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INVESTIGATIO

FORMVLAE INTEGRALIS

ALIS

CASV QVO POST INTEGRATIONEM STATIVITVR

$x = \infty$.

$$\int \frac{x^{m-1} dx}{(1+x^k)^n}$$

Iam scis notum est, huius formulae integrale partim logarithmicas hanc progressionem constitutre:

$$\begin{aligned} & -\frac{2}{k} \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} I' V \left(1 - 2x \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} + xx \right) \\ & - \frac{2}{k} \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} I' V \left(1 - 2x \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} + xx \right) \\ & - \frac{2}{k} \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} I' V \left(1 - 2x \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} + xx \right) \\ & - \dots \\ & - \frac{2}{k} \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} I' V \left(1 - 2x \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} + xx \right) \end{aligned}$$

vbi i denotat maximum numerum imparem ipso k non maiorem, hac tamen restrictione, vt i si k fuerit impar, idoque $i = k$, vltimum membrum ad dñidum reduci debat. Quoniam, si huius progressionis summan inuestigare velimus, duo casis erant constitendi: alter quo k est numerus par et $i = k - 1$, alter vero quo k est impar et $i = k$.

Euolutio casus prioris, quo k est numerus par et

$$i = k - 1.$$

§. 3. Hoc ergo casu, posito $x = \infty$, formula $-\frac{2}{k} \operatorname{cof}_{\frac{m}{k}}$

multiplicatur per hanc Cosinum seriem:
 $\operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} + \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} + \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} + \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} + \dots + \operatorname{cof}_{\frac{(k-1)m}{k}} \frac{\pi}{k}$,
 cuius summam statuamus $= S$. Dicamus hanc seriem in fin. $\frac{n\pi}{k}$, et cum in genere sit

$$\sin \frac{n\pi}{k} \operatorname{cof}_{\frac{n\pi}{k}} \frac{\pi}{k} = \frac{1}{i} \sin \frac{(i+1)n\pi}{k} - \frac{1}{i} \sin \frac{(i-n)n\pi}{k},$$

facta hac reductione habebimus

$S \sin \frac{n\pi}{k} = \frac{1}{i} \sin \frac{n\pi}{k} + \frac{1}{i} \sin \frac{2n\pi}{k} + \dots + \frac{1}{i} \sin \frac{(k-1)n\pi}{k}$
 $\quad - \frac{1}{i} \sin \frac{n\pi}{k} - \frac{1}{i} \sin \frac{2n\pi}{k} - \dots - \frac{1}{i} \sin \frac{(k-1)n\pi}{k}$

vbi i denotat numerum imparem non maiorem quam k . Hinc si k fuerit numerus par, exit $i = k - 1$; ac si k fuerit numerus impar, hanc progressionem continuari oportet usque ad $i = k$; eius vero coefficientis duplo minor capi debet, seu loco $-\frac{1}{i}$ tantum scribi dolet $-\frac{1}{i}$, cuius irregula- ritas ratio in Calculo Integrali est exposita.

§. 2.

§. 2.

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§. 2. Cum haec partes quante iam evanescant possit
to $x = 0$, statuamus tamen $x = \infty$, et cum in genere sit
 $V(1 - 2x \operatorname{cof}_{\frac{m}{k}} \omega + xx) = x - \operatorname{cof}_{\frac{m}{k}} \omega$, erit

$I'(1 - 2x \operatorname{cof}_{\frac{m}{k}} \omega + xx) = I(x - \operatorname{cof}_{\frac{m}{k}} \omega) = Ix - \frac{\operatorname{cof}_{\frac{m}{k}} \omega}{x} = Ix$, ob $\frac{\operatorname{cof}_{\frac{m}{k}} \omega}{x} = 0$;

enones ergo illi logarithmi reducuntur ad eandem formam Ix , quae multiplicanda est per hanc seriem:

$$-\frac{2}{k} \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} - \frac{1}{k} \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k} - \dots - \frac{1}{k} \operatorname{cof}_{\frac{m}{k}} \frac{\pi}{k},$$

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vbi omnes termini praeter ultimum manifeste se defrunt, ita vt sit

$$S \sin. \frac{m\pi}{k} = i \sin. m\pi.$$

Iam vero quia nostri coefficientes m et k supponuntur integrati, vique erit $\sin. m\pi = 0$, ideoque etiam $S = 0$, nisi forte etiam fierit $\sin. \frac{m\pi}{k} = 0$, qui autem casus locum habere nequit, quoniam in integratione formulae propriae $\frac{x^{m-1} dx}{1+x^2}$ semper astini soleat esse $m < k$. Hoc igitur modo euismus est, casu quo post integrationem statuitur $x = \infty$, omnes parres logarithmicas integralis se defrunt.

Evolutio casus alterius, quo est k numerus impar

et $i = k$.

§. 4. Hoc ergo casu, sumto $x = \infty$, formula Ix

multiplicator per hanc feriem :

$$-\frac{1}{k} \text{cof.} \frac{m\pi}{k} - \frac{1}{k} \text{cof.} \frac{m\pi}{k} - \frac{1}{k} \text{cof.} \frac{m\pi}{k} - \dots - i \text{cof.} \frac{im\pi}{k},$$

vbi terminus penultimus est $- \frac{1}{k} \text{cof.} \frac{(k-2)m\pi}{k}$, pro ultimo vero termino erit $\text{cof.} m\pi = \pm 1$, signo superiore valente si n sit numerus par, inferiore si impar; quare remoto termino ultimo pro reliquis ponamus

$$\text{cof.} \frac{m\pi}{k} + \text{cof.} \frac{m\pi}{k} + \text{cof.} \frac{m\pi}{k} + \dots + \text{cof.} \frac{(k-2)m\pi}{k} = S$$

ita vt multiplicator ipsius logarithmi x sit

$$- \frac{i\pi}{k} - i \text{cof.} m\pi.$$

Hinc procedendo vt ante fiet

$$S \sin. \frac{m\pi}{k} = i \sin. \frac{m\pi}{k} + i \sin. \frac{m\pi}{k} + i \sin. \frac{m\pi}{k} + \dots + i \sin. \frac{(k-1)m\pi}{k}$$

$$- i \sin. \frac{m\pi}{k} - i \sin. \frac{m\pi}{k} - i \sin. \frac{m\pi}{k} - \dots - i \sin. \frac{(k-1)m\pi}{k},$$

vbi

destrunt,

vbi iterum omnes termini praeter ultimum se mutuo tollunt, ita vt hinc prodeat

$$S \sin. \frac{m\pi}{k} = i \sin. \frac{(k-1)m\pi}{k} = i \sin. (m\pi - \frac{m\pi}{k});$$

at vero est

$$\sin. (m\pi - \frac{m\pi}{k}) = \sin. m\pi \text{cof.} \frac{m\pi}{k} - \text{cof.} m\pi \sin. \frac{m\pi}{k},$$

vbi notetur esse $\sin. m\pi = 0$, ob m numerum integrum; habebimus ergo

$$S \sin. \frac{m\pi}{k} = - i \text{cof.} m\pi \sin. \frac{m\pi}{k},$$

consequenter multiplicator ipsius Ix erit

$$= i \text{cof.} m\pi - i \text{cof.} m\pi = 0,$$

fique manifestum est, siue k sit numerus par siue impar, omnia membra logarithmica in nostro integrali se mutuo destruere, siquidem post integrationem statuamus $x = \infty$,

eris impar

, formula Ix

$$- i \text{cof.} \frac{m\pi}{k},$$

, pro ultimo eriore valente juare remoto

$$\frac{-2}{k} \sin. \frac{m\pi}{k} = S$$

§. 5. Consideremus nunc etiam partes a circulo pendentes, ex quibus integrale nostrae formulae componiatur.

Hae autem partes sequentem progressionem constituerent sunt comparatae :

$$\frac{2}{k} \sin. \frac{m\pi}{k} \text{Atang.} \frac{x \sin. \frac{\pi}{k}}{1-x \text{cof.} \frac{\pi}{k}} + \frac{2}{k} \sin. \frac{3m\pi}{k} \text{Atang.} \frac{x \sin. \frac{3\pi}{k}}{1-x \text{cof.} \frac{3\pi}{k}}$$

$$+ \frac{2}{k} \sin. \frac{5m\pi}{k} \text{Atang.} \frac{x \sin. \frac{5\pi}{k}}{1-x \text{cof.} \frac{5\pi}{k}} + \frac{2}{k} \sin. \frac{7m\pi}{k} \text{Atang.} \frac{x \sin. \frac{7\pi}{k}}{1-x \text{cof.} \frac{7\pi}{k}}$$

$$+ \dots + \frac{2}{k} \sin. \frac{im\pi}{k} \text{Atang.} \frac{x \sin. \frac{i\pi}{k}}{1-x \text{cof.} \frac{i\pi}{k}}$$

vbi in ultimo membro est vel $i = k - r$, vel $i = k$; prius dicilicet valet si i est numerus par, posterius si impar.

§. 6. Cum etiam omnia haec membra emanentia posito $x=0$, faciamus pro initio nostro $x=\infty$. In genere igitur fit

$$A \tan \frac{x \sin \frac{i\pi}{k}}{x - x \cos \frac{i\pi}{k}} = A \tan \left(- \tan \frac{i\pi}{k} \right).$$

Est vero

$$-\tan \frac{i\pi}{k} = +\tan \frac{(k-i)\pi}{k},$$

ex quo hic arcus fit $= \frac{(k-i)\pi}{k}$. Hinc ergo loco i scribendo successive numeros 1, 3, 5, 7 etc. istae partes nostri integrandis quaevis erunt

$$\begin{aligned} & \frac{ik-i\pi}{k} \sin \frac{im\pi}{k} + \frac{(k-i)\pi}{k} \sin \frac{im\pi}{k} + \frac{(k-i)\pi}{k} \sin \frac{im\pi}{k} \\ & + \frac{(k-i)\pi}{k} \sin \frac{im\pi}{k} + \frac{(k-i)\pi}{k} \sin \frac{im\pi}{k} + \dots \dots \end{aligned}$$

ubi casu, quo k est numerus par, progressi oportet usque ad $i=k$; ac si k sit numerus impar, usque ad $i=k$.

§. 7. Scavamus brevitas gratia

$$(k-1) \sin \frac{m\pi}{k} + (k-3) \sin \frac{im\pi}{k} + (k-5) \sin \frac{im\pi}{k} + \dots \dots$$

$$+ (k-i) \sin \frac{im\pi}{k} = S$$

ita ut integrale quedam sit $\frac{imS}{k^2}$, quandoquidem partes logarithmicas se nusso destruxerunt. Multiplicemus nunc triplete per $\sin \frac{m\pi}{k}$, et cum in genere fit

$$2 \sin \frac{m\pi}{k} \sin \frac{im\pi}{k} = \operatorname{cof} \frac{(k-1)m\pi}{k} - \operatorname{cof} \frac{(k-1)m\pi}{k},$$

falsa substitutione erit

S

membra evanescant nostro $x=\infty$. In

$$\begin{aligned} & 2S \sin \frac{m\pi}{k} = (k-1) \operatorname{cof} \frac{im\pi}{k} + (k-3) \operatorname{cof} \frac{im\pi}{k} + (k-5) \operatorname{cof} \frac{im\pi}{k} + \dots \dots \\ & -(k-1) \operatorname{cof} \frac{im\pi}{k} - (k-3) \operatorname{cof} \frac{im\pi}{k} - (k-5) \operatorname{cof} \frac{im\pi}{k} + \dots \dots \\ & \dots \dots + (k-i) \operatorname{cof} \frac{(i-1)m\pi}{k} \\ & - (k-i) \operatorname{cof} \frac{(i+1)m\pi}{k} \end{aligned}$$

quae series manifeste contrahitur in sequentem :

$$2S \sin \frac{m\pi}{k} = (k-1) \cdot 2 \operatorname{cof} \frac{im\pi}{k} - 2 \operatorname{cof} \frac{im\pi}{k} + \dots \dots + 2 \operatorname{cof} \frac{(k-1)m\pi}{k}$$

$$- (k-i) \operatorname{cof} \frac{(i+1)m\pi}{k}$$

tgo locc i scriben-
ae partes nostri in-

vbi, primo et ultimo membro sublati, regularem termini intermedii constituant seriem, pro cuius valore investigan-
do ponamus

$$T = \operatorname{cof} \frac{im\pi}{k} + \operatorname{cof} \frac{im\pi}{k} + \operatorname{cof} \frac{im\pi}{k} + \dots \dots + \operatorname{cof} \frac{(k-1)m\pi}{k},$$

ita vt sic

$$2S \sin \frac{m\pi}{k} = k - 1 - 2T - (k-i) \operatorname{cof} \frac{(i+1)m\pi}{k}.$$

Hic autem iterum conuenit dulos casus perpendere, prout k fuerit par vel impar.

Euolutio casus prioris, quo k est numerus par et

§. 8. Hoc ergo casus habebimus

$$T = \operatorname{cof} \frac{im\pi}{k} + \operatorname{cof} \frac{im\pi}{k} + \operatorname{cof} \frac{im\pi}{k} + \dots \dots + \operatorname{cof} \frac{(k-1)m\pi}{k}.$$

Multiplicemus denovo per $\sin \frac{m\pi}{k}$, et per reductiones supra indicatas habebimus

$$2T \sin \frac{m\pi}{k} = \sin \frac{im\pi}{k} + \sin \frac{im\pi}{k} + \sin \frac{im\pi}{k} + \dots \dots + \sin \frac{(k-1)m\pi}{k}$$

$$- \sin \frac{m\pi}{k} - \sin \frac{m\pi}{k} - \sin \frac{m\pi}{k} - \dots \dots - \sin \frac{(k-1)m\pi}{k};$$

deletis igitur terminis se mutuo tollentibus erit

T

$z T \sin. \frac{m\pi}{k} = - \sin. \frac{m\pi}{k} + \sin. \frac{(k-1)m\pi}{k}$

Eft vero

$\sin. \frac{(k-1)m\pi}{k} = \sin. (m\pi - \frac{\pi}{k}) = \sin. m\pi \cos. \frac{m\pi}{k} - \cos. m\pi \sin. \frac{m\pi}{k}$,
vbi $\sin. m\pi = 0$, quonobrem sicut $z T = -1 - \cos. m\pi$.

§. 9. Inuenio valore pro T colligetur fore

$z S \sin. \frac{m\pi}{k} = k$, ideoque $S = \frac{k}{z \sin. \frac{m\pi}{k}}$.

Denique vero ipfe valor formulae nostrae integrallis, quicun
quarumvis, erit $\frac{m\pi}{k}$, et nunc manifestum est, integrale nostrae
formulae, casu quo S est numerus par, fore $\overline{k \sin. \frac{m\pi}{k}}$, siqui-
dem post integrationem statuatur $x = \infty$.

Evolutio alterius casus, quo k est numerus impar et.

§. 10. Hoc ergo casu est

$T = \cos. \frac{m\pi}{k} + \cos. \frac{2m\pi}{k} + \cos. \frac{3m\pi}{k} + \dots + \cos. \frac{(k-1)m\pi}{k}$,
quae series multiplicata per z sin. $\frac{m\pi}{k}$ product ut ante
 $z T \sin. \frac{m\pi}{k} = \sin. \frac{m\pi}{k} + \sin. \frac{2m\pi}{k} + \sin. \frac{3m\pi}{k} + \dots + \sin. \frac{km\pi}{k}$,
 $= \sin. \frac{m\pi}{k} - \sin. \frac{m\pi}{k} - \sin. \frac{2m\pi}{k} - \sin. \frac{3m\pi}{k} - \dots - \sin. \frac{(k-2)m\pi}{k}$,
vnde de teis terminis se mutuo collentibus reperiuntur

$z T \sin. \frac{m\pi}{k} = - \sin. \frac{m\pi}{k} + \sin. m\pi$

ideoque

$z T = -1 + \sin. \frac{m\pi}{k} = 1$, ob $\sin. m\pi = 0$,

hincque

hincque porro sicut

$z S \sin. \frac{m\pi}{k} = k$;

quare cum valor integrallis quacunque sit $\frac{m\pi}{k}$. ut etiam hoc
casu integrale nostrum $= \overline{k \sin. \frac{m\pi}{k}}$, prouidus uti praecedente
casu. Hinc ergo deducimus sequens

fore

Theorem.

§. 11. Si habe formula differentialis: $\frac{x^{m-1} d x}{1+x^k}$, ita in-
tegratur, vt, posito $x=0$, integrata exangescat, ita vero si.
natur $x=\infty$, valer inde regula scilicet sicut $\overline{k \sin. \frac{m\pi}{k}}$,
sive k sit numerus par, sive impar. Huius Theorematis de-
monstratio ex praecedentibus est manifesta.

us impar et

§. 12. In evolutione huius formulae affutimus
esse $m < k$, quia alioquin membra logarithmica se non de-
trahent; at vero ne hac quidem limitatione nunc amplius
est opus. Casu enim quo fuerit $m=k$, integrale formulae
 $\frac{x^{m-1} d x}{1+x^k}$ efficit $\frac{1}{k}(1+x^k)$, quod factio $x=\infty$ fieret etiam
sive; verum hoc idem indicat, nostrum integrale utile $\overline{k \sin. \frac{m\pi}{k}}$.
Dummodo ergo m non fuerit maius quam k , nostra formula
veritati semper est conuenientia.

§. 13. Quoniam ne quidem necesse est ut expon-
entes m cu k sint numeri integri, dummodo non fuerit
 $m > k$; si enim fuerit $m = \frac{k}{l}$ et $k = \frac{x}{l}$, erit valor per no-
stram

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item formulam $\frac{\pi}{x} \lambda$, cuius centrum ita ostenditur. Quia

hoc casu formula integranda est $\int \frac{x^k}{1+x^k} \frac{dx}{x}$, ita autem $x = y^\lambda$, erit $\frac{dx}{x} = \frac{\lambda y^{\lambda-1}}{y} dy$ et formula fit:

$$\int \frac{y^k}{1+y^k} \frac{\lambda y^{\lambda-1}}{y} dy = \lambda \int \frac{y^{k+m-1}}{1+y^k} dy$$

cuius valor utique erit $\frac{\lambda \pi}{\lambda \ln \frac{y}{x}}$.

Alia demonstratio Theorematis.

§. 14. Denotet P valorem integralis $\int \frac{x^m}{1+x^k} \frac{dx}{x}$ a termino $x=0$ usque ad $x=1$; et Q valorem eiusdem integralis a termino $x=1$ usque ad $x=\infty$, ita ut $P+Q$ praecedit eum ipsum valorem, qui in theoremate contineatur. Nunc pro valore Q inveniendo statuatur $x=y$, unde fit $\frac{dx}{x} = -\frac{dy}{y}$, fieri que

$$Q = \int \frac{y^{k-m}}{1+y^k} \frac{-dy}{y} = - \int \frac{y^{k-m}}{1+y^k} \frac{dy}{y}$$

a termino $y=1$ usque ad $y=\infty$. Hinc igitur commutatis terminis erit $Q = + \int \frac{y^{k-m}}{1+y^k} \frac{dy}{y}$, a termino $y=0$ usque ad $y=1$. Iam quia hoc integrandi expeditio littera y ex calculo egreditur, loco y ferbere licet x , ita ut sit

$$Q = \int \frac{x^{k-m}}{1+x^k} \frac{dx}{x},$$

quae

quo facto habebimus

$$P+Q = \int \frac{x^m + x^{k-m}}{1+x^k} \frac{dx}{x}$$

item, quia P integratur a termino $x=0$ usque ad terminum $x=1$. Verum non

ia pridem demonstrauit, valorem huius formulae integralis inter terminos $x=0$ et $x=1$ contestum esse $= \frac{\pi}{k \ln \frac{1}{k}}$. Hinc igitur nascitur sequens Theorema non minus notandum.

Theorema.

§. 15. *Vale huius formulae integralis:*

$$\int \frac{x^m + x^{k-m}}{1+x^k} \frac{dx}{x}$$

inter terminos $x=0$ et $x=1$ continet, aequalis est valori *integrallis: $\int \frac{x^m}{1+x^k} \frac{dx}{x}$, inter terminos $x=0$ et $x=\infty$* *est integrallis: $\int \frac{x^{k-m}}{1+x^k} \frac{dx}{x}$, inter terminos $x=1$ et $x=\infty$* *continet.*

Item, quia hoc integrandi expedito littera y ex calculo

egreditur, loco y ferbere licet x , ita ut sit

$$Q = \int \frac{y^{k-m}}{1+y^k} \frac{dy}{y} = \int \frac{y^{k-m}}{1+y^k} \frac{dy}{x}$$

commutatis terminis erit $Q = + \int \frac{y^{k-m}}{1+y^k} \frac{dy}{x}$, a termino $y=1$ usque ad $y=\infty$.

Item, quia hoc integrandi expedito littera y ex calculo

$$Q = \int \frac{x^{k-m}}{1+x^k} \frac{dx}{x},$$

quae

quae asquatio, per x^{m-n} diuina ac per $(x^k - x^n)$ multiplicata, terminum negatuum a dextra ad sinistram transponendo, erit

$$\frac{x + \lambda k A x^k}{1 + x^n} = n A + B$$

quae aequatio manifesto subtiliter nequit, nisi sit $\lambda k A = 1$,
fue $A = \frac{1}{\lambda k}$, unde erit $x = n A + B = \frac{n}{\lambda k} + B$, sicque
erit $B = x - \frac{n}{\lambda k}$.

§. 17. Invenitis his valoribus pro literis A et B,
primum afflumus, integralia ita capi, ut euaneantur posito
 $x = 0$; tum vero posito $x = \infty$, quia exponentis n minor
supponitur quam k , membrum ab solutorum littera A affectum
sponte evanescit, ita ut hoc casu $x = \infty$ fiat

$$\int \frac{x^{m-n} dx}{(1+x^n)^{\frac{1}{n}}} = \left(x - \frac{n}{k} \right) \left(1 - \frac{n}{2k} \right) \left(x - \frac{n}{3k} \right) \dots \frac{\pi}{k \sin \frac{\pi}{n}}$$

Quod si iam primo capiamus $\lambda = 1$, quia ante inuenimus
pro codem casu $x = \infty$ esse

$$\int \frac{x^{m-n} dx}{1+x^n} = \frac{\pi}{k \sin \frac{\pi}{n}},$$

habebimus valorem itius integralis

$$\int \frac{x^{m-n} dx}{(1+x^n)^{\frac{1}{n}}} = \left(x - \frac{n}{k} \right) \frac{\pi}{k \sin \frac{\pi}{n}},$$

si quidem integrare etiam a termino $x = 0$ usque ad ter-
minum $x = \infty$ extendatur.

§. 18. Quod si iam simil modo ponamus $\lambda = 2$,
reperiatur pro iisdem terminis integrationis

$$\int x$$

ⁱ
initian trans-

$$\int \frac{x^{m-n} dx}{(1+x^n)^{\frac{2}{n}}} = \left(x - \frac{n}{k} \right) \left(x - \frac{n}{2k} \right) \frac{\pi}{k \sin \frac{\pi}{n}}$$

colem modo ii litterae λ continuo maiores valores trahit
autem, representantur sequentes integralium formae omni atref-
tione dignae:

$$\int \frac{x^{m-n} dx}{(1+x^n)^{\frac{3}{n}}} = \left(x - \frac{n}{k} \right) \left(x - \frac{n}{2k} \right) \left(x - \frac{n}{3k} \right) \frac{\pi}{k \sin \frac{\pi}{n}}$$

$$\int \frac{x^{m-n} dx}{(1+x^n)^{\frac{4}{n}}} = \left(x - \frac{n}{k} \right) \left(x - \frac{n}{2k} \right) \left(x - \frac{n}{3k} \right) \left(x - \frac{n}{4k} \right) \frac{\pi}{k \sin \frac{\pi}{n}}$$

$$\int \frac{x^{m-n} dx}{(1+x^n)^{\frac{5}{n}}} = \left(x - \frac{n}{k} \right) \left(x - \frac{n}{2k} \right) \left(x - \frac{n}{3k} \right) \left(x - \frac{n}{4k} \right) \left(x - \frac{n}{5k} \right) \frac{\pi}{k \sin \frac{\pi}{n}}$$

etc.
etc.

§. 19. Quare si littera n denotet numerum, quem-
unque integrum, pro formula in titulo expressa, si eius
integrale a termino $x = 0$ usque ad $x = \infty$ extendatur, eius
valor sequenti modo se habebit:

$$\left(x - \frac{n}{k} \right) \left(x - \frac{n}{2k} \right) \left(x - \frac{n}{3k} \right) \left(x - \frac{n}{4k} \right) \dots \left(x - \frac{n}{(n-1)k} \right) \frac{\pi}{k \sin \frac{\pi}{n}}$$

qui ergo conueniet hanc formulae integrali:

$$\int \frac{x^{m-n} dx}{(1+x^n)^{\frac{n}{n}}}.$$

vsque ad ter-
minum $x = \infty$

§. 20. Hic quidem necessario pro n ali numeri
praeter integros accipi non licet: ut vero per methodum
interpolationum, quac fuisse iam passim est explicata, hanc
integrationem etiam ad catus, quibus exponentis n est num-
erus fragiis, extendere licet. Quod si enim sequentes formu-
lae

Nec integrals a termino $y=0$ usque ad $y=1$ extenduntur, in genere valer notiae formulae propriae ita representari poterit:

$$\int \frac{x^{n-1} dx}{(1+x^k)^{\frac{m}{k}}} = \frac{\pi}{k} \int y^{nk-m-1} dy \left(\frac{1-y^k}{1-y^k}\right)^{\frac{m}{k}-1}$$

Vnde si fuerit $m=1$ et $k=2$, sequitur fore

$$\int \frac{dx}{(1+x^2)^{\frac{m}{2}}} = \frac{\pi}{2} \int y^{k-m-1} dy \left(\frac{1-y^2}{1-y^2}\right)^{\frac{m}{2}-1}$$

Ita si $n=1$ erit

$$\int \frac{dx}{(1+x^k)^{\frac{1}{k}}} = \int \frac{y dy}{\sqrt{1-y^k}}$$

cuius veritas sponte elucet, quia integrare prius generatum est $\int \frac{dx}{\sqrt{1-x^2}}$, postea vero $= 1 - \sqrt{(1-y^k)}$, quae, falso $x=\infty$ et $y=1$, viisque sunt aequalia. Ceterum pro hac integratione generali notasse iutabatur, exponentem unitate minorem accipi non posse, quia aliquin valores amborum integralium in infinitum excrescent.

tenduntur, in
representari

VALORIS INTEGRALIS

$$\int \frac{y^{(n-1)} dy}{\sqrt{1-y^k}} = \int \frac{y^{m-1} dy}{1-x^k \cos \theta + x^k}$$

A TERMINO $x=0$ VSQUE AD $x=\infty$ EXTENSI.

s generatim est
e, falso $x=\infty$
o hac integra-

itate minorem
rum integrali-

§. 1.

Quaternus primo integrare formulae proposatae indefini- tum, arque adeo omnes operationes ex primis Analy- ficos principiis repeatamus. Ac primo quidem, quoniam de- nominator in factores reales simplices resolut nequit, sit in genere eius factor duplicitus quicunque $x-a$ et $x-b$; evidens enim est, denominatorem fore productum ex k hu- viismodi factoribus duplicatis. Cum igitur, proflo hoc factore nominator duplief modo etancere debet, siue si ponatur

$$x = \operatorname{cof.} \omega + \gamma - 1 \sin. \omega, \quad \text{siue}$$

$$x = \operatorname{cof.} \omega - \gamma - 1 \sin. \omega.$$

Constat autem omnes potestates harum formulaarum ita con-

node exprimi posse, ut si

$$(\operatorname{cof.} \omega \pm \gamma - 1 \sin. \omega)^k = \operatorname{cof.} k \omega \pm \gamma - 1 \sin. k \omega;$$

hinc igitur erit

$$x^k = \operatorname{cof.} k \omega + \gamma - 1 \sin. k \omega \pm \gamma - 1 \sin. 2k \omega,$$