PLENIOR EXPLICATIO
CIRCA COMPARATIONEM QVANTITATVM IN FORMVLA INTEGRALI

$$\int \frac{Z\,dz}{\sqrt{(1+mzz+nz^4)}}$$

CONTENTARVM, DENOTANTE Z FVNCTIONEM QVAMCVNQVE RATIONALEM IPSIVS zz.

Auctore

L. EVLERO.

§. 1.

Etsi hoc argumentum iam saepius tractaui atque Illustrissimus *La Grange* plures egregias obseruationes super huiusmodi formulis cum publico communicauit: id tamen neutiquam adhuc satis exploratum, multo minus exhaustum est censendum, sed plurima adhuc maxime abscondita inuoluere videtur, quae profundissimam indagationem requirunt atque insignia incrementa Analyseos pollicentur. Imprimis autem ipsae operationes analyticae, quae me primum ad hanc inuestigationem perduxerunt, ita sunt comparatae, vt non nisi per plures ambages totum negotium conficiant, vnde merito etiamnunc methodus directa ad easdem comparationes perducens maxime est desideranda. Praeterea vero vniuersa haec inuestigatio multo latius patet, quam eas formulas integrales, quas primo sum contemplatus, vbi pro littera Z tantum vel quantitatem constantem vel functionem integram ipsius zz huius formae: $F + Gzz + Hz^4 + Iz^6 + Kz^8$ etc. assumsi, quibus casibus ostendi, propositis duabus quibuscunque quantitatibus huius generis, semper tertiam eiusdem generis inueniri posse, quae a summa illarum discrepet quantitate algebraica, quae quidem euanescat casu quo Z est tantum quantitas constans.

§. 2. Nunc autem obseruaui, easdem comparationes institui posse, si pro Z accipiatur functio quaecunque rationalis ipsius zz, quae scilicet habeat huiusmodi formam:

$$\frac{F + Gzz + Hz^4 + Iz^6 + Kz^8 + etc.}{\mathfrak{F} + \mathfrak{G}zz + \mathfrak{H}z^4 + \mathfrak{I}z^6 + \mathfrak{K}z^8 + etc.}$$

vbi quidem hoc discrimen occurrit, quod differentia inter summam duarum huiusmodi formularum et tertiam formulam eiusdem generis inueniendam non amplius sit quantitas algebraica, veruntamen per logarithmos et arcus circulares semper exhiberi possit; ita vt nunc ista inuestigatio multo latius pateat, quam eam adhuc eram complexus. Atque hinc fortasse, si omnes operationes, quae ad hunc scopum manuducunt, debita attentione perpendantur, faciliorem viam apperire poterunt ad methodum directam perueniueniendi totumque hoc argumentum maxime abſtruſum feliciori ſucceſſu perſcrutandi.

§. 3. Quo autem haec omnia clarius perſpici queant, denotet iſte character $\Pi : z$ eam quantitatem tranſcendentem, quae ex integratione formulae propoſitae $\int \frac{Z\,dz}{\sqrt{(1+mzz+nz^4)}}$ naſcitur, dum integrale ita capi aſſumitur, vt euaneſcat poſito $z = 0$; vnde ſtatim manifeſtum eſt, fore quoque $\Pi : 0 = 0$. Deinde cum Z inuoluat tantum pares poteſtates ipſius z, cuiusmodi etiam in formula radicali inſunt, euidens eſt, ſi loco z ſcribatur $-z$, tum valorem quoque iſtius formulae integralis ideoque etiam characteris $\Pi : z$ in ſui negatiuum abire, ita vt ſit $\Pi : (-z) = -\Pi : z$. His praenotatis ſi proponantur duae quaecunque huiusmodi quantitates $\Pi : p$ et $\Pi : q$, ſemper inuenire licet tertiam quantitatem eiusdem generis $\Pi : r$, quae a ſumma illarum formularum $\Pi : p + \Pi : q$ differat quantitate vel algebraica vel ſaltem per logarithmos et arcus circulares aſſignabili. Regula vero, qua ex datis litteris p et q tertia r elicitur, ſemper manet eadem, quaecunque functio per litteram Z deſignetur: ſemper enim erit

$$r = \frac{p\sqrt{(1+mqq+nq^4)} + q\sqrt{(1+mpp+np^4)}}{1-nppqq}.$$

Hinc autem pro ſequentibus combinationibus obſeruaſſe iuuabit fore $\sqrt{(1+mrr+nr^4)} =$

$$\frac{(mpq+\sqrt{(1+mpp+np^4)}\sqrt{(1+mqq+nq^4)})(1+nppqq)+2npq(pp+qq)}{(1-nppqq)^2}.$$

§. 4. Non ſolum autem haec inueſtigatio adſtringitur ad huiusmodi formulas $\Pi : p$ et $\Pi : q$ pro arbitrio accipiendas, ſed adeo ad quotcunque formulas datas poteſt extendi, ita vt, quotcunque huiusmodi formulae fuerint propositae, scilicet:

$$\Pi : f + \Pi : g + \Pi : h + \Pi : i + \Pi : k + \text{etc.}$$

semper noua huiusmodi formula $\Pi : r$ assignari possit, quae ab illarum summa discrepet quantitate vel algebraica vel saltem per logarithmos et arcus circulares assignabili. Quin etiam formulas illas, quas tanquam datas spectauimus, ita definire licebit, vt discrimen illud, siue algebraicum siue a logarithmis arcubusque circularibus pendens, prorsus euanescat, ita vt futurum sit:

$$\Pi : r = \Pi : f + \Pi : g + \Pi : h + \Pi : i + \Pi : k + \text{etc.}$$

Atque haec fere sunt, ad quae hanc inuestigationem generaliorem, quam hic exponere constitui, mihi quidem extendere licuit; quamobrem singulas operationes, quae me huc perduxerunt, succincte sum propositurus.

Operatio I.

§. 5. Vniuersam hanc inuestigationem inchoaui a consideratione huius aequationis algebraicae:

$$\alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy = 0,$$

ex qua, cum sit quadratica, tam pro x quam pro y radicem extrahendo, colligitur vel

$$y = \frac{-\delta x + \sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4)}}{\gamma + \zeta xx} \text{ vel}$$

$$y = \frac{-\delta y + \sqrt{(\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4)}}{\gamma + \zeta yy},$$

ita vt hinc fiat

$$\sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4} = \gamma y + \delta x - \zeta xxy \text{ et}$$

$$\sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4} = \gamma x + \delta y + \zeta xyy).$$

§. 6.

§. 6. Nunc litteras α, γ, δ, ζ, ita definio, vt ambae formulae radicales ad formam

$$\sqrt{(1+mxx+na^4)} \text{ et } \sqrt{(1+myy+ny^4)},$$

reducantur, quem in finem facio

$$1^\circ.\ -\alpha\gamma = k,$$

$$2^\circ.\ \delta\delta - \gamma\gamma - \alpha\zeta = mk \text{ et}$$

$$3^\circ.\ -\gamma\zeta = nk,$$

ex quarum aequalitatum prima fit $\alpha = -\frac{k}{\gamma}$, ex tertia $\zeta = \frac{-nk}{\gamma}$, qui valores in fecunda fubftituti praebent

$$\delta\delta = \gamma\gamma + \frac{nkk}{\gamma\gamma} + mk,$$

ideoque

$$\delta = \sqrt{(\gamma\gamma + \frac{nkk}{\gamma\gamma} + mk)} = \frac{1}{\gamma}\sqrt{(\gamma^4 + m\gamma\gamma k + nkk)},$$

vnde aequatio noftra nunc erit

$$-k + \gamma\gamma(xx+yy) + 2xy\sqrt{(\gamma^4 + m\gamma\gamma k + nkk)} - nkxxyy = 0,$$

hinc igitur ambae noftrae formulae irrationales erunt

$$\sqrt{k(1+mxx+nx^4)} = \gamma y + \frac{1}{\gamma}x\sqrt{(\gamma^4+m\gamma\gamma k+nkk)} - \frac{nk}{\gamma}xxy,$$

$$\sqrt{k(1+myy+ny^4)} = \gamma x + \frac{1}{\gamma}y\sqrt{(\gamma^4+m\gamma\gamma k+nkk)} - \frac{nk}{\gamma}xyy.$$

§. 7. Cum nunc ambae quantitates x et y ita a fe inuicem pendeant, quemadmodum aequatio affumta declarat, litteras adhuc indefinitas γ et k ita definiamus, vt pofito $x = 0$ fiat $y = a$. Oportebit igitur effe $-k + \gamma\gamma aa = 0$, ideoque $k = \gamma\gamma aa$, quo valore fubftituto aequatio noftra erit

$$0 = \gamma\gamma(xx+yy-aa) + 2\gamma\gamma xy\sqrt{(1+maa+na^4)}$$
$$- n\gamma\gamma aaxxyy,$$

hincque fiet per $\gamma\gamma$ diuidendo

$$0=(xx+yy-aa)+2xy\sqrt{(1+maa+na^4)}-naaxxyy.$$

Tum vero ambae noſtrae formulae radicales ita exprimentur:

$$\sqrt{(1+mxx+nx^4)}=\tfrac{y}{a}+\tfrac{x}{a}\sqrt{(1+maa+na^4)}-naxxy$$
$$\sqrt{(1+myy+ny^4)}=\tfrac{x}{a}+\tfrac{y}{a}\sqrt{(1+maa+na^4)}-naxyy.$$

§. 8. Quo has formulas tractatu faciliores reddamus, ponamus $\sqrt{(1+maa+na^4)}=\mathfrak{A}$, ſimilique modo $\sqrt{(1+mxx+nx^4)}=\mathfrak{X}$ et $\sqrt{(1+myy+ny^4)}=\mathfrak{Y}$, et aequatio noſtra erit

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0,$$

vnde reperitur

$$y=-\frac{\mathfrak{A}x+a\mathfrak{X}}{1-naaxx} \text{ et } x=-\frac{\mathfrak{A}y+a\mathfrak{Y}}{1-naayy},$$

vnde patet ſi fuerit $y=0$ fore $x=a$; tum vero erunt formulae radicales

$$\sqrt{(1+mxx+nx^4)}=\mathfrak{X}=\tfrac{y}{a}+\tfrac{\mathfrak{A}x}{a}-naxxy,$$
$$\sqrt{(1+myy+ny^4)}=\mathfrak{Y}=\tfrac{x}{a}+\tfrac{\mathfrak{A}y}{a}-naxyy.$$

§. 9. Quemadmodum autem tam y per x quam x per y exprimere licuit, ita etiam $\mathfrak{Y}$ per ſolum x et $\mathfrak{X}$ per ſolum y exprimere licebit. Calculo autem inſtituto reperietur fore.

$$\mathfrak{X}=\frac{(-may+\mathfrak{A}\mathfrak{Y})(1+naayy)-2nay(aa+yy)}{(1-naayy)^2},$$
$$\mathfrak{Y}=\frac{(-max+\mathfrak{A}\mathfrak{X})(1+naaxx)-2nax(aa+xx)}{(1-naaxx)^2}.$$

§. 10. Praecipue autem circa noſtram aequationem

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0$$

notari-

notari meretur, quod ternae quantitates xx, yy, aa perfecte inter se sint permutabiles. Si enim membrum irrationale ad alteram partem transferatur, vt sit

$$xx + yy - aa - naaxxyy = - 2\mathfrak{A}xy,$$

et quadrata sumantur, restituendo pro $\mathfrak{A}^2$ valorem suum $1 + maa + na^4$, prodibit ista aequatio:

$$\left.\begin{array}{ll} + x^4 - 2xxyy - 4maaxxyy - 2na^4xxyy & + nna^4x^4y^4 \\ + y^4 - 2aaxx & - 2naax^4yy \\ + a^4 - 2aayy & - 2naaxxy^4 \end{array}\right\} = 0,$$

vbi permutabilitas litterarum a, x, y manifesto in oculos incurrit. In ipsis quidem formulis superioribus, vbi ipsa quantitas a ingreditur, permutabilitas non adeo est manifesta, sed prorsus elucebit, si loco a scribamus $-b$, itemque $\mathfrak{B}$ loco $\mathfrak{A}$; tum enim, quemadmodum erat

$$y = -\frac{x\mathfrak{B} - b\mathfrak{X}}{1 - nbbxx} \text{ et } x = -\frac{y\mathfrak{B} - b\mathfrak{Y}}{1 - nbbyy},$$

ita erit $b = \frac{-x\mathfrak{Y} - y\mathfrak{X}}{1 - nxxyy}$; similique modo pro formulis radicalibus seu litteris maiusculis erit

$$\mathfrak{Y} = \frac{(mbx + \mathfrak{B}\mathfrak{X})(1 + nbbxx) + 2nbx(bb + xx)}{(1 - nbbxx)^2},$$

$$\mathfrak{X} = \frac{(mby + \mathfrak{B}\mathfrak{Y})(1 + nbbyy) + 2nby(bb + yy)}{(1 - nbbyy)^2},$$

$$\mathfrak{B} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2},$$

sicque perfecta permutabilitas perspicitur.

Operatio II.

§. 11. Differentiemus nunc nostram aequationem algebraicam assumtam, quae est

$$xx + yy - aa + 2\mathfrak{A}xy - naaxxyy = 0,$$

 et

et aequatio differentialis erit

$$dx(x+\mathfrak{A}y-naaxyy)+dy(y+\mathfrak{A}x-naaxxy)=0,$$

siue

$$\frac{dx}{y+\mathfrak{A}x-naaxxy}+\frac{dy}{x+\mathfrak{A}y-naaxyy}=0.$$

Ex superioribus autem constat esse

$$y+\mathfrak{A}x-naaxyy=a\mathfrak{X} \text{ et}$$
$$x+\mathfrak{A}y-naaxyy=a\mathfrak{Y},$$

vnde aequatio differentialis hanc induet formam:

$$\frac{dx}{a\mathfrak{X}}+\frac{dy}{a\mathfrak{Y}}=0, \text{ siue}$$
$$\frac{dx}{\sqrt{(1+mxx+nx^4)}}+\frac{dy}{\sqrt{(1+myy+ny^4)}}=0.$$

§. 12. Inuenta igitur hac aequatione differentiali denotet iste character $\Gamma:x$ integrale $\int\frac{dx}{\mathfrak{X}}$, et character $\Gamma:y$ integrale $\int\frac{dy}{\mathfrak{Y}}$, vtroque integrali ita sumto, vt euanescat posito vel $x=0$ vel $y=0$, atque aequationem illam differentialem integrando fiet $\Gamma:x+\Gamma:y=C$. Cum autem sumto $x=0$ fiat etiam $\Gamma:x=0$ et $y=a$, erit constans illa $C=\Gamma:a$, ita vt habeamus hanc aequationem: $\Gamma:x+\Gamma:y=\Gamma:a$.

§. 13. Quoniam hic nulla amplius variabilitatis ratio tenetur, patet, sumtis binis litteris x et y pro lubitu, litteram a ita semper definiri posse, vt fiat

$$\Gamma:a=\Gamma:x+\Gamma:y.$$

Si enim in §. 10. loco b scribatur $-a$, sumi debet

$$a=\frac{x\mathfrak{Y}+y\mathfrak{X}}{1-nxxyy},$$

quae comparatio iam casum constituit specialem inuestigationis

tionis generalis, quam ſuſcepimus. Si enim loco x et y ſcribamus p et q, at r loco a, tum vero $\mathfrak{P}$, $\mathfrak{Q}$ et $\mathfrak{R}$ loco $\mathfrak{X}$, $\mathfrak{Y}$ et $\mathfrak{A}$, atque ſi, ſumtis pro lubitu quantitatibus p, q, capiatur $r = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$; tum vtique erit $\Gamma : r = \Gamma : p + \Gamma : q$, ita vt hoc caſu diſcrimen illud inter $\Gamma : r$ et ſummam $\Gamma : p + \Gamma : q$ plane euaneſcat. Sicque iam euoluimus caſum, quo in noſtra forma generali $\int \frac{Z\,dz}{\sqrt{(1 + mzz + nz^4)}}$ pro Z ſumitur quantitas conſtans.

Operatio III.

§. 14. Quo nunc propius ad noſtrum inſtitutum accedamus, ſint X et Y tales functiones ipſarum x et y, qualem volumus eſſe Z ipſius z, et quoniam modo inuenimus

$$\frac{dx}{\sqrt{(1 + mxx + nx^4)}} + \frac{dy}{\sqrt{(1 + myy + ny^4)}} = 0,$$

ponamus eſſe

$$\frac{X\,dx}{\sqrt{(1 + mxx + nx^4)}} + \frac{Y\,dy}{\sqrt{(1 + myy + ny^4)}} = dV,$$

ita vt, ſi X et Y eſſent quantitates conſtantes, foret $dV = 0$. Hinc ergo ſi loco $\frac{dy}{\sqrt{(1 + myy + ny^4)}}$ ſcribamus $\frac{-dx}{\sqrt{(1 + mxx + nx^4)}}$, fiet $dV = \frac{(X - Y)\,dx}{\sqrt{(1 + mxx + nx^4)}}$, vel etiam

$$dV = \frac{(Y - X)\,dy}{\sqrt{(1 + myy + ny^4)}}.$$

At ſi loco radicalium ſuos valores rationales ſcribamus, erit

$$dV = \frac{a(X - Y)\,dx}{y + \mathfrak{A}x - naaxxy} \text{ vel}$$

$$dV = \frac{a(Y - X)\,dy}{x + \mathfrak{A}y - naaxyy}.$$

§. 15. Cum autem nulla ſit ratio, cur iſtud differentiale dV potius per dx quam per dy exprimamus;

 conſul-

consultum erit nouam quantitatem in calculum introducere, quae aeque referatur ad x et ad y. Hunc in finem faciamus productum $xy = u$, ac statuamus

$$\frac{dx}{y + \mathfrak{A}x - naaxxy} = - \frac{dy}{x + \mathfrak{A}y - naaxyy} = s\,du.$$

Hinc igitur erit

$$dx = s\,du\,(y + \mathfrak{A}x - naaxxy) \text{ et}$$
$$dy = - s\,du\,(x + \mathfrak{A}y - naaxyy),$$

ex quibus colligimus

$$y\,dx + x\,dy = s\,du\,(yy - xx) = du,$$

sicque habebimus $s = \frac{1}{yy - xx}$, ita vt habeamus:

$$\frac{dx}{y + \mathfrak{A}x - naaxxy} = - \frac{dy}{x + \mathfrak{A}y - naaxyy} = \frac{du}{yy - xx}.$$

Hoc igitur valore substituto nanciscimur:

$$dV = \frac{a\,(X - Y)\,du}{yy - xx} = \frac{-\,a\,du\,(X - Y)}{xx - yy}.$$

§. 16. Cum autem X et Y sint functiones rationales pares ipsarum x et y, in quibus tantum insunt potestates pares harum litterarum; facile intelligitur, formulam $X - Y$ semper esse diuisibilem per $xx - yy$ et quotum praeter productum $xy = u$ insuper inuoluere summam quadratorum $xx + yy$; quamobrem statuamus $xx + yy = t$, et cum aequatio nostra fundamentalis fiat

$$t - aa + 2\mathfrak{A}u - naauu = 0,$$

ex ea fit

$$t = aa - 2\mathfrak{A}u + naauu,$$

ita vt t aequetur functioni rationali ipsius u. Quod si ergo hic valor vbique loco t scribatur, differentiale nostrum quaesitum dV per solam variabilem u exprimetur, ita vt

posito

posito $dV = U du$ semper sit U functio rationalis ipsius u, quae ergo si fuerit integra, tum V aequabitur functioni algebraicae ipsius u; sin autem sit functio fracta, tum integrale $V = \int U du$ semper per logarithmos et arcus circulares exhiberi poterit. Hoc ergo integrale si ita capiatur, vt euanescat posito $u = xy = 0$, id etiam euanescet posito $x = 0$ vel $y = 0$. Atque hinc integrando impetrabimus:

$$\int \frac{X dx}{\sqrt{(1 + mxx + nx^4)}} + \int \frac{Y dy}{\sqrt{(1 + myy + ny^4)}} = C + V = C + \int U du.$$

§. 17. Quod si igitur characteres $\Pi : x$ et $\Pi : y$ denotent valores horum integralium, ita vt vtrumque euanescat sumto vel $x = 0$ vel $y = 0$, quoniam facto $x = 0$ per hypothesin fit $y = a$, manifestum est constantem hanc fore $\Pi : a$, sicque aequatio finita resultabit ista:

$$\Pi : x + \Pi : y = \Pi : a + \int U du.$$

§. 18. Accuratius autem in valores huius fractionis U pro quouis casu inquiramus. Ac primo quidem, si sumatur

$$Z = \alpha + \beta zz + \gamma z^4 + \delta z^6 + \text{etc.}$$

erit simili modo

$$X = \alpha + \beta xx + \gamma x^4 + \delta x^6 + \text{etc. et}$$
$$Y = \alpha + \beta yy + \gamma y^4 + \delta y^6 + \text{etc.},$$

quare cum inuenerimus

$$dV = U du = -\frac{a du (X - Y)}{xx - yy}, \text{ erit}$$
$$U = -\frac{a(X - Y)}{xx - yy}, \text{ ideoque}$$
$$U = -\frac{a(\beta(xx - yy) + \gamma(x^4 - y^4) + \delta(x^6 - y^6))}{xx - yy},$$

 vnde

vnde fit

$$U = -a\beta - a\gamma(xx + yy) - a\delta(x^4 + xxyy + y^4).$$

Cum igitur fit $xx + yy = t$ et $xy = u$, erit

$$U = -a\beta - a\gamma t - a\delta(tt - uu),$$

vnde cum fit

$$t = aa - 2\mathfrak{A}u + naauu,$$

calculo fubducto fiet

$$\begin{aligned} \int U\,du = & -a\beta u - a\gamma(aau - \mathfrak{A}uu + \tfrac{1}{3}naau^3) \\ & -a\delta(a^4uu - 2aa\mathfrak{A}uu + \tfrac{2}{3}na^4u^3 - n\mathfrak{A}a^2u^4 + \tfrac{1}{5}n^2a^4u^5 \\ & \qquad + \tfrac{4}{3}\mathfrak{A}^2u^3 \\ & \qquad + \tfrac{1}{3}u^5). \end{aligned}$$

Atque hinc intelligitur, fi functio Z ad altiores poteftates exfurgat, quomodo valor integralis ipfius $\int U\,du$ inde inueniri queat.

§. 19. Sin autem Z fuerit functio fracta, fcilicet

$$Z = \frac{\alpha + \beta zz + \gamma z^4}{\zeta + \eta zz + \theta z^4}, \text{ hincque}$$

$$X = \frac{\alpha + \beta xx + \gamma x^4}{\zeta + \eta xx + \theta x^4} \text{ et } Y = \frac{\alpha + \beta yy + \gamma y^4}{\zeta + \eta yy + \theta y^4}, \text{ erit}$$

$$X - Y = \frac{(\beta\zeta - \alpha\eta)(xx - yy) + (\gamma\zeta - \alpha\theta)(x^4 - y^4) + (\gamma\eta - \beta\theta)x^2y^2(x^2 - y^2)}{\zeta\zeta + \zeta\eta(xx + yy) + \zeta\theta(x^4 + y^4) + \eta^2x^2y^2 + \eta\theta x^2y^2(xx + yy) + \theta\theta x^4y^4}.$$

Hinc igitur introductis litteris t et u erit

$$\frac{X - Y}{xx - yy} = \frac{\beta\zeta - \alpha\eta + (\gamma\zeta - \alpha\theta)t + (\gamma\eta - \beta\theta)uu}{\zeta\zeta + \zeta\eta t + \zeta\theta(tt - 2uu) + \eta\eta uu + \eta\theta tuu + \theta\theta u^4},$$

quam ob rem cum fit

$$U = -\frac{a(X - Y)}{xx - yy}, \text{ ob}$$

$$t = aa - 2\mathfrak{A}u + naauu,$$

manifeftum eft, integrale formulae $\int U\,du$ nifi fuerit algebraicum, femper, conceffis logarithmis et arcubus circularibus, exhiberi poffe. Sicque per has tres operationes omnia prae-

praeſtitimus, quibus opus eſt ad omnia problemata huc ſpectantia ſoluenda.

Problema 1.

§. 20. *Si* $\Pi : x$ *et* $\Pi : y$ *denotent valores formularum integralium:*

$$\int \frac{X\,dx}{\sqrt{(1 + mxx + nx^4)}} \quad et \quad \int \frac{Y\,dy}{\sqrt{(1 + myy + ny^4)}},$$

vbi X et Y ſint functiones pares ſimiles ipſarum x et y, atque dentur binae huiusmodi formulae $\Pi : x$ *et* $\Pi : y$; *inuenire tertiam formulam eiusdem generis* $\Pi : z$, *vt ſit* $\Pi : z = \Pi : x + \Pi : y + W$, *ita vt W ſit functio vel algebraica vel per logarithmos et arcus circulares aſſignabilis.*

Solutio.

Cum dentur binae quantitates x et y, ex iis formentur radicales

$$\mathfrak{X} = \sqrt{(1 + mxx + nx^4)} \text{ et } \mathfrak{Y} = \sqrt{(1 + nyy + ny^4)},$$

ex quibus definiatur quantitas z, eodem modo quo ſupra litteram a per x et y definire docuimus, ita vt ſit $z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}$ eiusque valor irrationalis

$$\mathfrak{Z} = \sqrt{(1 + mzz + nz^4)} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2},$$

tum in ſuperioribus formulis vbique loco a et $\mathfrak{A}$ ſcribamus z et Z et capiatur $= -\frac{z(X - Y)}{xx - yy}$, quam quantitatem vidimus ſemper reduci poſſe ad functionem ipſius u, exiſtente $u = xy$, ac ponatur $V = \int U\,du$, in qua integratione quantitates z et $\mathfrak{Z}$ pro conſtantibus ſunt habendae, ita vt littera V ſpectari poſſit tanquam functio ipſius $u = xy$, quandoquidem etiam z et $\mathfrak{Z}$ per x et y determinantur.

Probe

Probe autem teneatur, in ista formula integrali solam quantitatem a vt variabilem esse tractandam. Hac igitur quantitate V inuenta erit $\Pi : x + \Pi : y = \Pi : z + V$, vnde cum debeat esse $\Pi : z = \Pi : x + \Pi : y + W$, patet esse $W = - V$, ideoque quantitatem vel algebraicam, vel per logarithmos et arcus circulares assignabilem.

Corollarium 1.

§. 21. Totum ergo negotium hic redit ad integrationem formulae $U\,du$, existente $u = xy$ et $U = -\frac{z(X - Y)}{xx - yy}$, quam supra vidimus semper per u exprimi posse, siquidem in hac integratione litterae z et $\mathfrak{Z}$ vt quantitates constantes tractentur.

Corollarium 2.

§. 22. Cum igitur pro data indole binarum functionum X et Y haec integratio nulla laboret difficultate, ipsumque integrale per u, hoc est per xy exprimatur, cuius valorem ex datis quantitatibus x et y semper exhibere liceat, loco quantitatis V scribemus in posterum characterem $\Phi : xy$, vnde pro quibusque aliis litteris loco x et y assumtis intelligitur significatus characterum $\Phi : pq$, $\Phi : ab$ etc.

Corollarium 3.

§. 33. Hoc igitur charactere recepto, si pro datis quantitatibus x et y capiamus $z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}$, vnde fit

$$\mathfrak{Z} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2},$$

erit $\Pi : z = \Pi : x + \Pi : y - \Phi : xy$.

Pro-

Problema 2.

§. 24. *Seruatis omnibus characteribus, quos hactenus explicauimus, si dentur ternae formulae*, $\Pi : p$, $\Pi : q$, $\Pi : r$; *inuenire quartam eiusdem generis* $\Pi : z$, *vt fiat*

$$\Pi : z = \Pi p + \Pi : q + \Pi : r + W,$$

ita vt W *sit quantitas algebraica, vel per logarithmos arcusue circulares assignabilis.*

Solutio.

Ex datis binis quantitatibus p et q, ideoque etiam $\mathfrak{P}$ et $\mathfrak{Q}$ inde oriundis, capiatur $x = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$, simulque

$$\mathfrak{X} = \frac{(mpq + \mathfrak{P}\mathfrak{Q})(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2}.$$

Tum vero etiam colligatur valor characteris $\Phi : pq$, eritque per praecedentia

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq, \text{ siue}$$

$$\Pi : p + \Pi : q = \Pi : x + \Phi : pq,$$

quo valore substituto erit

$$\Pi : z = \Pi : x + \Pi : r + \Phi : pq + \mathrm{W}.$$

Ex praecedente autem problemate, si loco y hic scribamus r et capiamus $z = \frac{x\mathfrak{R} + r\mathfrak{X}}{1 - nrrxx}$, vnde fit

$$\mathfrak{Z} = \frac{mrx + \mathfrak{R}\mathfrak{X}(1 + nrrxx) + 2nrx(rr + xx)}{(1 - nrrxx)^2}, \text{ erit}$$

$$\Pi : z = \Pi : x + \Pi : r - \Phi : rx,$$

qua forma cum praecedente collata colligitur

$$\mathrm{W} = -\Phi : pq - \Phi : rx, \text{ ita vt sit}$$

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r - \Phi : pq - \Phi : rx.$$

Problema 3.

§. 25. *Propositis huiusmodi formulis*: $\Pi : p$, $\Pi : q$, $\Pi : r$, $\Pi : s$, *inuenire quintam eiusdem generis* $\Pi : z$, *vt fiat* $\Pi : z = \Pi : p + \Pi : q + \Pi : r + \Pi\, s + W$, *ita vt* W *sit quantitas algebraica, vel per logarithmos arcusue circulares assignabilis.*

Solutio.

Ex datis binis p et q quaeratur x, vt sit $x = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$, item

$$\mathfrak{X} = \frac{(mpq + \mathfrak{P}\mathfrak{Q})(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2},$$

eritque $\Pi : x = \Pi : p + \Pi : q - \Phi : pq$. Simili modo ex binis datis r et s quaeratur y, vt sit $y = \frac{r\mathfrak{S} + s\mathfrak{R}}{1 - nrrss}$, eritque

$$\mathfrak{Y} = \frac{(mrs + \mathfrak{R}\mathfrak{S})(1 + nrrss) + 2nrs(rr + ss)}{(1 - nrrss)^2},$$

tum vero

$$\Pi : y = \Pi : r + \Pi : s - \Phi : rs.$$

Nunc denique ex inuentis x et y sumatur $z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}$ et

$$\mathfrak{Z} = \frac{(mxy + \mathfrak{Y}\mathfrak{X})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2}$$

eritque $\Pi : z = \Pi : x + \Pi : y - \Phi : xy$. Quod si ergo loco $\Pi : x$ et $\Pi : y$ valores modo inuenti substituantur, fiet

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + \Pi : s - \Phi : pq - \Phi : rs - \Phi\, xy.$$

Corollarium 1.

§. 26. Hinc iam abunde intelligitur, si proponantur quotcunque huiusmodi formulae, quemadmodum nouam eiusdem generis $\Pi : z$ inuestigari oporteat, quae ab illis iunctim sumtis discrepet quantitate algebraica, vel per logarithmos arcusue circulares assignabili.

Corollarium 2.

§. 27. Quod si omnes illae formulae fuerint inter se aequales earumque numerus $= \lambda$, semper noua formula $\Pi : z$ inueniri poterit, vt sit $\Pi : z = \lambda \Pi : p + W$, existente W quantitate vel algebraica, vel per logarithmos arcusue circulares assignabili. Quin etiam, duabus huiusmodi formulis $\Pi : p$ et $\Pi : q$ propositis, inueniri poterit $\Pi : z$, vt sit $\Pi : z = \lambda : \Pi : p + \mu . \Pi : q + W$.

Scholion.

§. 28. Hoc igitur modo non solum principia et fundamenta, quibus hoc argumentum innititur, succincte ac dilucide mihi quidem exposuisse videor: sed hoc argumentum etiam multo latius amplificaui quam adhuc est factum. Semper autem maxime est optandum, vt via magis directa detegatur, quae ad easdem inuestigationes perducat. Nemo enim certe dubitabit, quin hinc maxima in vniuersam Analysin incrementa essent redundatura.

Applicatio.

ad quantitates transcendentes in forma $\int \frac{dz(\alpha + \beta zz)}{\sqrt{(1 + mzz + nz^4)}} = \Pi : z$ contentas.

§. 29. Cum igitur hic sit $Z = \alpha + \beta zz$, propositis duabus formulis huius generis $\Pi : x$ et $\Pi : y$ sumtoque $z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}$, hincque

$$\mathfrak{Z} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(x^2 + y^2)}{(1 - nxxyy)^2},$$

ex §. 18. vbi $u = xy$ et $a = z$, erit

$$\Pi : z = \Pi : x + \Pi : y + \beta x y z,$$

ita vt character ante adhibitus $\Phi : y$ hoc caſu accipiat valorem $\beta x y z$. Huius igitur regulae ope propoſitis duabus huiusmodi formulis $\Pi : x$ et $\Pi : y$, tertia $\Pi : z$ ſemper reperiri poteſt, quae a ſumma illarum differat quantitate algebraica $\beta x y z$.

§. 30. Ponamus igitur quotcunque huiusmodi formulas tranſcendentes proponi:

$$\Pi : a,\ \Pi : b,\ \Pi : c,\ \Pi : d,\ \Pi : e,\ \Pi : f,\ \Pi : g, \text{ etc.}$$

et ex ſingulis quantitatibus a, b, c, d, e, colligi valores irrationales litteris germanicis inſignitas:

$$\mathfrak{A} = \sqrt{(1 + m a a + n a^4)};\quad \mathfrak{B} = \sqrt{(1 + m b b + n b^4)};$$
$$\mathfrak{C} = \sqrt{(1 + m c c + n c^4)};\quad \mathfrak{D} = \sqrt{(1 + m d d + n d^4)};$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

ſemper noua formula eiusdem generis exhiberi poterit, quae a ſumma earum diſcrepet quantitate algebraica, quantuscunque etiam fuerit earum formularum datarum numerus. Operationes autem ad hunc finem perducentes commodiſſime ſequenti modo inſtituentur.

§. 31. Primo ſcilicet ex binis datarum a et b quaeratur p, vt ſit

$$p = \frac{a\mathfrak{B} + b\mathfrak{A}}{1 - n a a b b} \text{ et } \mathfrak{P} = \frac{(m a b + \mathfrak{A}\mathfrak{B})(1 + n a a b b) + 2 n a b (a a + b b)}{(1 - n a a b b)^2}.$$

Deinde ex hac quantitate p, cum datarum tertia c iuncta, definiatur q, vt ſit

$$q = \frac{p\mathfrak{C} + c\mathfrak{P}}{1 - n c c p p} \text{ et } \mathfrak{Q} = \frac{(m c p + \mathfrak{C}\mathfrak{P})(1 + n c c p p) + 2 n c p (c c + p p)}{(1 - n c c p p)^2}.$$

Tertio ex hac quantitate q cum quarta datarum d iuncta quae-

quaeratur r, vt ſit

$$r = \frac{q\mathfrak{D} + d\mathfrak{Q}}{1 - nddqq} \text{ et } \mathfrak{R} = \frac{(mdq + \mathfrak{D}\mathfrak{Q})(1 + nddqq) + 2ndq(dd + qq)}{(1 - nddqq)^2}.$$

Quarto ex iſta quantitate r cum quinta datarum e definiatur s, vt ſit

$$s = \frac{r\mathfrak{E} + e\mathfrak{R}}{1 - neerr} \text{ et } \mathfrak{S} = \frac{(mer - \mathfrak{E}\mathfrak{R})(1 + neerr) + 2ner(ee + rr)}{(1 - neerr)^2}.$$

Haeque operationes continuentur donec omnes quantitates datae in computum fuerint ductae.

§. 32. His autem omnibus valoribus inuentis, ſequentes comparationes deſideratae ordine ita ſe habebunt:

$$\text{I. } \Pi : p = \Pi : a + \Pi : b + \beta abp$$

$$\text{II. } \Pi : q = \Pi : a + \Pi : b + \Pi : c + \beta abp + \beta cpq$$

$$\text{III. } \Pi : r = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \beta abp + \beta cpq + \beta dqr$$

$$\text{IV. } \Pi : s = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e + \beta abp + \beta cpq + \beta dqr + \beta ers$$

$$\text{V. } \Pi : t = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e + \Pi : f + \beta abp + \beta cpq + \beta dqr + \beta ers + \beta fst.$$

etc. etc.

§. 33. Cum igitur iſta formula tranſcendens:

$\Pi : z = \int \frac{dz(\alpha + \beta zz)}{\sqrt{(1 + mzz + z^4)}}$,

in

in se contineat arcus omnium sectionum conicarum a vertice sumtos, harum formularum ope, quotcunque proponantur arcus in quauis sectione conica, qui omnes a vertice sint sumti, semper nouus in eadem sectione conica arcus pariter a vertice abscindi poterit, qui a summa illorum arcuum datorum discrepet quantitate algebraice assignabili. Quin etiam nihil impedit, quo minus aliqui inter arcus datos capiantur negatiui, quandoquidem iam annotauimus esse $\Pi : (-z) = -\Pi : z$, ita vt nostra determinatio etiam accommodari possit ad arcus inter terminos quoscunque interceptos. Hocque modo tractatio, quam nuper circa comparationem talium arcuum dedi, multo generalior reddi poterit.

§. 34. Ceterum, quemadmodum hoc casu, quo sumsimus $Z = \alpha + \beta z z$, character supra vsurpatus $\Phi : x y$ abiit in $\beta x y z$, dum scilicet ex binis quantitatibus x et y, secundum praecepta data tertia z determinatur: ita etiam, quaecunque alia functio loco Z adhibeatur, quoniam posuimus $\Phi : x y = a \int \frac{(X - Y)\, d u}{x x - y y}$, existente $u = x y$, integratione absoluta functio inde resultans tantum quantitatem u cum litteris a et $\mathfrak{A}$ continebit, quandoquidem littera t ita exprimebatur: $t = a a - 2 \mathfrak{A} u + n a a u u$, cum inuento integrali vbique loco u scribatur $x y$, at vero loco a et $\mathfrak{A}$ litterae z et $\mathfrak{Z}$; atque hoc modo obtinebitur valor characteris $\Phi : x y$ pro quouis casu proposito, quae functio, nisi fuerit algebraica, semper per logarithmos et arcus circulares exhiberi poterit, siquidem, vti assumsimus, littera Z fuerit functio rationalis par ipsius z.

VBE-