ex qua deducimus ipſam aequationem noſtram demonſtrandam

$$\frac{1}{s} = \cot. s + \tfrac{1}{3}\,\text{tag.}(30^\circ + \tfrac{s}{3}) + \tfrac{1}{9}\,\text{tag.}(30^\circ + \tfrac{s}{9}) + \tfrac{1}{27}\,\text{tag.}\,30^\circ + \tfrac{s}{27} + \text{etc.}$$
$$- \tfrac{1}{3}\,\text{tag.}(30^\circ - \tfrac{s}{3}) - \tfrac{1}{9}\,\text{tag.}(30^\circ - \tfrac{s}{9}) - \tfrac{1}{27}\,\text{tag.}\,30^\circ - \tfrac{s}{27} - \text{etc.}$$

§. 13. Quin etiam ſimili modo huiusmodi feries pro maioribus rationibus, quibus arcus s continuo diminuitur, exhibere licet. Cum enim ſit

$$\text{ſin}\ 4\Phi = 8\ \text{ſin.}\,\Phi\ \text{coſ.}(45^\circ + \Phi)\ \text{coſ.}(45^\circ - \Phi)\ \text{coſ.}\,\Phi$$

pro ratione quadrupla erit

$$\frac{1}{s} = \cot. s + \tfrac{1}{4}\,\text{tag.}\tfrac{s}{4} + \tfrac{1}{16}\,\text{tag.}\tfrac{s}{16} + \tfrac{1}{64}\,\text{tag.}\tfrac{s}{64} + \text{etc.}$$
$$+ \tfrac{1}{4}\,\text{tag.}(45^\circ + \tfrac{s}{4}) + \tfrac{1}{16}\,\text{tag.}(45^\circ + \tfrac{s}{16}) + \tfrac{1}{64}\,\text{tag}\,(45^\circ + \tfrac{s}{64}) + \text{etc.}$$
$$- \tfrac{1}{4}\,\text{tag.}(45^\circ - \tfrac{s}{4}) - \tfrac{1}{16}\,\text{tag.}(45^\circ - \tfrac{s}{16}) - \tfrac{1}{64}\,\text{tag.}(45^\circ - \tfrac{s}{64}) - \text{etc.}$$

Porro cum ſit

$$\text{ſin.}\,5\Phi = 16\,\text{ſin.}\,\Phi\,\text{coſ}(18^\circ + \Phi)\,\text{coſ.}(18^\circ - \Phi)\,\text{coſ.}(54^\circ + \Phi)\,\text{coſ.}(54^\circ - \Phi)$$

reperiemus pro ratione quintupla

$$\frac{1}{s} = \cot. s + \tfrac{1}{5}\,\text{tag.}(18^\circ + \tfrac{s}{5}) + \tfrac{1}{25}\,\text{tag.}(18^\circ + \tfrac{s}{25})$$
$$- \tfrac{1}{5}\,\text{tag.}(18^\circ - \tfrac{s}{5}) - \tfrac{1}{25}\,\text{tag.}(18^\circ - \tfrac{s}{25})$$
$$+ \tfrac{1}{5}\,\text{tag.}(54^\circ + \tfrac{s}{5}) + \tfrac{1}{25}\,\text{tag.}(54^\circ + \tfrac{s}{25})\quad \text{etc.}$$
$$- \tfrac{1}{5}\,\text{tag.}(54^\circ - \tfrac{s}{5}) - \tfrac{1}{25}\,\text{tag.}(54^\circ - \tfrac{s}{25}).$$

Pari modo vlterius progredi liceret, verum feries reſultarent nimis perplexae quam vt attentione dignae viderentur.

QVO-

m noſtram de-
tag. 30° + s/27 + etc.
tag. 30° − s/27 − etc.
huiusmodi feries
s continuo dimi-
(45° − Φ) coſ. Φ
tc.
tag (45° + s/64) + etc.
tag. (45° − s/64) − etc.
54° + Φ) coſ. (54° − Φ)
(18° + s/25)
(18° − s/25)
(54° + s/25) etc.
(54° − s/25).
verum feries re-
tione dignae vide-
QVO-

QVOMODO SINVS ET COSINVS

ANGVLORVM MVLTIPLORVM

PER PRODVCTA EXPRIMI QVEANT.

§. 1.

Propoſito angulo quocunque Φ ponatur breuitatis gratia:

$$\text{coſ.}\,\Phi + \sqrt{-1}\ \text{ſin.}\,\Phi = p \quad \text{et} \quad \text{coſ.}\,\Phi - \sqrt{-1}\ \text{ſin.}\,\Phi = q,$$

erit $pq = 1$; tum vero

$$p^n = \text{coſ.}\,n\Phi + \sqrt{-1}\ \text{ſin.}\,n\Phi \ \text{et}\ q^n = \text{coſ.}\,n\Phi - \sqrt{-1}\ \text{ſin.}\,n\Phi,$$

vnde fit

$$p^n + q^n = 2\,\text{coſ.}\,n\Phi \quad \text{et} \quad p^n - q^n = 2\sqrt{-1}\ \text{ſin.}\,n\Phi;$$

Res igitur eo redit, vt formulae $p^n + q^n$ et $p^n - q^n$ in factores reſoluantur.

§. 2. Conſideremus primo formulam

$$p^n + q^n = 2\,\text{coſ.}\,n\Phi,$$

quae, quoties n eſt numerus impar, factorem habet ſimplicem $p + q - 2\,\text{coſ.}\,\Phi$, ita vt his caſibus coſ. Φ ſit factor ipſius coſ. $n\Phi$: Pro reliquis factoribus autem ponamus factorem duplicem in genere eſſe, $pp - 2pq\,\text{coſ.}\,\omega + qq$, ita vt formula $p^n + q^n$ euaneſcat, poſito

pp –

ex qua deducimus ipſam aequationem noſtram demonſtrandam

$$\frac{1}{s} = \mathrm{cot.}\, s + \tfrac{1}{3}\,\mathrm{tag.}(30^\circ + \tfrac{s}{3}) + \tfrac{1}{9}\,\mathrm{tag.}(30^\circ + \tfrac{s}{9}) + \tfrac{1}{27}\,\mathrm{tag.}\, 30^\circ + \tfrac{s}{27} + \text{etc.}$$
$$- \tfrac{1}{3}\,\mathrm{tag.}(30^\circ - \tfrac{s}{3}) - \tfrac{1}{9}\,\mathrm{tag.}(30^\circ - \tfrac{s}{9}) - \tfrac{1}{27}\,\mathrm{tag.}\, 30^\circ - \tfrac{s}{27} - \text{etc.}$$

§. 13. Quin etiam ſimili modo huiusmodi ſeries pro maioribus rationibus, quibus arcus s continuo diminuitur, exhibere licet. Cum enim ſit

$$\mathrm{fin}\; 4\Phi = 8\,\mathrm{fin.}\,\Phi\,\mathrm{cof.}\,(45^\circ + \Phi)\,\mathrm{cof.}\,(45^\circ - \Phi)\,\mathrm{cof.}\,\Phi$$

pro ratione quadrupla erit

$$\frac{1}{s} = \mathrm{cot.}\, s + \tfrac{1}{4}\,\mathrm{tag.}\,\tfrac{s}{4} + \tfrac{1}{16}\,\mathrm{tag.}\,\tfrac{s}{16} + \tfrac{1}{64}\,\mathrm{tag.}\,\tfrac{s}{64} + \text{etc.}$$
$$+ \tfrac{1}{4}\,\mathrm{tag.}(45^\circ + \tfrac{s}{4}) + \tfrac{1}{16}\,\mathrm{tag.}(45^\circ + \tfrac{s}{16}) + \tfrac{1}{64}\,\mathrm{tag}\,(45^\circ + \tfrac{s}{64}) + \text{etc.}$$
$$- \tfrac{1}{4}\,\mathrm{tag.}(45^\circ - \tfrac{s}{4}) - \tfrac{1}{16}\,\mathrm{tag.}(45^\circ - \tfrac{s}{16}) - \tfrac{1}{64}\,\mathrm{tag.}\,(45^\circ - \tfrac{s}{64}) - \text{etc.}$$

Porro cum ſit

$$\mathrm{fin.}\,5\Phi = 16\,\mathrm{fin.}\,\Phi\,\mathrm{cof}(18^\circ + \Phi)\,\mathrm{cof.}(18^\circ - \Phi)\,\mathrm{cof.}(54^\circ + \Phi)\,\mathrm{cof.}(54^\circ - \Phi)$$

reperiemus pro ratione quintupla

$$\frac{1}{s} = \mathrm{cot.}\, s + \tfrac{1}{5}\,\mathrm{tag.}\,(18^\circ + \tfrac{s}{5}) + \tfrac{1}{25}\,\mathrm{tag.}\,(18^\circ + \tfrac{s}{25})$$
$$- \tfrac{1}{5}\,\mathrm{tag.}\,(18^\circ - \tfrac{s}{5}) - \tfrac{1}{25}\,\mathrm{tag.}\,(18^\circ - \tfrac{s}{25}) \quad \text{etc.}$$
$$+ \tfrac{1}{5}\,\mathrm{tag.}\,(54^\circ + \tfrac{s}{5}) + \tfrac{1}{25}\,\mathrm{tag.}\,(54^\circ + \tfrac{s}{25})$$
$$- \tfrac{1}{5}\,\mathrm{tag.}\,(54^\circ - \tfrac{s}{5}) - \tfrac{1}{25}\,\mathrm{tag.}\,(54^\circ - \tfrac{s}{25}).$$

Pari modo vlterius progredi liceret, verum ſeries reſultarent nimis perplexae quam vt attentione dignae viderentur.

QVO-

m noſtram de-
tag. 30° + s/27 + etc.
tag. 30° − s/27 − etc.
huiusmodi ſeries
s continuo dimi-
(45° − Φ) cof. Φ
tc.
tag (45° + s/64) + etc.
tag. (45° − s/64) − etc.
54° + Φ) cof. (54° − Φ)
(18° + s/25)
(18° − s/25) etc.
(54° + s/25)
(54° − s/25).
verum ſeries re-
tione dignae vide-
QVO-

QVOMODO SINVS ET COSINVS

ANGVLORVM MVLTIPLORVM PER PRODVCTA EXPRIMI QVEANT.

§. 1.

Propoſito angulo quocunque Φ ponatur breuitatis gratia:

$$\mathrm{cof.}\,\Phi + \sqrt{-1}\,\mathrm{fin.}\,\Phi = p \text{ et } \mathrm{cof.}\,\Phi - \sqrt{-1}\,\mathrm{fin.}\,\Phi = q,$$

erit $pq = 1$; tum vero

$$p^n = \mathrm{cof.}\,n\Phi + \sqrt{-1}\,\mathrm{fin.}\,n\Phi \text{ et } q^n = \mathrm{cof.}\,n\Phi - \sqrt{-1}\,\mathrm{fin.}\,n\Phi,$$

vnde fit

$$p^n + q^n = 2\,\mathrm{cof.}\,n\Phi \text{ et } p^n - q^n = 2\sqrt{-1}\,\mathrm{fin.}\,n\Phi;$$

Res igitur eo redit, vt formulae $p^n + q^n$ et $p^n - q^n$ in factores reſoluantur.

§. 2. Conſideremus primo formulam

$$p^n + q^n = 2\,\mathrm{cof.}\,n\Phi,$$

quae, quoties n eſt numerus impar, factorem habet ſimplicem $p + q - 2\,\mathrm{cof.}\,\Phi$, ita vt his caſibus $\mathrm{cof.}\,\Phi$ ſit factor ipſius $\mathrm{cof.}\,n\Phi$: Pro reliquis factoribus autem ponamus factorem duplicem in genere eſſe, $pp - 2pq\,\mathrm{cof.}\,\omega + qq$, ita vt formula $p^n + q^n$ euaneſcat, poſito

 pp-

$$pp - 2pq \cos.\omega + qq = 0,$$

tum autem erit vel

$$p = q(\cos.\omega + \sqrt{-1}\sin.\omega) \text{ vel } p = q(\cos.\omega - \sqrt{-1}\sin.\omega),$$

hinque

$$p^n = q^n(\cos.n\omega \pm \sqrt{-1}\sin.n\omega)$$

ſicque debebit eſſe

$$q^n(\cos.n\omega \pm \sqrt{-1}\sin.n\omega) + q^n = 0, \text{ ſiue}$$

$$\cos.n\omega \pm \sqrt{-1}\sin.n\omega + 1 = 0,$$

vnde fit $\sin.n\omega = 0$ et $\cos.n\omega = -1$, tum autem ſponte fit $\sin.n\omega = 0$.

§. 3. Quia igitur $\cos.n\omega = -1$, angulus $n\omega$ erit vel π, vel 3π, vel 5π, vel 7π, vel etc. Sicque ſi i denotet numerum imparem quemcunque, erit $n\omega = i\pi$, hincque $\omega = \frac{i\pi}{n}$, quocirca factor duplex in genere erit

$$pp - 2pq\cos.\tfrac{i\pi}{n} + qq.$$

§. 4. Cum nunc ſit $pp + qq = 2\cos.2\Phi$, ob $pq = 1$ erit iſte factor $2\cos.2\Phi - 2\cos.\frac{i\pi}{n}$, qui ſponte in duos factores reſoluitur. Cum enim ſit

$$\cos.A - \cos.B = 2\sin.\tfrac{B+A}{2}.\sin.\tfrac{B-A}{2}, \text{ erit}$$

$$\cos.2\Phi - \cos.\tfrac{i\pi}{n} = 2\sin.(\tfrac{i\pi}{2n}+\Phi).\sin.(\tfrac{i\pi}{2n}-\Phi)$$

ſicque vnus factor in genere erit

$$4\sin.(\tfrac{i\pi}{2n}+\Phi)\sin.(\tfrac{i\pi}{2n}-\Phi).$$

Hinc pro i ſucceſſiue numeros 1, 2, 3, 4, etc. ſcribendo, erit:

$$2\cos.n\Phi = 4\sin.(\tfrac{\pi}{2n}+\Phi)\sin.(\tfrac{\pi}{2n}-\Phi).$$
$$4\sin.(\tfrac{3\pi}{2n}+\Phi).\sin.(\tfrac{3\pi}{2n}-\Phi).4\sin.(\tfrac{5\pi}{2n}+\Phi).\sin.(\tfrac{5\pi}{2n}-\Phi),$$

donec omnino habeantur n factores.

§. 5.

ſi n $\sqrt{-1}\sin.\omega)$, ſi n ſi n ſi n ſiue ſi n ſi n utem ſponte lus. $n\omega$ erit cque ſi i de- $= i\pi$, hinc- erit $n d\Phi$ / cos ſ. 2Φ, ob qui ſponte erit $\frac{\pi}{n} - \Phi)$ endo, erit: n. $(\frac{5\pi}{2n}-\Phi)$, §. 5.

§. 5. Percurramus igitur hanc expreſſionem ſecundum ſingulos factores numeri n, eritque

ſi $n=1$ | $2\cos.\Phi = \sin.(\frac{\pi}{2}-\Phi)$
ſi $n=2$ | $2\cos\,2\Phi = \sin\,2^2(\frac{\pi}{4}-\Phi)\sin.(\frac{\pi}{4}+\Phi)$
ſi $n=3$ | $2\cos.3\Phi = 2^3\sin.(\frac{\pi}{6}-\Phi)\sin(\frac{\pi}{6}+\Phi)\sin.(\frac{3\pi}{6}-\Phi)$
ſi $n=4$ | $2\cos.4\Phi = 2^4\sin.(\frac{\pi}{8}-\Phi)\sin.(\frac{\pi}{8}+\Phi)\sin.(\frac{3\pi}{8}-\Phi).\sin(\frac{3\pi}{8}+\Phi)$
ſi $n=5$ | $2\cos.5\Phi = 2^5\sin.(\frac{\pi}{10}-\Phi)\sin.(\frac{\pi}{10}+\Phi)\sin(\frac{3\pi}{10}-\Phi)\sin.(\frac{3\pi}{10}+\Phi)\sin.(\frac{5\pi}{10}-\Phi)$
ſi $n=6$ | $2\cos.6\Phi = 2^6\sin.(\frac{\pi}{12}-\Phi)\sin.(\frac{\pi}{12}+\Phi)\sin.(\frac{3\pi}{12}-\Phi)\sin.(\frac{3\pi}{12}+\Phi)\sin.(\frac{5\pi}{12}-\Phi)\sin(\frac{5\pi}{12}+\Phi)$

Generaliter autem erit

$$\cos.n\Phi = 2^{n-1}\sin.(\tfrac{\pi}{2n}-\Phi)\sin(\tfrac{\pi}{2n}+\Phi)\sin.(\tfrac{3\pi}{2n}-\Phi)\sin.(\tfrac{3\pi}{2n}+\Phi) \text{ etc.}$$

donec habeantur n factores

§. 6. Sumendis igitur logarithmis erit

$$l\cos.n\Phi = l\,2^{n-1} + l\sin.(\tfrac{\pi}{2n}-\Phi) + l\sin.(\tfrac{\pi}{2n}+\Phi) + l\sin.(\tfrac{3\pi}{2n}-\Phi) + l\sin.(\tfrac{3\pi}{2n}+\Phi) + \text{etc.}$$

quae aequatio differentiata praebet

$$\frac{n\,d\Phi\sin.n\Phi}{\cos.n\Phi} = \frac{d\Phi\cos.(\frac{\pi}{2n}-\Phi)}{\sin.(\frac{\pi}{2n}-\Phi)} - \frac{d\Phi\cos.(\frac{\pi}{2n}+\Phi)}{\sin.(\frac{\pi}{2n}+\Phi)} + \frac{d\Phi\cos.(\frac{3\pi}{2n}-\Phi)}{\sin.(\frac{3\pi}{2n}-\Phi)} - \frac{d\Phi\cos.(\frac{3\pi}{2n}+\Phi)}{\sin.(\frac{3\pi}{2n}+\Phi)} + \text{etc.}$$

hoc eſt

$$n\,\mathrm{tag}.n\Phi = \cot.(\tfrac{\pi}{2n}-\Phi) - \cot.(\tfrac{\pi}{2n}+\Phi) + \cot.(\tfrac{3\pi}{2n}-\Phi) - \cot.(\tfrac{3\pi}{2n}+\Phi) + \text{etc.}$$

vnde deducuntur ſequentes aequalitates memoratu dignae

I° $\mathrm{tag}.\Phi = \cot.(\frac{\pi}{2}-\Phi)$

II° $2\,\mathrm{tag}.2\Phi = \cot(\frac{\pi}{4}-\Phi) - \cot.(\frac{\pi}{4}+\Phi) = \mathrm{tg}.(\frac{\pi}{4}+\Phi) - \mathrm{tg}.(\frac{\pi}{4}-\Phi)$

III° $\mathrm{tag}.3\Phi = \cot.(\frac{\pi}{6}-\Phi) - \cot.(\frac{\pi}{6}+\Phi) + \cot.(\frac{3\pi}{6}-\Phi)$, ſiue

$\mathrm{tag}.3\Phi = \mathrm{tag}.(\frac{\pi}{3}+\Phi) - \mathrm{tag}.(\frac{\pi}{3}-\Phi) + \mathrm{tag}.\Phi$

IV° $\mathrm{tag}.4\Phi = \cot.(\frac{\pi}{8}-\Phi) - \cot.(\frac{\pi}{8}+\Phi) + \cot.(\frac{3\pi}{8}-\Phi) - \cot.(\frac{3\pi}{8}+\Phi)$

ſiue $4\,\mathrm{tg}.4\Phi = \mathrm{tag}.(\frac{3\pi}{8}+\Phi) - \mathrm{tag}.(\frac{3\pi}{8}-\Phi) + \mathrm{tg}.(\frac{\pi}{8}+\Phi) - \mathrm{tag}.(\frac{\pi}{8}-\Phi)$

§. 7.

§. 7. Eodem modo tractemus formulam

$$p^n - q^n = 2\sqrt{-1}\,\sin n\phi$$

cujus factorem duplicem ſtatuamus

$$pp - 2pq\cos\omega + qq$$

quo poſito = 0 fit ut ante

$$p = q(\cos\omega \pm \sqrt{-1}\,\sin\omega)$$

hincque porro

$$p^n = q^n(\cos n\omega \pm \sqrt{-1}\,\sin n\omega)$$

ſicque debebit eſſe

$q^n(\cos n\omega \pm \sqrt{-1}\,\sin n\omega) - q^n = 0$, ſive $\cos n\omega \pm \sqrt{-1}\,\sin n\omega - 1 = 0$

Vnde fieri debet

$$\sin n\omega = 0 \text{ ac } \cos n\omega = 1$$

quam ob rem angulus $n\omega$ erit vel 0, vel 2π, vel 4π, vel 6π, vel in genere $2i\pi$, ideoque $\omega = \frac{2i\pi}{n}$, denotante i numeros omnes 1, 2, 3, 4, etc. Hinc igitur factor duplex in genere erit

$$pp - 2pq\cos\tfrac{2i\pi}{n} + qq = 2\cos 2\phi - 2\cos\tfrac{2i\pi}{n}$$

qui reſolvitur in hos factores:

$$2\sin(\tfrac{i\pi}{n} - \phi)\; 2\sin(\tfrac{i\pi}{n} + \phi);$$

praeterea autem formula $p^n - q^n$ habet factorem ſimplicem

$$p - q = 2\sqrt{-1}\,\sin\phi$$

conſequenter habebimus

$$\sin n\phi = \sin\phi.\; 2\sin(\tfrac{i\pi}{n} - \phi).\; 2\sin(\tfrac{i\pi}{n} + \phi) \text{ etc.}$$

ideoque

$$\sin n\phi = \sin\phi.\, 2\sin(\tfrac{\pi}{n}-\phi)\, 2\sin(\tfrac{\pi}{n}+\phi).\, 2\sin(\tfrac{2\pi}{n}-\phi)\, 2\sin(\tfrac{2\pi}{n}+\phi) \text{ etc.}$$

donec omnino prodeant n factores. Erit ergo

$$\sin n\phi = 2^{n-1}\sin\phi.\, \sin(\tfrac{\pi}{n}-\phi)\sin(\tfrac{\pi}{n}+\phi)\sin(\tfrac{2\pi}{n}-\phi)\sin(\tfrac{2\pi}{n}+\phi) \text{ etc.}$$

§. 8.

eamus
ſi n = 1 | ſir
ſi n = 2 | ſir
ſi n = 3 | ſir
ſi n = 4 | ſir
ſi n = 5 | ſii
ſi n = 6 | ſii
m
−ſin. n ω − 1 = 0
eritque
l l
el 6 π, vel in-
quae a
s 1, 2, 3, 4, etc.
n coſ. n
ſin. n
ſiue
n c
donec
1 ſimplicem
ſpecial
φ) etc.
ſi n = 1
ſi n = 2
(2π/n + φ) etc.
ſi n = 3
ſi n = 4
(2π/n + φ) etc.
§. 8.

§. 8. Iam ex hac forma generali ſequentes deducamus formas ſpeciales:

ſi $n = 1$ | $\sin\phi = 2^0 \sin\phi$

ſi $n = 2$ | $\sin 2\phi = 2\sin\phi\sin(\tfrac{\pi}{2} - \phi)$

ſi $n = 3$ | $\sin 3\phi = 4\sin\phi\sin(\tfrac{\pi}{3} - \phi).\sin(\tfrac{\pi}{3} + \phi)$

ſi $n = 4$ | $\sin 4\phi = 8\sin\phi\sin(\tfrac{\pi}{4} - \phi)\sin(\tfrac{\pi}{4} + \phi)\sin(\tfrac{2\pi}{4} - \phi)$

ſi $n = 5$ | $\sin 5\phi = 16\sin\phi\sin(\tfrac{\pi}{5} - \phi)\sin(\tfrac{\pi}{5} + \phi)\sin(\tfrac{2\pi}{5} - \phi)\sin(\tfrac{2\pi}{5} + \phi)$

ſi $n = 6$ | $\sin 6\phi = 32\sin\phi\sin(\tfrac{\pi}{6} - \phi)\sin(\tfrac{\pi}{6} + \phi)\sin(\tfrac{2\pi}{6} - \phi)\sin(\tfrac{2\pi}{6} + \phi)\sin(\tfrac{3\pi}{6} - \phi)$

§. 9. Sumamus hic etiam ut ante logarithmos, eritque

$$l\sin n\phi = l2^{n-1} + l\sin\phi + l\sin(\tfrac{\pi}{n} - \phi) + l\sin(\tfrac{\pi}{n} + \phi) + \text{etc.}$$

quae aequatio differentiata et per $d\phi$ diuiſa praebet

$$\frac{n\cos n\phi}{\sin n\phi} = \frac{\cos\phi}{\sin\phi} - \frac{\cos(\frac{\pi}{n} - \phi)}{\sin(\frac{\pi}{n} - \phi)} + \frac{\cos(\frac{\pi}{n} + \phi)}{\sin(\frac{\pi}{n} + \phi)} - \frac{\cos(\frac{2\pi}{n} - \phi)}{\sin(\frac{2\pi}{n} - \phi)} + \text{etc}$$

ſiue

$$n\cot n\phi = \cot\phi - \cot(\tfrac{\pi}{n} - \phi) + \cot(\tfrac{\pi}{n} + \phi) - \cot(\tfrac{2\pi}{n} - \phi) \text{ etc.}$$

donec habeantur n termini

§. 10. Hinc igitur ſequentes obtinebimus formas ſpeciales:

ſi $n = 1$ | $\cot\phi = \cot\phi$

ſi $n = 2$ | $2\cot 2\phi = \cot\phi - \cot(\tfrac{\pi}{2} - \phi)$

ſi $n = 3$ | $3\cot 3\phi = \cot\phi - \cot(\tfrac{\pi}{3} - \phi) + \cot(\tfrac{\pi}{3} + \phi)$

ſi $n = 4$ | $4\cot 4\phi = \cot\phi - \cot(\tfrac{\pi}{4} - \phi) + \cot(\tfrac{\pi}{4} + \phi) - \cot(\tfrac{2\pi}{4} - \phi)$

 §. 11.

§. 11. Si formulam pro cot. $n\Phi$ inuentam denuo differentiemus, ob d cot. $\theta = \frac{-d\theta}{\text{fin.}\,\theta^2}$, per $-\theta\Phi$ diuidendo habebimus,

$$\frac{nn}{\text{fin.}\,n\Phi^2} = \frac{1}{\text{fin.}\,\Phi^2} + \frac{1}{\text{fin.}(\frac{\pi}{n}-\Phi)^2} + \frac{1}{\text{fin.}(\frac{\pi}{n}+\Phi)^2} + \frac{1}{\text{fin.}(\frac{2\pi}{n}-\Phi)^2}\ \text{etc.}$$

donec habeantur n termini, vnde sequentes casus notentur:

$$\text{fi } n=1 \left| \frac{1}{\text{fin.}\,\Phi^2} = \frac{1}{\text{fin.}\,\Phi^2} \right.$$

$$\text{fi } n=2 \left| \frac{4}{\text{fin.}\,2\Phi^2} = \frac{1}{\text{fin.}\,\Phi^2} + \frac{1}{\text{fin.}(\frac{\pi}{2}-\Phi)^2} \right.$$

$$\text{fi } n=3 \left| \frac{9}{\text{fin}\,3\Phi^2} = \frac{1}{\text{fin}\,\Phi^2} + \frac{1}{\text{fin.}(\frac{\pi}{3}-\Phi)^2} + \frac{1}{\text{fin.}(\frac{\pi}{3}+\Phi)^2} \right.$$

$$\text{fi } n=4 \left| \frac{16}{\text{fin.}\,4\Phi^2} = \frac{1}{\text{fin.}\,\Phi^2} + \frac{1}{\text{fin.}(\frac{\pi}{4}-\Phi)^2} + \frac{1}{\text{fin.}(\frac{\pi}{4}+\Phi)^2} + \frac{1}{\text{fin.}(\frac{2\pi}{4}-\Phi)^2}\ \text{etc.} \right.$$

Euolutio formulae

$$p^{2n} - 2p^n q^n \cos.\,\theta + q^n.$$

§. 12. Sumamus hic ut ante

$$p = \cos.\,\Phi + \sqrt{-1}\,\text{fin.}\,\Phi \text{ et } q = \cos.\,\Phi - \sqrt{-1}\,\text{fin}\,\Phi$$

ita vt ista formula inuoluat hunc valorem:

$$2\cos.\,2n\Phi - 2\cos.\,\theta = 4\,\text{fin.}\,(n\Phi + \tfrac{1}{2}\theta)\,\text{fin.}\,(\tfrac{1}{2}\theta - n\Phi);$$

Iam sit $pp - 2pq\cos.\,\omega + qq$ factor duplex huius formulae, quae ergo euanescere debet posito $p = q(\cos.\,\omega \pm \sqrt{-1}\,\text{fin.}\,\omega)$, vnde facta substitutione prodibit

$$q^{2n}(\cos.\,2n\omega \pm \sqrt{-1}\,\text{fin.}\,2n\omega) - 2q^{2n}\cos.\,\theta(\cos.\,\omega\omega \pm \sqrt{-}\,\text{fin.}\,n\omega) + q^{m} = 0$$

hoc

inuentam denuo
) diuidendo habe-
$+ \frac{1}{\text{fin.}(\frac{2\pi}{n}-\Phi)^2}$ etc.
tentes casus no-
$-\Phi)^2$
$\frac{1}{\text{fin.}(\frac{2\pi}{4}-\Phi)^2}$ etc.
$\sqrt{-1}\,\text{fin}\,\Phi$
:
$(\tfrac{1}{2}\theta - n\Phi)$;
ex huius formu-
$\omega \pm \sqrt{-1}\,\text{fin.}\,\omega)$,
$\sqrt{-}\,\text{fin.}\,n\omega) + q^{m} = 0$
hoc

hoc est

$$\cos.\,2n\omega \qquad - 2\cos.\,\theta\cos.\,n\omega + 1 = 0$$
$$\pm \sqrt{-1}\,\text{fin.}\,2n\omega + 2\cos.\,\theta\sqrt{-1}\,\text{fin.}\,n\omega$$

vnde nascuntur hae duae aequationes:

$$\cos.\,2n\omega - 2\cos.\,\theta\cos.\,n\omega + 1 = 0 \text{ et}$$
$$\text{fin.}\,2n\omega - 2\cos.\,\theta\,\text{fin.}\,n\omega = 0$$

Cum nunc sit

$$\cos.\,2n\omega = 2\cos.\,n\omega^2 - 1 \text{ et fin.}\,2n\omega = 2\,\text{fin.}\,n\omega\cos.\,n\omega$$

hae duae aequationes erunt

$$2\cos.\,n\omega^2 - 2\cos.\,\theta\cos.\,n\omega = 0 \text{ et } 2\,\text{fin.}\,n\omega\cos.\,n\omega - \cos.\,\theta\,\text{fin.}\,n\omega = 0,$$

siue

$$\cos.\,n.\,\omega - \cos.\,\theta = 0 \text{ et } \cos.\,n\omega - \cos.\,\theta = 0$$

vnde sequitur $\cos.\,n\omega = \cos.\,\theta$. Erit ergo vel $n\omega = \theta$, vel $n\omega = 2\pi + \theta$, vel $4\pi + \theta$, vel $6\pi + \theta$, vel in genere $n\omega = 2i\pi + \theta$, vnde fit in genere $\omega = \frac{2i\pi \pm \theta}{n}$, ita vt i denotet numeros 0, 1, 2, 3, 4, etc.

§. 13. Formulae igitur nostrae factor duplex in genere erit

$$pp + qq - 2pq\cos.\,(\tfrac{2i\pi \pm \theta}{n}).$$

Est vero

$$pp + qq = 2\cos.\,2\Phi \text{ et } pq = 1,$$

vnde hic factor erit

$$2(\cos.\,2\Phi - \cos.\,(\tfrac{2i\pi \pm \theta}{n}))$$

qui reducitur ad hos factores simplices:

$$4\,\text{fin.}\,\frac{2i\pi + 2n\Phi \pm \theta}{2n}\,\text{fin.}\,\frac{2i\pi \pm \theta - 2n\Phi}{2n}.$$

vnde

vnde loco i ſcribendo numeros 1, 2, 3, 4, etc. factores noſtrae formulae erunt

$$4\,\text{ſin.}\tfrac{2n\Phi+\theta}{2n}\,\text{ſin.}\tfrac{\theta-2n\Phi}{2n}.\;4\,\text{ſin.}\tfrac{2\pi+2n\Phi+\theta}{2n}\,\text{ſin.}\tfrac{2\pi+\theta-2n\Phi}{2n}\;4\,\text{ſin.}\tfrac{4\pi+2n\Phi+\theta}{2n}.$$
$$\text{ſin.}\tfrac{4\pi+\theta-2n\Phi}{2n}.\;4\,\text{ſin.}\tfrac{6\pi+2n\Phi+\theta}{2n}\,\text{ſin.}\tfrac{6\pi+\theta-2n\Phi}{2n}$$

qui factores eouſque continuari debent, donec eorum numerus fiat $= n$

§. 14. Cum igitur hoc productum aequale ſit formulae $4\,\text{ſin}\,(n\Phi+\tfrac{1}{2}\theta)\,\text{ſin.}(\tfrac{1}{2}\theta-n\Phi)$, et in noſtro producto factor numericus ſit $4^n = 2^{2n}$, per 4 diuidendo habebimus hanc aequationem:

$$\text{ſin.}(n\Phi+\tfrac{1}{2}\theta)\,\text{ſin.}(\tfrac{1}{2}\theta-n\Phi) = 2^{2n-2}\,\text{ſin.}(\tfrac{2n\Phi+\theta}{2n})\,\text{ſin.}(\tfrac{\theta-2n\Phi}{2n}).$$
$$\text{ſin.}(\tfrac{2\pi+2n\Phi+\theta}{2n})\,\text{ſin.}(\tfrac{2\pi+\theta-2n\Phi}{2n}).$$
$$\text{ſin.}(\tfrac{4\pi+2n\Phi+\theta}{2n})\,\text{ſin.}(\tfrac{4\pi+\theta-2n\Phi}{2n})\ \text{etc.}$$

quae aequatio quo concinnior reddatur ponamus $\theta = 2n\alpha$ et erit

$$\text{ſin.}\,n\,(\alpha+\Phi)\,\text{ſin.}\,n\,(\alpha-\Phi) = 2^{2n-2}\,\text{ſin.}(\alpha+\Phi)\,\text{ſin.}(\alpha-\Phi).$$
$$\text{ſin.}(\tfrac{\pi}{n}+\alpha+\Phi)\,\text{ſin.}(\tfrac{\pi}{n}+\alpha-\Phi).$$
$$\text{ſin.}(\tfrac{2\pi}{n}+\alpha+\Phi)\ \text{ſin.}(\tfrac{2\pi}{n}+\alpha-\Phi)\ \text{etc.}$$

§. 15. Haec autem expreſſio non eſt noua, ſed iam in praecedente continetur, quae erat.

$$\text{ſin.}n\Phi = 2^{n-1}\text{ſin.}\Phi.\text{ſin.}(\tfrac{\pi}{n}-\Phi)\text{ſin.}(\tfrac{\pi}{n}+\Phi)\text{ſin}(\tfrac{2\pi}{n}-\Phi)\text{ſin.}(\tfrac{2\pi}{n}+\Phi)\ \text{etc.}$$

et cum ſit $\text{ſin}\,(o-\Phi) = \text{ſin.}\,(\pi-o+\Phi)$, erit,

$$\text{ſin}\,(\tfrac{\pi}{n}-\Phi) = \text{ſin.}(\tfrac{(n-1)\pi}{n}+\Phi);\ \text{ſin.}(\tfrac{2\pi}{n}-\Phi) = \text{ſin.}(\tfrac{(n-2)\pi}{n}+\Phi).$$
$$\text{ſin.}(\tfrac{3\pi}{n}-\Phi) = \text{ſin.}(\tfrac{(n-3)\pi}{n}+\Phi),$$

vnde illa expreſſio reducetur ad hanc formam:

ſin.

$$\text{ſin.}\,n\,\Phi = 2^{n-1}\,\text{ſin.}\Phi.\,\text{ſin.}(\tfrac{\pi}{n}+\Phi)\,\text{ſin.}(\tfrac{2\pi}{n}+\Phi)$$
$$\text{ſin.}(\tfrac{3\pi}{n}+\Phi)\ \ldots\ldots\ \text{ſin.}(\tfrac{(n-1)\pi}{n}+\Phi)$$

vbi arcus in progreſſione arithmetica continua prograediuntur. Quod ſi iam hic loco Φ ſcribamus primo $\alpha+\Phi$, deinde $\alpha-\Phi$, hinc duae formulae ſequentes naſcentur:

$$\text{ſin.}\,n\,(\alpha+\Phi) = 2^{n-1}\,\text{ſin.}(\alpha+\Phi)\,\text{ſin.}(\tfrac{\pi}{n}+\alpha+\Phi)$$
$$\text{ſin.}(\tfrac{2\pi}{n}+\alpha+\Phi)\,\text{ſin.}(\tfrac{3\pi}{n}+\alpha+\Phi)\ \text{etc.}$$
$$\text{ſin.}\,n(\alpha-\Phi) = 2^{n-1}\,\text{ſin}\,(\alpha-\Phi)\,\text{ſin.}(\tfrac{\pi}{n}+\alpha-\Phi)$$
$$\text{ſin.}(\tfrac{2\pi}{n}+\alpha-\Phi)\,\text{ſin.}(\tfrac{3\pi}{n}+\alpha-\Phi)\ \text{etc.}$$

quae duae aequationes in ſe inuicem ductae praebent

$$\text{ſin.}\,n\,(\alpha+\Phi)\ \text{ſin.}\,n\,(\alpha-\Phi) = 2^{2n-2}\,\text{ſin.}(\alpha+\Phi)$$
$$\text{ſin.}(\alpha-\Phi).\,\text{ſin.}(\tfrac{\pi}{n}+\alpha+\Phi)\,\text{ſin.}(\tfrac{\pi}{n}+\alpha-\Phi)$$
$$\text{ſin.}(\tfrac{2\pi}{n}+\alpha+\Phi)\,\text{ſin.}(\tfrac{2\pi}{n}+\alpha-\Phi)\,\text{ſin.}(\tfrac{3\pi}{n}+\alpha+\Phi)\ \text{etc.}$$

§. 16. Si nunc attendamus ad originem harum formularum, quandoquidem ex noſtra formula

$$p^{2n} - 2\,p^n\,q^n\,\text{coſ.}\,2\,n\,\alpha + q^{2n}$$

nata eſt haec:

$$4\,\text{ſin.}\,n\,(\alpha+\Phi)\ \text{ſin.}\,n\,(\alpha-\Phi)$$

exiſtente

$$p = \text{coſ.}\,\Phi + \sqrt{-1}\,\text{ſin.}\,\Phi\ \text{et}$$
$$q = \text{coſ.}\,\Phi - \sqrt{-1}\,\text{ſin.}\,\Phi,$$

ſi ponamus

$$f = \text{coſ.}\,(\alpha+\Phi) + \sqrt{-1}\,\text{ſin.}\,(\alpha+\Phi)\ \text{et}$$
$$g = \text{coſ.}\,(\alpha+\Phi) - \sqrt{-1}\,\text{ſin.}\,(\alpha+\Phi),$$

tum erit $f^n - g^n = 2\sqrt{-1}\,\text{ſin.}\,n\,(\alpha+\Phi)$.

 Dein-

Deinde ſi ponamus

$$b = \cos.(\alpha - \Phi) + \sqrt{-1}\,\sin(\alpha - \Phi) \text{ et}$$
$$k = \cos.(\alpha - \Phi) - \sqrt{-1}\,\sin.(\alpha - \Phi)$$

erit ſimili modo

$$b^n - k^n = 2\sqrt{-1}\,\sin.\,n(\alpha - \Phi),$$

unde erit

$$(f^n - g^n)(b^n - k^n) = -4\sin.\,n(\alpha + \Phi)\sin.\,n(\alpha - \Phi) =$$
$$-p^{2n} + 2p^n q^n \cos.\,2n\alpha - q^{2n}.$$

Ad hoc demonſtrandum notetur eſſe

$$f = p(\cos.\alpha + \sqrt{-1}\,\sin.\alpha);$$
$$g = q(\cos.\alpha - \sqrt{-1}\,\sin.\alpha);$$
$$b = q(\cos.\alpha + \sqrt{-1}\,\sin.\alpha);$$
$$k = p(\cos.\alpha - \sqrt{-1}\,\sin.\alpha);$$

vnde fit

$$f^n = p^n(\cos.n\alpha + \sqrt{-1}\,\sin.n\alpha);$$
$$g^n = q^n(\cos.n\alpha - \sqrt{-1}\,\sin.n\alpha);$$
$$b^n = q^n(\cos.n\alpha + \sqrt{-1}\,\sin.n\alpha);$$
$$k^n = p^n(\cos.n\alpha - \sqrt{-1}\,\sin.n\alpha);$$

Ponamus breuitatis ergo

$$\cos.n\alpha + \sqrt{-1}\,\sin.n\alpha = A; \quad \cos.n\alpha - \sqrt{-1}\,\sin.n\alpha = B$$

vt ſit

$$f^n = A p^n;\ g^n = B q^n;\ b^n = A q^n \text{ et } k^n = B p^n,$$

hincque porro

$$f^n - g^n = A p^n - B q^n \text{ et } b^n - k^n = A q^n - B p^n$$

quae duae formulae multiplicatae praebent

$$(f^n - g^n)(b^n - k^n) = (A^2 + B^2)p^n q^n - AB(p^{2n} + q^{2n});$$

vbi

) et
)
) ſin. $n(\alpha - \Phi) =$
$\sqrt{-1}\,\sin.n\alpha = B$
$k^n = B p^n,$
$A q^n - B p^n$
bent
$- AB(p^{2n} + q^{2n});$
vbi

vbi cum ſit

$$AB = 1 \text{ et } AA + BB = 2\cos.\,2n\alpha$$

hoc productum erit

$$-p^{2n} + 2p^n q^n \cos.\,2n\alpha - q^{2n}$$

quod eſt id ipſum quod inuenimus.

Corollarium.

Hinc igitur intelligimus, formulam

$$p^{2n} - 2p^n q^n \cos.\,2n\alpha + q^{2n}$$

reſolui in hos duos factores:

$$(A p^n - B q^n) \text{ et } (B p^n - A q^n)$$

exiſtente

$$A = \cos.\,n\alpha + \sqrt{-1}\,\sin.\,n\alpha$$
$$B = \cos.\,n\alpha - \sqrt{-1}\,\sin.\,n\alpha.$$