iae integrales a termino $y=0$ vsque ad $y=1$ extendantur, in genere valor nostrae formulae propositae ita repraesentari poterit:

$$\int \frac{x^{m-1}\,dx}{(1+x^k)^n} = \frac{\pi}{k \sin.\frac{m\pi}{k}} \cdot \frac{\int y^{nk-m-1}\,dy\,(1-y^k)^{\frac{m}{k}-1}}{\int y^{k-m-1}\,dy\,(1-y^k)^{\frac{m}{k}-1}}.$$

Vnde si fuerit $m=1$ et $k=2$, sequitur fore

$$\int \frac{dx}{(1+xx)^n} = \frac{\pi}{2} \int \frac{y^{2(n-1)}\,dy}{\sqrt{(1-yy)}} : \int \frac{dy}{\sqrt{(1-yy)}} = \int \frac{y^{2(n-1)}\,dy}{\sqrt{(1-yy)}}$$

Ita si $n=\frac{3}{2}$ erit

$$\int \frac{dx}{(1+xx)^{\frac{3}{2}}} = \int \frac{y\,dy}{\sqrt{(1-yy)}}$$

cuius veritas sponte elucet, quia integrale prius generatim est $\frac{x}{\sqrt{1+xx}}$, posterius vero $=1-\sqrt{(1-yy)}$, quae, facto $x=\infty$ et $y=1$, vtique fiunt aequalia. Caeterum pro hac integratione generali notasse iuuabit, exponentem vnitate minorem accipi non posse, quia alioquin valores amborum integralium in infinitum excrescerent.

IN:

tendantur, in repraesentari

$y^k)^{\frac{m}{k}-1}$ / $y^k)^{\frac{m}{k}-1}$.

$\int \frac{y^{2(n-1)}\,dy}{\sqrt{(1-yy)}}$

s generatim est e, facto $x=\infty$ o hac integra- nitate minorem rum integrali-

IN:

INVESTIGATIO
VALORIS INTEGRALIS

$$\int \frac{x^{m-1}\,dx}{1-2x^k \cos.\theta + x^{2k}}$$

A TERMINO $x=0$ VSQVE AD $x=\infty$ EXTENSI.

§. 1.

Quaeramus primo integrale formulae propositae indefinitum, atque adeo omnes operationes ex primis Analyseos principiis repetamus. Ac primo quidem, quoniam denominator in factores reales simplices resolui nequit, sit in genere eius factor duplicatus quicunque $1-2x\cos.\omega+xx$; euidens enim est, denominatorem fore productum ex k huiusmodi factoribus duplicatis. Cum igitur, posito hoc factore $=0$, fiat $x=\cos.\omega \pm \sqrt{-1}\sin.\omega$, etiam ipse denominator duplici modo euanescere debebit, siue si ponatur

$$x=\cos.\omega+\sqrt{-1}\sin.\omega, \text{ siue}$$
$$x=\cos.\omega-\sqrt{-1}\sin.\omega.$$

Constat autem omnes potestates harum formularum ita commode exprimi posse, vt sit

$$(\cos.\omega \pm \sqrt{-1}\sin.)^\lambda = \cos.\lambda\omega \pm \sqrt{-1}\sin.\lambda\omega;$$

hinc igitur erit

$$x^k = \cos.k\omega + \sqrt{-1}\sin.k\omega \text{ et}$$
$$x^{2k} = \cos.2k\omega \pm \sqrt{-1}\sin.2k\omega.$$

Sub-

ſubſtituamus ergo hos valores et denominator noſter euader

$$1 - 2\,\text{coſ.}\,\theta\,\text{coſ.}\,k\omega + \text{coſ.}\,2k\omega =$$
$$\pm\, 2\sqrt{-1}\,\text{coſ.}\,\theta\,\text{ſin.}\,k\omega + \sqrt{-1}\,\text{ſin.}\,2k\omega$$

§. 2. Perſpicuum igitur eſt huius aequationis tam terminos reales quam imaginarios ſeorſim ſe mutuo tollere debere, vnde naſcuntur hae duae aequationes:

$$\text{I. } 1 - 2\,\text{coſ.}\,\theta\,\text{coſ.}\,k\omega + \text{coſ.}\,2k\omega = 0$$

$$\text{II. } -2\,\text{coſ.}\,\theta\,\text{ſin.}\,k\omega + \text{ſin.}\,2k\omega = 0.$$

Cum igitur ſit

$$\text{ſin.}\,2k\omega = 2\,\text{ſin.}\,k\omega\,\text{coſ.}\,k\omega,$$

poſterior aequatio induet hanc formam:

$$-2\,\text{coſ.}\,\theta\,\text{ſin.}\,k\omega + 2\,\text{ſin.}\,k\omega\,\text{coſ.}\,k\omega,$$

quae per $2\,\text{ſin.}\,k\omega$ diuiſa dat $+\text{coſ.}\,k\omega = \text{coſ.}\,\theta$, ideoque

$$\text{coſ.}\,2k\omega = \text{coſ.}\,2\theta = \text{coſ.}\,\theta^2 - \text{ſin.}\,\theta^2 = 2\,\text{coſ.}\,\theta^2 - 1,$$

qui valores in aequatione priore ſubſtitui praebent aequationem identicam, ita vt vtrique aequationi ſatisfiat ſumendo $\text{coſ.}\,k\omega = \text{coſ.}\,\theta$.

§. 3. Pro ω igitur eiusmodi angulum aſſumi oportet, vt fiat $\text{coſ.}\,k\omega = \text{coſ.}\,\theta$, vnde quidem ſtatim deducitur $k\omega = \theta$, ideoque $\omega = \frac{\theta}{k}$. Verum quia infiniti dantur anguli eundem coſinum habentes, qui praeter ipſum angulum θ ſunt $2\pi \pm \theta$, $4\pi \pm \theta$, $6\pi \pm \theta$ etc. atque adeo in genere $2i\pi + \theta$, denotante i omnes numeros integros, quaeſito noſtro ſatisfiet, faciendo $k\omega = 2i\pi + \theta$, vnde colligitur angulus $\omega = \frac{2i\pi \pm \theta}{k}$, ſicque pro ω nanciſceremus innumerabiles angulos ſatisfacientes, quorum autem ſufficiet tot

noſter euader
ſin. 2 k ω
aequationis tam
ſe mutuo tollere
nes:
= o
= o.
θ,
coſ. θ, ideoque
= 2 coſ. θ² − 1,
praebent aequa-
ſatisfiat ſumendo
um aſſumi opor-
ſtatim deducitur
finiti dantur an-
ipſum angulum
atque adeo in
numeros integros,
iπ + θ, vnde
ω nanciſceremus
m autem ſufficiet
tot

tot aſſumſiſſe, quot exponens k continet vnitates; ſucceſſiue igitur angulo ω ſequentes tribuamus valores:

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \frac{8\pi+\theta}{k},\ \ldots\ \frac{2(k-1)\pi+\theta}{k}$$

Quod ſi ergo angulo ω ſucceſſiue ſingulos iſtos valores, quorum numerus eſt $= k$, tribuamus, formula $1 - 2x\,\text{coſ.}\,\omega + xx$ omnes ſuppeditabit factores duplicatos noſtri denominatoris $1 - 2x^k\,\text{coſ.}\,\theta + x^{2k}$, quorum numerus erit $= k$.

§. 4. Inuentis iam omnibus factoribus duplicatis noſtri denominatoris, fractio $\frac{x^{m-1}}{1 - 2x^k\,\text{coſ.}\,\theta + x^{2k}}$ reſolui debet in tot fractiones partiales, quarum denominatores ſint ipſi iſti factores duplicati, quorum numerus eſt k, ita vt in genere talis fractio partialis habitura ſit talem formam:

$$\frac{A + Bx}{1 - 2x\,\text{coſ.}\,\omega + xx}$$

quam inſuper reſoluamus in binas ſimplices, etſi imaginarias, et cum ſit

$$xx - 2x\,\text{coſ.}\,\omega + 1 = (x - \text{coſ.}\,\omega + \sqrt{-1}\,\text{ſin.}\,\omega)(x - \text{coſ.}\,\omega - \sqrt{-1}\,\text{ſin.}\,\omega),$$

ſtatuantur ambae iſtae fractiones partiales

$$\frac{f}{x - \text{coſ.}\,\omega - \sqrt{-1}\,\text{ſin.}\,\omega} + \frac{g}{x - \text{coſ.}\,\omega + \sqrt{-1}\,\text{ſin.}\,\omega},$$

ita vt totum reſolutionis negotium huc redeat, vt ambo numeratores f et g determinentur; iis enim inuentis habebitur ſumma ambarum fractionum

$$= \frac{fx + gx - (f+g)\,\text{coſ.}\,\omega + \sqrt{-1}\,(f-g)\,\text{ſin.}\,\omega}{xx - 2x\,\text{coſ.}\,\omega + 1},$$

vbi igitur erit

$$B = f + g \quad \text{et} \quad A = (f-g)\sqrt{-1}\,\text{ſin.}\,\omega - (f+g)\,\text{coſ.}\,\omega.$$

§. 5.

§. 5. Per methodum igitur fractiones quascunque in fractiones simplices resoluendi statuamus

$$\frac{x^{m-1}}{1-2x^k \operatorname{cos.}\theta + x^{2k}} = \frac{f}{x-\operatorname{cos.}\omega - \sqrt{-1}\operatorname{sin.}\omega} + R,$$

vbi R complectatur omnes reliquas fractiones partiales. Hinc per $x-\operatorname{cos.}\omega - \sqrt{-1}\operatorname{sin.}\omega$ multiplicando habebitur

$$\frac{x^m - x^{m-1}(\operatorname{cos.}\omega + \sqrt{-1}\operatorname{sin.}\omega)}{1-2x^k \operatorname{cos.}\theta + x^{2k}} = f R(x-\operatorname{cos.}\omega - \sqrt{-1}\operatorname{sin.}\omega),$$

quae aequatio cum vera esse debeat, quicunque valor ipsi x tribuatur, statuamus $x = \operatorname{cos.}\omega + \sqrt{-1}\operatorname{sin.}\omega$, vt membrum postremum prorsus e calculo tollatur; tum vero in parte sinistra, quia formula $x-\operatorname{cos.}\omega - \sqrt{-1}\operatorname{sin.}\omega$ simul est factor denominatoris, facta hac substitutione tam numerator quam denominator in nihilum abibunt, ita vt hinc nihil concludi posse videatur.

§. 6. Hic igitur vtamur regula notissima, et loco tam numeratoris quam denominatoris eorum differentialia scribamus, vnde nostra aequatio accipiet sequentem formam:

$$\frac{m x^{m-1} - (m-1)x^{m-2}(\operatorname{cos.}\omega + \sqrt{-1}\operatorname{sin.}\omega)}{-2k x^{k-1}\operatorname{cos.}\theta + 2k x^{2k-1}} =$$

$$\frac{m x^m - (m-1)x^{m-1}(\operatorname{cos.}\omega + \sqrt{-1}\operatorname{sin.}\omega)}{-2k x^k \operatorname{cos.}\theta + 2k x^{2k}} = f,$$

posito scilicet $x = \operatorname{cos.}\omega + \sqrt{-1}\operatorname{sin.}\omega$. Tum autem erit

$x^m = \operatorname{cos.} m\omega + \sqrt{-1}\operatorname{sin.} m\omega$ et

$x^{m-1}(\operatorname{cos.}\omega + \sqrt{-1}\operatorname{sin.}\omega) = x^m = \operatorname{cos.} m\omega + \sqrt{-1}\operatorname{sin.} m\omega$

et pro denominatore

$x^k = \operatorname{cos.} k\omega + \sqrt{-1}\operatorname{sin.} k\omega$ et

$x^{2k} = \operatorname{cos.} 2k\omega + \sqrt{-1}\operatorname{sin.} 2k\omega$;

vnde

quascunque
sin. ω + R,
artiales. Hinc
ebitur
√−1 sin. ω),
valor ipsi x
, vt mem-
tum vero in
sin. ω simul
e tam nume-
a vt hinc ni-
ssima, et loco
differentialia
ntem formam:
) =
) = f,
am autem erit
+√−1 sin. mω
vnde

vnde fit numerator

$$x^m = \operatorname{cos.} m\omega + \sqrt{-1}\operatorname{sin.} m\omega$$

et denominator

$$-2k \operatorname{cos.}\theta \operatorname{cos.} k\omega + 2k \operatorname{cos.} 2k\omega$$
$$-2k\sqrt{-1}\operatorname{cos.}\theta \operatorname{sin.} k\omega + 2k\sqrt{-1}\operatorname{sin.} 2k\omega.$$

§. 7. Pro denominatore reducendo recordemur, iam supra inuentum esse $\operatorname{cos.} k\omega = \operatorname{cos.}\theta$, vnde fit $\operatorname{sin.} k\omega = \operatorname{sin.}\theta$, tum vero

$$\operatorname{cos.} 2k\omega = \operatorname{cos.} 2\theta = 2\operatorname{cos.}\theta^2 - 1 \text{ et } \operatorname{sin.} 2k\omega = 2\operatorname{sin.}\theta\operatorname{cos.}\theta,$$

quibus valoribus adhibitis denominator noster erit

$$2k\operatorname{cos.}\theta^2 - 2k + 2k\sqrt{-1}\operatorname{sin.}\theta\operatorname{cos.}\theta = -2k\operatorname{sin.}\theta^2 + 2k\sqrt{-1}\operatorname{sin.}\theta\operatorname{cos.}\theta$$
$$= -2k\operatorname{sin.}\theta(\operatorname{sin.}\theta - \sqrt{-1}\operatorname{cos.}\theta),$$

quamobrem hoc valore adhibito habebimus

$$f = \frac{\operatorname{cos.} m\omega + \sqrt{-1}\operatorname{sin.} m\omega}{2k\operatorname{sin.}\theta(\sqrt{-1}\operatorname{cos.}\theta - \operatorname{sin.}\theta)}.$$

Simul vero hinc sine nouo calculo deducemus valorem g, quippe qui ab f ratione signi $\sqrt{-1}$ tantum discrepat, sicque erit

$$g = \frac{\operatorname{cos.} m\omega - \sqrt{-1}\operatorname{sin.} m\omega}{-2k\operatorname{sin.}\theta(\operatorname{sin.}\theta + \sqrt{-1}\operatorname{cos.}\theta)}.$$

§. 8. Inuentis autem his litteris f et g pro litteris A et C colligemus primo

$$f + g = \frac{\operatorname{cos.}\theta\operatorname{sin.} m\omega - \operatorname{sin.}\theta\operatorname{cos.} m\omega}{k\operatorname{sin.}\theta} = \frac{\operatorname{sin.}(m\omega - \theta)}{k\operatorname{sin.}\theta},$$

tum vero erit

$$f - g = -\frac{\sqrt{-1}\operatorname{cos.}(m\omega - \theta)}{k\operatorname{sin.}\theta}.$$

Ex his igitur reperiemus

g =

$B = \frac{\sin.(m\omega-\theta)}{k\sin.\theta}$ et

$$A = \frac{\sin.\omega\cos.(m\omega-\theta)-\cos.\omega\sin.(m\omega-\theta)}{k\sin.\theta} = -\frac{\sin.((m\omega-\theta)-\omega)}{k\sin.\theta};$$

vbi ergo imaginaria sponte se mutuo destruxerunt.

§. 9. Inuentis his valoribus A et B inuestigari oportet integrale partiale $\int \frac{(A+Bx)dx}{1-2x\cos.\omega+xx}$, vbi, cum denominatoris differentiale sit

$$2x\,dx - 2\,dx\cos.\omega = 2\,dx\,(x-\cos.\omega),$$

statuamus

$$A+Bx = B(x-\cos.\omega)+C, \text{ eritque } C = A+B\cos.\omega,$$

hinc igitur erit

$$C = \frac{\cos.\omega\sin.(m\omega-\theta)-\sin.((m\omega-\theta)-\omega)}{k\sin.\theta}.$$

Quia vero

$$-\sin.(m\omega-\theta-\omega) = -\sin.(m\omega-\theta)\cos.\omega+\cos.(m\omega-\theta)\sin.\omega, \text{ erit}$$

$$C = \frac{\sin.\omega\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Hac ergo forma adhibita formula integranda $\frac{(A+Bx)dx}{1-2x\cos.\omega+xx}$ discerpatur in has duas partes:

$$\frac{B(x-\cos.\omega)dx}{1-2x\cos.\omega+xx} + \frac{C\,dx}{1-2x\cos.\omega+xx}.$$

Hic igitur prioris partis integrale manifesto est

$$B\,l\sqrt{(1-2x\cos.\omega+xx)},$$

alterius vero partis facile patet integrale per arcum circuli expressum iri, cuius tangens sit $\frac{x\sin.\omega}{1-x\cos.\omega}$. Ad hoc integrale inueniendum ponamus

$$\int \frac{C\,dx}{1-2x\cos.\omega+xx} = D.\,A\,\text{tang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$

et

rum.
B inuestigari
cum deno-
$\omega)$,
$=A+B\cos.\omega$,
$-\theta)\sin.\omega$, erit
$\frac{(A+Bx)dx}{1-2x\cos.\omega+xx}$ di-
est
er arcum cir-
Ad hoc inte-
et

et sumtis differentialibus, quia $d.\,A\,\text{tang.}\,t$ aequale est $\frac{dt}{1+tt}$, habebimus

$$\frac{C\,dx}{1-2x\cos.\omega+xx} = D.\frac{dx\sin.\omega}{1-2x\cos.\omega+xx}$$

vnde manifesto fit

$$D = \frac{C}{\sin.\omega} = \frac{\cos.(m\omega-\theta)}{k\sin.\theta}.$$

§. 10. Substituamus igitur loco B et D valores modo inuentos et ex singulis factoribus denominatoris

$$1 - 2x^k\cos.\theta + x^{2k},$$

quorum forma est $1-2x\cos.\omega+xx$, oritur pars integralis constans ex membro logarithmico et arcu circulari, quae erit

$$\frac{\sin.(m\omega-\theta)}{k\sin.\theta}\,l\sqrt{(1-2x\cos.\omega+xx)} + \frac{\cos.(m\omega-\theta)}{k\sin.\theta}\,A\,\text{tang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$

quae euanescit sumto $x=0$. In hac igitur forma tantum opus est vt loco ω successiue scribamus valores supra indicatos, scilicet:

$$\omega = \frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi-\theta}{k},\ \frac{6\pi+\theta}{k},\ \text{etc.}$$

donec perueniatur ad $\frac{2(k-1)\pi+\theta}{k}$; tum enim summa omnium harum formarum praebebit totum integrale indefinitum formulae propositae.

§. 11. Postquam igitur integrale indefinitum eliciumus, nihil aliud superest, nisi vt in eo faciamus $x=\infty$, quo facto pars logarithmica, ob

$$\sqrt{(1-2x\cos.\omega+xx)} = x-\cos.\omega,$$

erit $B\,l(x-\cos.\omega)$. Est vero

$$l(x-\cos.\omega) = l\,x - \frac{\cos.\omega}{x} = l\,x, \text{ ob } \frac{\cos.\omega}{k} = 0;$$

quam-

quamobrem facto $x=\infty$ quaelibet pars logarithmica habebit hanc formam: $\frac{\cos(m\omega-\theta)}{k\sin\theta}\, l\, x$. Deinde pro partibus a circulo pendentibus, facto $x=\infty$ fit

$$\frac{x\sin\omega}{1-x\cos\omega}=-\tang.\,\omega=\tang.(\pi-\omega),$$

sicque arcus, cuius haec est tangens, erit $=\pi-\omega$, hincque pars circularis quaecunque fiet $\frac{\sin(m\omega-\theta)}{k\sin\theta}(\pi-\omega)$.

§. 12. Cum quilibet valor anguli ω in genere hanc habeat formam: $\frac{2i\pi+\theta}{k}$, erit angulus

$$m\omega-\theta=\frac{2im\pi-\theta(k-m)}{k} \text{ et } \pi-\omega=\frac{\pi(k-2i)-\theta}{k}.$$

Ponamus breuitatis gratia

$$\frac{\theta(k-m)}{k}=\zeta \text{ et } \frac{m\pi}{k}=\alpha, \quad \text{vt fit } m\omega-\theta=2i\alpha-\zeta,$$

vbi loco i scribi debent successiue numeri 0, 1, 2, 3, etc. vsque ad $k-1$. Hinc igitur si omnes partes logarithmicas in vnam summam colligamus, ea ita repraesentari poterit:

$$\frac{l\,x}{k\sin\theta}(-\sin\zeta+\sin(2\alpha-\zeta)+\sin(4\alpha-\zeta)+\sin(6\alpha-\zeta)$$
$$+\sin(8\alpha-\zeta)----+\sin(2(k-1)\alpha-\zeta));$$

vbi quidem ex iis, quae hactenus sunt tradita, facile suspicari licet, totam hanc progressionem ad nihilum redigi. Verum hoc ipsum firma demonstratione muniri necesse est.

§. 13. Ad hoc ostendendum ponamus

$$S=-\sin\zeta+\sin(2\alpha-\zeta)+\sin(4\alpha-\zeta)+----+\sin(2(k-1)\alpha-\zeta)$$

multiplicemus vtrinque per $2\sin\alpha$, et cum sit

$$2\sin\alpha\sin\varphi=\cos(\alpha-\varphi)-\cos(\alpha+\varphi)$$

huius reductionis ope obtinebimus sequentem expressionem:

rithmica habe-
partibus a cir-
$\pi-\omega$, hinc-
$(\pi-\omega)$.
in genere hanc
$\frac{(k-2i)-\theta}{k}$.
$\theta=2i\alpha-\zeta$,
1, 2, 3, etc. vs-
logarithmicas
sentari poterit:
$+\sin(6\alpha-\zeta)$
$(k-1)\alpha-\zeta)$);
facile suspicari
redigi. Verum
le est.
us
$(2(k-1)\alpha-\zeta)$
fit
$+\varphi)$
expressionem:
2 S

$$2S\sin\alpha=\cos(\alpha+\zeta)-\cos(\alpha-\zeta)+\cos(3\alpha-\zeta)$$
$$+\cos(\alpha-\zeta)\qquad-\cos(3\alpha-\zeta)$$
$$+\cos(5\alpha-\zeta)----\cos((2k-1)\alpha-\zeta)$$
$$-\cos(5\alpha-\zeta)------$$

vnde deletis terminis se mutuo destruentibus habebitur

$$2S\sin\alpha=\cos(\alpha+\zeta)-\cos((2k-1)\alpha-\zeta).$$

§. 14. Ponamus hos duos angulos, qui sunt relicti, $\alpha+\zeta=p$ et $(2k-1)\alpha-\zeta=q$, eritque eorum summa $p+q=2\alpha k$. Quia porro est $\alpha=\frac{m\pi}{k}$, erit $p+q=2m\pi$, hoc est multiplo totius circuli peripheriae, ob m numerum integrum. Quare cum sit $q=2m\pi-p$, erit $\cos q=\cos p$; vnde patet summam inuentam nihilo esse aequalem, sicque manifestum est, omnes partes logarithmicas, quae in integrale formulae nostrae ingrediuntur, casu $x=\infty$ se mutuo destruere.

§. 15. Progrediamur igitur ad partes circulares, quarum forma generalis, vt vidimus, est $\frac{\sin(m\omega-\theta)}{k\sin\theta}(\pi-\omega)$, quae posito $\alpha=\frac{m\pi}{k}$ et $\zeta=\frac{\theta(k-m)}{k}$ fit

$$\frac{\cos(2i\alpha-\zeta)}{k\sin\theta}\left(\pi-\frac{2i\pi-\theta}{k}\right)=\frac{\sin(2i\alpha-\zeta)}{k\sin\theta}\left(\pi-\frac{2i\pi}{k}-\frac{\theta}{k}\right).$$

Hic ponatur porro $\frac{\pi}{k}=\beta$ et $\pi-\frac{\theta}{k}=\gamma$, vt forma generalis fit $\frac{\cos(2i\alpha-\zeta)}{k\sin\theta}(\gamma-2i\beta)$. Quare si loco i scribamus ordine valores 0, 1, 2, 3, 4, vsque ad $k-1$, omnes partes circulares hanc progressionem constituent:

$$\frac{1}{k\sin\theta}(\gamma\cos\zeta+(\gamma-2\beta)\cos(2\alpha-\zeta)+(\gamma-4\beta)\cos(4\alpha-\zeta)$$
$$------(\gamma-2(k-1)\beta)\cos(2(k-1)\alpha-\zeta)).$$

Ponamus

Ponamus igitur

$$S = \gamma \operatorname{cof.} \zeta + (\gamma - 2\beta) \operatorname{cof.} (2\alpha - \zeta) + (\gamma - 4\beta) \operatorname{cof.} (4\alpha - \zeta)$$
$$- \cdots - (\gamma - 2(k-1)\beta) \operatorname{cof.} (2(k-1)\alpha - \zeta)$$

vt ſumma omnium partium circularium ſit $\frac{S}{k \sin.\theta}$, quae ergo praebebit valorem quaeſitum formulae integralis propoſitae, caſu quo poſt integrationem ſtatuitur $x = \infty$, ita vt totum negotium in inueſtigando valore ipſius S verſetur.

§. 16. Hunc in finem multiplicemus vtrinque per $2 \operatorname{fin.} \alpha$, et cum in genere ſit

$$2 \operatorname{fin.} \alpha \operatorname{cof.} \phi = \operatorname{fin.} (\alpha + \phi) - \operatorname{fin.} (\phi - \alpha)$$

hac reductione in ſingulis terminis facta perueniemus ad hanc aequationem:

$$2S \operatorname{fin.} \alpha = \gamma \operatorname{fin.} (\alpha + \zeta) + \gamma \operatorname{fin.} (\alpha - \zeta) + (\gamma - 2\beta) \operatorname{fin.} (3\alpha - \zeta)$$
$$- (\gamma - 2\beta) \operatorname{fin.} (\alpha - \zeta) - (\gamma - 4\beta) \operatorname{fin.} (3\alpha - \zeta)$$
$$+ (\gamma - 4\beta) \operatorname{fin.} (5\alpha - \zeta) - \cdots + (\gamma - 2(k-1)\beta) \operatorname{fin.} (2k-1)\alpha - \zeta)$$
$$- (\gamma - 6\beta) \operatorname{fin.} (7\alpha - \zeta) - \cdots$$

vbi praeter primum et vltimum terminum omnes reliqui contrahi poſſunt, ita vt prodeat

$$2S \operatorname{fin.} \alpha = \gamma \operatorname{fin.} (\alpha + \zeta) + 2\beta \operatorname{fin} (\alpha - \zeta) + 2\beta \operatorname{fin.} (3\alpha - \zeta) + 2\beta \operatorname{fin.} (5\alpha - \zeta)$$
$$- \cdots \operatorname{fin.} ((2k-1)\alpha - \zeta) +$$

§. 17. Iam pro hac ſerie ſummanda ponamus porro

$$T = 2 \operatorname{fin.} (\alpha - \zeta) + 2 \operatorname{fin.} (3\alpha - \zeta) + 2 \operatorname{fin.} (5\alpha - \zeta) + \cdots + 2 \operatorname{fin.} (2k-3)\alpha - \zeta)$$

vt habeamus

2 S

;β) cof.(4α−ζ)
)α−ζ)
, quae ergo
alis propoſitae,
, ita vt totum
ſetur.
vtrinque per
ζ)
perueniemus ad
(3α−ζ)
(3α−ζ)
)β) fin.(2k-1)α-ζ)
omnes reliqui
)+2β fin.(5α−ζ)
ponamus porro
2 fin.(2k-3)α-ζ)
2 S

$$2S \operatorname{fin.} \alpha = \gamma \operatorname{fin.} (\alpha + \zeta) + (\gamma - 2(k-1)\beta) \operatorname{fin.} (2k-1)\alpha - \zeta) + \beta T.$$

Iam multiplicemus, vt hactenus, per $\operatorname{fin.} \alpha$, et cum ſit

$$2 \operatorname{fin.} \alpha \operatorname{fin.} \phi = \operatorname{cof.} (\phi - \alpha) - \operatorname{cof.} (\phi + \alpha),$$

facta hac reductione naneiſcimur

$$T \operatorname{fin.} \alpha = + \operatorname{cof.} \zeta + \operatorname{cof.} (2\alpha - \zeta) + \operatorname{cof.} (4\alpha - \zeta) + \cdots + \operatorname{cof.} (2(k-2)\alpha - \zeta)$$
$$- \operatorname{cof.} (2\alpha - \zeta) - \operatorname{cof.} (4\alpha - \zeta) - \operatorname{cof.} (6\alpha - \zeta) - \cdots - \operatorname{cof.} (2(k-1)\alpha - \zeta)$$

vnde deletis terminis, quae ſe mutuo deſtruunt, remanebit tantum iſta expreſſio:

$$T \operatorname{fin.} \alpha = \operatorname{cof.} \zeta - \operatorname{cof.} (2(k-1)\alpha - \zeta).$$

Cum igitur ſit $\alpha = \frac{m\pi}{k}$ erit

$$2(k-1)\alpha = 2m\pi - \frac{2m\pi}{k},$$

cuius loco ſcribere licet $-\frac{2m\pi}{k}$, vnde ob $\zeta = \frac{\theta(k-m)}{k}$ erit

$$T \operatorname{fin.} \alpha = \operatorname{cof.} \frac{\theta(k-m)}{k} - \operatorname{cof.} \left(\frac{2m\pi + \theta(k-m)}{k}\right).$$

§. 18. Nunc vero notetur in genere eſſe

$$\operatorname{cof.} p - \operatorname{cof.} q = 2 \operatorname{fin.} \frac{q+p}{2} \operatorname{fin.} \frac{q-p}{2},$$

quare cum ſit

$$p = \frac{\theta(k-m)}{k} \text{ et } q = \frac{2m\pi + \theta(k-m)}{k}, \text{ erit}$$
$$\frac{q+p}{2} = \frac{m\pi + \theta(k-m)}{k} \text{ et } \frac{q-p}{2} = \frac{m\pi}{k},$$

vnde ſequitur fore

$$T \operatorname{fin.} \alpha = 2 \operatorname{fin.} \left(\frac{m\pi + \theta(k-m)}{k}\right) \operatorname{fin.} \frac{m\pi}{k}, \text{ ideoque}$$
$$T = 2 \operatorname{fin.} \left(\frac{m\pi + \theta(k-m)}{k}\right), \text{ ob } \alpha = \frac{m\pi}{k}.$$

§. 19. Hoc igitur valore T inuento reperiemus porro

$$2S \operatorname{fin.} \alpha = \gamma \operatorname{fin.} (\alpha + \zeta) + (\gamma - 2(k-1)\beta) \operatorname{fin.} (2k-1)\alpha - \zeta)$$
$$+ 2\beta \operatorname{fin.} \left(\frac{m\pi + \theta(k-m)}{k}\right)$$

quae ob $\frac{m\pi+\theta(k-m)}{k}=\alpha+\zeta$ reducitur ad hanc formam:

$$2\,S\,\text{fin.}\,\alpha=(\gamma+2\beta)\,\text{fin.}(\alpha+\zeta)+(\gamma-2(k-1)\beta)\,\text{fin.}((2k-1)\alpha-\zeta),$$

quae ita repraesentari potest:

$$2\,S\,\text{fin.}\,\alpha=(\gamma+2\beta)(\text{fin.}(\alpha+\zeta)+\text{fin.}((2k-1)\alpha-\zeta))-2k\beta\,\text{fin.}((2k-1)\alpha-\zeta),$$

vbi pro parte priore, ob

$$\text{fin.}\,p+\text{fin.}\,q=2\,\text{fin.}\,\tfrac{p+q}{2}\,\text{cof.}\,\tfrac{p-q}{2},\ \text{erit}$$

$$\tfrac{p+q}{2}=\alpha k\ \text{et}\ \tfrac{p-q}{2}=(k-1)\alpha-\zeta;$$

vnde pars ipsa prior fit

$$2(\gamma+2\beta)\,\text{fin.}\,\alpha k\,\text{cof.}((k-1)\alpha-\zeta),$$

vbi cum sit $\alpha k=m\pi$, erit $\text{fin.}\,\alpha k=0$, ita vt tantum superfit

$$2\,S\,\text{fin.}\,\alpha=-2\beta k\,\text{fin.}((2k-1)\alpha-\zeta)\ \text{hincque}$$

$$S=-\tfrac{\beta k\,\text{fin.}((2k-1)\alpha-\zeta)}{\text{fin.}\,\alpha}.\ \text{Est vero}$$

$$(2k-1)\alpha-\zeta=2m\pi-\tfrac{m\pi}{k}-\tfrac{\theta(k-m)}{k};$$

omisso termino $2m\pi$ erit igitur

$$S=+\frac{\pi\,\text{fin.}\frac{(m\pi+\theta(k-m))}{k}}{\text{fin.}\frac{m\pi}{k}}$$

ideoque valor quaesitus erit

$$\frac{S}{k\,\text{fin.}\,\theta}=+\frac{\pi\,\text{fin.}\frac{(m\pi+\theta(k-m))}{k}}{k\,\text{fin.}\,\theta\,\text{fin.}\frac{m\pi}{k}}$$

quae forma reducitur ad hanc:

$$\frac{\pi\,\text{fin.}\frac{(m(\pi-\theta)+\theta)}{k}}{k\,\text{fin.}\,\theta\,\text{fin.}\frac{m\pi}{k}}.$$

§. 20.

ad hanc formam:
)β) fin. (2k-1)α-ζ),
)-2kβ fin.(2k-1)α-ζ),
, erit
-ζ;
)α-ζ),
, ita vt tantum su-
)α-ζ) hincque
-θ(k-m)/k;
§. 20.

§. 20 Contemplemur hic ante omnia casum quo $\theta=\frac{\pi}{2}$, et formula integralis proposita abit in hanc:

$$\int\frac{x^{m-1}\,dx}{1+x^{2k}}$$

cuius ergo valor, si post integrationem ponatur $x=\infty$, euadet

$$=\frac{\pi\,\text{fin.}\left(\frac{\pi}{2}+\frac{m\pi}{2k}\right)}{k\,\text{fin.}\frac{m\pi}{k}}=\frac{\pi\,\text{cof.}\frac{m\pi}{2k}}{k\,\text{fin.}\frac{m\pi}{k}}.$$

Quia igitur est

$$\text{fin.}\tfrac{m\pi}{k}=2\,\text{fin.}\tfrac{m\pi}{2k}\,\text{cof.}\tfrac{m\pi}{2k},$$

prodibit iste valor $=\frac{\pi}{2k\,\text{fin.}\frac{m\pi}{2k}}$, qui valor egregie conuenit cum eo, quem non ita pridem pro formula $\int\frac{x^{m-1}\,dx}{1+x^{k}}$ assignauimus, si quidem loco k scribatur $2k$.

§. 21. Euoluamus etiam casum quo $\theta=\pi$ et formula nostra integralis $\int\frac{x^{m-1}\,dx}{(1+x^{k})^{2}}$, cuius ergo, facto $x=\infty$, valor erit

$$\frac{\pi\,\text{fin.}\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{k\,\text{fin.}\,\theta\,\text{fin.}\frac{m\pi}{k}}=\frac{\pi}{k\,\text{fin.}\frac{m\pi}{k}}\cdot\frac{\text{fin.}\frac{m(\pi-\theta)}{k}+\theta}{\text{fin.}\,\theta}.$$

Huius autem posterioris fractionis, casu $\theta=\pi$, tam numerator quam denominator euanescit; quare, vt eius verus valor eruatur, loco vtriusque eius differentiale scribamus, quo facto ista fractio abibit in hanc: $\frac{d\theta\left(1-\frac{m}{k}\right)\text{cof.}\left(\frac{m(\pi-\theta)}{k}+\theta\right)}{d\theta\,\text{cof.}\,\theta}$, cuius valor facto $\theta=\pi$ nunc manifesto est $1-\frac{m}{k}$; sicque valor in-

integralis quaesitus erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k\,\text{sin.}\frac{m\pi}{k}}$, prorsus vti in superiore dissertatione inuenimus.

§. 22. Quo autem valorem generalem inuentum commodiorem reddamus, ponamus $\pi-\theta=\eta$, fietque $\text{sin.}\theta=\text{sin.}\eta$ et $\text{cos.}\theta=-\text{cos.}\eta$; tum vero erit angulus

$$\frac{m(\pi-\theta)}{k}+\theta=\frac{m\pi}{k}+\pi-\eta,$$

cuius sinus est $\text{sin.}\left(1-\frac{m}{k}\right)\eta$, vnde valor quaesitus nostrae formulae erit $\frac{\pi\,\text{sin.}\left(1-\frac{m}{k}\right)\eta}{k\,\text{sin.}\eta\,\text{sin.}\frac{m\pi}{k}}$, atque hinc tandem sequens adepti sumus

Theorema.

§. 23. *Si haec formula integralis:*

$$\int\frac{x^{m-1}\,dx}{1+2x^k\,\text{cos.}\eta+x^{2k}}$$

a termino $x=0$ *vsque ad terminum* $x=\infty$ *extendatur, eius valor erit* $=\frac{\pi\,\text{sin.}\left(1-\frac{m}{k}\right)\eta}{k\,\text{sin.}\eta\,\text{sin.}\frac{m\pi}{k}}$, *siue cum sit*

$$\text{sin.}\left(1-\frac{m}{k}\right)\eta=\text{sin.}\eta\,\text{cos.}\frac{m\eta}{k}-\text{cos.}\eta\,\text{sin.}\frac{m\eta}{k},$$

iste valor etiam hoc modo exprimi potest:

$$\frac{\pi\,\text{cos.}\frac{m\eta}{k}}{k\,\text{sin.}\frac{m\pi}{k}}-\frac{\pi\,\text{sin.}\frac{m\eta}{k}}{k\,\text{tang.}\eta\,\text{sin.}\frac{m\pi}{k}}.$$

§. 24

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§. 24

§. 24. Consideremus nunc alio modo hanc formulam integralem: $\int\frac{x^{m-1}\,dx}{1+2x^k\,\text{cos.}\eta+x^{2k}}$, cuius valor a termino $x=0$ vsque ad $x=1$ ponatur $=P$, eiusdem vero valor ab $x=1$ vsque ad $x=\infty$ ponatur $=Q$, ita vt $P+Q$ exhibere debeat ipsum valorem ante inuentum. Nunc vero pro valore Q inueniendo ponamus $x=\frac{1}{y}$ et formula nostra ita repraesentata:

$$\frac{x^m}{1+2x^k\,\text{cos.}\eta+x^{2k}}\frac{dx}{x},$$

ob $\frac{dx}{x}=-\frac{dy}{y}$ induet hanc formam:

$$-\int\frac{y^{-m}}{1+2y^{-k}\,\text{cos.}\eta+y^{-2k}}\frac{dy}{y}=-\int\frac{y^{2k-m-1}\,dy}{y^{2k}+2y^k\,\text{cos.}\eta+1}$$

cuius valor a termino $y=1$ vsque ad $y=0$ extendi debet. Commutatis igitur his terminis habebimus

$$Q=+\int\frac{y^{2k-m-1}\,dy}{y^{2k}+2y^k\,\text{cos.}\eta+1}$$

a termino $y=0$ vsque ad $y=1$.

§. 25. Quia in vtraque forma pro P et Q eadem conditio integrationis praescribitur, a termino 0 vsque ad 1, nihil impedit quo minus in posteriore loco y scribamus x, vnde pro $P+Q$ habebimus hanc formam integralem:

$$\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^k\,\text{cos.}\eta+x^{2k}}dx,$$

cuius valor, a termino $x=0$ vsque ad $x=1$ extensus, aequabitur huic expressioni: $\frac{\pi\,\text{sin.}\left(1-\frac{m}{k}\right)\eta}{k\,\text{sin.}\eta\,\text{sin.}\frac{m\pi}{k}}$. Comparatis igitur

 his

his binis formulis integralibus nancifcemur fequens Theorema notatu maxime dignum.

Theorema.

§. 26. *Haec formula integralis:*

$$\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \,\text{cof.}\, \eta + x^{2k}} dx,$$

a termino $x = 0$ *vsque ad terminum* $x = 1$ *extenfa, aequalis eſt huic formulae integrali:* $\int \frac{x^{m-1} dx}{1 + 2x^k \,\text{cof.}\, \eta + x^{2k}}$, *a termino* $x = 0$ *vsque ad terminum* $x = \infty$ *extenfae: vtriusque enim valor erit* $\frac{\pi \,\text{fin.}\, (1 - \frac{m}{k}) \eta}{k \,\text{fin.}\, \eta \,\text{fin.}\, \frac{m\pi}{k}}$.

§. 27. Quod fi hanc fractionem: $\frac{\text{fin.}\, \eta}{1 + 2x^k \,\text{cof.}\, \eta + x^{2k}}$ in feriem infinitam euoluamus, quae fit,

$$\text{fin.}\, \eta + A x^k + B x^{2k} + C x^{3k} + D x^{4k} + E x^{5k} \text{ etc.}$$

per denominatorem multiplicando peruenienus ad hanc expreffionem infinitam:

$$\begin{aligned} \text{fin.}\,\eta = \text{fin.}\,\eta &+ A x^k + B x^{2k} + C x^{3k} + D x^{4k} + E x^{5k} + F x^{6k} + \text{etc.} \\ &+ 2\,\text{fin.}\,\eta\,\text{cof.}\,\eta + 2A\,\text{cof.}\,\eta + 2B\,\text{cof.}\,\eta + 2C\,\text{cof.}\,\eta + 2D\,\text{cof.}\,\eta + 2E\,\text{cof.}\,\eta + \text{etc.} \\ &+ \text{fin.}\,\eta + A + B + C + D \text{ etc.} \end{aligned}$$

vnde fingulis terminis ad nihilum reductis reperiemus

1° $A + 2\,\text{fin.}\,\eta\,\text{cof.}\,\eta = 0$, hincque $A = -\,\text{fin.}\,2\eta$,
2° $B + 2A\,\text{cof.}\,\eta + \text{fin.}\,\eta = 0$, vnde fit $B = \text{fin.}\,3\eta$,
3° $C + 2B\,\text{cof.}\,\eta + A = 0$, vnde fit $C = -\,\text{fin.}\,4\eta$,
4° $D + 2C\,\text{cof.}\,\eta + B = 0$, vnde fit $D = \text{fin.}\,5\eta$
etc. etc.

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− fin. 4 η,
= fin. 5 η
ita

ita vt noftra fractio $\frac{\text{fin.}\, \eta}{1 + 2x^k \,\text{cof.}\, \eta + x^{2k}}$ refoluatur in hanc feriem:

$$\text{fin.}\,\eta - x^k\,\text{fin.}\,2\eta + x^{2k}\,\text{fin.}\,3\eta - x^{3k}\,\text{fin.}\,4\eta + x^{4k}\,\text{fin.}\,5\eta \text{ etc.}$$

§. 28. Multiplicemus nunc hanc feriem per $x^{m-1} dx + x^{2k-m-1} dx$ et poſt integrationem faciamus $x = 1$, vt obtineamus valorem huius formulae:

$$\text{fin.}\,\eta \int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \,\text{cof.}\, \eta + x^{2k}} dx$$

pro cafu $x = 1$, hocque modo peruenienus ad geminas fequentes feries:

$$\frac{\text{fin.}\,\eta}{m} - \frac{\text{fin.}\,2\eta}{m+k} + \frac{\text{fin.}\,3\eta}{m+2k} - \frac{\text{fin.}\,4\eta}{m+3k} + \frac{\text{fin.}\,5\eta}{m+4k} - \text{etc.}$$
$$\frac{\text{fin.}\,\eta}{2k-m} - \frac{\text{fin.}\,2\eta}{3k-m} + \frac{\text{fin.}\,3\eta}{4k-m} - \frac{\text{fin.}\,4\eta}{5k-m} + \frac{\text{fin.}\,5\eta}{6k-m} - \text{etc.}$$

Aggregatum igitur harum duarum feriesum iunctim fumtarum aequabitur huic valori: $\frac{\pi \,\text{fin.}\, (1 - \frac{m}{k}) \eta}{k \,\text{fin.}\, \frac{m\pi}{k}}$, vnde fubiungamus adhuc iftud Theorema:

Theorema.

§. 29. *Si* η *denotet angulum quemcunque, litterae vero* m *et* k *pro lubitu accipiantur, ex iisque binae fequentes feries formentur:*

$$P = \frac{\text{fin.}\,\eta}{m} - \frac{\text{fin.}\,2\eta}{m+k} + \frac{\text{fin.}\,3\eta}{m+2k} - \frac{\text{fin.}\,4\eta}{m+3k} + \frac{\text{fin.}\,5\eta}{m+4k} - \text{etc.}$$
$$Q = \frac{\text{fin.}\,\eta}{2k-m} - \frac{\text{fin.}\,2\eta}{3k-m} + \frac{\text{fin.}\,3\eta}{4k-m} - \frac{\text{fin.}\,4\eta}{5k-m} + \frac{\text{fin.}\,5\eta}{6k-m} - \text{etc.}$$

neutrius quidem fumma exhiberi poteſt, vtriusque autem iunctim fumtae fumma erit

P +

$$P + Q = \frac{\pi \sin.(1 - \frac{m}{k})\eta}{k \sin.\frac{m\pi}{k}}.$$

Corollarium.

§. 30. Quod si ergo angulum η infinite paruum capiamus, vt fiat

$$\sin.\eta = \eta;\ \sin.2\eta = 2\eta;\ \sin.3\eta = 3\eta;\ \text{etc.}$$

quia in formula summae fiet

$$\sin.(1 - \tfrac{m}{k})\eta = (1 - \tfrac{m}{k})\eta;$$

si vtrinque per η diuidamus, obtinebimus sequentem seriem geminatam:

$$\frac{1}{m} - \frac{2}{m+k} + \frac{3}{m+2k} - \frac{4}{m+3k} + \frac{5}{m+4k} - \text{etc.}$$

$$\frac{1}{k-m} - \frac{2}{2k-m} + \frac{3}{3k-m} - \frac{4}{4k-m} + \frac{5}{5k-m} - \text{etc.}$$

cuius ergo summa erit $(1 - \frac{m}{k})\frac{\pi}{k \sin.\frac{m\pi}{k}}$; vbi notetur, ambas istas series non incongrue in hanc simplicem contrahi posse:

$$\frac{2k}{m(k-m)} - \frac{8k}{(m+k)(2k-m)} + \frac{18k}{(m+2k)(3k-m)} - \frac{32k}{(m+3k)(4k-m)} + \text{etc.}$$

vbi numeratores sunt numeri quadrati duplicati.

§. 31. Formulae autem, quarum valores hactenus inuenimus, multo concinnius et elegantius exprimi possunt, si loco exponentis m scribamus $k - n$, tum enim in valore integrali inuento fiet $(1 - \frac{m}{k})\eta = \frac{n\eta}{k}$; at vero pro denominatore fiet $\frac{m\pi}{k} = \pi - \frac{n\pi}{k}$, cuius sinus erit $\sin.\frac{n\pi}{k}$; sicque nostra formula inuenta hanc induet formam: $\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}}$, quae ergo exprimet valorem huius formulae integralis:

m η infinite paruum

$3\eta = 3\eta$; etc.

mus sequentem seriem

— etc.

— etc.

; vbi notetur, am-

mplicem contrahi posse:

$- \frac{32k}{(m+3k)(4k-m)} + \text{etc.}$

duplicati.

rum valores hactenus

tius exprimi possunt, si

tum enim in valore

; at vero pro deno-

us erit $\sin.\frac{n\pi}{k}$; sicque

formam: $\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}}$,

ormulae integralis:

$$\int \frac{x^{k-n-1}dx}{1 + 2x^k \cos.\eta + x^{2k}},$$

ab $x = 0$ vsque ad $x = \infty$, vt et huius formulae:

$$\int \frac{x^{k-n-1} + x^{k+n-1}dx}{1 + x^k \cos.\eta + x^{2k}},$$

a termino $x = 0$ vsque ad terminum $x = 1$; et quia vtriusque valor est $\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}}$, perspicuum est eum manere eundem, etsi loco n scribatur $-n$, ex quo prior formula ita repraesentari poterit:

$$\int \frac{x^{k \pm n-1}}{1 + x^k \cos.\eta + x^{2k}};$$

at posterior formula ob hanc ambiguitatem nullam plane mutationem patitur.

§. 32. Ponendo $m = k - n$ etiam series nostra geminata pulchriorem accipiet faciem; habebitur enim

$$\frac{\sin.\eta}{k-n} - \frac{\sin.2\eta}{2k-n} + \frac{\sin.3\eta}{3k-n} - \frac{\sin.4\eta}{4k-n} + \text{etc.}$$

$$\frac{\sin.\eta}{k+n} - \frac{\sin.2\eta}{2k+n} + \frac{\sin.3\eta}{3k+n} - \frac{\sin.4\eta}{4k+n} + \text{etc}$$

cuius ergo summa erit $\frac{\pi \sin.\frac{n\eta}{k}}{k \sin.\frac{n\pi}{k}}$. Tum vero si hae geminae series in vnam contrahantur et vtrinque per $2k$ diuidatur, obtinebitur sequens summatio memoratu digna:

$$\frac{\pi \sin.\frac{n\eta}{k}}{2kk \sin.\frac{n\pi}{k}} = \frac{\sin.\eta}{kk-nn} - \frac{2\sin.2\eta}{4kk-nn} + \frac{3\sin.3\eta}{9kk-nn} - \frac{4\sin.4\eta}{16kk-nn} + \text{etc.}$$

§. 33.

§. 33. Quod si haec postrema series differentietur, sumendo solum angulum η variabilem, ob

$d.\sin.\frac{n\eta}{k} = \frac{n\,d\eta}{k}\cos.\frac{n\eta}{k}$ habebimus

$$\frac{\pi n\cos.\frac{n\eta}{k}}{2k^2\sin.\frac{n\pi}{k}} = \frac{\cos.\eta}{kk-nn} - \frac{4\cos.2\eta}{4kk-nn} + \frac{9\cos.3\eta}{9kk-nn} - \frac{16\cos.4\eta}{16kk-nn} + \text{etc.}$$

Vnde si sumatur $\eta = 0$, orietur ista summatio:

$$\frac{\pi n}{2k^2\sin.\frac{n\pi}{k}} = \frac{1}{kk-nn} - \frac{4}{4kk-nn} + \frac{9}{9kk-nn} - \frac{16}{16kk-nn} + \text{etc.}$$

sin autem sumatur $\eta = 90° = \frac{\pi}{2}$, erit $\cos.\eta = 0$, $\cos.2\eta = -1$, $\cos.3\eta = 0$, $\cos.4\eta = +1$ etc. vnde nascitur sequens series:

$$\frac{n\pi\cos.\frac{n\pi}{2k}}{2k^2\sin.\frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}$$

Quia autem $\sin.\frac{n\pi}{k} = 2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k}$, erit eiusdem seriei summa $\frac{n\pi}{4k^2\sin.\frac{n\pi}{2k}}$.

§. 34. At si series illa §. 32 exhibita in $d\eta$ ducatur et integretur, ob

$\int d\eta\sin.\frac{n\eta}{k} = -\frac{k}{n}\cos.\frac{n\eta}{k}$, erit

$$C - \frac{\pi\cos.\frac{n\eta}{k}}{2nk\sin.\frac{n\pi}{k}} = -\frac{\cos.\eta}{kk-nn} + \frac{\cos.2\eta}{4kk-nn} - \frac{\cos.3\eta}{9kk-nn} + \frac{\cos.4\eta}{16kk-nn} + \text{etc.}$$

Vt autem hic constantem addendam C definiamus, sumamus $\eta = 0$, fietque

$$C - \frac{\pi}{2nk\sin.\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.}$$

quare

quare si huius seriei summa aliunde pateat, constans C definiri poterit. Series autem haec in sequentem geminatam resolui potest:

$$2nC - \frac{\pi}{k\sin.\frac{n\pi}{k}} = \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.}$$

$$-\frac{1}{k-n} + \frac{1}{2k-n} - \frac{1}{3k-n} + \frac{1}{4k-n} - \text{etc.}$$

§. 35. Cum igitur in *Introductione in Analysin Infinitorum* pag. 142. ad hanc peruenissem seriem:

$$\frac{1}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \text{etc.}$$

$$= \frac{\pi}{2kn\sin.\frac{n\pi}{k}} - \frac{1}{2nn}$$

(hic scilicet loco litterarum ibi adhibitarum m et n scripsi n et k) hoc valore adhibito nostra aequatio erit

$$C - \frac{\pi}{2nk\sin.\frac{n\pi}{k}} = \frac{1}{2nn} - \frac{\pi}{2nk\sin.\frac{n\pi}{k}},$$

vnde fit $C = \frac{1}{2nn}$. Hinc ergo habebimus istam summationem:

$$\frac{\pi\cos.\frac{n\eta}{k}}{2nk\sin.\frac{n\pi}{k}} - \frac{1}{2nn} = \frac{\cos.\eta}{kk-nn} - \frac{\cos.2\eta}{4kk-nn}$$

$$+ \frac{\cos.3\eta}{9kk-nn} - \frac{\cos.4\eta}{16kk-nn} + \text{etc.}$$

quae series vtique omni attentione digna videtur.