \frac{(\gamma n + 3\gamma + c)(\alpha n + 4\alpha + a)}{(\alpha n + 4\alpha + a) T'''' : T'''''};$$

vnde manifestum est, si in prima formula continuo sequentes valores ordine substituantur, prodituram esse fractionem continuam, cuius valor aequalis erit formulae $(\alpha n + a) \frac{T}{T'}$.

§. 28. Quod si ergo loco n successiue scribamus numeros 1, 2, 3, 4, etc., sequens problema circa fractiones continuas resoluere poterimus.

 Pro-

Problema.

Propofita fractione continua huius formae:

$$\beta+b+\cfrac{(\gamma+c)(2\alpha+a)}{2\beta+b+\cfrac{(2\gamma+c)(3\alpha+a)}{3\beta+b+\cfrac{(3\gamma+c)(4\alpha+a)}{4\beta+b+\cfrac{(4\gamma+c)(5\alpha+a)}{5\beta+b+\cfrac{(5\gamma+c)(6\alpha+a)}{6\beta+b+\text{ etc.}}}}}}$$

eius valorem inueftigare.

Solutio.

Confideretur in genere ifta relatio inter ternas quantitates fibi fuccedentes T, T′, T″, quae fit

$$(\alpha n+a)T=(\beta n+b)T'+(\gamma n+c)T'',$$

atque ex praecedente Problemate quaeratur valor ipfius T, fiquidem fieri poteft, hoc modo expreffus:

$$T=\int x^{n-1}\, dv=\int \frac{x^{n-1}Q\,dx}{\alpha-\beta x-\gamma xx},$$

cuius integrale ab $x=0$ vsque ad $x=f$ extendatur, quae formula inuenta ponatur

$$\int \frac{Q\,dx}{\alpha-\beta x-\gamma xx}=A \text{ et } \int \frac{nQ\,dx}{\alpha-\beta x-\gamma xx}=B,$$

ita vt A et B fint valores ipfius T, pro cafibus $n=1$ et $n=2$, quibus definitis fractionis continuae propofitae valor per praecedentia erit $=\frac{(\alpha+a)A}{B}$. Hanc igitur inueftigationem ad fequentia exempla accommodemus.

Exem-

m
q
Q
ft
er ernas quan-
fra
fac ″,
r ipfius T,
vn
vn
har idatur, quae
qua is $n=1$ et
pofitae valor
vnd inueftigatio-
Exem-

Exemplum 1.

§. 29. *Inueftigare valorem fractionis continuae notiffimae, quam olim* Broucherus *pro quadratura circuli protulit, quae eft*

$$2+\cfrac{1\cdot 1}{2+\cfrac{3\cdot 3}{2+\cfrac{5\cdot 5}{2+\text{ etc.}}}}$$

Quia omnes partes integrae laeuam refpicientes funt conftantes $=2$, pro noftra forma generali fiet

$$\beta+b=2,\ 2\beta+b=2,\ 3\beta+b=2 \text{ etc.}$$

erit ergo $\beta=0$ et $b=2$; at pro numeratoribus fequentium fractionum, quandoquidem conftant binis factoribus, erit pro factoribus prioribus

$$\gamma+c=1,\ 2\gamma+c=3,\ 3\gamma+c=5,\ 4\gamma+c=7,$$

vnde concluditur $\gamma=2$ et $c=-1$, pro alteris vero erit

$$2\alpha+a=1,\ 3\alpha+a=3,\ 4\alpha+a=5 \text{ etc.}$$

vnde $\alpha=2$ et $a=-3$. Ex his autem valoribus colligimus hanc aequationem

$$\frac{dQ}{Q}=-\frac{dx(3+2x-xx)}{2x(1-xx)},$$

quae per $1+x$ depreffa praebet

$$\frac{dQ}{Q}=-\frac{dx(3-x)}{2x(1-x)},$$

vnde integrando fit

$$lQ=-\tfrac{3}{2}lx+(1-x) \text{ et hinc } Q=\frac{1-x}{x^{\frac{3}{2}}},$$

ex

ex quo valore porro sequitur,

$$A = \int \frac{(1-x)\,dx}{2x^{\frac{3}{2}}(1-xx)} = \int \frac{dx}{2x(1+x)\sqrt{x}}$$

$$B = \int \frac{(1-x)\,dx}{2x^{\frac{1}{2}}(1-xx)} = \int \frac{dx}{2(1+x)\sqrt{x}}.$$

§. 30. In his autem valoribus istud incommodum deprehenditur, quod prius integrale euanescens reddi nequit posito $x = 0$. Hoc autem incommodum facile remoueri potest, si fractionem continuam supremo membro truncemus et quaeramus valorem istius fractionis:

$$2 + \cfrac{3.\,3}{2 + \cfrac{5.\,5}{2 + \text{etc.}}}$$

qui si repertus fuerit $= s$, erit ipsius propositae valor $= b + \frac{1}{s}$. Nunc vero, comparatione instituta, fit quidem vt ante $\beta = 0$ et $b = 2$, tum vero $\gamma = 2$ et $c = +1$, $\alpha = 2$ et $a = -1$, vnde sequitur

$$\frac{dQ}{Q} = -\frac{dx(1+2x+xx)}{2x(1-xx)} = -\frac{dx(1+x)}{2x(1-x)},$$

vnde integrando fit

$$lQ = -\tfrac{1}{2}lx + l(1-x) \text{ ideoque } Q = \tfrac{1-x}{\sqrt{x}},$$

ex quo valore iam habebimus

$$A = \int \tfrac{(1-x)\,dx}{2(1-xx)\sqrt{x}} = \tfrac{1}{2}\int \tfrac{dx}{(1+x)\sqrt{x}} \text{ et}$$

$$B = \tfrac{1}{2}\int \tfrac{dx\sqrt{x}}{1+x}$$

vbi cum sit $Q = \frac{1-x}{\sqrt{x}}$, eius valor manifesto euanescit posito $x = 1$, quamobrem illa integralia a termino $x = 0$, vsque ad $x = 1$ sunt extendenda.

§. 31.

mmodum
ddi nequit
noueri po-
ncemus et
lor $= b + \frac{1}{s}$
ante $\beta = 0$
et $a = -1$,
$\frac{-x}{\sqrt{x}}$,
nescit posito
$= 0$, vsque
§. 31.

§. 31. Quo nunc haec integralia facilius eruamus, statuamus $x = zz$, ita vt termini integrationis etiamnunc sint $z = 0$ et $z = 1$, eritque

$$A = \int \tfrac{dz}{1+zz} = \text{A tang. } z = \tfrac{\pi}{4} \text{ et}$$

$$B = \int \tfrac{zz\,dz}{1+zz} = 1 - \tfrac{\pi}{4},$$

sicque habebimus $s = \frac{\pi}{4-\pi}$, quocirca ipsius fractionis *Broun-cherianae* valor est $1 + \frac{4}{\pi}$, omnino vti olim *Brounckerus* iam inuenerat.

Exemplum 2.

§. 31. *Inuestigare valorem huius fractionis continuae* Brouncheriana *latius patentis:*

$$b + \cfrac{1.\,1}{b + \cfrac{3.\,3}{b + \cfrac{5.\,5}{b + \text{etc.}}}}$$

Vt hic incommodum superius euitemus, omittamus membrum supremum et quaeramus

$$s = b + \cfrac{3.\,3}{b + \cfrac{5.\,5}{b + \text{etc.}}}$$

quandoquidem tum erit valor quaesitus $= b + \frac{1}{s}$. Nunc igitur erit $\beta = 0$ et $b = b$, $\gamma = 2$, $c = 1$, $\alpha = 2$ et $a = -1$, vnde fit

$$\frac{dQ}{Q} = -\frac{dx(1+bx+xx)}{2x(1-xx)}, \text{ ac proinde}$$

$$lQ = -\tfrac{1}{2}lx - \tfrac{b-2}{4}l(1+x) + \tfrac{b+2}{4}l(1-x),$$

Q =

hincque

$$Q = \frac{(1-x)^{\frac{b+2}{4}}}{(1+x)^{\frac{b-2}{4}}\sqrt{x}};$$

quae formula manifesto fit $=0$ ponendo $x=1$, siquidem $b+2$ fuerit numerus positiuus, vnde fit

$$dv = \frac{(1-x)^{\frac{b-2}{4}}}{2(1+x)^{\frac{b+2}{4}}\sqrt{x}}.$$

Hinc autem colligetur

$$A = \tfrac{1}{2}\int \frac{(1-x)^{\frac{b-2}{4}}\,dx}{(1+x)^{\frac{b+2}{4}}\sqrt{x}} \quad \text{et}$$

$$B = \tfrac{1}{2}\int \frac{(1-x)^{\frac{b-2}{4}}\,dx\sqrt{x}}{(1+x)^{\frac{b+2}{4}}},$$

siue ponendo $x=zz$ habebimus

$$A = \int \frac{(1-zz)^{\frac{b-2}{4}}\,dz}{(1+zz)^{\frac{b+2}{4}}} \quad \text{et}$$

$$B = \int \frac{(1-zz)^{\frac{b-2}{4}}\,zz\,dz}{(1+zz)^{\frac{b+2}{4}}},$$

quae ambo integralia a $z=0$ vsque ad $z=1$ sunt extendenda. Ex his autem valoribus A et B erit $s=\frac{A}{B}$; ipsius igitur fractionis propositae valor erit $=b+\frac{1}{s}=b+\frac{B}{A}$.

§. 42.

= 1, siquidem

= 1 sunt exten-
t $s=\frac{A}{B}$; ipsius
$+\frac{1}{s}=b+\frac{B}{A}$.
§. 42.

§. 32. Quod si hic ponatur $b=2$, prodit casus ante expositus a quadratura circuli pendens, quippe quo casu formula fit rationalis. Quando autem exponentes $\frac{b-2}{4}$ et $\frac{b+2}{4}$ non sunt numeri integri, tum litteras A et B neque per arcus circulares, neque per logarithmos exprimere licet. Veluti si fuerit $b=4$, erit

$$A = \int \frac{dz\sqrt{(1-zz)}}{(1+zz)^{\frac{3}{2}}},$$

cuius valor per arcus ellipticos exhiberi posset. At si b fuerit numerus impar, hi valores multo magis euadunt transcendentes, ita vt his ipsis litteris A et B debeamus esse contenti. Contra autem si exponentes illi fiant numeri integri, totum negotium per arcus circulares expedire licebit.

§. 33. Exponentes autem illi $\frac{b-2}{4}$ et $\frac{b+2}{4}$ erunt numeri integri, quoties fuerit b numerus huius formae:

$$b=4i+2$$

tum enim erit

$$A = \int \frac{(1-zz)^i\,dz}{(1+zz)^{i+1}} \quad \text{et}$$

$$B = \int \frac{(1-zz)^i\,zz\,dz}{(1+zz)^{i+1}};$$

quos ergo casus quomodo euolui oporteat operae pretium erit docere, quoniam *Wallisius* eos iam est contemplatus.

§. 34. Quoniam hoc negotium totum redit ad reductionem huiusmodi formularum integralium ad formas

sim-

simpliciores, consideremus in genere formam $P = \frac{z^m}{(1+zz)^n}$, cuius differentiale sub sequentibus formis exhiberi potest:

$$1^\circ)\ dP = \frac{m z^{m-1} dz}{(1+zz)^n} - \frac{2n z^{m+1} dz}{(1+zz)^{n+1}}$$

$$2^\circ)\ dP = \frac{m z^{m-1} dz}{(1+zz)^{n+1}} - \frac{(2n-m) z^{m+1} dz}{(1+zz)^{n+1}}$$

$$3^\circ)\ dP = -\frac{(2n-m) z^{m-1} dz}{(1+zz)^n} - \frac{2n z^{m-1} dz}{(1+zz)^{n+1}}$$

vnde hanc triplicem reductionem integralium deducimus:

$$\text{I. } \int \frac{z^{m+1} dz}{(1+zz)^{n+1}} = \frac{m}{2n} \int \frac{z^{m-1} dz}{(1+zz)^n} - \frac{1}{2n} \cdot \frac{z^m}{(1+zz)^n}$$

$$\text{II. } \int \frac{z^{m+1} dz}{(1+zz)^{n+1}} = \frac{m}{2n-m} \int \frac{z^{m-1} dz}{(1+zz)^{n+1}} - \frac{1}{2n-m} \cdot \frac{z^m}{(1+xz)^n}$$

$$\text{III. } \int \frac{z^{m-1} dz}{(1+zz)^{n+1}} = \frac{2n-m}{2n} \int \frac{z^{m-1} dz}{(1+zz)^n} + \frac{1}{2n} \cdot \frac{z^m}{(1+zz)^n}$$

quarum reductionum ope casibus $b = 4i + 2$ totum negotium absolui et ad formulam $\frac{\pi}{4}$ reduci poterit, siquidem post integrationem sumatur $z = 1$.

§. 35. Sit $i = 1$ ideoque $b = 6$ eritque
$A = \int \frac{(1-zz) dz}{(1+zz)^2}$ et $B = \int \frac{(1-zz) zz dz}{(1+zz)^2}$.

Nunc igitur reperiemus per reductionem tertiam

$$\int \frac{dz}{(1+zz)^2} = \frac{1}{2} \int \frac{dz}{1+zz} + \frac{1}{2} \cdot \frac{z}{1+zz} = \frac{\pi}{8} + \frac{1}{4}$$

et per reductionem primam

$$\int \frac{zz\,dz}{(1+zz)^2} = \frac{1}{2} \int \frac{dz}{1+zz} - \frac{1}{2} \cdot \frac{z}{1+zz} = \frac{\pi}{8} - \frac{1}{4}, \text{ porro}$$

$$\int \frac{z^4 dz}{(1+zz)^2} = \frac{3}{2} \int \frac{zz\,dz}{1+zz} - \frac{1}{2} \cdot \frac{z^3}{1+zz} = \frac{5}{4} - \frac{3\pi}{8}.$$

Ex

$P = \frac{z^m}{(1+zz)^n}$,
hiberi potest:
deducimus:
+ 2 totum negotium
poterit, siquidem
ritque
rtiam
porro
Ex

Ex his iam valoribus colligitur $A = \frac{1}{2}$ et $B = \frac{\pi}{2} - \frac{3}{2}$, ideoque $\frac{B}{A} = \pi - 3$, quocirca orietur ista summatio:

$$3 + \pi = 6 + \cfrac{1 \cdot 1}{6 + \cfrac{3 \cdot 3}{6 + \cfrac{5 \cdot 5}{6 + \cfrac{7 \cdot 7}{6 + \text{etc.}}}}}$$

§. 36. Sit nunc $i = 2$ et $b = 10$, eritque
$A = \int \frac{(1-zz)^2 dz}{(1+zz)^3}$ et $B = \int \frac{zz(1-zz)^2 dz}{(1+zz)^3}$.

Quo harum integralium valores inuestigemus, sequentes euoluamus formulas:

$$\int \frac{dz}{(1+zz)^3} = \frac{3}{4} \int \frac{dz}{(1+zz)^2} + \frac{1}{4} \cdot \frac{z}{(1+zz)^2} = \frac{3\pi}{32} + \frac{1}{4}$$

$$\int \frac{zz\,dz}{(1+zz)^3} = \frac{1}{4} \int \frac{dz}{(1+zz)^2} - \frac{1}{4} \cdot \frac{z}{(1+zz)^2} = \frac{\pi}{32}$$

$$\int \frac{z^4 dz}{(1+zz)^3} = \frac{3}{4} \int \frac{zz\,dz}{(1+zz)^2} - \frac{1}{4} \cdot \frac{z^3}{(1+zz)^2} = \frac{3\pi}{32} - \frac{1}{4}$$

$$\int \frac{z^6 dz}{(1+zz)^3} = \frac{5}{4} \int \frac{z^4 dz}{(1+zz)^2} - \frac{1}{4} \cdot \frac{z^5}{(1+zz)^2} = \frac{3}{2} - \frac{15\pi}{32}.$$

Ex quibus iam valoribus deducitur $A = \frac{\pi}{8}$ et $B = 2 - \frac{5\pi}{8}$, ideoque $\frac{B}{A} = \frac{16 - 5\pi}{\pi}$, vnde emergit sequens summatio:

$$\frac{5\pi + 16}{\pi} = 10 + \cfrac{1 \cdot 1}{10 + \cfrac{3 \cdot 3}{10 + \cfrac{5 \cdot 5}{10 + \text{etc.}}}}$$

§. 37. Si b esset numerus negatiuus, inuestigatio nulla prorsus laboraret difficultate. Si enim in genere fuerit

s =

$$s = -a + \cfrac{\alpha}{-b + \cfrac{\beta}{-c + \cfrac{\gamma}{-d + \cfrac{\delta}{-e + \text{etc.}}}}}$$

semper erit

$$s = a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{\varepsilon + \text{etc.}}}}}$$

vnde si habeatur valor istius expressionis, idem negatiue sumtus dabit valorem illius.

Exemplum 3.

§. 38. *Proposita sit fractio continua, cuius valorem inuestigari oporteat, ista:*

$$1 + \cfrac{1.1}{3 + \cfrac{3.3}{5 + \cfrac{5.5}{7 + \cfrac{7.7}{9 + \text{etc.}}}}}$$

Quo fractiones supra allegatas, omisso membro supremo, sint

$$3 + \cfrac{3.3}{5 + \cfrac{5.5}{7 + \cfrac{7.7}{9 + \text{etc.}}}}$$

eritque $\beta + b = 3$, $2\beta + b = 5$, ideoque $\beta = 2$ et $b = 1$; tum vero vt ante $\alpha = 2$, $a = -1$, $\gamma = 2$ et $c = +1$;

Inuento

m negatiue
ius valorem
upremo, sint
2 et $b = 1$;
et $c = +1$;
Inuento

inuento autem s erit valor quaesitus $= 1 + \frac{1}{s}$. Nunc igitur habebimus

$$\frac{dQ}{Q} = -\frac{dx(1+x+xx)}{2x(1-x-xx)}. \quad \text{Est vero}$$

$$\frac{1+x+xx}{x(1-x-xx)} = \frac{1}{x} + \frac{2+2x}{1-x-xx}, \text{ vnde fit}$$

$$lQ = -\tfrac{1}{2}lx - \int\frac{dx(1+x)}{1-x-xx}.$$

Porro vero pro formula $\int\frac{dx(1+x)}{1-x-xx}$ inuenienda, statuamus denominatorem

$$1 - x - xx = (1 - fx)(1 - gx)$$

eritque $f + g = 1$ et $fg = -1$, vnde fit

$$f = \frac{1+\sqrt{5}}{2} \text{ et } g = \frac{1-\sqrt{5}}{2}.$$

Nunc statuatur

$$\frac{1+x}{1-x-xx} = \frac{\mathfrak{A}}{1-fx} + \frac{\mathfrak{B}}{1-gx},$$

vnde reperietur

$$\mathfrak{A} = \frac{1+f}{f-g} \text{ et } \mathfrak{B} = -\frac{(1+g)}{f-g}$$

siue substitutis pro f et g valoribus supra datis erit

$$\mathfrak{A} = \frac{\sqrt{5}+1}{2\sqrt{5}} \text{ et } \mathfrak{B} = \frac{\sqrt{5}-1}{2\sqrt{5}},$$

quibus inuentis erit

$$\int\frac{dx(1+x)}{1-x-xx} = -\frac{\mathfrak{A}}{f}l(1-fx) - \frac{\mathfrak{B}}{g}l(1-gx) = -\frac{(1+\sqrt{5})}{2\sqrt{5}}l(1-fx) - \frac{(\sqrt{5}-1)}{2\sqrt{5}}l(1-gx)$$

quocirca fiet

$$lQ = -\tfrac{1}{2}lx + \frac{(\sqrt{5}+1)}{2\sqrt{5}}l(1-fx) + \frac{(\sqrt{5}-1)}{2\sqrt{5}}l(1-gx)$$

consequenter

$$Q = \frac{(1-fx)^{\frac{\sqrt{5}+1}{2\sqrt{5}}}(1-gx)^{\frac{\sqrt{5}-1}{2\sqrt{5}}}}{\sqrt{x}},$$

qui

qui valor duobus casibus euanescit: altero quo

$$x = \frac{1}{f} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2},$$

altero vero quo $x = \frac{1}{g} = -\frac{1-\sqrt{5}}{2}$; vtrouis autem vtamur, res eodem redibit.

§. 39. Ex hoc autem valore habebimus

$$A = \int \frac{Q\,dx}{1-x-xx} \text{ et } B = \int \frac{Q\,x\,dx}{1-x-xx};$$

vnde porro deducitur

$$s = (\alpha + a)\frac{A}{B} = \frac{A}{B}$$

hinc propositae fractionis summa erit $1 + \frac{B}{A}$. Hinc autem nihil vlterius concludere licet, ob formulas differentiales non solum irrationales, sed etiam vere transscendentes ob exponentes surdos.

Exemplum 4.

§. 40. *Proposita sit haec fractio continua:*

$$b + \cfrac{1.1}{b + \cfrac{2.2}{b + \cfrac{3.3}{b + \cfrac{4.4}{b + \text{etc.}}}}}$$

vbi est $\beta = 0$, $\mathfrak{b} = b$.
Nunc consideremus hanc formam:

$$s = b + \cfrac{2.2}{b + \cfrac{3.3}{b + \text{etc.}}},$$

quippe quo valore inuento quaesitus erit $= b + \frac{1}{s}$. Habebimus igitur $\gamma + c = 2$, $2\gamma + c = 3$, ideoque $\gamma = 1$ et $c = 1$,

$c = 1$, deinde erit $\alpha = \gamma = 1$, $a = 0$ et $c = 1$. Hinc igitur colligimus

$$\frac{dQ}{Q} = -\frac{dx\,(bx + xx)}{x\,(1-xx)} = -\frac{dx\,(b+x)}{1-xx}, \text{ ideoque}$$

$$lQ = -\frac{b}{2}\,l\,\frac{1+x}{1-x} + \frac{1}{2}\,l\,(1-xx) \text{ hincque}$$

$$Q = \frac{(1-x)^{\frac{b}{2}}\sqrt{(1-xx)}}{(1+x)^{\frac{b}{2}}} = \frac{(1-x)^{\frac{b+1}{2}}}{(1+x)^{\frac{b-1}{2}}},$$

quae quantitas manifesto euanescit posito $x = 1$. Hinc igitur fiet

$$A = \int \frac{Q\,dx}{1-xx} = \int \frac{(1-x)^{\frac{b+1}{2}}\,dx}{(1+x)^{\frac{b-1}{2}}(1-xx)} = \int \frac{(1-x)^{\frac{b-1}{2}}\,dx}{(1+x)^{\frac{b+1}{2}}} \text{ et}$$

$$B = \int \frac{x\,(1-x)^{\frac{b-1}{2}}\,dx}{(1+x)^{\frac{b+1}{2}}},$$

tum autem erit $s = (\alpha + a)\frac{A}{B} = \frac{A}{B}$ ideoque summa quaesita $b + \frac{B}{A}$.

§. 41. Percurramus nunc casus praecipuos: ac primo sit $b = 1$ eritque

$$A = \int \frac{dx}{1+x} = l\,(1+x) = l\,2 \text{ et}$$

$$B = \int \frac{x\,dx}{1+x} = x - \int \frac{dx}{1+x} = 1 - l\,2,$$

ideoque $b + \frac{B}{A} = \frac{1}{l\,2}$; ergo hinc prodiit ista summatio:

$\frac{1}{l\,2} =$

$$\frac{1}{l2} = 1 + \cfrac{1.1}{1 + \cfrac{2.2}{1 + \cfrac{3.3}{1 + \text{etc.}}}}$$

§. 42. Sit nunc $b = 2$ eritque

$$A = \int \frac{dx\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}} \text{ et } B = \int \frac{x\,dx\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}.$$

Ad has formulas rationales reddendas ſtatuamus

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} = z, \text{ eritque } x = \frac{1-zz}{1+zz},$$

vnde terminis integrationis $x = 0$ et $x = 1$ reſpondebunt $z = 1$ et $z = 0$; tum vero erit

$$1 + x = \frac{2}{1+zz} \text{ et } dx = -\frac{4z\,dz}{(1+zz)^2},$$

hincque colligitur

$$A = -2\int \frac{zz\,dz}{1+zz} = -2z + 2\,\text{A tang.}\,z + 2 - \frac{\pi}{2} = 2 - \frac{\pi}{2},$$

porro fit

$$B = -2\int \frac{zz\,dz}{(1+zz)^2} + 2\int \frac{z^4\,dz}{(1+zz)^2}.$$

Per reductiones igitur ſupra § 35 monſtratas, ſi hic ſcilicet terminos integrationis $z = 1$ et $z = 0$ permutemus, vt habeamus

$$B = +2\int \frac{zz\,dz}{(1+zz)^2} - 2\int \frac{z^4\,dz}{(1+zz)^2}, \text{ erit}$$

$$B = 2\left(\frac{\pi}{8} - \frac{1}{4}\right) - 2\left(\frac{3}{4} - \frac{\pi}{4}\right) = \pi - 3,$$

vnde ſequitur iſta ſummatio:

$\frac{2}{4-\pi}$

$$\frac{2}{4-\pi} = 2 + \cfrac{1.1}{2 + \cfrac{3.3}{2 + \cfrac{4.4}{2 + \text{etc.}}}}$$

quae *Brouncherianae* ſimplicitate nihil cedit.

§. 43. Si ponamus $b = 0$, fractio continua abit in ſequens continuum productum:

$$\frac{1.1}{2.2}\cdot\frac{3.3}{4.4}\cdot\frac{5.5}{6.6}\cdot\frac{7.7}{8.8}\cdot \text{etc.}$$

hoc autem caſu fit

$$A = \int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2} \text{ et } B = \int \frac{x\,dx}{\sqrt{(1-xx)}} = 1$$

vnde iſtius producti valor colligitur $\frac{2}{\pi}$, id quod egregie convenit cum iam dudum cognitis, quandoquidem hoc productum eſt ipſa progreſſio *Walliſiana*.

Exemplum 5.

§. 44. *Propoſita ſit haec fractio continua, vbi* $\beta = 0$, $b = b$ *et numeratores numeri trigonales:*

$$b + \cfrac{1}{b + \cfrac{3}{b + \cfrac{6}{b + \cfrac{10}{b + \text{etc.}}}}}$$

Omiſſo ſupremo membro ſtatuamus

$$s = b + \cfrac{3}{b + \cfrac{6}{b + \text{etc.}}}$$

et primo numeratores per producta repraesentemus, hoc modo:

$$3 = 2 \cdot \tfrac{3}{2}, \quad 6 = 3 \cdot \tfrac{4}{2}, \quad 10 = 4 \cdot \tfrac{5}{2}$$

quorum priores comparentur cum formulis

$$\gamma + c, \; 2\gamma + c, \; 3\gamma + c,$$

posteriores vero cum formulis $2\alpha + a, 3\alpha + a, 4\alpha + a$, eritque $\gamma = 1$, $c = 1$, $\alpha = \frac{1}{2}$, $a = \frac{1}{2}$, vnde erit

$$\frac{dQ}{Q} = \frac{dx(\frac{1}{2} - bx - xx)}{x(\frac{1}{2} - xx)} = \frac{dx(1 - 2bx - 2xx)}{x(1 - 2xx)}$$

siue $$\frac{dQ}{Q} = \frac{dx}{x} - \frac{2b\,dx}{1 - 2xx}$$

cuius integrale est

$$lQ = lx - \frac{b}{\sqrt{2}}\, l\frac{1 + x\sqrt{2}}{1 - x\sqrt{2}}, \text{ ergo}$$

$$Q = \frac{x(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}}{(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

quae formula euanescit casu $x = \frac{1}{\sqrt{2}}$. Hinc igitur erit

$$dv = \frac{2x(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}\,dx}{(1 - 2xx)(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

Sit $\frac{b}{\sqrt{2}} = \lambda$ eritque

$$A = 2\int \frac{x(1 - x\sqrt{2})^{\lambda}\,dx}{(1 - 2xx)(1 + x\sqrt{2})^{\lambda}} = 2\int \frac{x(1 - x\sqrt{2})^{\lambda - 1}\,dx}{(1 + x\sqrt{2})^{\lambda + 1}}$$

et

$$B = 2\int \frac{xx(1 - x\sqrt{2})^{\lambda - 1}\,dx}{(1 + x\sqrt{2})^{\lambda + 1}},$$

vbi post integrationem statuitur $x = \frac{1}{\sqrt{2}}$; cum autem fit $s = \frac{A}{B}$, hincque valor fractionis propositae $= b + \frac{B}{A}$.

§. 45.

hoc modo:
+a, erit-
τ − 2 x x)
. x x)
r erit
√ 2)^{λ−1} dx
x √ 2)^{λ+1}
m fit s = A/B,
§. 45.

§. 45. Nisi igitur fuerit $\lambda = \frac{b}{\sqrt{2}}$ numerus rationalis, hos valores commode assignare non licet. Sit igitur $b = \sqrt{2}$, siue $\lambda = 1$, eritque

$$A = 2\int \frac{x\,dx}{(1 + x\sqrt{2})^2} \text{ et } B = 2\int \frac{xx\,dx}{(1 + x\sqrt{2})^2}.$$

Hinc integrando colligitur

$$A = l(1 + x\sqrt{2}) - \frac{x\sqrt{2}}{1 + x\sqrt{2}}$$

ideoque posito $x\sqrt{2} = 1$ fiet $A = l2 - \frac{1}{2}$; tum vero reperitur

$$B = \frac{1}{2\sqrt{2}} - \sqrt{2}.\, l2,$$

quare ob $b = \sqrt{2}$ erit $b + \frac{B}{A} = \frac{1}{\sqrt{2}(2l2 - 1)}$, vnde sequitur haec summatio:

$$\frac{1}{\sqrt{2}(2l2 - 1)} = \sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{3}{\sqrt{2} + \cfrac{6}{\sqrt{2} + \text{etc.}}}}$$

Scholion.

§. 46. Fractiones autem continuae, ad quas plerumque calculo numerico deducimur, huiusmodi formam habere solent:

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \text{etc.}}}}}$$

vbi omnes numeratores sunt vnitates, denominatores vero a, b, c, d, etc. numeri integri. Verum ope nostrae methodi difficulter talium formarum valores eruere licet, etiamsi numeri

meri a, b, c, d, e progressionem arithmeticam constituant, id quod sequenti exemplo ostendamus.

Exemplum.

§. 47. *Proposita sit ista fractio continua:*

$$\beta+b+\cfrac{1}{2\beta+b+\cfrac{1}{3\beta+b+\cfrac{1}{4\beta+b+\cfrac{1}{5\beta+b+\text{etc.}}}}}$$

vbi $\alpha=0$, $\gamma=0$, $a=1$, $c=1$.
Hinc fit

$$\frac{dQ}{Q}=-\frac{dx(1+bx-xx)}{\beta xx}, \text{ vnde}$$

$$lQ=\frac{1}{\beta x}+\frac{b}{\beta}lx+\frac{x}{\beta} \text{ et}$$

$$Q=e^{\frac{1+xx}{\beta x}}.x^{\frac{b}{\beta}},$$

quae autem expressio nullo casu euanescere potest, etiamsi per x^n multiplicetur, siquidem β fuerit numerus positiuus. Verum si pro β sumamus numeros negatiuos puta, $\beta=-m$, tum valor $Q=x^{\frac{-b}{m}}\times e^{\frac{-(1+xx)}{mx}}$, manifesto euanescit, tam si $x=0$, quam si $x=\infty$. Hinc autem erit

$$dv=\frac{x^{\frac{-b}{m}}.e^{\frac{-1-xx}{mx}}dx}{mxx},$$

quamobrem habebimus

$$A=\frac{1}{m}\int\frac{dx}{x^{2+\frac{b}{m}}.e^{\frac{1+xx}{mx}}} \text{ et}$$

B =

constituant,
b + etc.
, etiamsi per
..nus. Verum
= − m, tum
..am si x = 0;

$$B=\frac{1}{m}\int\frac{dx}{x^{1+\frac{b}{m}}.e^{\frac{1+xx}{mx}}}.$$

His valoribus inuentis formula $\frac{A}{B}$ exprimet summam huius fractionis continuae:

$$-m+b+\cfrac{1}{-2m+b+\cfrac{1}{-3m+b+\cfrac{1}{-4m+b+\cfrac{1}{-5m+b+\text{etc.}}}}}$$

quamobrem formula illa negatiue sumta $-\frac{A}{B}$ exprimet valorem huius fractionis continuae:

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

quem igitur assignare liceret, si modo formulae integrales A et B expediri et a termino $x=0$ ad $x=\infty$ extendi possent. Verum istae formulae ita sunt comparatae, vt earum integratio nullo plane casu per quantitates cognitas exprimi queat, quod tamen non impedit, quo minus fractio $\frac{A}{B}$ valores satis cognitos inuoluere queat, etiamsi eos nullo adhuc modo assignare valeamus.

§. 49. Talium autem fractionum continuarum mihi quidem binae sequentes innotuere, quarum valores commode exhibere licet:

$n+1$

$$\cfrac{n+1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}} = \frac{e^{\frac{2}{n}}}{e^{\frac{2}{n}}-1} \text{ et}$$

$$\cfrac{n-1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\cfrac{1}{9n-\text{etc.}}}}} = \cot.\frac{1}{n}$$

Harum fractionum prior cum formulis postremi exempli collata praebet $m-b=n$, $2m-b=3n$, ideoque $m=2n$ et $b=n$, vnde fit

$$A=\frac{1}{2n}\int\frac{dx}{x^{\frac{1}{2}}\,e^{\frac{1+xx}{2nx}}} \text{ et}$$

$$B=\frac{1}{2n}\int\frac{dx}{x^{\frac{1}{2}}\,e^{\frac{1+xx}{2n}}},$$

vnde iam discimus si hae duae formulae integrentur à termino $x=0$ vsque ad terminum $x=\infty$, tum fore

$$\frac{A}{B}=\frac{1+e^{\frac{2}{n}}}{1-e^{\frac{2}{n}}},$$

quanquam nulla adhuc via analytica patet, hanc conuenientiam demonstrandi.

SVM-

exempli
$m=2n$
ur à ter-
re
conueni-
SVM-

SVMMATIO FRACTIONIS CONTINVAE, CVIVS INDICES PROGRESSIONEM ARITHMETICAM CONSTITVVNT, DVM NVMERATORES OMNES SVNT VNITATES; VBI SIMVL RESOLVTIO AEQVATIONIS RICCATIANAE PER HVIVSMODI FRACTIONES DOCETVR.

§. 1.

Cum in praecedente dissertatione methodum exposuissem, fractiones continuas ad duas formulas integrales reducendi, ea quidem infinitis casibus feliciter successit: at vero casus, qui simplicissimus videtur, vbi omnes numeratores inter se ponuntur aequales, ad eiusmodi formulas integrales perduxit, quas nullo adhuc modo euoluere et inter se comparare licuit, cum tamen ex hoc genere binae fractiones continuae habeantur, quarum valores satis commode exhiberi possunt:

$$\cfrac{n+1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\text{etc.}}}} = \frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1} \text{ et}$$