E592

DE RELATIONE FRACTIONVM TRANSCENDENTIVM IN INFINITAS FRACTIONES SIMPLICES.

§. 1.

Proposita fractione quacunque algebraica $\frac{P}{Q}$, cuius tam numerator P quam denominator Q sint functiones rationales integrae quantitatis z, iam pridem ostendi quomodo ea in fractiones simplices resolui possit, quarum denominatores aequentur factoribus simplicibus denominatoris Q, numeratores vero sint constantes, siquidem variabilis z in denominatore Q plures habeat dimensiones quam in numeratore P. Quin etiam ostendi, quemadmodum pro quolibet factore simplici denominatoris fractio simplex respondens reperiri queat, fine vllo respectu ad reliquos factores habito. Ita si constet, denominatorem Q factorem complecti simplicem $z - a$, fractio simplex inde nata, quae erit huius formae: $\frac{\alpha}{z-a}$, facillime hoc modo definitur. Statuatur $\frac{P}{Q} = \frac{\alpha}{z-a} + R$, vbi R complectatur omnes fractiones simplices ex reliquis oriendas. Multiplicetur vtrinque per $z - a$, vt fiat

$$\frac{P(z-a)}{Q} = \alpha + R(z - a),$$

et

et quia α est quantitas constans, ea semper eundem retinebit valorem, quicunque valor variabili z tribuatur; quamobrem fiat vbique $z = a$, vt reliquarum fractionum simplicium ratio ex calculo excedat, et habebitur $\alpha = \frac{P(z-a)}{Q}$, siquidem in hac formula fiat $z = a$, tum autem numerator $P(z-a)$ in nihilum abit; verum, quia $z - a$ est factor denominatoris Q, etiam denominator Q in nihilum abibit. Hinc igitur per regulam consuetam loco numeratoris ac denominatoris sua differentialia substituantur, quandoquidem etiamnunc erit $\frac{P\,dz + (z-a)\,dP}{dQ} = \alpha$, siquidem hic vbique loco z scribatur a. Ponamus igitur hoc casu $z = a$ fieri $P = A$ et $\frac{dQ}{dz} = C$, quae ergo quantitates A et C facilime inueniuntur, tum igitur prodibit numerator quaesitus $\alpha = \frac{A}{C}$, ita vt fractio simplex ex denominatoris factore $z - a$ oriunda fit $= \frac{A}{C}(z - a)$, ita vt non opus sit reliquos factores denominatoris nosse. Simili autem modo pro singulis reliquis factoribus fractiones simplices respondentes determinabuntur, quarum omnium summa aequabitur fractioni propositae $\frac{P}{Q}$, dummodo variabilis z pauciores habeat dimensiones in numeratore P quam in denominatore Q.

§. 2. Haec igitur principia sequentes pro denominatore Q eiusmodi assumamus functiones transcendentes, quas in infinitos factores simplices resoluere liceat, id quod euenit, si eae infinitis casibus nihilo aequales euadant. Praeterea vero necesse est vt omnes isti factores inter se sint inaequales, quandoquidem factores aequales peculiarem resolutionem postulant. Imprimis autem requiritur, vt productum omnium talium factorum ipsam functionem Q penitus exhauriat, quoniam quandoque factores imaginarii se

inter-

intermiscere possent. Veluti si sumatur $Q = \tang. \phi$, ea vtique omnibus iisdem casibus euanescit, quibus haec functio: $\sin. \phi$, hincque ambae istae functiones eosdem factores simplices inuoluunt, etiamsi inter se neutiquam sint aequales. Deinde vero numeratorem P ita comparatum esse oportet vt cum denominatore Q nullos habeat factores communes. Imprimis autem cauendum est, ne quantitas variabilis in numeratore ad totidem vel plures dimensiones assurgat quam in denominatore. Cum autem ea in denominatore ad infinitas dimensiones assurgere sit censenda, istud incommodum non erit pertimescendum, quamdiu variabilis in numeratore tantum finito dimensionum numero continetur. Sin autem eius potestates etiam in infinitum ascendant, saepenumero difficile erit iudicare, num dimensionum numerus maior sit vel minor quam in denominatore. Interim tamen etiam his casibus fractio proposita $\frac{P}{Q}$ omnes continebit fractiones simplices, ad quas methodus nostra perducit. Verum euenire potest vt praeter eas etiam quasdam partes quasi integras inuoluat. His igitur praenotatis sequentes casus euoluamus.

I. Sumatur $Q = \sin. \phi$, vt fractio resoluenda sit $\frac{P}{\sin. \phi}$.

§. 3. Quoniam formula $\sin. \phi$, denotante π semiperipheriam circuli cuius radius $= 1$, seu angulum duobus rectis aequalem, omnibus his casibus euanescit:

$$\phi = 0,\ \phi = \pm \pi,\ \phi = \pm 2\pi,\ \phi = \pm 3\pi, \text{ etc.}$$

et in genere $\phi = \pm i\pi$, eius factores numero infiniti erunt

$$\phi, (\phi \pm \pi), (\phi \pm 2\pi), (\phi + 3\pi) \text{ et in genere } (\phi + i\pi).$$

Aliunde autem certum est, hanc formulam $\sin. \phi$ praeter istos factores nullos alios siue reales siue imaginarios inuoluere; cum

cum enim sit

$$\sin. \phi = \phi - \frac{\phi^3}{1.2.3} + \frac{\phi^5}{1.2.3.4.5} - \text{ etc.}$$

constat hanc seriem aequari huic producto infinito:

$$\phi(1 - \tfrac{\phi}{\pi})(1 + \tfrac{\phi}{\pi})(1 - \tfrac{\phi}{2\pi})(1 + \tfrac{\phi}{2\pi})(1 - \tfrac{\phi}{3\pi})(1 + \tfrac{\phi}{3\pi}) \text{ etc.}$$

§. 4. Consideremus igitur nostri denominatoris $Q = \sin. \phi$ factorem, quemcunque $\phi + i\pi$, vbi i denotet omnes plane numeros integros, tam positiuos quam negatiuos, ciphra non excepta; sitque fractio partialis hinc oriunda $\frac{\alpha}{\phi + i\pi}$. Ad eius numeratorem α inueniendum statuatur primo in numeratore P vbique $\phi = i\pi$, sitque quantitas inde resultans $= A$; deinde cum sit $Q = \sin. \phi$, erit $dQ = d\phi \cos. \phi$, siue $\frac{dQ}{d\phi} = \cos. \phi$, vbi loco ϕ itidem scribi oportet $i\pi$, vt obtineamus C, vnde patet fore $C = \cos. i\pi$, ita vt sit $C = \pm 1$, vbi signum $+$ valebit pro numeris paribus, signum vero $-$ pro imparibus numeris loco i assumtis. Hoc igitur modo numerator fractionis nostrae erit $\alpha = \pm A$ ipsaque fractio quaesita $\pm \frac{A}{\phi - i\pi}$. Hinc autem vlterius progredi non licet, quamdiu numeratorem in genere spectamus, vnde eius loco plures valores determinatos accipiamus, singulosque in sequentibus exemplis euoluamus.

1°. Sit $P = 1$ et fractio proposita $\frac{1}{\sin. \phi}$.

§. 5. Hic igitur semper erit $A = 1$ et fractio simplex quaecunque $= \frac{\pm 1}{\phi - i\pi}$, vbi signum superius valet si i numerus par, inferius vero si impar. Substituamus igitur successiue pro i omnes eius valores ordine

$$0, +1, -1, +2, -2, +3, -3, +4, -4 \text{ etc.}$$

et

et resolutio nostrae fractionis $\frac{1}{\sin.\phi}$ in fractiones simplices ita se habebit:

$$\frac{1}{\sin.\phi}=+\frac{1}{\phi}-\frac{1}{\phi-\pi}-\frac{1}{\phi+\pi}+\frac{1}{\phi-2\pi}+\frac{1}{\phi+2\pi}-\frac{1}{\phi-3\pi}-\frac{1}{\phi+3\pi}+\text{etc.}$$

quae in hanc formam reducatur:

$$\frac{1}{\sin.\phi}=\frac{1}{\phi}+\frac{1}{\pi-\phi}-\frac{1}{\pi+\phi}-\frac{1}{2\pi-\phi}+\frac{1}{2\pi+\phi}+\frac{1}{3\pi-\phi}-\frac{1}{3\pi+\phi}-\text{etc.}$$

Contrahantur post primum terminum bini sequentium, vt nanciscamur hanc seriem:

$$\frac{1}{\sin.\phi}=\frac{1}{\phi}+\frac{2\phi}{\pi\pi-\phi\phi}-\frac{2\phi}{4\pi\pi-\phi\phi}+\frac{2\phi}{9\pi\pi-\phi\phi}-\frac{2\phi}{16\pi\pi-\phi\phi}+\text{etc.}$$

vnde deducitur sequens series memoratu digna

$$\frac{1}{2\phi\sin.\phi}-\frac{1}{2\phi\phi}=\frac{1}{\pi\pi-\phi\phi}-\frac{1}{4\pi\pi-\phi\phi}+\frac{1}{9\pi\pi-\phi\phi}-\frac{1}{16\pi\pi-\phi\phi}\ \text{etc.}$$

§. 6. Has quidem series iam olim fusius sum prosecutus: interim tamen pro sequentibus casibus haud inutile erit sequentes transformationes hic repetere. Ponamus igitur primo $\phi=\lambda\pi$, vt littera π ex seriebus elidatur, atque hinc nanciscemur

$$\frac{\pi}{\sin.\lambda\pi}=\frac{1}{\lambda}-\frac{1}{\lambda-1}-\frac{1}{\lambda+1}+\frac{1}{\lambda-2}+\frac{1}{\lambda+2}-\frac{1}{\lambda-3}-\frac{1}{\lambda+3}+\text{etc. et}$$

$$\frac{\pi}{2\lambda\sin.\lambda\pi}-\frac{1}{2\lambda\lambda}=\frac{1}{1-\lambda\lambda}-\frac{1}{4-\lambda\lambda}+\frac{1}{9-\lambda\lambda}-\frac{1}{16-\lambda\lambda}+\text{etc.}$$

atque hinc per differentiationem, spectando λ tanquam quantitatem variabilem, infinitas alias series notatu dignissimas elicere poterimus. Ex priore scilicet nanciscemur

$$\frac{\pi\pi\cos.\lambda\pi}{\sin.\lambda\pi^2}=\frac{1}{\lambda\lambda}-\frac{1}{(\lambda-1)^2}-\frac{1}{(\lambda+1)^2}+\frac{1}{(\lambda-2)^2}+\frac{1}{(\lambda+2)^2}-\frac{1}{(\lambda-3)^2}-\text{etc.}$$

Hinc igitur sequitur, si $\lambda=\frac{1}{2}$ fore

$$0=1-1-\frac{1}{9}+\frac{1}{9}+\frac{1}{25}-\frac{1}{25}-\text{etc.}$$

quod quidem est manifestum. At si $\lambda=\frac{1}{3}$ erit

$$\frac{2\pi\pi}{27}=1-\frac{1}{4}-\frac{1}{16}+\frac{1}{25}+\frac{1}{49}-\frac{1}{64}-\frac{1}{100}+\frac{1}{121}+\frac{1}{169}-\text{etc.}$$

Si

Si $\lambda=\frac{2}{3}$, oritur series praecedens.

Si $\lambda=\frac{1}{4}$, prodit haec summatio:

$$\frac{\pi\pi}{8\sqrt{2}}=1-\frac{1}{9}-\frac{1}{25}+\frac{1}{49}+\frac{1}{81}-\frac{1}{121}-\frac{1}{169}+\text{etc.}$$

Quod si denuo differentiemus obtinebitur sequens summatio:

$$\frac{\pi^3}{\sin.\lambda\pi^3}-\frac{\pi^3}{2\sin.\lambda\pi}=\frac{1}{\lambda^3}-\frac{1}{(\lambda-1)^3}-\frac{1}{(\lambda+1)^3}+\frac{1}{(\lambda-2)^3}+\frac{1}{(\lambda+2)^3}-\text{etc.}$$

sicque continuo vlterius progredi licet.

§. 7. Simili modo etiam alteram formam differentiemus, quae reducta praebet

$$\frac{1}{2\lambda^4}-\frac{\pi}{4\lambda^3\sin.\lambda\pi}-\frac{\pi\pi\cos.\lambda\pi}{4\lambda\lambda\sin.\lambda\pi^2}=\frac{1}{(1-\lambda\lambda)^2}-\frac{1}{(4-\lambda\lambda)^2}+\frac{1}{(9-\lambda\lambda)^2}-\frac{1}{(16-\lambda\lambda)^2}\ \text{etc.}$$

Quod si nunc sumamus $\lambda=\frac{1}{2}$, prodibit ista summatio:

$$\frac{1}{2}-\frac{\pi}{8}=\frac{1}{3^2}-\frac{1}{15^2}+\frac{1}{35^2}-\frac{1}{63^2}+\frac{1}{99^2}-\text{etc.},$$

quae series prorsus noua omnem attentionem meretur; neque autem opus est hinc nouam differentiationem instituere.

§. 8. Posteriorem autem summationem

$$\frac{\pi}{2\lambda\sin.\lambda\pi}-\frac{1}{2\lambda\lambda}=\frac{1}{1-\lambda\lambda}-\frac{1}{4-\lambda\lambda}+\frac{1}{9-\lambda\lambda}-\frac{1}{16-\lambda\lambda}+\text{etc.}$$

accuratius perpendamus, ac primo quidem cum ea semper debeat esse vera, quicquid pro λ assumatur, sumamus $\lambda=0$. Quia autem hoc casu membrum sinistrum abit in $\infty-\infty$, tractetur λ vt quantitas quam minima, et cum sit $\lambda\pi=\lambda\pi-\frac{1}{6}\lambda^3\pi^3$, istud membrum euadet

$$\frac{\pi}{2\lambda(\lambda\pi-\frac{1}{6}\lambda^3\pi^3)}-\frac{1}{2\lambda\lambda},$$

quae fractiones ad communem denominatorem perductae dant

$$\frac{1-1+\frac{1}{6}\lambda\lambda\pi\pi}{2\lambda\lambda(1-\frac{1}{6}\lambda\lambda\pi\pi)}=\frac{\pi\pi}{12-2\lambda\lambda\pi\pi}.$$

O 2

Nunc

Nunc igitur facto $\lambda = 0$, eius factor erit $= \frac{\pi\pi}{12}$, feries autem ipfa hoc cafu euadet

$$1 + \tfrac{1}{4} - \tfrac{1}{9} + \tfrac{1}{16} - \tfrac{1}{25} + \text{etc.}$$

cuius fummam conftat effe $\frac{\pi\pi}{12}$.

§. 9. Manifeftum porro eft, quoties pro λ accipiatur numerus integer, vnum terminum feriei, ideoque etiam ipfam feriem fieri infinitam, quod egregie conuenit cum fumma inuenta, quandoquidem hoc cafu fit $\mathrm{fin.}\,\lambda\pi = 0$. Atque hinc nata eft ifta quaeftio: fi ille terminus feriei in infinitum abiens ad finiftram partem transferatur, quanta futura fit reliquorum terminorum fumma. Ponamus fcilicet effe $\lambda = 1$, et primus feriei terminus euadet infinitus, qui ergo ad finiftram partem translatus dabit

$$\frac{\pi}{2\lambda\,\mathrm{fin.}\,\lambda\pi} - \frac{1}{2\lambda\lambda} - \frac{1}{1-\lambda\lambda} = -\tfrac{1}{3} + \tfrac{1}{8} - \tfrac{1}{15} + \tfrac{1}{24} - \tfrac{1}{35} + \tfrac{1}{48} - \text{etc.}$$

Nunc ad valorem huius feriei inueftigandum ftatuatur λ vnitati tantum proxime aequale, ponendo $\lambda = 1 - \omega$, eritque

$$\mathrm{fin.}\,\lambda\pi = \mathrm{fin.}(\pi - \pi\omega) = \mathrm{fin.}\,\pi\omega;\ \text{eft vero}$$

$$\mathrm{fin.}\,\pi\omega = \pi\omega - \tfrac{1}{6}\pi^3\omega^3,$$

quo valore fubftituto prodibit

$$\frac{1}{2(1-\omega)\omega(1-\frac{1}{6}\pi^2\omega^2)} - \frac{1}{2(1-\omega)^2} - \frac{1}{2\omega - \omega\omega}.$$

Primum autem membrum

$$\frac{1}{2(1-\omega)\omega(1-\frac{1}{6}\pi^2\omega^2)},\ \text{ob}$$

$$\frac{1}{1-\omega} = 1 + \omega + \omega^2 \ \text{et}$$

$$\frac{1}{1-\frac{1}{6}\pi^2\omega^2} = 1 + \tfrac{1}{6}\pi^2\omega^2,$$

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negligendo poteftates ipfius ω quadrato altiores, transmutatur in hanc formam:

$$\tfrac{1}{2\omega}(1 + \omega + \omega^2 + \tfrac{1}{6}\pi\pi\omega\omega);$$

tertium autem membrum

$$-\frac{1}{2\omega(1-\frac{1}{2}\omega)},\ \text{ob}\ \frac{1}{1-\frac{1}{2}\omega} = 1 + \tfrac{1}{2}\omega + \tfrac{1}{4}\omega\omega$$

abit in

$$-\tfrac{1}{2\omega}(1 + \tfrac{1}{2}\omega + \tfrac{1}{4}\omega\omega),$$

vnde primum et tertium membrum fimul faciunt

$$\tfrac{1}{2\omega}(\tfrac{1}{2}\omega + \tfrac{3}{4}\omega\omega + \tfrac{1}{6}\pi\pi\omega\omega) = \tfrac{1}{4} + \tfrac{3}{8}\omega + \tfrac{1}{12}\pi\pi\omega$$

qui valor pofito $\omega = 0$ fit $= \frac{1}{4}$, vnde fecundum membrum, quod erit $-\frac{1}{2}$, iunctum dabit totam fummam quaefitam $-\frac{1}{4}$, ita vt fit mutatis fignis

$$\tfrac{1}{4} = \tfrac{1}{3} - \tfrac{1}{8} + \tfrac{1}{15} - \tfrac{1}{24} + \tfrac{1}{35} - \tfrac{1}{48} + \tfrac{1}{63}\ \text{etc.}$$

cuius ratio eft manifefta, cum fit

$$\tfrac{1}{3} = \tfrac{1}{2}(1 - \tfrac{1}{3});\ \tfrac{1}{8} = \tfrac{1}{2}(\tfrac{1}{2} - \tfrac{1}{4});\ \tfrac{1}{15} = \tfrac{1}{2}(\tfrac{1}{3} - \tfrac{1}{5});\ \tfrac{1}{24} = \tfrac{1}{2}(\tfrac{1}{4} - \tfrac{1}{6});\ \text{etc.}$$

his enim valoribus fubftitutis et fublatis terminis fe deftruentibus fiet $\frac{1}{4} = \frac{1}{2} - \frac{1}{2 \cdot 2} = \frac{1}{4}$.

§. 10. Circa eandem autem feriem quaeftio magis ardua occurrit, qua quaeritur fumma feriei, fi $\lambda\lambda$ fuerit numerus negatiuus, ideoque λ quantitas imaginaria. Ponatur igitur $\lambda\lambda = -\mu\mu$, fiue $\lambda = \mu\sqrt{-1}$, ac feries nihilo minus erit realis, fcilicet

$$\frac{1}{1+\mu\mu} - \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} - \frac{1}{16+\mu\mu} + \frac{1}{25+\mu\mu} - \text{etc}$$

cuius ergo fumma erit

$$\frac{\pi}{2\mu\sqrt{-1}\,\mathrm{fin.}\,\pi\mu\sqrt{-1}} + \frac{1}{2\mu\mu},$$

cuius

cuius ergo valor realis quaeritur, ſiquidem nullum eſt dubium quin ſeriei valor fiat realis.

§. 11. In doctrina angulorum oſtendi ſolet eſſe

$$\text{ſin.}\,\varphi = \frac{e^{\varphi\sqrt{-1}} - e^{-\varphi\sqrt{-1}}}{2\sqrt{-1}}.$$

Fiat igitur $\varphi = \mu\pi\sqrt{-1}$, eritque

$$\varphi\sqrt{-1} = -\mu\pi \text{ et } -\varphi\pi\sqrt{-1} + = \mu\pi,$$

vnde concluditur

$$\text{ſin.}\,\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}$$

vnde ſumma quaeſita erit

$$\frac{\pi}{\mu(e^{-\mu\pi} - e^{+\mu\pi})} + \frac{1}{2\mu\mu}.$$

Erit igitur

$$\frac{1}{1+\mu\mu} - \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} - \frac{1}{16+\mu\mu} + \frac{1}{25+\mu\mu} - \text{etc.}$$
$$= \frac{1}{2\mu^2} - \frac{\pi}{\mu(e^{\mu\pi} - e^{-\mu\pi})}.$$

2°. Sit $P = \varphi$ et fractio propoſita $\frac{\varphi}{\text{ſin.}\,\varphi}$.

§. 12. Hic ergo ob numeratorem $P = \varphi$ factor denominatoris primus φ tollitur, quemadmodum etiam noſtra reſolutio numeratorem ipſi reſpondentem nihilo praebet aequalem. Hic igitur pro denominatore $\varphi - i\pi$ fit numerator $\frac{i\pi}{\text{coſ.}\, i\pi} = \pm i\pi$, vbi ſignum ſuperius valet ſi i numerus par, inferius ſi impar. Quod ſi ergo fuerit $i = 2n$, fractio inde nata erit $\frac{2n\pi}{\varphi - 2n\pi}$; at ſi $i = -2n$, fractio erit

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— ... um eſt du- deniq ... venta ... ſolet eſſe ſiue vnde ... $\mu\pi$, quae Ac ſ vnde ... etc. atque dedu ... $\frac{\varphi}{\text{ſ}.\varphi}$. $= \varphi$ factor etiam no- ... nihilo praebet ... r fit nume- erit ... et ſi i nu- pari ... erit $i = 2n$, ſcriba ... fractio erit — 2

$-\frac{2n\pi}{\varphi + 2n\pi}$, at ſi fuerit $i = 2n - 1$, fractio erit $-\frac{(2n-1)\pi}{\varphi - (2n-1)\pi}$; denique ex $i = -2n - 1$ oritur $\frac{(2n-1)\pi}{\varphi + (2n-1)\pi}$, quocirca ſeries inventa erit

$$\frac{\varphi}{\text{ſin.}\,\varphi} = -\frac{\pi}{\varphi - \pi} + \frac{\pi}{\varphi + \pi} + \frac{2\pi}{\varphi - 2\pi} - \frac{2\pi}{\varphi + 2\pi} - \frac{3\pi}{\varphi - 3\pi} + \frac{3\pi}{\varphi + 3\pi} - \frac{4\pi}{\varphi + 4\pi} + \text{etc.}$$

ſiue

$$\frac{\varphi}{\text{ſin.}\,\varphi} = \frac{\pi}{\pi - \varphi} + \frac{\pi}{\pi + \varphi} - \frac{2\pi}{2\pi - \varphi} - \frac{2\pi}{2\pi + \varphi} + \frac{3\pi}{3\pi - \varphi} + \frac{3\pi}{3\pi + \varphi} - \frac{4\pi}{4\pi - \varphi} - \text{etc.}$$

vnde ſi bini termini in vnum contrahantur, erit

$$\frac{\varphi}{\text{ſin.}\,\varphi} = \frac{2\pi\pi}{\pi\pi - \varphi\varphi} - \frac{8\pi\pi}{4\pi\pi - \varphi\varphi} + \frac{18\pi\pi}{9\pi\pi - \varphi\varphi} - \frac{32\pi\pi}{16\pi\pi - \varphi\varphi} + \text{etc.}$$

quae per $2\pi\pi$ diuiſa producit hanc ſummationem:

$$\frac{\varphi}{2\pi\pi\,\text{ſin.}\,\varphi} = \frac{1}{\pi\pi - \varphi\varphi} - \frac{4}{4\pi\pi - \varphi\varphi} + \frac{9}{9\pi\pi - \varphi\varphi} - \frac{16}{16\pi\pi - \varphi\varphi} + \text{etc.}$$

Ac ſi ponatur $\varphi = \lambda\pi$, prodibit

$$\frac{\lambda\pi}{2\,\text{ſin.}\,\lambda\pi} = \frac{1}{1 - \lambda\lambda} - \frac{4}{4 - \lambda\lambda} + \frac{9}{9 - \lambda\lambda} - \frac{16}{16 - \lambda\lambda} + \text{etc.}$$

vnde ſi fuerit $\lambda\lambda = -\mu\mu$, ſeu $\lambda = \mu\sqrt{-1}$, ob

$$\text{ſin.}\,\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}, \text{ erit}$$

$$\frac{\mu\pi}{e^{\mu\pi} - e^{-\mu\pi}} = \frac{1}{1+\mu\mu} - \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} - \frac{1}{16+\mu\mu} + \text{etc.}$$

atque hinc per differentiationem infinitas alias ſummationes deducere licebit.

3°. Sit numerator $P = \varphi^2$ et fractio $\frac{\varphi\varphi}{\text{ſin.}\,\varphi}$.

§. 13. Pro denominatore igitur $\varphi - i\pi$ numerator erit $\pm ii\pi\pi$, vbi ſignum ſuperius valet pro i numero pari, inferius vero pro impari. Hinc ſi loco i ſucceſſiue ſcribantur numeri

$$+1 - 1 + 2 - 2 + 3 - 3 + \text{etc.}$$

ſeries

feries refultans erit

$$\frac{\phi\phi}{\text{fin.}\,\phi} = -\frac{\pi\pi}{\phi-\pi} - \frac{\pi\pi}{\phi+\pi} + \frac{4\pi\pi}{\phi-2\pi} + \frac{4\pi\pi}{\phi+2\pi} - \frac{9\pi\pi}{\phi-3\pi} - \frac{9\pi\pi}{\phi+3\pi} + \text{etc.}$$

fiue

$$\frac{\phi\phi}{\text{fin.}\,\phi} = \frac{\pi\pi}{\pi-\phi} - \frac{\pi\pi}{\pi+\phi} - \frac{4\pi\pi}{2\pi-\phi} + \frac{4\pi\pi}{2\pi+\phi} + \frac{9\pi\pi}{3\pi-\phi} - \frac{9\pi\pi}{3\pi+\phi} - \text{etc.}$$

Contractis igitur binis terminis fiet

$$\frac{\phi\phi}{\text{fin.}\,\phi} = \frac{2\pi\pi\phi}{\pi\pi-\phi\phi} - \frac{8\pi\pi\phi}{4\pi\pi-\phi\phi} + \frac{18\pi\pi\phi}{9\pi\pi-\phi\phi} - \frac{32\pi\pi\phi}{16\pi\pi-\phi\phi}, \text{ fiue}$$

$$\frac{\phi}{2\,\text{fin.}\,\phi} = \frac{\pi\pi}{\pi\pi-\phi\phi} - \frac{4\pi\pi}{4\pi\pi-\phi\phi} + \frac{9\pi\pi}{9\pi\pi-\phi\phi} - \frac{16\pi\pi}{16\pi\pi-\phi\phi} + \text{etc.}$$

Quod fi nunc quilibet terminus huius feriei in duas partes difcerpatur, quarum prior femper eft 1, binae fequentes feries nafcentur:

$$\frac{\phi}{2\,\text{fin.}\,\phi} = \begin{cases} +1 \quad -1 \quad +1 \quad -1 \quad +1 \quad -\text{etc.} \\ +\frac{\phi\phi}{\pi\pi-\phi\phi} - \frac{\phi\phi}{4\pi\pi-\phi\phi} + \frac{\phi\phi}{9\pi\pi-\phi\phi} - \frac{\phi\phi}{16\pi\pi-\phi\phi} + \frac{\phi\phi}{25\pi\pi-\phi\phi} - \text{etc.} \end{cases}$$

Notum autem eft, feriei

$$+1-1+1-1+1-1 \text{ etc.}$$

fummam effe $= \frac{1}{2}$, qua ad alteram partem translata et per $\phi\phi$ diuifa prodibit

$$\frac{1}{2\phi\,\text{fin.}\,\phi} - \frac{1}{2\phi\phi} = \frac{1}{\pi\pi-\phi\phi} - \frac{1}{4\pi\pi-\phi\phi} + \frac{1}{9\pi\pi-\phi\phi} - \text{etc.}$$

quae prorfus conuenit cum ferie in §. 5 inuenta.

4°. Sit $P = \phi^\gamma$, denotante γ numerum imparem quemcunque pofitiuum; vt fractio propofita fit

$$\frac{\phi^\gamma}{\text{fin.}\,\phi}.$$

§. 14. Cum igitur pro denominatore $\phi - i\pi$ fiat $A = i^\gamma \pi^\gamma$ et $C = +1$, erit numerator $\pm i^\gamma \pi^\gamma$, vnde cum γ fit numerus impar, figna noftrorum terminorum eadem

dem lege procedent atque in cafu $P = \phi$, vbi $\gamma = 1$; vnde feries hinc nata erit

$$\frac{\phi^\gamma}{\text{fin.}\,\phi} = \frac{\pi^\gamma}{\pi-\phi} + \frac{\pi^\gamma}{\pi+\phi} - \frac{2^\gamma\pi^\gamma}{2\pi-\phi} - \frac{2^\gamma\pi^\gamma}{2\pi+\phi} + \frac{3^\gamma\pi^\gamma}{3\pi-\phi} + \frac{3^\gamma\pi^\gamma}{3\pi+\phi} - \text{etc.}$$

quae per π^γ diuifa praebet

$$\frac{\phi^\gamma}{\pi^\gamma\,\text{fin.}\,\phi} = \frac{1}{\pi-\phi} + \frac{1}{\pi+\phi} - \frac{2^\gamma}{2\pi-\phi} - \frac{2^\gamma}{2\pi+\phi} + \frac{3^\gamma}{3\pi-\phi} + \frac{3^\gamma}{3\pi+\phi} - \text{etc.}$$

et binis terminis contractis erit

$$\frac{\phi^\gamma}{2\pi^{\gamma+1}\,\text{fin.}\,\phi} = \frac{1}{\pi\pi-\phi\phi} - \frac{2^\gamma}{4\pi\pi-\phi\phi} + \frac{3^\gamma}{9\pi\pi-\phi\phi} - \frac{4^\gamma}{16\pi\pi-\phi\phi} + \text{etc.}$$

Statuamus nunc $\phi = \lambda\pi$ eritque

$$\frac{\lambda^\gamma\pi}{2\,\text{fin.}\,\phi\pi} = \frac{1}{1-\lambda\lambda} - \frac{2^\gamma}{4-\lambda\lambda} + \frac{3^\gamma}{9-\lambda\lambda} - \frac{4^\gamma}{16-\lambda\lambda} + \text{etc.}$$

§. 15. Hinc fi fuerit $\lambda = \mu\sqrt{-1}$, erit vt ante

$$\text{fin.}\,\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}.$$

Pro valore autem poteftatis λ^γ duos cafus euolui oportet, prouti fuerit $\gamma = 4n+1$ vel $\gamma = 4n-1$. Priore cafu erit

$$\lambda^\gamma = (\mu\sqrt{-1})^{4n+1} = (\mu\sqrt{-1})^{4n} \times \mu\sqrt{-1}.$$

 Eft

Eſt vero $(\mu\sqrt{-1})^{4n} = \mu^{4n}$, vnde erit $\lambda^{\gamma} = \mu^{4n+1}\sqrt{-1}$, hincque prodit ſequens ſummatio realis:

$$\frac{\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}} = \frac{1^{4n+1}}{1+\mu\mu} - \frac{2^{4n+1}}{4+\mu\mu} + \frac{3^{4n+1}}{9+\mu\mu} - \frac{4^{4n+1}}{16+\mu\mu} + \text{etc.}$$

Altero autem caſu, quo $\gamma = 4n-1$, prius membrum capi debet negatiue, eritque

$$-\frac{\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}} = \frac{1}{1+\mu\mu} - \frac{2^{4n-1}}{4+\mu\mu} + \frac{4^{4n-1}}{9+\mu\mu} - \text{etc.}$$

Has autem ſummationes facile patet veras eſſe non poſſe, niſi γ ſit numerus integer impar et quidem poſitiuus.

5. Sit numerator $P = \phi^{\delta}$, denotante δ numerum parem poſitiuum quemcunque, et fractio $\frac{\phi^{\delta}}{\text{fin.}\,\phi}$.

§. 15. Pro denominatore ergo $\phi - i\pi$ numerator erit $\pm i^{\delta}\pi^{\delta}$, ambiguitate ſignorum eandem legem tenente. Hoc igitur caſu ratio ſignorum perinde ſe habebit ac caſu $P = \phi\phi$, eritque idcirco.

$$\frac{\phi^{\delta}}{\text{fin.}\,\phi} = \frac{\phi^{\delta}}{\phi} - \frac{\pi^{\delta}}{\pi-\phi} - \frac{\pi^{\delta}}{\pi+\phi} - \frac{2^{\delta}\pi^{\delta}}{2\pi-\phi} + \frac{2^{\delta}\pi^{\delta}}{2\pi+\phi} + \frac{3^{\delta}\pi^{\delta}}{3\pi-\phi} - \frac{3^{\delta}\pi^{\delta}}{3\pi+\phi} + \text{etc.}$$

Quare ſi ponamus $\phi = \lambda\pi$ erit haec ſeries

$$\frac{\lambda^{\delta}\pi}{\text{fin.}\lambda\pi} = \frac{1}{1-\lambda} - \frac{1}{1+\lambda} - \frac{2^{\delta}}{2-\lambda} + \frac{2^{\delta}}{2+\lambda} + \frac{3^{\delta}}{3-\lambda} - \frac{3^{\delta}}{3+\lambda} - \text{etc.}$$

hinc binis terminis in vnum contrahendis fiet

$$\frac{\lambda^{\delta-1}\pi}{2\,\text{fin.}\,\lambda\pi} = \frac{1^{\delta}}{1-\lambda\lambda} - \frac{2^{\delta}}{4-\lambda\lambda} + \frac{3^{\delta}}{3-\lambda\lambda} - \frac{4^{\delta}}{4-\lambda\lambda} + \text{etc.}$$

§. 16.

√−1, + etc. um capi — etc. oſſe, niſi um pa- φ. umerator tenente. ac caſu 3^δ π^δ / 3π−φ + etc. +λ — etc. λλ + etc. §. 16.

§. 16. Statuamus nunc etiam $\lambda = \mu\sqrt{-1}$, vt ſit

$$\text{fin.}\,\lambda\pi = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}.$$

Pro valore autem ipſius $\lambda^{\delta-1}$ duos iterum caſus euolui oportet, prout fuerit vel $\delta = 4n$, vel $\delta = 4n+2$. Priore caſu, quo $\delta = 4n$, erit $\lambda^{4n} = \mu^{4n}$, ideoque $\lambda^{4n-1} = \frac{\mu^{4n-1}}{\sqrt{-1}}$; atque hinc orietur iſta ſummatio:

$$\frac{-\mu^{4n-1}\pi}{e^{\mu\pi}-e^{-\mu\pi}} = \frac{1^{4n}}{1+\mu\mu} - \frac{2^{4n}}{4+\mu\mu} + \frac{3^{4n}}{9+\mu\mu} - \frac{4^{4n}}{16+\mu\mu} + \text{etc.}$$

Pro altero autem caſu $\delta = 4n+2$ ſummatio ita ſe habebit:

$$\frac{+\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}} = \frac{1^{4n+2}}{1+\mu\mu} - \frac{2^{4n+2}}{4+\mu\mu} + \frac{3^{4n+2}}{9+\mu\mu} - \text{etc.}$$

§. 17. Hae autem ſummationes eatenus tantum veritati erunt conſentaneae, quatenus pro exponentibus γ et δ numeri integri, prouti ſunt definiti, accipiantur, nihilque impedit quo minus quantumuis magni aſſumantur. Cum enim denominator

$$\text{fin.}\,\phi = \phi - \tfrac{1}{6}\phi^3 + \tfrac{1}{120}\phi^5 - \text{etc.}$$

ad dimenſiones infinitas ipſius ϕ aſſurgat, dummodo maxima poteſtas in numeratore non fiat infinita, reſolutio in fractiones ſemper ad veritatem perducit. Sin autem exponentes illi non eſſent integri poſitiui, ſed fracti, vel adeo negatiui, reſolutio in fractiones partiales locum plane habere nequit. Quamobrem ſi loco numeratoris P eiusmodi functiones ipſius ϕ ſtatuamus, quae etiam ad infinitum dimenſionum numerum adſurgant, tum de ſumma inuenta non amplius erimus

mus certi. Verum fieri poteſt, vt ad fractiones partiales inuentas inſuper quaedam partes integrae adiici debeant. Huiusmodi igitur caſus aliquos euoluamus.

6°. Sit numerator $P = \cos.\varphi$ et fractio $= \frac{\cos.\varphi}{\sin.\varphi}$.

§. 18. Cum ſit

$$\cos.\varphi = 1 - \tfrac{1}{2}\varphi\varphi + \tfrac{1}{24}\varphi^4 - \tfrac{1}{720}\varphi^6 + \text{etc.}$$

poteſtates ipſius φ in numeratore aeque in infinitum exſurgunt atque in denominatore; vnde fieri poſſet, vt haec fractio partem integram inuolueret, quae cum reperiatur ſi ſumatur $\varphi = \infty$, foret iſta pars integra $= \frac{\cos.\infty}{\sin.\infty} = \cot.\infty$, quae autem in ſe prorſus eſt indeterminata. Interim tamen, quia totidem caſibus euadere poteſt negatiua atque poſitiua, medium ſumendo valor recte videri poteſt $= 0$; ceterum dubium per ſequentem euolutionem tolletur. Cum pro denominatore $\varphi - i\pi$ fiat $A = \cos.i\pi$ et $C = \cos.i\pi$, erit numerator huius fractionis $= 1$; hinc ergo naſcetur ſequens feries:

$$\frac{\cos.\varphi}{\sin.\varphi} = \frac{1}{\varphi} + \frac{1}{\varphi-\pi} + \frac{1}{\varphi+\pi} + \frac{1}{\varphi-2\pi} + \frac{1}{\varphi+2\pi} + \text{etc. ſiue.}$$

$$\text{cotag.}\,\varphi = \frac{1}{\varphi} - \frac{1}{\pi-\varphi} + \frac{1}{\pi+\varphi} - \frac{1}{2\pi-\varphi} + \frac{1}{2\pi+\varphi} - \text{etc.}$$

Poſito igitur $\varphi = \lambda\pi$, haec ſeries induet hanc formam:

$$\pi\cot.\lambda\pi = \frac{1}{\lambda} - \frac{1}{1-\lambda} + \frac{1}{1+\lambda} - \frac{1}{2-\lambda} + \frac{1}{2+\lambda} - \frac{1}{3-\lambda} + \text{etc.}$$

quae ſummatio an vera ſit per caſus inueſtigemus. Ac primo quidem ſi λ denotet numerum integrum, veritas confirmatur; ſemper enim aliquis ſeriei terminus ſit infinitus; ſumma vero quoque ſit infinita. Sumamus autem $\lambda = \frac{1}{2}$, erit $\pi\cot.\frac{\pi}{2} = 0$, ipſa autem ſeries prodit

$$\tfrac{2}{1} - \tfrac{2}{1} + \tfrac{2}{3} - \tfrac{2}{3} + \tfrac{2}{5} - \tfrac{2}{5} + \text{etc.}$$

vbi

ones partiales
diici debeant.
$) = \frac{\cos.\varphi}{\sin.\varphi}$.
etc.
cum exſurgunt
fractio partem
ſumatur $\varphi = \infty$,
e autem in ſe
ia totidem ca-
medium ſumen-
dubium per
denominato-
erit numera-
ſequens feries:
$\frac{1}{+2\pi}$ + etc. ſiue.
$\frac{1}{2\pi+\varphi}$ − etc.
ic formam:
$- \frac{1}{3-\lambda}$ + etc.
mus. Ac primo
ritas confirma-
ſit infinitus;
tem $\lambda = \frac{1}{2}$, erit
vbi

vbi omnes termini ſe manifeſto deſtruunt. Sumamus autem inſuper $\lambda = \frac{1}{4}$, prodibitque

$$\pi = \tfrac{4}{1} - \tfrac{4}{3} + \tfrac{4}{5} - \tfrac{4}{7} + \tfrac{4}{9} - \tfrac{4}{11} + \text{etc.}$$

quae eſt ſeries notiſſima Leibniziana. Sicque omne dubium circa veritatem huius ſummationis euaneſcit.

§. 19. Contrahamus binos terminos, primo excepto, in ſingulos, et obtinebimus

$$\pi\cot.\lambda\pi = \frac{1}{\lambda} - \frac{2\lambda}{1-\lambda\lambda} - \frac{2\lambda}{4-\lambda\lambda} - \frac{2\lambda}{9-\lambda\lambda} - \frac{2\lambda}{16-\lambda\lambda} - \text{etc.}$$

quae ſeries reducitur ad hanc formam:

$$\frac{1}{2\lambda\lambda} - \frac{\pi\cot.\lambda\pi}{2\lambda} = \frac{1}{1-\lambda\lambda} + \frac{1}{4-\lambda\lambda} + \frac{1}{9-\lambda\lambda} + \frac{1}{16-\lambda\lambda} + \text{etc.}$$

Quod ſi hic iterum ſtatuamus $\lambda = \mu\sqrt{-1}$, ob

$$\cos.\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} + e^{\mu\pi}}{2} \text{ et}$$

$$\sin.\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}},$$

haec obtinebitur ſummatio:

$$-\frac{1}{2\mu\mu} + \frac{\pi(e^{+\mu\pi} + e^{-\mu\pi})}{(e^{\mu\pi} - e^{-\mu\pi})2\mu} = \frac{1}{1+\mu\mu} + \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} + \frac{1}{16+\mu\mu} + \text{etc.}$$

Nunc autem per ſe eſt manifeſtum, per differentiationem ſimili modo vt ſupra infinitas alias ſummationes obtineri poſſe.

II.

II. Sumatur $Q = \cos.\zeta - \cos.\phi$, vt fractio resoluenda fit $\frac{P}{\cos.\zeta - \cos.\phi}$.

§. 20. Cum sit denominator $Q = \cos.\zeta - \cos.\phi$, vbi angulus ζ vt datus et constans spectatur, is sequentibus casibus euanescit,

$$\phi = \pm\zeta;\ \phi = \pm 2\pi \pm \zeta;\ \phi = \pm 4\pi \pm \zeta;$$
$$\phi = \pm 6\pi \pm \zeta;\ \phi = \pm 8\pi \pm \zeta;\ \text{etc.}$$

ideoque in genere $\phi = \pm i\pi \pm \zeta$, vbi i denotat omnes numeros pares tam negatiuos quam positiuos; vnde denominatores fractionum simplicium quas quaerimus erunt

$$\phi-\zeta,\ \phi+\zeta,\ \phi-2\pi-\zeta,\ \phi-2\pi+\zeta,\ \phi+2\pi-\zeta,\ \phi+2\pi+\zeta,\ \text{etc.}$$

hocque modo omnes fractiones simplices reperiemus, quarum omnium summa aequalis esse debet fractioni propositae $\frac{P}{\cos.\zeta - \cos.\phi}$.

§. 21. Consideremus nunc primo denominatorem simplicem in genere $\phi - i\pi - \zeta$, ac posito $\phi = i\pi + \zeta$ abeat numerator P in A. Deinde cum ex denominatore fiat $\frac{dQ}{d\phi} = \sin.\phi$, erit $C = \sin.(i\pi + \zeta) = \sin.\zeta$, vnde numerator huius fractionis erit $\frac{A}{C} = \frac{A}{\sin.\zeta}$, ideoque fractio hinc nata $\frac{A}{\sin.\zeta(\phi - i\pi - \zeta)}$. At vero pro denominatore $\phi - i\pi + \zeta$, si in numeratore P ponatur $\phi = i\pi - \zeta$, prodit quantitas B; ex denominatore autem fiet

$$C = \sin.(i\pi - \zeta) = -\sin.\zeta,$$

vnde orietur ista fractio: $-\frac{B}{\sin.\zeta(\phi - i\pi + \zeta)}$. Nunc igitur tantum opus est vt loco i successiue omnes numeri pares tam positiui quam negatiui substituantur.

1°.

oluenda
– cos. φ, quentibus
± ζ;
omnes nu- lenomina-
π+ζ, etc.
ius, qua- propositae
minatorem π + ζ ab- natore fiat numerator hinc nata
- i π + ζ, t quantitas
Nunc igi- numeri pa-
1°.

1°. Sit numerator $P = 1$ et fractio proposita $\frac{1}{\cos.\zeta - \cos.\phi}$.

§. 22. Pro binis igitur formulis generalibus erit tam $A = 1$ quam $B = 1$, vnde istae fractiones generales erunt

$$\frac{1}{\sin.\zeta(\phi - i\pi - \zeta)} - \frac{1}{\sin.\zeta(\phi - i\pi + \zeta)} = \frac{2\zeta}{\sin.\zeta((\phi - i\pi)^2 - \zeta\zeta)},$$

consequenter hinc deducemus sequentem summationem:

$$\frac{1}{\cos.\zeta - \cos.\phi} = \frac{2\zeta}{\sin.\zeta(\phi\phi - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\phi - 2\pi)^2 - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\phi + 2\pi)^2 - \zeta\zeta)}$$
$$+ \frac{2\zeta}{\sin.\zeta((\phi - 4\pi)^2 - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\phi + 4\pi)^2 - \zeta\zeta)} + \text{etc.}$$

siue habebimus

$$\frac{\sin.\zeta}{2\zeta(\cos.\zeta - \cos.\phi)} = \frac{1}{\phi\phi - \zeta\zeta} + \frac{1}{(\phi - 2\pi)^2 - \zeta\zeta} + \frac{1}{(\phi + 2\pi)^2 - \zeta\zeta}$$
$$+ \frac{1}{(\phi - 4\pi)^2 - \zeta\zeta} + \frac{1}{(\phi + 4\pi)^2 - \zeta\zeta} + \text{etc.}$$

§. 23. Quod si ergo fuerit $\zeta = 0$, erit

$$\frac{1}{2 - 2\cos.\phi} = \frac{1}{\phi^2} + \frac{1}{(\phi - 2\pi)^2} + \frac{1}{(\phi + 2\pi)^2} + \frac{1}{(\phi - 4\pi)^2} + \frac{1}{(\phi + 4\pi)^2} + \text{etc.}$$

Sit nunc porro $\phi = \frac{\pi}{2}$, erit haec summatio:

$$\frac{\pi\pi}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{121} + \text{etc.}$$

vti quidem satis constat. Ponamus porro $\Phi = \pi$, prodibitque haec summatio:

$$\frac{\pi\pi}{4} = 1 + 1 + \frac{1}{9} + \frac{1}{9} + \frac{1}{25} + \frac{1}{25} + \frac{1}{49} + \frac{1}{49} + \text{etc.}$$

quae series cum praecedente congruit. Sin autem ponamus $\Phi = \lambda\pi$, erit

$$\frac{\pi\pi}{2(1 - \cos.\lambda\pi)} = \frac{1}{\lambda\lambda} + \frac{1}{(\lambda - 2)^2} + \frac{1}{(\lambda + 2)^2} + \frac{1}{(\lambda - 4)^2} + \frac{1}{(\lambda + 4)^2} + \text{etc.}$$

quae summa etiam est $\frac{\pi\pi}{4(\sin.\frac{1}{2}\lambda\pi)^2}$.

§. 24.

§. 24. Ponamus autem in genere $\zeta = \alpha\pi$ et $\Phi = \lambda\pi$, vt obtineatur iſta ſummatio:

$$\frac{\pi \sin. \alpha\pi}{2\alpha(\cos. \alpha\pi - \cos. \lambda\pi)} = \frac{1}{\lambda\lambda - \alpha\alpha} + \frac{1}{(\lambda-2)^2 - \alpha\alpha} + \frac{1}{(\lambda+2)^2 - \alpha\alpha} + \frac{1}{(\lambda-4)^2 - \alpha\alpha} + \frac{1}{(\lambda+4)^2 - \alpha\alpha} + \text{etc.}$$

Quod ſi iam α fuerit quantitas imaginaria, ſiue $\alpha = \beta\sqrt{-1}$, ſummatio haec erit

$$\frac{\pi(e^{\beta\pi} - e^{-\beta\pi})}{2\alpha(e^{\beta\pi} + e^{-\beta\pi}) - 2\cos. \lambda\pi} = \frac{1}{\lambda\lambda + \beta\beta} + \frac{1}{(\lambda-2)^2 + \beta\beta} + \frac{1}{(\lambda+2)^2 + \beta\beta} + \frac{1}{(\lambda-4)^2 + \beta\beta} + \frac{1}{(\lambda+4)^2 + \beta\beta} + \text{etc.}$$

§. 25. Hinc ſi proponatur haec fractio in ſeriem reſoluenda: $\frac{1}{a - \cos. \Phi}$, ſiue $\frac{1}{a - \cos. \lambda\pi}$, duos caſus conſiderari oportet, prouti a fuerit vel vnitate minor vel maior. Sit $a < 1$, vt fieri queat $a = \cos. \alpha\pi$, vnde fit $\alpha = \frac{A \cos. a}{\pi}$, atque invento α reperietur

$$\frac{1}{a - \cos. \lambda\pi} = \frac{2\alpha}{\pi\sqrt{(1-aa)}}\left(\frac{1}{\lambda\lambda - \alpha\alpha} + \frac{1}{(\lambda-2)^2 - \alpha\alpha} + \frac{1}{(\lambda+2)^2 - \alpha\alpha} + \frac{1}{(\lambda-4)^2 - \alpha\alpha}\right) + \text{etc.})$$

Sin autem fuerit $a > 1$, quaeri debet β, vt fiat

$$\frac{e^{\beta\pi} + e^{-\beta\pi}}{2} = a.$$

Hinc ergo fiet $e^{+2\beta\pi} + 1 = 2ae^{\beta\pi}$, vnde radice extracta reperitur $e^{\beta\pi} = a + \sqrt{(aa-1)}$ hincque $e^{-\beta\pi} = a - \sqrt{(aa-1)}$, vnde porro fiet

$$\beta\pi = l(a + \sqrt{(aa-1)}), \text{ ergo}$$

$$\beta = \tfrac{1}{\pi} l(a + \sqrt{(aa-1)}).$$

Inuento igitur hoc numero β poſtrema formula praebet hanc ſeriem:

$$\frac{\pi\sqrt{(aa-1)}}{2\beta(a - \cos. \lambda\pi)} = \frac{1}{\lambda\lambda + \beta\beta} + \frac{1}{(\lambda-2)^2 + \beta\beta} + \frac{1}{(\lambda+2)^2 + \beta\beta} + \frac{1}{(\lambda-4)^2 + \beta\beta} + \text{etc.}$$

conſe-

et $\Phi = \lambda\pi$,
etc.
$= \beta\sqrt{-1}$,
in ſeriem
rari opor-
Sit $a < 1$,
atque in-
) + etc.)
xtracta re-
$\sqrt{(aa-1)}$,
a praebet
+ etc.
conſe-

conſequenter habebimus

$$\frac{1}{a - \cos. \lambda\pi} = \frac{2\beta}{\pi\sqrt{(aa-1)}}\left(\frac{1}{\lambda\lambda + \beta\beta} + \frac{1}{(\lambda-2)^2 + \beta\beta} + \frac{1}{(\lambda+2)^2 + \beta\beta} + \frac{1}{(\lambda-4)^2 + \beta\beta} + \text{etc.}\right)$$

caſu autem medio, quo $a = 1$, fit $\alpha = 0$; tum vero ponatur $a = 1 - \omega$, eritque

$$A \cos. (1 - \omega) = A \sin. \sqrt{(2\omega - \omega\omega)} = \sqrt{(2\omega - \omega\omega)}.$$

Eſt vero etiam

$$\sqrt{(1 - aa)} = \sqrt{(2\omega - \omega\omega)},$$

vnde pro hoc caſu ſeriei ſummatio erit

$$\frac{1}{1 - \cos. \lambda\pi} = \frac{2}{\pi\pi}\left(\frac{1}{\lambda\lambda} + \frac{1}{(\lambda-2)^2} + \frac{1}{(\lambda+2)^2} + \frac{1}{(\lambda-4)^2} + \frac{1}{(\lambda+4)^2} + \text{etc.}\right)$$

Cum igitur ſit

$$1 - \cos. \lambda\pi = 2 \sin. \tfrac{1}{2}\lambda\pi^2,$$

habebimus hanc ſummationem:

$$\frac{\pi\pi}{4 \sin. \frac{1}{2}\lambda\pi^2} = \frac{1}{\lambda\lambda} + \frac{1}{(\lambda-2)^2} + \frac{1}{(\lambda+2)^2} + \frac{1}{(\lambda-4)^2} + \text{etc.}$$

quae ſeries iam ſupra § 23 eſt inuenta.

2°. Sit nunc $P = \sin. \Phi$ et fractio propoſita $\frac{\sin. \Phi}{\cos. \zeta - \cos. \Phi}$.

§. 26. Cum igitur ſit $P = \sin. \Phi$ ſumto, $\Phi = i\pi + \zeta$ erit $A = \sin. (i\pi + \zeta) = \sin. \zeta$; at poſito $\Phi = i\pi - \zeta$ prodit $B = -\sin. \zeta$; hinc binae fractiones inde reſultantes erunt

$$\frac{1}{\Phi - i\pi - \zeta} + \frac{1}{\Phi - i\pi + \zeta} = \frac{2\Phi - 2i\pi}{(\Phi - i\pi)^2 - \zeta\zeta}.$$

Quare ſi loco i ſucceſſiue omnes eius valores ſcribamus, nanciſcemur ſequentem ſeriem:

$$\frac{\sin. \Phi}{\cos. \zeta - \cos. \Phi} = \frac{2\Phi}{\Phi\Phi - \zeta\zeta} + \frac{2(\Phi - 2\pi)}{(\Phi - 2\pi)^2 - \zeta\zeta} + \frac{2(\Phi + 2\pi)}{(\Phi + 2\pi)^2 - \zeta\zeta} + \frac{2(\Phi - 4\pi)}{(\Phi - 4\pi)^2 - \zeta\zeta} + \text{etc.}$$

 ſiue

ſiue

$$\frac{\sin.\Phi}{2(\cos.\zeta-\cos.\Phi)}=\frac{\Phi}{\Phi\Phi-\zeta\zeta}+\frac{\Phi-2\pi}{(\Phi-2\pi)^2-\zeta\zeta}+\frac{\Phi+2\pi}{(\Phi+2\pi)^2-\zeta\zeta}+\frac{\Phi-4\pi}{(\Phi-4\pi)^2-\zeta\zeta}+\text{etc.}$$

§. 27. Hinc ſi fuerit $\zeta=0$, erit

$$\frac{\sin.\Phi}{2(1-\cos.\Phi)}=\frac{1}{\Phi}+\frac{1}{\Phi-2\pi}+\frac{1}{\Phi+2\pi}+\frac{1}{\Phi-4\pi}+\frac{1}{\Phi+4\pi}+\text{etc.}$$

cuius igitur ſeriei ſumma eſt $\frac{1}{2}\cot.\frac{1}{2}\phi$. Hinc ſi ponamus $\phi=\lambda\pi$, erit

$$\tfrac{1}{2}\pi\cot.\tfrac{1}{2}\lambda\pi=\frac{1}{\lambda}+\frac{1}{\lambda-2}+\frac{1}{\lambda+2}+\frac{1}{\lambda-4}+\frac{1}{\lambda+4}+\text{etc.}$$

et contrahendis binis terminis

$$\tfrac{1}{2}\pi\cot.\tfrac{1}{2}\lambda\pi=\frac{1}{\lambda}+\frac{2\lambda}{\lambda\lambda-4}+\frac{2\lambda}{\lambda\lambda-16}+\frac{2\lambda}{\lambda\lambda-36}+\text{etc.}$$

hincque

$$\frac{1}{2\lambda\lambda}-\frac{\pi\cot.\frac{1}{2}\lambda\pi}{4\lambda}=\frac{1}{4-\lambda\lambda}+\frac{1}{16-\lambda\lambda}+\frac{1}{36-\lambda\lambda}+\text{etc.}$$

Quod ſi hic loco λ ſcribamus 2λ, habebimus

$$\frac{1}{8\lambda\lambda}-\frac{\pi\cot.\lambda\pi}{8\lambda}=\frac{1}{4-4\lambda\lambda}+\frac{1}{16-4\lambda\lambda}+\frac{1}{36-4\lambda\lambda}+\text{etc.}$$

ſiue

$$\frac{1}{2\lambda\lambda}-\frac{\pi\cot.\lambda\pi}{2\lambda}=\frac{1}{1-\lambda\lambda}+\frac{1}{4-\lambda\lambda}+\frac{1}{9-\lambda\lambda}+\frac{1}{16-\lambda\lambda}+\text{etc.}$$

quae ſeries eſt plane eadem, quam ſupra § 19 inuenimus.

§. 28. Ponamus nunc vt ſupra $\zeta=\alpha\pi$ et $\phi=\lambda\pi$, vt obtineatur ſequens ſeries:

$$\frac{\pi\sin.\lambda\pi}{2(\cos.\alpha\pi-\cos.\lambda\pi)}=\frac{\lambda}{(\lambda\lambda-\alpha\alpha)}+\frac{\lambda-2}{(\lambda-2)^2-\alpha\alpha}+\frac{\lambda+2}{(\lambda+2)^2-\alpha\alpha}+\frac{\lambda-4}{(\lambda-4)^2-\alpha\alpha}+\text{etc.}$$

Sin autem hic ponatur $\alpha=\beta\surd-1$, iſta ſeries ſequentem induet formam:

$$\frac{\pi\sin.\lambda\pi}{e^{\beta\pi}+e^{-\beta\pi}-2\cos.\lambda\pi}=\frac{\lambda}{\lambda\lambda+\beta\beta}+\frac{\lambda-2}{(\lambda-2)^2+\beta\beta}+\frac{\lambda+2}{(\lambda+2)^2+\beta\beta}+\frac{\lambda-4}{(\lambda-4)^2+\beta\beta}+\text{etc.}$$

§. 29.

+ etc.
+ etc.
ponamus
+ etc.
+ etc.
+ etc.
+ etc.
+ etc.
inuenimus.
et $\phi=\lambda\pi$,
+ etc.
ſequentem in-
+ etc.
§. 29.

§. 29. Quod ſi igitur propoſita fuerit haec fractio: $\frac{\sin.\Phi}{a-\cos.\Phi}$ ſiue $\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi}$, iterum duos caſus euolui conuenit, alterum quo $a<1$, alterum quo $a>1$. Priore quidem caſu, quo $a<1$, ſtatuatur $\cos.\alpha\pi=a$, vnde fit $\alpha=\frac{A\,\cos.a}{\pi}$, quo inuento erit

$$\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi}=\frac{2}{\pi}\left(\frac{\lambda}{\lambda\lambda-\alpha\alpha}+\frac{\lambda-2}{(\lambda-2)^2-\alpha\alpha}+\frac{\lambda+2}{(\lambda+2)^2-\alpha\alpha}+\text{etc.}\right)$$

Sin autem $a>1$, quaeri debet β, ita vt ſit vt ante

$$\beta=\tfrac{1}{\pi}\,l\,(a+\surd(aa-1))$$

quo valore inuento erit

$$\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi}=\frac{2}{\pi}\left(\frac{\lambda}{\lambda\lambda+\beta\beta}+\frac{\lambda-2}{(\lambda-2)^2+\beta\beta}+\frac{\lambda+2}{(\lambda+2)^2+\beta\beta}+\text{etc.}\right)$$

Sin autem fuerit $a=1$, tum fit tam $\alpha=0$ quam $\beta=0$, eademque ſeries reſultat, quam ſupra ex caſu $\zeta=0$ elicuimus. Hinc ergo ſi ſumatur $\lambda=\frac{1}{2}$ prodibit ſeries

$$\frac{\pi}{2\cos.\alpha\pi}=\frac{2}{1-4\alpha\alpha}-\frac{6}{9-4\alpha\alpha}+\frac{10}{25-4\alpha\alpha}-\frac{14}{49-4\alpha\alpha}+\text{etc.}$$

vel etiam haec:

$$\frac{\pi}{e^{\beta\pi}-e^{-\beta\pi}}=\frac{2}{1+4\beta\beta}-\frac{6}{9+4\beta\beta}+\frac{10}{25+4\beta\beta}-\frac{14}{49+4\beta\beta}+\text{etc.}$$

Ceterum per ſe intelligitur, per differentiationem plurimas alias ſeries formari poſſe.

III. Sit fractio reſoluenda $\frac{1}{\cos.\Phi-\cos.2\Phi}$.

§. 30. Ante omnia igitur hic quaeri debet, quibusnam caſibus iſte denominator euaneſcat. Cum igitur in genere ſit $\cos.\phi=\cos.(i\pi\pm\Phi)$, denotante i numerum parem, ſimilique modo $\cos.2\phi=\cos.(i'\pi+2\Phi)$, habebimus $i\pi\pm\phi=i'\pi\pm2\Phi$, vnde ob ambiguitatem ſignorum ſequentes caſus eruuntur: $\phi=i\pi$, $\phi=\frac{i\pi}{3}$. Hic autem pro-

be

be eſt obſeruandum, caſus priores bis occurrere, ſeu factores hinc natos $\varphi - i\pi$ bis eſſe collocandos, ita vt factor denominatoris ſit $(\varphi - i\pi)^2$. Quod cum minus clare appareat, ita oſtendamus: quoniam in genere eſt

$$\cos a - \cos b = 2 \sin \tfrac{a+b}{2} \sin \tfrac{b-a}{2}$$

erit noſter denominator $2 \sin \frac{1}{2}\varphi \sin \frac{3}{2}\varphi$, qui igitur euaneſcit tam quando $\sin \frac{1}{2}\varphi = 0$ quam quando $\sin \frac{3}{2}\varphi = 0$. Fit autem $\sin \frac{\varphi}{2} = 0$ quoties $\frac{\varphi}{2} = i\pi$, denotante i omnes numeros integros, ideoque $\varphi = 2i\pi$. Similique modo $\sin \frac{3\varphi}{2}$ euaneſcit, ſi $\frac{3\varphi}{2} = i\pi$ ideoque $\varphi = \frac{2i\pi}{3}$, quae poſterior formula, quoties $\frac{i}{3}$ eſt numerus integer, priores caſus ſuppeditat; ſicque manifeſtum eſt, in factoribus occurrere omnia quadrata $(\varphi - i\pi)^2$. Reliqui vero factores $\varphi - \frac{2i\pi}{3}$, quando i per 3 non eſt diuiſibile, erunt ſimplices.

§. 31. Cum igitur formula $(\varphi - 2i\pi)^2$ ſit factor noſtri denominatoris $\cos \varphi - \cos 2\varphi$, ſecundum regulam pro huiusmodi caſibus ſtatuamus

$$\frac{1}{\cos \varphi - \cos 2\varphi} = \frac{\alpha}{(\varphi - 2i\pi)^2} + \frac{\beta}{\varphi - 2i\pi} + R,$$

vbi R complectitur omnes reliquas fractiones. Nunc vtrinque multiplicemus per $(\varphi - 2i\pi)^2$ et habebimus

$$\frac{(\varphi - 2i\pi)^2}{\cos \varphi - \cos 2\varphi} = \alpha + \beta(\varphi - 2i\pi) + R(\varphi - 2i\pi)^2$$

Faciamus $\varphi = 2i\pi$ fietque $\alpha = \frac{(\varphi - 2i\pi)^2}{\cos \varphi - \cos 2\varphi}$, cuius fractionis numerator et denominator euaneſcent, hinc differentialibus ſubſtitutis fiet $\alpha = \frac{2(\varphi - 2i\pi)}{-\sin \varphi + 2 \sin 2\varphi}$, vbi cum numerator et denominator iterum euaneſcant, denuo eorum loco differentialia ſcribantur eritque $\alpha = \frac{2}{-\cos \varphi + 4 \cos 2\varphi}$. Nunc igitur poſito $\varphi = 2i\pi$ reperietur $\alpha = \frac{2}{3}$.

§. 32.

occurrere, ſeu facto-
andos, ita vt factor
m minus clare ap-
nere eſt
, qui igitur euaneſcit
o ſin. $\frac{3}{2}\varphi = 0$. Fit
tante i omnes nume-
nilique modo ſin. $\frac{3\varphi}{2}$
, quae poſterior for-
priores caſus ſuppe-
us occurrere omnia
ctores $\varphi - \frac{2i\pi}{3}$, quan-
implices.
$\varphi - 2i\pi)^2$ ſit factor
, ſecundum regulam
+ R,
ctiones. Nunc vtrin-
t habebimus
) + R $(\varphi - 2i\pi)^2$
cuius fractionis
hinc differentialibus ſub
numerator et denomina-
loco differentialia ſcri-
Nunc igitur poſito
§. 32.

§. 32. Iam in aequatione

$$\frac{(\varphi - 2i\pi)^2}{\cos \varphi - \cos 2\varphi} = \alpha + \beta(\varphi - 2i\pi) + R(\varphi - 2i\pi)^2$$

terminus $\alpha = \frac{2}{3}$ ad alteram partem transferatur et ad eandem denominationem reducatur et reſultabit haec aequatio:

$$\frac{(\varphi - 2i\pi)^2 - \frac{2}{3}(\cos \varphi - \cos 2\varphi)}{\cos \varphi - \cos 2\varphi} = \beta(\varphi - 2i\pi) + R(\varphi - 2i\pi)^2$$

vnde per $\varphi - 2i\pi$ diuidendo fiet

$$\frac{(\varphi - 2i\pi)^2 - \frac{2}{3}(\cos \varphi - \cos 2\varphi)}{(\varphi - 2i\pi)(\cos \varphi - \cos 2\varphi)} = \beta + R(\varphi - 2i\pi)$$

Quod ſi iam ſtatuatur $\varphi = 2i\pi$, β aequabitur fractioni, cuius tam numerator quam denominator ter euaneſcit, ita vt triplici differentiatione ſit opus.

Prima autem differentiatio dabit:

$$\beta = \frac{2(\varphi - 2i\pi) + \frac{2}{3}(\sin \varphi - 2 \sin 2\varphi)}{\cos \varphi - \cos 2\varphi - (\varphi - 2i\pi)(\sin \varphi - 2 \sin 2\varphi)}.$$

Secunda differentiatio dabit:

$$\beta = \frac{2 + \frac{2}{3}(\cos \varphi - 4 \cos 2\varphi)}{-2 \sin \varphi + 4 \sin 2\varphi - (\varphi - 2i\pi)(\cos \varphi - 4 \cos 2\varphi)}.$$

Tertia denique differentiatio dat:

$$\beta = \frac{-\frac{2}{3}(\sin \varphi - 8 \sin 2\varphi)}{-3 \cos \varphi + 12 \cos 2\varphi + (\varphi - 2i\pi)(\sin \varphi - 8 \sin 2\varphi)}.$$

Nunc autem facto $\varphi = 2i\pi$ numerator quidem iterum euaneſcit, denominator vero euadit 9, ita vt ſit $\beta = 0$.

§. 33. At vero iſte valor pro β ſine differentiatione facilius erui poteſt ponendo $\varphi = 2i\pi + \omega$, exiſtente ω infinite paruo; tum autem erit

cos.

$$\text{cos.}\,\Phi = \text{cos.}\,\omega \text{ et cos.}\,2\Phi = \text{cos.}\,2\,\omega;$$

aequatio autem fiet

$$\frac{\omega\omega}{\text{cos.}\,\omega - \text{cos.}\,2\,\omega} = \tfrac{2}{3} + \beta\,\omega + R\,\omega\,\omega.$$

Nunc ambos cosinus proxime exhibeamus vsque ad quartam potestatem ipsius ω procedendo, et cum sit

$$\text{cos.}\,\omega = 1 - \tfrac{1}{2}\,\omega\,\omega + \tfrac{1}{24}\,\omega^4 \text{ et}$$
$$\text{cos.}\,2\,\omega = 1 - 2\,\omega\,\omega + \tfrac{16}{24}\,\omega^4, \text{ erit}$$
$$\text{cos.}\,\omega - \text{cos.}\,2\,\omega = \tfrac{3}{2}\,\omega\,\omega - \tfrac{5}{8}\,\omega^4 = \tfrac{3}{2}\,\omega\,\omega\,(1 - \tfrac{5}{12}\,\omega\,\omega),$$

quo valore substituto habebimus

$$\frac{2}{3\,(1 - \frac{5}{12}\,\omega\,\omega)} = \tfrac{2}{3}(1 + \tfrac{5}{12}\,\omega\,\omega) = \tfrac{2}{3} + \beta\,\omega + R\,\omega\,\omega,$$

hincque fit $\beta = \frac{5}{18}\,\omega$; sicque facto $\omega = 0$ erit etiam $\beta = 0$.

§. 34. Hanc obrem pro denominatoris factore quadrato $(\Phi - 2\,i\,\pi)^2$ ob $\alpha = \frac{2}{3}$ fractio inde nata erit $\frac{2}{3(\Phi - 2\,i\,\pi)^2}$. Pro reliquis autem factoribus simplicibus $\Phi - \frac{2}{3}\,i\,\pi$ statuamus

$$\frac{1}{\text{cos.}\,\Phi - \text{cos.}\,2\,\Phi} = \frac{\alpha}{\Phi - \frac{2}{3}\,i\,\pi} + R,$$

quae aequatio multiplicetur per $\Phi - \frac{2}{3}\,i\,\pi = \omega$, vt prodeat

$$\frac{\omega}{\text{cos.}\,\Phi - \text{cos.}\,2\,\Phi} = \alpha + R\,\omega.$$

Vbi notetur numerum i non esse per 3 diuisibilem, vnde $\frac{2\,i\,\pi}{3}$ sequentes angulos exprimet:

$$\tfrac{2}{3}\pi,\ \tfrac{4}{3}\pi,\ \tfrac{8}{3}\pi,\ \tfrac{10}{3}\pi,\ \tfrac{14}{3}\pi,$$

at anguli $\frac{4\,i\,\pi}{3}$ valores sunt $\frac{4\pi}{3}, \frac{8\pi}{3}, \frac{16\pi}{3}$ quorum angulorum cosinus est idem $-\frac{1}{2}$, sinus autem horum angulorum sunt sin. $\frac{2\,i\,\pi}{3} = \pm\frac{\sqrt{3}}{2}$, vbi signum superius valet, si i sit $3\,n + 1$, inferius vero si fuerit $i = 3\,n + 2$. At vero sin. $\frac{4\,i\,\pi}{3}$ semper est $\mp\frac{\sqrt{3}}{2}$, vbi iterum signum superius valet si

si
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gau
sque ad quar-
n sit
$1 - \frac{5}{12}\,\omega\,\omega)$,
vnd
$+ R\,\omega\,\omega$,
vbi rit etiam $\beta = 0$.
feri
nor oris factore qua-
ara erit $\frac{2}{3(\Phi - 2\,i\,\pi)^2}$.
$- \frac{2}{3}\,i\,\pi$ statuamus
vnd
$\alpha =$
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iuisibilem, vnde
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nur quorum angu-
mei n horum angu-
seq iperius valet, si
$3\,n + 2$. At ve-
um superius valet
si

si $i = 3\,n + 1$, inferius vero si $i = 3\,n + 2$. Haecque regula semper valet, siue n sit numerus positiuus siue negatiuus.

§. 35. His praenotatis erit

$$\text{cos.}\,\varphi = -\tfrac{1}{2}\,\text{cos.}\,\omega \mp \tfrac{\sqrt{3}}{2}\,\text{sin.}\,\omega \text{ et}$$
$$\text{cos.}\,2\,\varphi = -\tfrac{1}{2}\,\text{cos.}\,2\,\omega \pm \tfrac{\sqrt{3}}{2}\,\text{sin.}\,2\,\omega,$$

vnde vero proxime habebimus

$$\text{cos.}\,\Phi = -\tfrac{1}{2}(1 - \tfrac{1}{2}\,\omega\,\omega) \mp \tfrac{\sqrt{3}}{2}\,\omega \text{ et}$$
$$\text{cos.}\,2\,\varphi = -\tfrac{1}{2}(1 - 2\,\omega\,\omega) \pm \tfrac{\sqrt{3}}{2}.\,2\,\omega,$$

vbi perpetuo signa superiora valent si $i = 3\,n + 1$, inferiora autem si $i = 3\,n + 2$. Hinc igitur erit noster denominator

$$\text{cos.}\,\varphi - \text{cos.}\,2\,\varphi = -\tfrac{3}{4}\,\omega\,\omega \mp \tfrac{3\sqrt{3}}{2}.\,\omega$$

vnde fit $\frac{1}{-\frac{3}{4}\,\omega \mp \frac{3\sqrt{3}}{2}} = \alpha$. Posito igitur $\omega = 0$ erit $\alpha = \mp\frac{2}{3\sqrt{3}}$, ita vt ex factore $\varphi - \frac{2\,i\,\pi}{3}$ nascatur ista fractio:

$$\mp\frac{2}{3\sqrt{3}}\left(\frac{1}{\varphi - \frac{2\,i\,\pi}{3}}\right) = \mp\frac{2}{(3\,\varphi - 2\,i\,\pi)\sqrt{3}}.$$

§. 36. Euoluamus igitur primo omnes terminos seriei ex factoribus geminatis $(\varphi - 2\,i\,\pi)^2$ natos, et cum numerator fuisset $\frac{2}{3}$, si loco i successiue omnes scribamus numeros integros tam positiuos quam negatiuos, series orietur sequens:

$$\frac{2}{3\Phi\Phi} + \frac{2}{3(\Phi - 2\pi)^2} + \frac{2}{3(\Phi + 2\pi)^2} + \frac{2}{3(\Phi - 4\pi)^2} + \frac{2}{3(\Phi + 4\pi)^2} + \frac{2}{3(\Phi - 6\pi)^2} + \text{ etc.}$$

Pro

Pro altera ſerie ſit 1°) $i = 3n + 1$, hincque fractio fiet $-\frac{2}{(3\Phi - 2(3n+1)\pi)\sqrt{3}}$; ſin autem ſit $i = -3n - 1$ ſignum inferius valebit et fractio erit $+\frac{2}{(3\Phi + 2(3n+1)\pi)\sqrt{3}}$, qui duo termini contracti praebent $-\frac{8(3n+1)\pi}{(9\Phi\Phi - 4(3n+1)^2\pi\pi)\sqrt{3}}$; ſin autem fuerit $i = 3n + 2$, tum vero etiam $i = -3n - 2$, binae fractiones in vnam contractae praebent

$$+\frac{8(3n+2)\pi}{(9\Phi\Phi - 4(3n+2)^2\pi\pi)\sqrt{3}}.$$

Quare cum valores negatiuos ipſius i iam ſimus complexi, loco n tantum omnes numeros poſitiuos 0, 1, 2, 3, 4, 5 etc. poni oportet, vnde ſequens reſultabit ſeries:

$$-\frac{8\pi}{(9\Phi\Phi - 4\pi\pi)\sqrt{3}} - \frac{8 \cdot 4\pi}{(9\Phi\Phi - 4 \cdot 16\pi\pi)\sqrt{3}} - \frac{8 \cdot 7\pi}{(9\Phi\Phi - 4 \cdot 49\pi\pi)\sqrt{3}}$$
$$+\frac{16 \cdot \pi}{(9\Phi\Phi - 16\pi\pi)\sqrt{3}} + \frac{8 \cdot 5\pi}{(9\Phi\Phi - 4 \cdot 25\pi\pi)\sqrt{3}} + \frac{8 \cdot 8\pi}{(9\Phi\Phi - 4 \cdot 64\pi\pi)\sqrt{3}}.$$

§. 37. Propoſita igitur fractio $\frac{1}{\cos.\Phi - \cos.2\Phi}$ reſoluitur in has duas ſeries

$$\tfrac{2}{3}\left(\frac{1}{\Phi^2} + \frac{1}{(\Phi - 2\pi)^2} + \frac{1}{(\Phi + 2\pi)^2} + \frac{1}{(\Phi - 4\pi)^2} + \frac{1}{(\Phi + 4\pi)^2} + \text{etc.}\right)$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{1}{9\Phi\Phi - 4 \cdot 1\pi\pi} + \frac{4}{9\Phi\Phi - 4 \cdot 4^2\pi\pi} + \frac{7}{9\Phi\Phi - 4 \cdot 7^2\pi\pi} + \frac{101}{9\Phi\Phi - 4 \cdot 10^2\pi\pi} + \text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{2}{9\Phi\Phi - 4 \cdot 2^2\pi\pi} + \frac{5}{9\Phi\Phi - 4 \cdot 5^2\pi\pi} + \frac{8}{9\Phi\Phi - 4 \cdot 8^2\pi\pi} + \frac{11}{9\Phi\Phi - 4 \cdot 11^2\pi\pi} + \text{etc.}\right)$$

Hinc ergo ſi faciamus vt ſupra $\Phi = \lambda\pi$, erit fractio

$$\frac{\pi\pi}{\cos.\lambda\pi - \cos.2\lambda\pi} =$$
$$\tfrac{2}{3}\left(\frac{1}{\lambda^2} + \frac{1}{(\lambda - 2)^2} + \frac{1}{(\lambda + 2)^2} + \frac{1}{(\lambda - 4)^2} + \frac{1}{(\lambda + 4)^2} + \frac{1}{(\lambda - 6)^2} + \frac{1}{(\lambda + 6)^2} + \text{etc.}\right)$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{1}{9\lambda\lambda - 4 \cdot 1^2} + \frac{4}{9\lambda\lambda - 4 \cdot 4^2} + \frac{7}{9\lambda\lambda - 4 \cdot 7^2} + \frac{10}{9\lambda\lambda - 4 \cdot 10^2} + \text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{2}{9\lambda\lambda - 4 \cdot 2^2} + \frac{5}{9\lambda\lambda - 4 \cdot 5^2} + \frac{8}{9\lambda\lambda - 4 \cdot 8^2} + \frac{11}{9\lambda\lambda - 4 \cdot 11^2} + \text{etc.}\right)$$

§. 38. Vt exemplum afferamus ſit $\lambda = \frac{1}{3}$, vt fiat $9\lambda\lambda = 1$ ac prodibit iſta ſummatio:

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. 3, 4, 5 etc.
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ractio
+ etc.)
+ etc.)
+ etc.)
= ⅓, vt fiat
ππ

$$\pi\pi = \tfrac{2}{3}\left(\tfrac{9}{1^2} + \tfrac{9}{5^2} + \tfrac{9}{7^2} + \tfrac{9}{11^2} + \tfrac{9}{13^2} + \tfrac{9}{17^2} + \text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{1}{4 \cdot 1^2 - 1} + \frac{4}{4 \cdot 4^2 - 1} + \frac{7}{4 \cdot 7^2 - 1} + \frac{10}{4 \cdot 10^2 - 1} + \frac{13}{4 \cdot 13^2 - 1} + \text{etc.}\right)$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{2}{4 \cdot 2^2 - 1} + \frac{5}{4 \cdot 5^2 - 1} + \frac{8}{4 \cdot 8^2 - 1} + \frac{11}{4 \cdot 11^2 - 1} + \frac{14}{4 \cdot 14^2 - 1} + \text{etc.}\right)$$

quae ſummatio etiam hoc modo referri poteſt:

$$\pi\pi = 6\left(\tfrac{1}{1^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{11^2} + \tfrac{1}{13^2} + \tfrac{1}{17^2} + \tfrac{1}{19^2} + \text{etc.}\right)$$
$$+\frac{4\pi}{\sqrt{3}}\left(\frac{2}{2^2 - 1} - \frac{4}{4^2 - 1} + \frac{8}{8^2 - 1} - \frac{10}{10^2 - 1} + \frac{14}{14^2 - 1} - \frac{16}{16^2 - 1} + \text{etc.}\right)$$

elegantior autem forma erit ſequens:

$$\pi\pi = 6\left(\tfrac{1}{1^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{11^2} + \tfrac{1}{13^2} + \tfrac{1}{17^2} + \tfrac{1}{19^2} + \text{etc.}\right)$$
$$+\frac{2\pi}{\sqrt{3}}\left(1 - \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} - \text{etc.}\right)$$

§. 39. Quoniam hoc caſu occurrerunt factores quadrati, etiam eiusmodi fractiones reſoluere poterimus, quarum denominatores ipſi ſunt quadrati, ideoque meros factores ſimplices quadratos inuoluunt. Atque adeo hanc reſolutionem extendere licebit ad denominatores cubicos aliorumque poteſtatum, ſi modo in ſubſidium vocentur ea praecepta, quae pro huiusmodi reſolutionibus olim dedi.

IV. Sit fractio reſoluenda propoſita $\frac{1}{\sin.\Phi^2}$.

§. 40. Cum igitur hic omnes factores quadrati denominatoris in hac forma contineantur: $\frac{1}{(\Phi - i\pi)^2}$, denotante i omnes numeros integros tam poſitiuos quam negatiuos, ponamus pro reſolutione generali

$$\frac{1}{\sin.\Phi^2} = \frac{\alpha}{(\Phi - i\pi)^2} + \frac{\beta}{\Phi - i\pi} + R$$

vbi R complectitur omnes fractiones reliquas. Hinc per $(\Phi - i\pi)^2$ multiplicando erit

$$\frac{(\Phi - i\pi)^2}{\sin.\Phi^2} = \alpha + \beta(\Phi - i\pi) + R(\Phi - i\pi)^2.$$

 Iam

Iam fiat $\phi = i\pi$, et quoniam hoc casu numerator ac denominator nostrae fractionis euanescunt, statuamus $\Phi - i\pi = \omega$, eritque

$\sin.\Phi = \sin.(i\pi+\omega) = \sin.i\pi\cos.\omega + \sin.\omega\cos.i\pi = \pm\sin.\omega$

ob $\sin.i\pi = 0$ et $\cos.i\pi = \pm 1$; vbi signum superius valet si i sit numerus par, inferius vero si impar, quod tamen discrimen hic non in censum venit, cum sit $\sin.\Phi^2 = \sin.\omega^2$. Hinc igitur erit

$$\frac{\omega\omega}{\sin.\omega^2} = \alpha + \beta\omega + R\omega\omega.$$

Cum igitur sit

$$\sin.\omega = \omega - \tfrac{1}{6}\omega^3 = \omega(1 - \tfrac{1}{6}\omega\omega), \text{ erit}$$

$$\frac{1}{(1-\frac{1}{6}\omega\omega)^2} = 1 + \tfrac{1}{3}\omega\omega = \alpha + \beta\omega + R\omega\omega$$

vnde fit statim $\alpha = 1$. Tum vero aequatio erit $\frac{1}{3}\omega = \beta + R\omega$, sicque facto $\omega = 0$ fit $\beta = 0$, consequenter ex denominatoris factore $(\Phi - i\pi)^2$ oritur haec fractio $\frac{1}{(\Phi-i\pi)^2}$.

§. 41. Tribuantur nunc ipsi i omnes valores debiti ac reperietur haec series:

$$\frac{1}{\sin.\Phi^2} = \frac{1}{\Phi\Phi} + \frac{1}{(\Phi-\pi)^2} + \frac{1}{(\Phi+\pi)^2} + \frac{1}{(\Phi-2\pi)^2} + \frac{1}{(\Phi+2\pi)^2} + \frac{1}{(\Phi-3\pi)^2} + \text{etc.}$$

quae quidem series deduci potuisset ex § 18, vbi inuenimus

$$\frac{\cos.\Phi}{\sin.\Phi} = \frac{1}{\Phi} + \frac{1}{\Phi-\pi} + \frac{1}{\Phi+\pi} + \frac{1}{\Phi-2\pi} + \frac{1}{\Phi+2\pi} + \text{etc.},$$

vnde per differentiationem signis mutatis ea ipsa oritur series quam hic inuenimus.

§. 42. Quod si fractio fuisset proposita $\frac{\cos.\Phi^2}{\sin.\Phi^2}$ et eodem modo resolutio instituatur, ob

cos.

imerator ac deno- amus $\Phi - i\pi = \alpha$, $i\pi = \pm \sin.\omega$ ium superius valet par, quod tamen it $\sin.\Phi^2 = \sin.\omega$. , erit $+ R\omega\omega$ erit $\frac13\omega = \beta + R\omega$, er ex denomina- $\frac{1}{(\Phi-i\pi)^2}$. nnes valores de- + etc. 18, vbi inuenimus + etc., ea ipsa oritur se- proposita $\frac{\cos.\Phi^2}{\sin.\Phi^2}$ et cos.
qu nit qu abl ref ria in pa eu cas Por (vb aeq vnd ficq non frat

$$\cos.(i\pi + \omega) = \pm\cos.\omega \text{ ideoque}$$
$$\cos.\Phi^2 = \cos.\omega^2 = 1 - \omega\omega$$

quoniam secundae potestates ipsius ω non in computum veniunt, numerator foret vt in casu praecedente $= 1$, ideoque eadem plane series prodiisset, id quod vtique foret absurdum. Supra autem iam animaduertimus, huiusmodi resolutiones veritati non esse consentaneas, nisi quantitas variabilis Φ in numeratore pauciores habeat dimensiones quam in denominatore, quia alioquin praeter seriem fractionum partes integrae essent accessurae, id quod hoc casu manifesto euenit, cum sit $\frac{\cos.\Phi^2}{\sin.\Phi^2} = \frac{1}{\sin.\Phi^2} - 1$, ita vt pars integra hoc casu sit $= -1$.

V. Sit fractio resoluenda $= \frac{1}{\sin.\Phi^3}$.

§. 43. Pro hoc ergo casu poni oportebit

$$\frac{1}{\sin.\Phi^3} = \frac{\alpha}{(\Phi-i\pi)^3} + \frac{\beta}{(\Phi-i\pi)^2} + \frac{\gamma}{\Phi-i\pi} + R.$$

Ponamus nunc iterum $\Phi = i\pi + \omega$, et cum sit

$$\frac{1}{\sin.\Phi^3} = \pm\frac{1}{\omega^3}(1 + \tfrac{1}{2}\omega\omega)$$

(vbi ratio signorum legem supra datam seruet) haec resultat aequatio, postquam per ω^3 fuerit multiplicata:

$$\pm\frac{\omega^3(1+\frac{1}{2}\omega\omega)}{\omega^3} = \alpha + \beta\omega + \gamma\omega\omega + R\omega^3 = 1 + \tfrac{1}{2}\omega\omega,$$

vnde manifesto fit $\alpha = \pm 1$, tum vero $\beta + \gamma\omega + R\omega\omega = \frac{1}{2}\omega$ sicque erit $\beta = 0$ et $\gamma = \pm\frac{1}{2}$. Hoc igitur modo ex denominatoris factore cubico $(\Phi - i\pi)^3$ nascentur hae duae fractiones: $\frac{1}{(\Phi-i\pi)^3} \pm \frac{1}{2(\Phi-i\pi)}$.

R 2 §. 44.

§. 44. Tribuamus igitur litterae i fucceffiue omnes valores tam pofitiuos quam negatiuos atque obtinebimus fequentem refolutionem:

$$\frac{1}{\sin.\Phi^3} = \frac{1}{\Phi^3} - \frac{1}{(\Phi-\pi)^3} - \frac{1}{(\Phi+\pi)^3} + \frac{1}{(\Phi-2\pi)^3} + \frac{1}{(\Phi+2\pi)^3} - \text{etc.}$$
$$+ \frac{1}{2\Phi} - \frac{1}{2(\Phi-\pi)} - \frac{1}{2(\Phi+\pi)} + \frac{1}{2(\Phi-2\pi)} + \frac{1}{2(\Phi+2\pi)} - \text{etc.}$$

Hic obferuaffe iuuabit, inferiorem feriem iam fupra in primo exemplo effe inuentam; vnde intelligimus fore fummam huius feriei $= \frac{1}{2\sin.\Phi}$: quamobrem feries fuperior cuborum fola aequabitur huic formulae: $\frac{1}{\sin.\Phi^3} - \frac{1}{2\sin.\Phi}$.

§. 45. Egregie hoc quoque conuenit cum principiis fupra ftabilitis, ex quibus per differentiationem continuo alias nouas feries eruere docuimus. Cum enim fit

$$\frac{1}{\sin.\Phi} = \frac{1}{\Phi} - \frac{1}{\Phi-\pi} - \frac{1}{\Phi+\pi} + \frac{1}{\Phi-2\pi} + \frac{1}{\Phi+2\pi} + \text{etc.}$$

hinc deducitur differentiando

$$-\frac{\cos.\Phi}{\sin.\Phi^2} = -\frac{1}{\Phi^2} + \frac{1}{(\Phi-\pi)^2} + \frac{1}{(\Phi+\pi)^2} - \frac{1}{(\Phi-2\pi)^2} - \frac{1}{(\Phi+2\pi)^2} + \text{etc.}$$

atque hinc denuo differentiando

$$\frac{1}{\sin.\Phi} + \frac{2\cos.\Phi^2}{\sin.\Phi^3} = \frac{2}{\Phi^3} - \frac{2}{(\Phi-\pi)^3} - \frac{2}{(\Phi+\pi)^3} + \frac{2}{(\Phi-2\pi)^3} + \frac{2}{(\Phi+2\pi)^3} + \text{etc.}$$

quae reducitur ad hanc formam: $\frac{2}{\sin.\Phi^3} - \frac{1}{\sin.\Phi}$, id quod egregie conuenit cum valore praecedente.

VI. Sit fractio propofita refoluenda $= \frac{1}{\tang.\Phi - \sin.\Phi}$.

§. 46. Denominator ifte tang. Φ − fin. Φ manifefto euanefcit cafibus quibus $\Phi = i\pi$, denotante i omnes numeros integros tam pofitiuos quam negatiuos, vnde fractiones fimplices, quarum denominatores continent iftum factorem $\Phi - i\pi$, euadunt infiniti cafu $\Phi = i\pi$, dum reliquae fractiones

ffiue omnes
nebimus fe-
etc.
etc.
upra in pri-
re fummam
r cuborum
um princi-
m continuo
fit
+ etc.
+ etc.
+ etc.
quod egre-
manifefto
omnes nu-
de fractiones
m factorem
eliquae fra-
ctiones

ctiones retinent valorem finitum. Haecque confideratio nobis aperit nouam methodum, omnes fractiones fimplices inueftigandi. Pro quouis enim tali factore euanefcente quaeratur valor ipfius fractionis propofitae, qui cum fiat infinitus, ei aequales effe debebunt ii feriei termini, qui eodem cafu euadunt infiniti. Hanc ob rem ftatui oportebit $\Phi - i\pi = \omega$; denotante ω angulum infinite paruum. Quo facto fractio propofita fiet certa quaedam functio ipfius ω, quam fecundum eius dimenfiones euolui conueniet.

§. 47. Hanc igitur ideam fequentes, duos cafus diftinguere debemus, prouti i fuerit numerus vel par vel impar, quoniam priore cafu fit fin. Φ = fin. ω, pofteriore vero cafu fit fin. Φ = − fin. ω, dum vtroque cafu manet tang. Φ = tang. ω.

Sit igitur primo i numerus impar et erit cafu $\Phi = i\pi$ noftra fractio $\frac{1}{\tang.\omega + \sin.\omega}$. Eft vero proxime tang. $\omega = \omega + \frac{1}{3}\omega^3$ et fin. $\omega = \omega - \frac{1}{6}\omega^3$, vnde ifta fractio fiet

$$\frac{1}{2\omega + \frac{1}{6}\omega^3} = \frac{1}{2\omega(1 + \frac{1}{12}\omega\omega)} = \frac{1}{2\omega}(1 - \frac{1}{12}\omega\omega).$$

Haec iam expreffio fponte praebet has duas fractiones $\frac{1}{2\omega} - \frac{1}{24}\omega$, vnde ob $\omega = \Phi - i\pi$ pro ifto factore oritur haec fractio fimplex $\frac{1}{2(\Phi - i\pi)}$, quia altera pars euanefcit. Quare fi nunc loco i ordine fcribamus numeros impares, fequentem fractionum feriem adipifcemur:

$$\frac{1}{2(\Phi-\pi)} + \frac{1}{2(\Phi+\pi)} + \frac{1}{2(\Phi-3\pi)} + \frac{1}{2(\Phi+3\pi)} + \frac{1}{2(\Phi-5\pi)} + \text{etc.}$$

§. 48. Sit nunc etiam i numerus par, vnde fit tang. Φ = tang. ω et fin. Φ = fin. ω, hinc fractio noftra erit $\frac{1}{\tang.\omega - \sin.\omega}$, vbi facta euolutione p[illegible]imi termini ω fe se tollunt, ita vt in hoc denominatore infima potestas ipsius ω futura sit ω^3. Atque ob hanc caussam approximationem vlterius continuari oportet quam casu praecedente. Hunc in finem loco tang. ω scribamus $\frac{\sin.\omega}{\cos.\omega}$, vt fractio nostra fit $\frac{\cos.\omega}{\sin.\omega - \sin.\omega\cos.\omega}$. Cum iam sit

$$\sin.\omega\cos.\omega = \tfrac{1}{2}\sin.2\omega, \text{ erit per series}$$

$$\sin.\omega = \omega - \tfrac{1}{6}\omega^3 + \tfrac{1}{120}\omega^5 \text{ et}$$

$$\sin.2\omega = 2\omega - \tfrac{4}{3}\omega^3 + \tfrac{32}{120}\omega^5$$

vnde totus denominator erit

$$+\tfrac{1}{2}\omega^3 - \tfrac{1}{8}\omega^5 = \tfrac{1}{2}\omega^3(1 - \tfrac{1}{4}\omega\omega)$$

numerator vero est cos. $\omega = 1 - \frac{1}{2}\omega\omega$, vnde tota fractio nostra erit

$$\frac{1 - \frac{1}{2}\omega\omega}{\frac{1}{2}\omega^3(1 - \frac{1}{4}\omega\omega)} = \frac{1 - \frac{1}{4}\omega\omega}{\frac{1}{2}\omega^3};$$

hincque partes resultantes erunt $\frac{2}{\omega^3} - \frac{1}{2\omega}$, quae ambae casu $\omega = 0$ fiunt infinitae. Facile autem patet, si approximationem vlterius extendissemus, in sequenti termino litteram ω iam in numeratorem transituram fuisse. Scribatur igitur $\Phi - i\pi$ loco ω, et partes ex hoc factore denominatoris oriundae erunt $\frac{2}{(\Phi - i\pi)^3} - \frac{1}{2(\Phi - i\pi)}$, vnde loco i successiue omnes numeros pares scribendo ista prodibit series geminata:

$$\frac{2}{\Phi^3} + \frac{2}{(\Phi-2\pi)^3} + \frac{2}{(\Phi+2\pi)^3} + \frac{2}{(\Phi-4\pi)^3} + \frac{2}{(\Phi+4\pi)^3} + \frac{2}{(\Phi-6\pi)^3} + \text{etc.}$$

$$-\frac{1}{2\Phi} - \frac{1}{2(\Phi-2\pi)} - \frac{1}{2(\Phi+2\pi)} - \frac{1}{2(\Phi-4\pi)} - \frac{1}{2(\Phi+4\pi)} - \text{etc.}$$

§. 49. Iungamus igitur has series ex vtroque casu deductas et fractio proposita $\frac{1}{\tang.\Phi - \sin.\Phi}$ resolui reperitur in ternas series sequentes:

a potestas ipn approximapraecedente. vt fractio no-
s
tota fractio
ie ambae casu approximationo litteram ω cribatur igitur denominatoris successiue omies geminata:
$\frac{2}{(\Phi-6\pi)^3}$ + etc.
etc.
vtroque casu resolui reperitur
I 2

$$\frac{1}{2(\Phi-\pi)} + \frac{1}{2(\Phi+\pi)} + \frac{1}{2(\Phi-3\pi)} + \frac{1}{2(\Phi+3\pi)} + \frac{1}{2(\Phi-5\pi)} + \text{etc.}$$

$$-\frac{1}{2\Phi} - \frac{1}{2(\Phi-2\pi)} - \frac{1}{2(\Phi+2\pi)} - \frac{1}{2(\Phi-4\pi)} - \frac{1}{2(\Phi+4\pi)} - \text{etc.}$$

$$+\frac{2}{\Phi^3} + \frac{2}{(\Phi-2\pi)^3} + \frac{2}{(\Phi+2\pi)^3} + \frac{2}{(\Phi-4\pi)^3} + \frac{2}{(\Phi+4\pi)^3} + \text{etc.}$$

§. 50. Qnilibet hic facile sentiet, istam methodum non parum antecellere illi, qua ante vsi sumus, quandoquidem hoc modo statim fractiones ex quolibet denominatoris factore oriundas nacti sumus, neque opus fuerat earum numeratores per litteras indefinitas designare. Praeterea etiam hac ratione non opus erat sollicite inquirere, quoties singuli factores simplices in denominatore contineantur, siquidem nostra methodus hoc sponte declarat.

§. 51. In huiusmodi autem seriebus generalibus, vbi quorundam terminorum denominatores certo casu euanescunt ideoque hi termini in infinitum excrescunt, quaeri solet, his terminis sublatis, quanta futura sit summa reliquorum terminorum. Ita pro casu quo i est numerus impar, terminus $\frac{1}{2(\Phi - i\pi)}$ fit infinitus casu $\Phi = i\pi$. Hoc igitur termino deleto quaeritur, quanta futura sit summa reliquorum terminorum casu $\Phi = i\pi$. Ad hanc quaestionem soluendam ponatur $\Phi - i\pi = \omega$, atque ex §47 patet fore

$$\frac{1}{2\omega} - \frac{1}{24}\omega = \frac{1}{2(\Phi - i\pi)} + R$$

vbi R complectitur omnes reliquos terminos, quorum summa desideratur casu $\Phi = i\pi$. Transferatur igitur terminus $\frac{1}{2(\Phi - i\pi)} = \frac{1}{2\omega}$ in alteram partem ac statim elucet fore

$$R = -\tfrac{1}{24}\omega = 0 \text{ ob } \omega = 0,$$

ita vt omisso termino illo infinito summa omnium reliquorum casu $\Phi = i\pi$ semper sit 0.

§. 52. Quando autem i est numerus par, eadem conclusio locum habebit, ad quod ostendendum necesse est approximationem adhibitam vlterius continuare. Tum autem erit numerator

$$\cos. \omega = 1 - \tfrac{1}{2}\omega\omega + \tfrac{1}{24}\omega^4;$$

pro denominatore vero

$$\sin. \omega = \omega - \tfrac{1}{6}\omega^3 + \tfrac{1}{120}\omega^5 - \tfrac{1}{5040}\omega^7 \text{ et}$$

$$\sin. 2\omega = 2\omega - \tfrac{8}{6}\omega^3 + \tfrac{32}{120}\omega^5 - \tfrac{128}{5040}\omega^7,$$

vnde fit ipse denominator

$$\tfrac{1}{2}\omega^3 - \tfrac{1}{8}\omega^5 + \tfrac{1}{80}\omega^7 = \tfrac{1}{2}\omega^3(1 - \tfrac{1}{4}\omega\omega + \tfrac{1}{40}\omega^4);$$

hinc factor posterior in numeratorem translatus praebet

$$1 + \tfrac{1}{4}\omega\omega + \tfrac{3}{80}\omega^4$$

hincque tota fractio iam erit

$$\frac{1 - \frac{1}{4}\omega\omega - \frac{11}{240}\omega^4}{\frac{1}{2}\omega^3}$$

quae aequari debet toti seriei posito $\Phi = i\pi$, hoc est terminis inuentis $\frac{2}{(\Phi - i\pi)^3} - \frac{1}{2(\Phi - i\pi)}$ cum omnibus reliquis R, vnde elicitur $R = -\frac{11}{120}\omega = 0$; vnde patet etiam his casibus summam omnium reliquorum esse $= 0$.

§. 53. Quod si ergo sumamus $\Phi = 0$ et terminos in infinitum excrescentes deleamus, termini remanentes erunt

$$-\tfrac{1}{2\pi} + \tfrac{1}{2\pi} - \tfrac{1}{6\pi} + \tfrac{1}{6\pi} - \tfrac{1}{10\pi} + \tfrac{1}{10\pi} - \text{etc.}$$

$$+\tfrac{1}{4\pi} - \tfrac{1}{4\pi} + \tfrac{1}{8\pi} - \tfrac{1}{8\pi} + \tfrac{1}{12\pi} - \tfrac{1}{12\pi} + \text{etc.}$$

$$-\tfrac{2}{8\pi^3} + \tfrac{2}{8\pi^3} - \tfrac{2}{64\pi^3} + \tfrac{2}{64\pi^3} - \tfrac{2}{216\pi^3} + \tfrac{2}{216\pi^3} - \text{etc.}$$

vbi omnes termini manifesto se tollunt, id quod etiam omnibus reliquis casibus, quibus ponitur $\Phi = i\pi$, contingit.

rus par, eadem
ndum necesse est
uare. Tum au-
π, hoc est ter-
reliquis R, vnde
iam his casibus
= o et termi-
rmini remanentes
etc.
etc.
$\frac{2}{216\pi^3}$ − etc.
quod etiam om-
= $i\pi$, contingit.
§. 54.

§. 54. Sin autem binos terminos contiguos contraxissemus, hae series prodiissent:

$$-\tfrac{1}{2\Phi} + \tfrac{\Phi}{\Phi\Phi - \pi\pi} - \tfrac{\Phi}{\Phi\Phi - 4\pi\pi} + \tfrac{\Phi}{\Phi\Phi - 9\pi\pi} - \tfrac{\Phi}{\Phi\Phi - 16\pi\pi} + \tfrac{\Phi}{\Phi\Phi - 25\pi\pi} - \text{etc.}$$

$$+\tfrac{2}{\Phi^3} + \tfrac{4\Phi(\Phi\Phi + 3\cdot 4\pi\pi)}{(\Phi\Phi - 4\pi\pi)^3} + \tfrac{4\Phi(\Phi\Phi + 3\cdot 16\pi\pi)}{(\Phi\Phi - 16\pi\pi)^3} + \tfrac{4\Phi(\Phi\Phi + 3\cdot 36\pi\pi)}{(\Phi\Phi - 36\pi\pi)^3} + \text{etc.}$$

quarum serierum summa est $\frac{1}{\tang. \Phi - \sin. \Phi}$. Quod si nunc hic ponamus $\Phi = 0$, siue $\Phi = \omega$, quoniam omnes termini sunt diuisibiles per $\Phi = 0$, eorumque autem summa inuenta est $-\frac{11}{120}\omega$, si vtrinque per ω diuidamus, summa erit $= -\frac{11}{120}$; ipsae autem series euadent

$$-\tfrac{1}{\pi\pi} + \tfrac{1}{4\pi\pi} - \tfrac{1}{9\pi\pi} + \tfrac{1}{16\pi\pi} - \tfrac{1}{25\pi\pi} + \text{etc.}$$

$$-\tfrac{3\cdot 4}{(4\pi\pi)^2} - \tfrac{3\cdot 4}{(16\pi\pi)^2} - \tfrac{3\cdot 4}{(36\pi\pi)^2} - \tfrac{3\cdot 4}{(64\pi\pi)^2} - \text{etc.}$$

§. 55. Mutatis igitur signis et reductis terminis ad formam simplicissimam impetrabimus hanc summationem:

$$\tfrac{11}{120} = \tfrac{1}{\pi\pi}(1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \tfrac{1}{36} + \text{etc.})$$

$$+\tfrac{3}{4\pi^4}(1 + \tfrac{1}{4^2} + \tfrac{1}{9^2} + \tfrac{1}{16^2} + \tfrac{1}{25^2} + \tfrac{1}{36^2} + \text{etc.})$$

Notum autem est esse

$$1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} \text{ etc.} = \tfrac{\pi\pi}{12} \text{ et}$$

$$1 + \tfrac{1}{4^2} + \tfrac{1}{9^2} + \tfrac{1}{16^2} + \tfrac{1}{25^2} + \tfrac{1}{36^2} = \tfrac{\pi^4}{90}$$

vnde haec aequalitas manifesto in oculos incurrit.