OBSERVATIONES
IN ALIQVOT
THEOREMATA
ILLVSTR. DE LA GRANGE.

Postquam aliquod Theorema, ex iis quae non ita pridem demonstraui, quo ostendi, formulae integralis $\int \frac{(x-1)dx}{lx}$, si post integrationem ponatur $x = 1$, valorem esse $= l2$, cum illustri Domino *de la Grange* communicassem, is nouitate huius argumenti permotus, non solum felicissimo successu eius demonstrationem penetrauit, sed etiam plurima alia praeclara inuenta inde deduxit, quorum vberior enucleatio scientiae analyticae maxima incrementa polliceri videtur, ex quo genere aliquot praeclarissima specimina mecum beneuole communicauit, quae statim summo studio sum perscrutatus; et quoniam haec materia attentionem mereri videtur, meas meditationes, quae se mihi hac occasione obtulerunt, fusius sum expositurus. Cum autem hoc quasi nouum Analyseos genus potissimum in eiusmodi formulis integralibus versetur, in quibus variabili post integrationem certus valor determinatus tribuitur, ad taediosas verborum ambages euitandas, quas perpetua talium conditionum commemoratio postularet, peculiarem signandi modum adhibebo, quem ante omnia accuratius explicare necesse erit.

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Hypothesis.

§. 1. Hac signandi ratione:

$$\int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = b \end{bmatrix}$$

declaratur, integrale $\int P\,dx$ ita esse assumtum, vt euanescat posito $x = a$, tum vero statui $x = b$; quo pacto manifestum est eius valorem penitus fore determinatum.

Scholion.

§. 2. Quo indoles huius determinationis clarius perspiciatur, quoniam P denotat functionem aliquam ipsius x, eius naturam repraesentemus linea quadam curua $i\,x\,a\,b\,c\,o$, super axe IO exstructa, cuius quaecunque applicata Xx, abscissae $IX = x$ respondens, exhibeat ipsam functionem P, ita vt formula integralis $\int P\,dx$ indefinite exprimat aream huius curuae. Quod si iam capiantur abscissae $IA = a$, $IB = b$, quibus respondeant applicatae Aa et Bb, formula proposita exprimet aream $AaBb$, inter applicatas Aa et Bb interceptam. Eodem modo, si alia quaepiam abscissa statuatur $IC = c$ area $AaCc$ exprimetur hac formula: Tab. I. Fig. 1.

$$\int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = c \end{bmatrix};$$

area autem $BbCc$ ista formula:

$$\int P\,dx \begin{bmatrix} \text{ab } x = b \\ \text{ad } x = c \end{bmatrix};$$

tum vero, ab initio I incipiendo, area $IiAa$ indicabitur per hanc formulam:

$$\int P\,dx \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = a \end{bmatrix},$$

vnde

vnde ſponte fluunt ſequentia lemmata ita ſuccincte expreſſa:

§. 3. *Lemma I.*

$$\int P\,dx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} = -\int P\,dx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=a \end{bmatrix}.$$

Quoniam enim, ſi b vt maius ſpectetur quam a, formula poſterior

$$\int P\,dx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=a \end{bmatrix}$$

eandem aream A a B b refert quam prior, ſed ordine retrogrado, iſta expreſſio pro negatiua erit habenda, ſicque erit quoque

$$\int P\,dx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} + \int P\,dx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=a \end{bmatrix} = 0.$$

§. 4. *Lemma II.*

$$\int P\,dx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} + \int P\,dx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=c \end{bmatrix} = \int P\,dx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=c \end{bmatrix}$$

quemadmodum inſpectio figurae manifeſto declarat.

§. 5. *Lemma III.*

$$\int P\,dx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=c \end{bmatrix} - \int P\,dx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} = \int P\,dx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=c \end{bmatrix};$$

vbi in binis prioribus formulis idem occurrit terminus *a quo*, ſcilicet $x=a$, terminorum vero *ad quem*, ſcilicet $x=c$ et $x=b$, poſterior $x=b$ dat pro tertia formula terminum *a quo*, prior vero terminum *ad quem*.

§. 6.

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§. 6.

§. 6. *Lemma IV.*

$$\int P\,dx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=c \end{bmatrix} - \int P\,dx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=c \end{bmatrix} = \int P\,dx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix};$$

vbi notetur, binas formulas priores eundem habere terminum *ad quem*, ſcilicet $x=c$, terminorum autem *a quo* priorem $x=a$ dare in tertia terminum *a quo*, poſteriorem vero terminum *ad quem*.

§. 7. *Lemma V.*

$$\int P\,dx \begin{bmatrix} \text{ab } x=a \\ \text{ad } x=b \end{bmatrix} + \int P\,dx \begin{bmatrix} \text{ab } x=b \\ \text{ad } x=c \end{bmatrix} + \int P\,dx \begin{bmatrix} \text{ab } x=c \\ \text{ad } x=a \end{bmatrix} = 0.$$

Scholion.

§. 8. His igitur, quae per ſe ſunt maxime perſpicua, praemiſſis, argumenta praecipua, quae celeb. *de la Grange* mihi perſcripſit ordine percurram. Primo autem mentionem inſignis paradoxi facit, cuius indolem ipſe non ſatis perſpicere fatetur, a quo igitur meas meditationes inchoabo.

Reſolutio inſignis Paradoxi.

§. 9. Cum Vir celeb. etiam inueniſſet hoc theorema generale

$$\int \frac{x^n - x^m}{lx} \frac{dx}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = l\frac{n}{m}$$

cuius veritatem non ita pridem pluribus demonſtrationibus adſtruxi, poſuit $x^n = z$ et $x^m = y$: quo facto pars prior $\int \frac{x^{n-1}\,dx}{lx}$ transformatur in hanc: $\int \frac{dz}{lz}$; ſimili vero modo al-

tera

tera $\int \frac{x^{m-1}\,dx}{lx}$ in hanc: $\int \frac{dy}{ly}$; vnde his partibus feorfim pofitis fequitur fore

$$\int \frac{dz}{lz}\begin{bmatrix}\text{a} & z=0\\ \text{ad} & z=1\end{bmatrix} - \int \frac{dy}{ly}\begin{bmatrix}\text{ab} & y=0\\ \text{ad} & y=1\end{bmatrix} = l\frac{n}{m}.$$

Quare cum hae duae formulae omnino fint fimiles, atque iisdem terminis integrationis contentae, quis non crederet eos etiam inter fe perfecte fore aequales, fiue effe

$$\int \frac{dz}{lz}\begin{bmatrix}\text{a} & z=0\\ \text{ad} & z=1\end{bmatrix} = \int \frac{dy}{ly}\begin{bmatrix}\text{ab} & y=0\\ \text{ad} & y=1\end{bmatrix}?$$

Interim tamen vidimus, differentiam inter has formulas effe $l\frac{n}{m}$. Hic igitur fe offert quaeftio maximi momenti: quemadmodum iftam manifeftam contradictionem dirimere oporteat?

§. 10. Primo autem hic obferuari conuenit, ambas quantitates y et z certo quodam modo a fe inuicem pendere. Cum enim fit $y = x^m$ et $z = x^n$, erit $y^n = z^m$, quo tamen nexu non impeditur, quo minus, pofito fiue $y = 0$, fiue $y = 1$, etiam fiat $z = 0$, fiue $z = 1$. Interim tamen hinc neutiquam patet, cur ob hanc rationem iftae binae formulae:

$$\int \frac{dy}{ly}\begin{bmatrix}\text{ab} & y=0\\ \text{ad} & y=1\end{bmatrix} \text{ et } \int \frac{dz}{lz}\begin{bmatrix}\text{a} & z=0\\ \text{ad} & z=1\end{bmatrix}$$

difpares prodire queant; vnde haec obferuatio ad dubium foluendum nihil plane conferre videtur.

§. 11. Quin etiam nullo prorfus dubio obnoxia videtur haec aequatio multo generalior:

$$\int \frac{dy}{ly}\begin{bmatrix}\text{ab} & y=a\\ \text{ad} & y=b\end{bmatrix} = \int \frac{dz}{lz}\begin{bmatrix}\text{a} & z=a\\ \text{ad} & z=b\end{bmatrix};$$

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quandoquidem nihil plane impedit, quo minus loco z fcribamus y, vel viciffim; verum plurima phaenomena in analyfi obferuata fatis luculenter docent, huiusmodi aequalitates interdum exceptionem pati, quando valores euadunt infiniti. Haec autem circumftantia noftro cafu vtique locum habet, cum formula integralis $\int \frac{dy}{ly}$, fi ab $y = 0$ ad $y = 1$ extendatur, vtique in infinitum excrefcat, quod etiam de altera: $\int \frac{dz}{lz}$, eft tenendum. Si enim fiat $= 1$, applicata noftrae curuae, quae eft $\frac{1}{lz}$, manifefto fit infinite magna, vnde fuperior aequalitas generalis

$$\int \frac{dy}{ly}\begin{bmatrix}\text{ab} & y=a\\ \text{ad} & y=b\end{bmatrix} - \int \frac{dz}{lz}\begin{bmatrix}\text{ab} & z=a\\ \text{ad} & z=b\end{bmatrix} = 0$$

hanc reftrictionem poftulat, nifi vel a fit $= 1$, vel $b = 1$, quippe quibus cafibus vtraque formula fit infinita.

§. 12. His perpenfis nullum plane dubium mihi quidem fupereffe videtur, quin in hac circumftantia vera folutio propofiti paradoxi fit quaerenda, quae fcilicet in eo verfatur, quod fit tam $\int \frac{dy}{ly}\begin{bmatrix}\text{ab} & y=0\\ \text{ad} & y=1\end{bmatrix} = \infty$, quam $\int \frac{dz}{lz}\begin{bmatrix}\text{a} & z=0\\ \text{ad} & z=1\end{bmatrix} = \infty$, ita vt horum infinitorum differentia poffit aequari quantitati finitae cuicunque, ideoque in fe fpectata prorfus non determinetur; quod autem ifta differentia noftro cafu fit $l\frac{n}{m}$, ideoque determinata, inde venit quod fit $y^n = z^m$.

§. 13. Simile aliquid euenire poteft in formulis fimplicioribus, quales funt $\int \frac{dy}{y}$ et $\int \frac{dz}{z}$, quippe quarum valores, a termino $y = 0$ et $z = 0$ fumti, funt infiniti, vnde, etiamfi

poft

poſt integrationem idem terminus ad quem ſtatuatur, ſcilicet $y = 1$ et $z = 1$, tamen hinc nullo modo ſequitur, differentiam abſolute nihilo aequari, quin potius tanquam indeterminata ſpectari debebit, cum quidem pro aliis terminis integrationis certo ſit

$$\int \frac{dy}{y} \begin{bmatrix} \text{ab } y = a \\ \text{ad } y = b \end{bmatrix} = \int \frac{dz}{z} \begin{bmatrix} \text{a } z = a \\ \text{ad } z = b \end{bmatrix},$$

dummodo neque a neque b fuerit $= 0$ vel $= \infty$.

§. 14. Atque hinc etiam paradoxon propoſito penitus ſimile proferri poteſt, quod ita ſe habet:

$$\int \frac{dz}{z} \begin{bmatrix} \text{a } z = 0 \\ \text{ad } z = \infty \end{bmatrix} - \int \frac{dy}{y} \begin{bmatrix} \text{ab } y = 0 \\ \text{ad } y = \infty \end{bmatrix} = l\,a,$$

cuius veritas cum in aprico ſit poſita, ſi quidem accipiatur $z = a\,y$, etiam paradoxon propoſitum rite dilutum erit cenſendum.

Obſeruationes in hoc Theorema D. de la Grange.

$$\int \frac{(x^n - x^m)}{l\,x} \cdot \frac{dx}{x} \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = b \end{bmatrix} = \int (b^y - a^y) \frac{dy}{y} \begin{bmatrix} \text{ab } y = m \\ \text{ad } y = n \end{bmatrix}.$$

§. 15. Cum equidem ante aliquod tempus reductiones huiusmodi formularum tractaſſem, alios terminos integrationis, praeterquam ab $x = 0$ ad $x = 1$, non ſum contemplatus, vnde hoc Theorema mihi ſtatim altioris indaginis eſt viſum, atque omnino dignum quod ſumma cura expendatur. Primum igitur in eius veritatem per ſeries inquirere conſtitui, quod negotium ſequenti modo peregi.

§. 16.

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§. 16.

§. 16. Cum ſit

$$x^\alpha = e^{\alpha l x} = 1 + \alpha\,l\,x + \frac{(\alpha\,l\,x)^2}{1.\,2} + \frac{(\alpha\,l\,x)^3}{1.\,2.\,3} \text{ etc. erit}$$

$$x^n - x^m = (n-m)\frac{l\,x}{1} + (n^2 - m^2)\frac{(l\,x)^2}{1.\,2} + \frac{(n^3 - m^3)(l\,x)^3}{1.\,2.\,3} + \text{etc.}$$

Hanc ergo ſeriem ducamus in $\frac{dx}{x\,l\,x}$, et quia in genere

$$\int (l\,x)^\lambda \frac{dx}{x\,l\,x} \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = b \end{bmatrix} = \frac{(l\,b)^\lambda - (l\,a)^\lambda}{\lambda}$$

formulae ad ſiniſtram partem ſcriptae valor per hanc ſeriem infinitam exprimetur:

$$\frac{(n-m)}{1}\frac{(l\,b - l\,a)}{1} + \frac{(n^2 - m^2)}{1.\,2}\frac{(l\,b)^2 - (l\,a)^2}{2} + \frac{(n^3 - m^3)}{1.\,2.\,3}\frac{(l\,b)^3 - (l\,a)^3}{3} + \text{etc.}$$

§. 17. Simili modo pro formula ad dextram poſita per ſeriem infinitam erit

$$b^y - a^y = y\,\frac{(l\,b - l\,a)}{1} + y^2\,\frac{(l\,b)^2 - (l\,a)^2}{1.\,2} + y^3\,\frac{(l\,b)^3 - (l\,a)^3}{1.\,2.\,3}$$

quae ergo ducatur in $\frac{dy}{y}$, et quia in genere eſt

$$\int y^\lambda \frac{dy}{y} \begin{bmatrix} \text{ab } y = m \\ \text{ad } y = n \end{bmatrix} = \frac{n^\lambda - m^\lambda}{\lambda},$$

valor iſtius formulae per ſeriem hanc infinitam exprimetur:

$$\frac{(n-m)}{1}\frac{(l\,b - l\,a)}{1} + \frac{(n^2 - m^2)}{2}\frac{(l\,b^2 - l\,a^2)}{1.\,2} + \frac{(n^3 - m^3)}{3}\frac{(l\,b^3 - l\,a^3)}{1.\,2.\,3} + \text{etc.}$$

Quia igitur haec ſeries cum praecedente perfecte congruit, veritas theorematis firmiter eſt euicta.

§. 18. Verum hinc neutiquam perſpicitur, quomodo ſagaciſſimus auctor ad hoc Theorema ſit perductus, quam obrem, rebus probe perpenſis, viam inueni, ex iisdem principiis, quibus antehac ſum vſus, ad easdem formulas perueniendi. Inchoandum autem eſt ab hac forma ſimpliciſſima:

$\int x$

$$\int x^{\lambda}\frac{dx}{x}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\frac{b^{\lambda}-a^{\lambda}}{\lambda},$$

vbi vtrinque per $d\lambda$ multiplicans denuo integrationem instituo, et cum, vti iam passim demonstratum reperitur, sit

$$\int d\lambda\int x^{\lambda}\tfrac{dx}{x}=\int\tfrac{dx}{x}\int x^{\lambda}\,d\lambda,$$

quaeri tantum debet hoc integrale: $\int x^{\lambda}\,d\lambda$, spectata quantitate x vt constante, ita vt sola λ sit variabilis. Est vero

$$\int x^{\lambda}\,d\lambda=\frac{x^{\lambda}}{lx}+C,$$

quemadmodum ex elementis calculi exponentialis liquet. Hic vero cardo rei in hoc versatur, vt istud integrale certa lege definiatur, quam deinceps etiam in altera parte obseruari oportet. Statuamus ergo talia integralia ita capi, vt euanescant posito $\lambda=0$, eritque

$$\int x^{\lambda}\,d\lambda=\frac{x^{\lambda}-1}{lx},$$

quo pacto pro sinistra parte habebimus

$$\int d\lambda\int x^{\lambda}\frac{dx}{x}=\int\frac{dx}{x}\frac{(x^{\lambda}-1)}{lx}.$$

§. 19. Pro parte autem dextra habebimus

$$\int\tfrac{d\lambda}{\lambda}(b^{\lambda}-a^{\lambda}),$$

qua formula eadem lege integrata, vt facto $\lambda=0$ prodeat nihilum, hunc valorem more hic recepto repraesentare licebit:

$$\int\frac{dy}{y}(b^{y}-a^{y})\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=\lambda\end{bmatrix}$$

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Hic enim nil aliud fecimus, nisi quod pro λ scripsimus y, et facta integratione loco y eius valorem λ restitui assumsimus, sicque assecuti sumus sequentem formulam:

$$\int(x^{\lambda}-1)\frac{dx}{x\,lx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\int\frac{dy}{y}(b^{y}-a^{y})\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=\lambda\end{bmatrix},$$

quam tanquam Theorema vtilissimum spectare licet.

§. 20. Vi ergo huius Theorematis nanciscimur sequentes reductiones:

$$\int(x^{n}-1)\frac{dx}{x\,lx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\int\frac{dy}{y}(b^{y}-a^{y})\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=n\end{bmatrix}\text{ et}$$

$$\int(x^{m}-1)\frac{dx}{x\,lx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\int\frac{dy}{y}(b^{y}-a^{y})\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=m\end{bmatrix};$$

quare si formula posterior a priore subtrahatur, erit

$$\int(x^{n}-x^{m})\frac{dx}{x\,lx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\int\frac{dy}{y}(b^{y}-a^{y})\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=n\end{bmatrix}$$

$$-\int\frac{dy}{y}(b^{y}-a^{y})\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=m\end{bmatrix};$$

verum ista formula ad dextram posita per reductionem in lemmate 3° ostensam reuocatur ad hanc formam simpliciorem:

$$\int\frac{dy}{y}(b^{y}-a^{y})\begin{bmatrix}\text{ab } y=m\\ \text{ad } y=n\end{bmatrix};$$

vnde patet, hoc modo ipsum hoc insigne Theorema etiam ex nostris principiis inuestigari potuisse.

 §. 21.

§. 21. Hoc autem Theoremate generalissimo vir ingeniosissimus est vsus ad Theorema meum demonstrandum, quo ostendi esse

$$\int (x^n - x^m)\frac{dx}{x\,l\,x}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=1\end{matrix}\right]=l\frac{n}{m};$$

tantum enim opus erat, vt caperetur $a=0$ et $b=1$, quo pacto formula ad dextram posita integralis abit in

$$\int \frac{dy}{y}\left[\begin{matrix}\text{ab } y=m\\ \text{ad } y=n\end{matrix}\right],$$

cuius valor manifesto fit $l\,n - l\,m = l\frac{n}{m}$, quae est noua demonstratio mei Theorematis, cuiusmodi quidem dudum plures alias dederam.

Obseruationes in Theorema D. de la Grange.

$$\int\frac{x^n - x^m\,dx}{(1+x^r)\,l\,x}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=\infty\end{matrix}\right]=l\left(\frac{\text{tag.}\frac{(n+1)\pi}{2r}}{\text{tag.}\frac{(m+1)\pi}{2r}}\right).$$

§. 22. Quia hic ambo exponentes m et n neque a se inuicem neque ab exponente r pendent, manifestum est, pro vtraque potestate x^m et x^n seorsim integrale talem formam habere debere:

$$\int\frac{x^n\,dx}{(1+x^r)\,l\,x}=l\,\text{tag.}\frac{(m+1)\pi}{2r}+C \text{ et}$$

$$\int\frac{x^m\,dx}{(1+x^r)\,l\,x}=l\,\text{tag.}\frac{(m+1)\pi}{2r}+C.$$

Si enim posterior forma a priore subtrahatur, constans C ex

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ex calculo egreditur, et ipsum integrale propositum resultat. Hic igitur plurimum intererit valorem istius constantis C determinasse.

§. 23. Inter formulas integrales, quarum valores pro casu, quo post integrationem variabilis infinita statuitur, ex primis principiis calculi integralis assignaui, reperitur ista:

$$\int\frac{x^{k+n}}{1+x^{2k}}\frac{dx}{x}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=\infty\end{matrix}\right]=\frac{\pi}{2k\,\text{cos}\frac{n\pi}{2k}}=\frac{\pi}{2k\,\text{sin}\frac{(k+n)\pi}{2k}},$$

vbi autem assumitur, exponentem n non maiorem capi quam k. Quod si iam hic exponens n vt variabilis tractetur, spectata ipsa x vt constante, et vtrinque per dn multiplicetur denuoque integretur, formula finitra erit

$$\int dn\int\frac{x^{k+n}}{1+x^{2k}}\frac{dx}{x}=\int\frac{dx}{x(1+x^{2k})}\int x^{k+n}\,dn,$$

vbi postremum integrale fit

$$\int x^{k+n}\,dn=\frac{x^{k+n}}{l\,x}+C.$$

Vt autem hoc integrale determinetur, constantem ita definimus, vt id euanescat posito $n=0$, vnde obtinetur

$$\int x^{k+n}\,dn=\frac{x^{k+n}-x^k}{l\,x},$$

ita vt formula integralis ad finistram posita futura sit

$$\int\frac{x^{k+n}-x^k}{1+x^{2k}}\cdot\frac{dx}{x\,l\,x}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=\infty\end{matrix}\right].$$

§. 24. Pro parte dextra autem habebimus hoc integrale: $\int\frac{\pi\,dn}{2k\,\text{sin.}\frac{(k+n)\pi}{2k}}$, etiam ita sumendum, vt euanescat

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posito $n = 0$. Hunc in finem statuamus angulum $\frac{(k+n)\pi}{2k} = \phi$, et quia hinc erit $d\phi = \frac{\pi\, dn}{2k}$, formula nostra integranda erit $\int \frac{d\phi}{\sin.\phi}$, cuius integrale per regulas notas in genere est:

$$l\,\mathrm{tag}.\tfrac{1}{2}\phi + C = l\,\mathrm{tag}.\frac{(k+n)\pi}{4k} + C,$$

quod, facto $n = 0$, abit in $l\,\mathrm{tag}\frac{\pi}{4} + C$. Quare cum $\mathrm{tag}.\frac{\pi}{4} = 1$ et $l\,1 = 0$, euidens est constantem C fore $= 0$, ita vt integrale hoc quaesitum sit $l\,\mathrm{tag}.\frac{(k+n)\pi}{4k}$. Hinc ergo assecuti sumus istam reductionem generalem:

$$\int \frac{x^{k+n} - x^k}{1 + x^{2k}} \cdot \frac{dx}{x\,lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right] = l\,\mathrm{tag}.\frac{(k+n)\pi}{4k},$$

vbi autem probe notari oportet, exponentes m et n maiores capi non licere quam k.

§. 25. Cum igitur, loco n alium numerum m sumendo, simili modo fit:

$$\int \frac{x^{k+m} - x^k}{1 + x^{2k}} \cdot \frac{dx}{x\,lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right] = l\,\mathrm{tag}.\frac{(k+m)\pi}{4k},$$

subtrahatur ista formula a praecedente, et obtinebitur ista:

$$\int \frac{x^{k+n} - x^{k+m}}{1 + x^{2k}} \cdot \frac{dx}{x\,lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right] = l\left(\frac{\mathrm{tag}.\frac{(k+n)\pi}{4k}}{\mathrm{tag}.\frac{(k+m)\pi}{4k}}\right),$$

quae manifesto cum forma proposita congruit, si modo loco $k + n - 1$ scribatur n et m loco $k + m - 1$, at loco exponentis $2k$ scribatur r, tum enim manifesto fiet:

$$\int \frac{x^n - x^m}{(1+x^r)} \cdot \frac{dx}{l\,x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right] = l\left(\frac{\mathrm{tag}.\frac{(n+1)\pi}{2r}}{\mathrm{tag}.\frac{(m+1)\pi}{2r}}\right).$$

§. 26.

§. 26. Quoniam ista analysis nos perduxit ad hanc formam:

$$\int \frac{x^{k+n} - x^k}{1 + x^{2k}} \cdot \frac{dx}{x\,lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right] = l\,\mathrm{tang}.\frac{(k+n)\pi}{4k},$$

hic maximi momenti erit obseruasse, semper fore

$$\int \frac{x^k}{1 + x^{2k}} \cdot \frac{dx}{x\,lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right] = 0$$

id quod ita ostendere possum: Ponatur $x^k = z$, erit

$$x^{k-1}\,dx = \frac{dz}{k} \text{ et } lx = \frac{lz}{k},$$

sicque ista formula induet hanc formam: $\int \frac{dz}{(1+zz)\,lz}$, vbi termini integrationis etiamnunc sunt $z = 0$ et $z = \infty$. Fiat porro $z = \mathrm{tang}.\phi$, vnde termini integrationis erunt $\phi = 0$ et $\phi = \frac{\pi}{2}$; hinc autem ob $d\phi = \frac{dz}{1+zz}$ nascetur ista formula:

$$\int \frac{d\phi}{l\,\mathrm{tang}.\phi} \left[\begin{array}{l} \text{a } \phi = 0 \\ \text{ad } \phi = \frac{\pi}{2} \end{array}\right]$$

cuius valorem in nihilum abire ostendi debet.

§. 27. Ad hoc demonstrandum statuatur axis $I\Pi = \frac{\pi}{2}$, super quo ab initio I sumta abscissa indefinita $I\,p = \phi$, applicata sit $= \frac{1}{l\,\mathrm{tang}.\phi}$. Quod si ergo hic axis $I\,\Pi$ in O bisecetur, vt sit $I\,O = \frac{\pi}{4}$, in hoc puncto applicata erit Tab. I. Fig. 2.

$$= \frac{1}{l\,\mathrm{tang}.\frac{\pi}{4}} = \infty.$$

Iam ab hoc puncto O vtrinque capiantur interualla aequalia $O\,p = O\,q = \omega$, et pro puncto p erit $\phi = \frac{\pi}{4} - \omega$, sicque in hoc puncto p applicata erit $\frac{1}{l\,\mathrm{tang}.(\frac{\pi}{4} - \omega)}$; est vero

ro

$\mathrm{tang.}(\frac{\pi}{4}-\omega)=\cot.(\frac{\pi}{4}+\omega)$, quare cum sit $l\cot.=-l\,\mathrm{tang.}$ applicata in hoc puncto p erit $\frac{-1}{l\,\mathrm{tang.}(\frac{\pi}{4}+\omega)}$; at quia est $lq=(\frac{\pi}{4}+\omega)$, erit applicata in puncto $q=\frac{+1}{l\,\mathrm{tang.}(\frac{\pi}{4}+\omega)}$, sicque aequalis est applicatae in p, sed in contrarium vergens. Ita si applicata sursum directa fuerit qQ, in puncto p eadem applicata deorsum erit directa $pP=qQ$.

§. 28. Quod si ergo talis curua super axe $I\Pi=\frac{\pi}{2}$ exstruatur, ita vt abscissae ϕ respondeat applicata $\frac{1}{l\,\mathrm{tang.}\,\phi}$, haec curua ex duabus portionibus inter se perfecte aequalibus constabit, circa punctum medium O ita dispositis, vt curva sinistra sit I P M in infinitum descendens ad asymptotam O m, pars autem dextra similimodo a Π sinistrorsum sursum ascendet ad asymptotam O n. Quare cum formula integralis $\int\frac{d\phi}{l\,\mathrm{tang.}\,\phi}$, a $\phi=0$ ad $\phi=\frac{\pi}{2}$ extensa, exprimat totius huius curuae ab I vsque ad Π protensae aream, euidens est, totam hanc aream ad nihilum redigi, quia portio eius negatiue sumenda perfecte similis est portioni positiue sumendae.

§. 29. Sic igitur per demonstrationem omnino singularem euictum est, semper esse:

$$\int\frac{x^k}{1+x^{2k}}\frac{dx}{x\,lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=0,$$

quod certe est Theorema in hoc genere maxime notatu dignum. Quod si ergo cum illustri D. *de la Grange* statuamus $2k=r$, erit $\int\frac{x^{\frac{1}{2}r-1}\,dx}{(1+x^r)\,lx}=0$; praeterea vero pro nostra

ot. $=-l$tang.
; at quia est
$+1$
$.(\frac{\pi}{4}+\omega)$,
arium ver-
qQ, in
$pP=qQ$.
axe $I\Pi=\frac{\pi}{2}$
$\frac{1}{l\,\mathrm{tang.}\,\phi}$, haec
aequalibus
s, vt cur-
ad asympto-
Π sinistror-
Quare cum
sa, exprimat
am, euidens
portio eius
positiue su-
omnino sin-
xime notatu
Grange sta-
ea vero pro
nostra

nostra formula §. 24. exhibita, ob

$$\int\frac{x^k\,dx}{(1+x^{2k})\,x\,lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=0,$$

deducitur istud Theorema omnino notabile:

$$\int\frac{x^{k+n}}{1+x^{2k}}\frac{dx}{x\,lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\mathrm{tang.}\frac{(k+n)\pi}{4k},$$

quod more D. *de la Grange* ita proponi potest:

$$\int\frac{x^n\,dx}{(1+x^r)\,lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=l\,\mathrm{tang.}\frac{(n+1)\pi}{2r}$$

sicque patet constantem illam supra §. 22. a nobis inductam reuera nihilo aequari.

§. 30. Quoniam Demonstratio huius Theorematis methodo satis insueta innititur, eius veritatem per series ostendisse iuuabit. Ad hoc autem valorem formulae

$$\int\frac{x^{\lambda-1}\,dx}{(1+x^r)\,lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}$$

in duas partes diuelli necesse est (scilicet loco n scribendo $\lambda-1$), quae sint

$$P=\int\frac{x^{\lambda-1}\,dx}{(1+x^r)\,lx}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix} \text{ et}$$

$$Q=\int\frac{x^{\lambda-1}\,dx}{(1+x^r)\,lx}\begin{bmatrix}\text{ab } x=1\\ \text{ad } x=\infty\end{bmatrix}$$

ita vt P + Q exprimat valorem quem quaerimus. Nunc in posteriore parte loco x scribamus $\frac{1}{z}$, fietque

$$Q=\int\frac{z^{-\lambda}}{(1+z^{-r})}\frac{dz}{z\,lz}\begin{bmatrix}\text{a } z=1\\ \text{ad } z=0\end{bmatrix}=\int\frac{z^{r-\lambda}}{(1+z^r)}\frac{dz}{z\,lz}\begin{bmatrix}\text{a } z=1\\ \text{ad } z=0\end{bmatrix}$$

et

et commutatis terminis integrationis

$$Q = -\int \frac{z^{r-\lambda}}{1+z^r} \cdot \frac{dz}{z\,l\,z} \left[\begin{array}{l} \text{a } z=0 \\ \text{ad } z=1 \end{array}\right].$$

Nunc autem loco z ſcribamus x, quia termini integrationis vtrinque ſunt iidem, erit

$$P+Q = \int \frac{x^\lambda - x^{r-\lambda}}{1+x^r} \cdot \frac{dx}{x\,l\,x} \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=1 \end{array}\right]$$

cuius ergo valor formulae propoſitae eſt aequalis.

§. 31. Iam fractionem $\frac{1}{1+x^r}$ in ſeriem infinitam conuertamus

$$1 - x^r + x^{2r} - x^{3r} + x^{4r} - \text{etc.}$$

cuius ſinguli termini in $\frac{dx}{x\,l\,x}(x^\lambda - x^{r-\lambda})$ ducti producunt

$$\tfrac{dx}{x\,l\,x}(x^\lambda - x^{r-\lambda}) - \tfrac{dx}{x\,l\,x}(x^{r+\lambda} - x^{2r-\lambda}) + \tfrac{dx}{x\,l\,x}(x^{2r+\lambda} - x^{3r-\lambda})$$
$$- \tfrac{dx}{x\,l\,x}(x^{3r+\lambda} - x^{4r-\lambda}) + \text{etc.}$$

Cum autem per Theorema principale in hoc genere ſit

$$\int \frac{dx}{x\,l\,x}(x^\alpha - x^\gamma) \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=1 \end{array}\right] = l\frac{\alpha}{\beta},$$

ſingulis membris hoc modo integratis prodibit

$$P+Q = l\frac{\lambda}{r-\lambda} - l\frac{r+\lambda}{2r-\lambda} + l\frac{2r+\lambda}{3r-\lambda} - l\frac{3r+\lambda}{4r-\lambda} + \text{etc.}$$

§. 32. Omnes hos logarithmos in vnicum compingere licebit, ratione habita ſigni cuiusque, hocque modo reperietur fore

$$P+Q = l\frac{\lambda}{r-\lambda} \cdot \frac{3r-\lambda}{r+\lambda} \cdot \frac{2r+\lambda}{4r-\lambda} \cdot \frac{4r-\lambda}{3r+\lambda} \cdot \frac{4r+\lambda}{5r-\lambda} \cdot \frac{6r-\lambda}{5r+\lambda} \text{ etc.}$$

At

integrationis
infinitam
ducunt
$+\lambda - x^{3r-\lambda})$
nere ſit
+ etc.
m compin-
e modo re-
$\frac{6r-\lambda}{5r+\lambda}$ etc.
At

At vero in *Introductione in Analyſin Infinitorum* pag. 147. oſtendi eſſe

$$\text{tang.}\frac{m\pi}{2n} = \frac{m}{n-m} \cdot \frac{2n-m}{n+m} \cdot \frac{2n+m}{3n-m} \cdot \frac{4n-m}{3n+m} \cdot \frac{4n+m}{5n-m} \text{ etc.}$$

quae ſeries manifeſto in inuentam transformatur, ſtatuendo $m=\lambda$ et $n=r$, ita vt nunc ſit $P+Q = l\,\text{tang.}\frac{\lambda\pi}{2r}$, prorſus vti ſupra eſt inuentum.

Additamentum.

§. 33. In diſſertatione Actorum Tomo V. parte I. inſerta, vnde deſumſi hoc theorema:

$$\int \frac{x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x} \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=\infty \end{array}\right] = \frac{\pi}{2k\,\text{coſ.}\frac{n\pi}{2k}}$$

ſimul occurrunt ſequentia:

$$\int \frac{x^{k-n} + x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x} \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=\infty \end{array}\right] = \frac{\pi}{k\,\text{coſ.}\frac{n\pi}{2k}}$$

$$\int \frac{x^{k-n} + x^{k+n}}{1+x^{2k}} \cdot \frac{dx}{x} \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=1 \end{array}\right] = \frac{\pi}{2k\,\text{coſ.}\frac{n\pi}{2k}}$$

$$\int \frac{x^{k-n} - x^{k+n}}{1-x^{2k}} \cdot \frac{dx}{x} \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=\infty \end{array}\right] = \frac{\pi}{k}\,\text{tang.}\frac{n\pi}{2k}$$

$$\int \frac{x^{k-n} - x^{k+n}}{1-x^{2k}} \cdot \frac{dx}{x} \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=1 \end{array}\right] = \frac{\pi}{2k}\,\text{tang.}\frac{n\pi}{2k}$$

$$\int \frac{x^{k-n} + x^{k+n}}{1+2x^k\,\text{coſ.}\,\eta + x^{2k}} \cdot \frac{dx}{x} \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=\infty \end{array}\right] = \frac{2\pi\,\text{ſin.}\frac{n\eta}{k}}{k\,\text{ſin.}\,\eta\,\text{ſin.}\frac{n\pi}{k}}$$

$$\int \frac{x^{k-n} + x^{k+n}}{1+2x^k\,\text{coſ.}\,\eta + x^{2k}} \cdot \frac{dx}{x} \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=1 \end{array}\right] = \frac{\pi\,\text{ſin.}\frac{n\eta}{k}}{k\,\text{ſin.}\,\eta\,\text{ſin.}\frac{n\pi}{k}}$$

 ∫x

$$\int \frac{x^{k\pm n-1}}{1+2x^k \cos.\eta + x^{2k}} \frac{dx}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{2\pi \sin.\frac{n\eta}{k}}{k \sin.\eta \sin.\frac{n\pi}{k}}$$

quas formulas ergo simili modo tractare operae pretium erit.

§. 34. Incipiamus igitur a formula

$$\int \frac{x^{k-n}+x^{k+n}}{1+x^{2k}} \frac{dx}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = \frac{\pi}{2k \cos.\frac{n\pi}{2k}}$$

quia praecedens cum formula iam tractata prorsus conueniret, quae si ducatur in dn et ita integretur, vt integrale euanescat posito $n=0$, quoniam est

$$\int x^{k-n}\, dn = -\frac{x^{k-n}+x^k}{lx} \text{ et}$$

$$\int x^{k+n}\, dn = \frac{x^{k+n}-x^k}{lx},$$

tum vero, vt ante vidimus,

$$\int \frac{\pi\, dn}{2k} \cos.\frac{n\pi}{2k} = l \tang.\frac{(k+n)\pi}{4k},$$

prodibit haec integratio:

$$\int \frac{x^{k+n}-x^{k-n}}{(1+x^{2k})} \frac{dx}{x\,lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix} = l \tang.\frac{(k+n)\pi}{4k},$$

qui ergo valor prorsus conuenit cum eo, quem pro formula

$$\int \frac{x^{k+n}}{1+x^{2k}} \frac{dx}{x\,lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix}$$

inuenimus.

§. 35. Simili modo tractemus sequentem formam:

$$\int \frac{x^{k-n}-x^{k+n}}{1-x^{2k}} \frac{dx}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{\pi}{k} \tang.\frac{n\pi}{2k},$$

quae

quae, ducta in dn et vt supra integrata, praebet a parte sinistra

$$\int \frac{2x^k - x^{k-n} - x^{k+n}}{1-x^{2k}} \frac{dx}{x\,lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix},$$

a parte autem dextra

$$\int \frac{\pi\, dn}{k} \tang.\frac{n\pi}{2k} = \int \frac{\pi\, dn \sin.\frac{n\pi}{2k}}{k \cos.\frac{n\pi}{2k}}.$$

Ad hoc integrandum fiat $\frac{n\pi}{2k} = \phi$, eritque $\frac{\pi dn}{k} = 2d\phi$, sicque formula integranda erit

$$2\int \frac{d\phi \sin.\phi}{\cos.\phi} = -2l\cos.\phi + C = 2l\cos.\frac{n\pi}{2k} + C.$$

Fiat igitur $n=0$, esseque debebit $2l1 + C = 0$, ideoque constans $C=0$, quocirca haec integratio nobis suppeditat sequentem formulam:

$$\int \frac{2x^k - x^{k-n} - x^{k+n}}{1-x^{2k}} \frac{dx}{x\,lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = 2l\cos.\frac{n\pi}{2k}$$

sequens autem formula $\begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=1 \end{bmatrix}$ singulari euolutione non indiget, cum eius valor sit huius semissis.

§. 36. Euoluamus casum quo $k=2$ et $n=1$, et ex parte sinistra habemus

$$-\int \frac{(1-x)^2}{1-x^4} \frac{dx}{lx} = -\int \frac{(1-x)}{(1+x)(1+xx)} \frac{dx}{lx} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix};$$

at vero ex dextra parte: $2l\cos.\frac{\pi}{4} = -2l\sqrt{2} = -l2$. Verum fractio $\frac{1-x}{(1+x)(1+xx)}$ resoluitur in has duas: $\frac{1}{1+x} - \frac{x}{1+xx}$, vnde formula nostra resoluitur in has duas:

$$\int \frac{dx}{(1+x)lx} - \int \frac{x\,dx}{(1+xx)lx} = l2.$$

Sed

Sed ex formâ generali

$$\int \frac{x^{\lambda-1}\,dx^{r}}{(1+x^{r})\,lx} = l\,\text{tang.}\,\frac{\lambda\pi}{2r},$$

vtriusque formulae valor in infinitum excrefcit, ficque nihil impedit, quo minus differentia $= l\,2$.

§. 37. Quod fi hic in pofteriore formula ftatuamus $xx = z$, ea abibit in hanc: $\int \frac{dz}{(1+z)\,lz}$, quae priori omnino eft fimilis atque fub iisdem terminis integrationis continetur. Hic igitur iterum occurrit Paradoxon prorfus fimile illi, quod ab Ill. *de la Grange* fuit memoratum: duae fcilicet hic habentur formae prorfus pares: $\int \frac{dx}{(1+x)\,lx}$ et $\int \frac{dz}{(1+z)\,lz}$, quarum vtramque a termino o ad ∞ integrari oportet, nihilo tamen minus earum differentia non eft nulla, fed vti vidimus $= l\,2$. Atque hinc Solutio huius Paradoxi in eo manifefto eft fita, quod vtriusque integralis valor in infinitum excrefcit.

§. 38. Quod fi binas poftremas formulas eodem modo tractare et per dn multiplicatas integrare velimus, a parte finiftra refultat ifta formula integralis:

$$\int \frac{x^{k+n} - x^{k-n}}{(1 + 2x^{k}\,\text{cof.}\,\eta + x^{2k})}\,\frac{dx}{x\,lx}\left[\begin{array}{l}\text{ab } x = 0\\ \text{ad } x = \infty\end{array}\right];$$

pro dextra autem parte nancifcimur hanc formulam integralem:

$$\int \frac{2\pi\,dn}{k\,\text{fin.}\,\eta}\,\frac{\text{fin.}\,\frac{n\eta}{k}}{\text{fin.}\,\frac{n\pi}{k}}$$

a termino $n = 0$ extendendam. Verum haec integratio nullo modo fuccedit; fi enim ponamus $\frac{n\pi}{k} = \Phi$, fiet $\frac{n\eta}{k} = \frac{\eta\Phi}{\pi} = \alpha\Phi$, ponendo

que nihil
a ftatua-
priori om-
nis con-
rfus fimi-
n: duae
et $\int \frac{dz}{(1+z)\,lz}$,
ortet, ni-
fed vti
xi in eo
infinitum
s eodem
limus, a
lam inte-
atio nullo
$\frac{\eta\Phi}{\pi} = \alpha\Phi$,
ponendo

ponendo $\frac{\eta}{\pi} = \alpha$, vnde formula integranda erit $\frac{2}{l.\,\text{fin.}\,\eta}\int \frac{d\Phi\,\text{fin.}\,\alpha\Phi}{\text{fin.}\,\Phi}$, cuius valor aliter nifi per fignum fummatorium exprimi non poteft, ficque nulla concinna Theoremata hinc deriuare licet.

§. 39. Quemadmodum autem hic, exponentem n vt variabilem fpectando, transformationes per integrationem inftituimus, ita etiam differentiatio egregias transformationes fuppeditabit, quod argumentum vnica formula principali illuftraffe fufficiet. Confideremus fcilicet hanc formulam:

$$\int \frac{x^{k+n}}{1+x^{2k}}\,\frac{dx}{x}\left[\begin{array}{l}\text{ab } x = 0\\ \text{ad } x = \infty\end{array}\right] = \frac{\pi}{2k\,\text{cof.}\,\frac{n\pi}{2k}},$$

quae, fumto exponente n vt folo variabili, continuo differentietur, vbi notandum eft effe $d.x^{k+n} = x^{k+n}\,dn\,lx$. At vero pro formula $\frac{\pi}{2k\,\text{cof.}\,\frac{n\pi}{2k}}$ fcribamus litteram ν, quae ergo fpectanda erit tanquam functio ipfius n, cuius ergo differentialia cuiusque ordinis funt in noftra poteftate. Hinc igitur fequentes reductiones confequemur:

$$\int \frac{x^{k+n}}{1+x^{2k}}\,\frac{dx}{x}\,lx = \frac{d\nu}{dn} \text{ fiue}$$

$$\int \frac{x^{k+n-1}\,dx\,lx}{1+x^{2k}}\left[\begin{array}{l}\text{ab } x = 0\\ \text{ad } x = \infty\end{array}\right] = \frac{d\nu}{dn}$$

$$\int \frac{x^{k+n-1}\,dx\,(lx)^{2}}{1+x^{2k}}\left[\begin{array}{l}\text{ab } x = 0\\ \text{ad } x = \infty\end{array}\right] = \frac{dd\nu}{dn^{2}}$$

$$\int \frac{x^{k+n-1}\,dx\,(lx)^{3}}{1+x^{2k}}\left[\begin{array}{l}\text{ab } x = 0\\ \text{ad } x = \infty\end{array}\right] = \frac{d^{3}\nu}{dn^{3}}$$

$$\int \frac{x^{k+n-1}\,dx\,(lx)^{4}}{1+x^{2k}}\left[\begin{array}{l}\text{ab } x = 0\\ \text{ad } x = \infty\end{array}\right] = \frac{d^{4}\nu}{dn^{4}}$$

 $\int x$

$$\int \frac{x^{\lambda+n-1}\,dx\,(lx)^i}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d^i v}{dn^i}.$$

§. 40. Cum igitur hinc totum negotium ad differentialia continua ipsius v reducatur, ea sequenti modo commodissime reperire licebit. Cum enim sit

$$v = \frac{\pi}{2k \operatorname{cos.} \frac{n\pi}{2k}}, \text{ erit } v \operatorname{cos.} \frac{n\pi}{2k} = \frac{\pi}{2k},$$

hincque continuo differentiando obtinebimus sequentes formulas:

$$\frac{dv}{dn} \operatorname{cos.} \frac{n\pi}{2k} - \frac{\pi}{2k} v \operatorname{sin.} \frac{n\pi}{2k} = 0$$
$$\frac{ddv}{dn^2} \operatorname{cos.} \frac{n\pi}{2k} - \frac{2\pi}{2k} \frac{dv}{dn} \operatorname{sin.} \frac{n\pi}{2k} - \frac{\pi\pi}{4kk} v \operatorname{cos.} \frac{n\pi}{2k} = 0$$
$$\frac{d^3v}{dn^3} \operatorname{cos.} \frac{n\pi}{2k} - \frac{3\pi}{2k} \frac{ddv}{dn^2} \operatorname{sin.} \frac{n\pi}{2k} - \frac{3\pi\pi}{4kk} \frac{dv}{dn} \operatorname{cos.} \frac{n\pi}{2k} + \frac{\pi^3}{8k^3} v \operatorname{sin.} \frac{n\pi}{2k} = 0$$
$$\frac{d^4v}{dn^4} \operatorname{cos.} \frac{n\pi}{2k} - \frac{4\pi}{2k} \frac{d^3v}{dn^3} \operatorname{sin.} \frac{n\pi}{2k} - \frac{6\pi\pi}{4kk} \frac{ddv}{dn^2} \operatorname{cos.} \frac{n\pi}{2k} + \frac{4\pi^3}{8k^3} \frac{dv}{dn} \operatorname{sin.} \frac{n\pi}{2k}$$
$$+ \frac{\pi^4}{16k^4} v \operatorname{cos.} \frac{n\pi}{2k} = 0$$

etc. etc.

vnde singula differentialia altiora ex inferioribus formari possunt.

§. 41. Quo autem hae operationes magis sublevetur, statuamus breuitatis gratia $\frac{\pi}{2k} = \alpha$, vt sit $v = \frac{\alpha}{\operatorname{cos.} \alpha n}$, atque singula differentialia ex superioribus aequationibus sequenti modo determinabuntur:

$$\frac{dv}{dn} = \alpha v \operatorname{tang.} \alpha n$$
$$\frac{ddv}{dn^2} = 2\alpha \frac{dv}{dn} \operatorname{tang.} \alpha n + \alpha\alpha v$$
$$\frac{d^3v}{dn^3} = 3\alpha \frac{ddv}{dn^2} \operatorname{tang.} \alpha n + 3\alpha\alpha \frac{dv}{dn} - \alpha^3 v \operatorname{tang.} \alpha n$$

$d^4 v$

ad diffe-
nodo com-
entes for-
fin. $\frac{6\pi}{2k}$ = o
$\frac{dv}{dn}$ fin. $\frac{n\pi}{2k}$
is formari
sublcuen-
= $\frac{\alpha}{\operatorname{cos.} \alpha n}$, at-
onibus fe-
α n
$d^4 v$

$$\frac{d^4v}{dn^4} = 4\alpha \frac{d^3v}{dn^3} \operatorname{tang.} \alpha n + 6\alpha\alpha \frac{ddv}{dn^2} - 4\alpha^3 \frac{dv}{dn} \operatorname{tang.} \alpha n - \alpha^4 v$$
$$\frac{d^5v}{dn^5} = 5\alpha \frac{d^4v}{dn^4} \operatorname{tang.} \alpha n + 10\alpha\alpha \frac{d^3v}{dn^3} - 10\alpha^3 \frac{ddv}{dn^2} \operatorname{tang.} \alpha n - 5\alpha^4 \frac{dv}{dn}$$
$$+ \alpha^5 v \operatorname{tang.} \alpha n$$

etc. etc.

Quod si breuitatis gratia insuper statuamus $\operatorname{tang.} \alpha n = t$, et praecedentes valores in sequentibus substituamus, reperiemus:

$$\frac{dv}{dn} = \alpha v . t$$
$$\frac{ddv}{dn^2} = \alpha\alpha v (2tt + 1)$$
$$\frac{d^3v}{dn^3} = \alpha^3 v (6t^3 + 5t)$$
$$\frac{d^4v}{dn^4} = \alpha^4 v (24t^4 + 28tt + 5)$$
$$\frac{d^5v}{dn^5} = \alpha^5 v (120t^5 + 180t^3 + 61t)$$
$$\frac{d^6v}{dn^6} = \alpha^6 v (720t^6 + 1320t^4 + 662tt + 61).$$

§. 42. Ex consideratione harum expressionum facilis erui potest operatio, cuius ope ex qualibet earum expressionum sequens colligi potest. Sit enim pro differentiali ordinis indefiniti

$$\frac{d^\lambda v}{dn^\lambda} = \alpha^\lambda v . P$$

at pro ordine sequente

$$\frac{d^{\lambda+1} v}{dn^{\lambda+1}} = \alpha^{\lambda+1} v . Q$$

et quoniam vidimus valorem ipsius P talem habere formam:

$$P = A t^\lambda + B t^{\lambda-2} + C t^{\lambda-4} + D t^{\lambda-6};$$

tum

tum valor ipsius Q ex sequentibus binis seriebus erit compositus:

$$Q = (\lambda+1) A t^{\lambda+1} + (\lambda-1) B t^{\lambda-1} + (\lambda-3) C t^{\lambda-3} + (\lambda-5) D t^{\lambda-5} \text{ etc.}$$
$$+ \lambda A t^{\lambda-1} + (\lambda-2) B t^{\lambda-3} + (\lambda-4) C t^{\lambda-5} + \text{ etc.}$$

vnde patet hanc determinationem ita repraesentari posse, vt sit

$$Q = \frac{t.d.Pt}{dt} + \frac{dP}{dt}.$$

§. 43. Haec vero formula, qua ex cognito valore P sequens Q deriuatur, etiam ex ipsa natura rei sequenti modo ostendi potest. Cum per hypothesin sit

$$\frac{d^\lambda v}{d n^\lambda} = \alpha^\lambda v P,$$

erit differentiando

$$\frac{d^{\lambda+1} v}{d n^\lambda} = \alpha^\lambda P\, dv + \alpha^\lambda v\, dP;$$

initio autem vidimus esse $\frac{dv}{dn} = \alpha v.t$, siue $dv = \alpha v t.dn$ quo valore substituto fit

$$\frac{d^{\lambda+1} v}{d n^{\lambda+1}} = \alpha^{\lambda+1} v P t + \alpha^\lambda v \frac{dP}{dn};$$

tum vero assumsimus $t = \text{tang.}\, \alpha n$, vnde differentiando fit $\alpha\, dn = \frac{dt}{1+tt}$, quo valore in postremo termino substituto obtinebitur

$$\frac{d^{\lambda+1} v}{d n^{\lambda+1}} = \alpha^{\lambda+1} v P t + \alpha^{\lambda+1} v \frac{dP(1+tt)}{dt} = \alpha^{\lambda+1} v \left(Pt + \frac{dP(1+tt)}{dt}\right)$$

quae forma manifesto reducitur ad hanc:

$$\frac{d^{\lambda+1} v}{d n^{\lambda+1}} = \alpha^{\lambda+1} v. \frac{t.d.Pt + dP}{dt},$$

ita

ebus erit com-
(λ-5) D t^{λ-5} etc.
(λ-4)C t^{λ-5} + etc.
ntari posse, vt sit
cognito valore
rei sequenti
tir
dv = α v t. dn
differentiando fit
no substituto ob-
Pt + dP(1+tt)/dt)
ita

ita vt sit

$$Q = \frac{t.d.Pt + dP}{dt} = Pt + \frac{dP(1+tt)}{dt};$$

vnde intelligitur, si sumatur $tt + 1 = 0$, quo facto in nostris formulis signa terminorum alternabuntur, et omissa litera t, fieri $Q = P$; vnde patet, hoc casu omnes formulas superiores eundem valorem esse adepturas, id quod etiam ex formulis supra exhibitis manifestum est, ex quibus erit $2 - 1 = 1$; $6 - 5 = 1$; $24 - 28 + 5 = 1$; $120 - 180 + 61 = 1$; $720 - 1320 + 662 - 61 = 1$; etc. vnde insigne criterium obtinetur, verum formulae istae recte sint per calculum definitae.

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