

figitur $m = \frac{1}{2}$; $n = \frac{1}{2}$, ut sit $l = \frac{1}{2}$, unde quatuor numeri priores erunt

$$a = \frac{1}{2}; b = \frac{1}{2}; c = 6 \text{ et } d = 48;$$

unde porro deducimus:

$$p = 57; q = 451\frac{1}{2}; r = 931\frac{1}{2} \text{ et } s = 360;$$

ex his ergo deducitur:

$$z = \frac{4 \cdot 931\frac{1}{2} + 114 \cdot 361}{359^2} = \frac{44880}{128881}$$

qui numeri multo sunt minores quam precedentes.

latur numeri

$$360;$$

$$\frac{80}{881}$$

edentes.

VARIAE OBSERVATIONES

CIRCA ANGILOS

IN PROGRESSIONE GEOMETRICA PROGREDIENTES.

§. I.

Cum pleraeque infignes proprietates, quae adhuc circa angulos, siue arcus, eorumque sinus, cosinus, tangentes, cotangentes, secantes et cosecantes sunt investigatae, ex consideratione arcuum in arithmetica progressionem crecentium sint derivatae: non minus notata dignae videntur illae proprietates, quas ex consideratione arcuum in geometrica progressionem procedentium deducere licet; imprimis cum earum veritas plerumque multo magis abscoudita videatur, quocirca hoc loco plures eiusmodi proprietates evolvere constitui.

§. 2. Primum fontem ad huiusmodi speculationes nobis aperit notissima formula: $\sin. 2\Phi = 2 \sin. \Phi \cdot \cos. \Phi$ unde, si s denotet arcum siue angulum quemcumque, erit

$\sin. s = 2 \sin. \frac{1}{2}s \cdot \cos. \frac{1}{2}s$; tum vero simili modo erit

$\sin. \frac{1}{2}s = 2 \sin. \frac{1}{4}s \cdot \cos. \frac{1}{4}s$, qui valor ibi substitutus praebet

$\sin. s = 4 \sin. \frac{1}{4}s \cdot \cos. \frac{1}{4}s \cdot \cos. \frac{1}{4}s$. Deinde quia porro est

$\sin. \frac{1}{4}s = 2 \sin. \frac{1}{8}s \cdot \cos. \frac{1}{8}s$, hoc valore substituto erit

$\sin. s = 8 \sin. \frac{1}{8}s \cdot \cos. \frac{1}{8}s \cdot \cos. \frac{1}{8}s \cdot \cos. \frac{1}{8}s$. Pari modo progrediendo est

$$\sin. s = 16 \sin. \frac{1}{16}s \cdot \cos. \frac{1}{16}s \cdot \cos. \frac{1}{16}s \cdot \cos. \frac{1}{16}s \cdot \cos. \frac{1}{16}s$$

Euleri *Opusc. Anal. Tom. I.*

X x

atque

VARIAE

VARIAE

atque si hoc modo in infinitum progrediatur, denotante i numerum infinitum, seu potius infinitissimam potestatem ipsius 2 , habebimus

$$\sin s = i \sin \frac{s}{2}, \text{ cof. } \frac{s}{2}, \text{ cof. } \frac{s}{4}, \text{ cof. } \frac{s}{8}, \text{ etc.}$$

Vbi quia arcus i est infinite parvus, erit $\sin i = i$, ideoque $i \sin \frac{s}{2} = i$, unde adipiscimur hanc insignem proprietatem, ut sit

$$\sin s = s \text{ cof. } \frac{s}{2}, \text{ cof. } \frac{s}{4}, \text{ cof. } \frac{s}{8}, \text{ cof. } \frac{s}{16}, \text{ etc. in infinitum.}$$

§. 3. Hinc igitur ipse arcus s per eius sinum et cosinus arcuum continuo in ratione dupla decreverunt ita pulcherrime desinitur, ut sit

$$s = \frac{\text{cof. } \frac{s}{2}, \text{ cof. } \frac{s}{4}, \text{ cof. } \frac{s}{8}, \text{ cof. } \frac{s}{16}, \text{ cof. } \frac{s}{32}, \text{ etc.}}{\sin s.}$$

at quia $\frac{1}{\sin s} = \text{sec. } \Phi$, erit per expressionem integram

$$s = \sin s. \text{sec. } \frac{s}{2}, \text{ sec. } \frac{s}{4}, \text{ sec. } \frac{s}{8}, \text{ sec. } \frac{s}{16}, \text{ sec. } \frac{s}{32}, \text{ etc.}$$

quae expressio satis commode geometricè repraesentari potest, quemadmodum iam alio loco ostendi.

§. 4. Quia hic arcus s per productum exprimitur, sumendis logarithmis habebimus,

$$Ls = L \sin s + L \text{sec. } \frac{s}{2} + L \text{sec. } \frac{s}{4} + L \text{sec. } \frac{s}{8} + L \text{sec. } \frac{s}{16} + L \text{sec. } \frac{s}{32} + \text{etc.}$$

$$\text{Unde si accipiamus } s = \frac{\pi}{2} = 90^\circ \text{ fiet}$$

$$L \frac{\pi}{2} = 0 + L \text{sec. } 45^\circ + L \text{sec. } 22^\circ 30' + L \text{sec. } 11^\circ 15' + L \text{sec. } 5^\circ 37 \frac{1}{2}' \text{ etc.}$$

Unde calculo instituto erit

$$L \text{sec.}$$

denotante i potestatem

s , etc.

$\frac{s}{2}$, ideoque proprietatem,

$\frac{s}{2} s$, etc. in

us sinum et ecreverunt

. etc.

tegram

. sec. $\frac{s}{2} s$, etc.

ascutari po-

exprimitur,

sec. $\frac{s}{2} s + \text{etc.}$

$0, 5^\circ 37 \frac{1}{2}'$, etc.

$L \text{sec.}$

$L \text{sec } 45^\circ$	$= 0,1505150$
$L \text{sec } 22^\circ 30'$	$= 0,0343847$
$L \text{sec } 11^\circ 15'$	$= 0,0084261$
$L \text{sec } 5^\circ 37 \frac{1}{2}'$	$= 0,0020963$
$L \text{sec } 2^\circ 48'$	$= 0,0005435$
$L \text{sec } 1^\circ 24'$	$= 0,0001809$
$L \text{sec } 0^\circ 42'$	$= 0,0000327$
$L \text{sec } 0^\circ 21'$	$= 0,0000082$
reliqui omnes	$= 0,0000027$
$L \frac{\pi}{2}$	$= 0,1961201$
$L 2$	$= 0,3010300$
$L \pi$	$= 0,4971501$

hincque $\pi = 3,1415926535$ satis exacte, uti constat.

§. 5. Quo autem hinc novas relationes deducamus, differentiemus potestatem aequationem logarithmicam, et cum sit $d. L \text{sec. } \Phi = \frac{d\Phi}{\sin \Phi} = d\Phi \text{tag. } \Phi$, oriens per $d s$ dividendo sequens aequatio:

$$\frac{1}{s} = \text{cot. } s + \frac{1}{2} \text{tag. } \frac{s}{2} + \frac{1}{4} \text{tag. } \frac{s}{4} + \frac{1}{8} \text{tag. } \frac{s}{8} + \frac{1}{16} \text{tag. } \frac{s}{16} + \text{etc.}$$

quae series quam citissime convergit, id quod sequenti exemplo clarus patebit, in quo sumamus $s = 90^\circ = \frac{\pi}{2}$, unde fiet

$$\frac{\pi}{2} = \frac{1}{2} \text{tag. } 45^\circ + \frac{1}{4} \text{tag. } 22^\circ 30' + \frac{1}{8} \text{tag. } 11^\circ 15' + \frac{1}{16} \text{tag. } 5^\circ 37 \frac{1}{2}' + \text{etc.}$$

qui valores ex tabulis desumpti dabunt.

$\frac{1}{2} \text{tag. } 45^\circ$	$= 0,5000000$
$\frac{1}{4} \text{tag. } 22^\circ 30'$	$= 0,1035534$
$\frac{1}{8} \text{tag. } 11^\circ 15'$	$= 0,0244840$
$\frac{1}{16} \text{tag. } 5^\circ 37 \frac{1}{2}'$	$= 0,0061557$
$\frac{1}{32} \text{tag. } 2^\circ 48'$	$= 0,0015352$
$\frac{1}{64} \text{tag. } 1^\circ 24'$	$= 0,0003836$
pro reliquis	$= 0,0001278$

$$\frac{\pi}{2} = 0,6366197, \text{ hinc } \pi = \frac{1}{0,6366197} = \frac{1}{0,6366197} \quad \text{§. 6.}$$

§. 6. Si potteremam acquatorem deuto differ-
tiam, ad seriem perueniemus multo magis convergentem ;
cum enim fit $d \cdot \text{cof. } \Phi = \frac{d^2}{\text{fin. } \Phi^2}$ et $d \cdot \text{tag. } \Phi = \frac{d^2}{\text{cof. } \Phi^2} = d \Phi \text{ sec. } \Phi^2$;
experiemus

$$-\frac{1}{5} = \frac{-1}{\text{fin. } \frac{1}{2}} + \frac{1}{3} \text{ sec. } \frac{1}{2} s^2 - \frac{1}{5} \text{ sec. } \frac{1}{2} s^2 + \frac{1}{7} \text{ sec. } \frac{1}{2} s^2 + \dots \text{ etc.}$$

siue $\frac{1}{2} \text{ sec. } \frac{1}{2} s^2 + \frac{1}{2} \text{ sec. } \frac{1}{2} s^2 + \frac{1}{2} \text{ sec. } \frac{1}{2} s^2 + \frac{1}{2} \text{ sec. } \frac{1}{2} s^2 = \frac{1}{\text{fin. } \frac{1}{2}} - \frac{1}{5}$.

§. 7. Accommodemus eadem ratiocinia ad ratio-
nem tripliam, secundum quam arcus decrefcant ; hunc in
finem consideremus formulam

$$\text{fin. } 3 \Phi = 4 \text{ fin. } \Phi \text{ cof. } \Phi^2 = \text{fin. } \Phi (4 - 4 \text{ fin. } \Phi^2),$$

$$\text{quae dat fin. } 3 \Phi = 3 \text{ fin. } \Phi (1 - \frac{1}{3} \text{ fin. } \Phi^2); \text{ vnde si } s \text{ deno-}$$

$$\text{tet arcum quemcumque, erit fin. } s = 3 \text{ fin. } \frac{1}{3} s (1 - \frac{1}{3} \text{ fin. } \frac{s^2}{9});$$

$$\text{similique modo erit fin. } \frac{1}{3} s = 3 \text{ fin. } \frac{s}{9} (1 - \frac{1}{3} \text{ fin. } \frac{s^2}{81}), \text{ ita vt}$$

$$\text{nunc fit fin. } s = 9 \text{ fin. } \frac{s}{9} (1 - \frac{1}{3} \text{ fin. } \frac{s^2}{81})$$

Si tales substitutiones in infinitum continuentur,
peruenietur tandem vt ante ad hanc expressionem :

$$\text{fin. } s = s (1 - \frac{1}{3} \text{ fin. } \frac{s^2}{9}) (1 - \frac{1}{3} \text{ fin. } \frac{s^2}{81}) \text{ etc.}$$

§. 8. Factores hos nimis complicatos sequenti
modo in simplices resolvere licet ; namque forma gene-
ralis $1 - \frac{1}{3} \text{ fin. } \Phi^2$, ob $\text{fin } \Phi^2 = 1 - \frac{1}{3} \text{ cof. } 2 \Phi$, reducitur ad
hanc: $\frac{1}{2} + \frac{1}{2} \text{ cof. } 2 \Phi$, quam hoc modo referre licet: $\frac{\text{tag. cof. } 2 \Phi + \text{cof. cof. } \Phi}{2}$.

Cum iam fit

$$\text{cof. } a + \text{cof. } b = 2 \text{ (cof. } \frac{a+b}{2} \text{) (cof. } \frac{a-b}{2} \text{)}, \text{ erit}$$

$$\text{cof. } 60^\circ + \text{cof. } 2 \Phi = 2 \text{ cof. } (30^\circ + \Phi) \text{ cof. } (30^\circ - \Phi)$$

quae forma in $\frac{1}{2}$ ducta praebit

$$1 - \frac{1}{2} \text{ fin. } \Phi^2 = \frac{1}{2} \text{ cof. } (30^\circ + \Phi) \text{ cof. } (30^\circ - \Phi).$$

Quare

no differ-
entem ;
app $d \Phi \text{ sec. } \Phi^2$;

$$-\frac{1}{5} s^2 + \dots \text{ etc.}$$

quo $\frac{1}{\text{fin. } \frac{1}{2}} - \frac{1}{5}$

a ad ratio-
nem ; hunc in

qui in Φ^2 ;

si s deno-
tet arcum quemcumque, erit $\text{fin. } s = 3 \text{ fin. } \frac{1}{3} s (1 - \frac{1}{3} \text{ fin. } \frac{s^2}{9})$;
ita vt

continuentur,
peruenietur tandem vt ante ad hanc expressionem :

$$\text{fin. } s = s (1 - \frac{1}{3} \text{ fin. } \frac{s^2}{9}) (1 - \frac{1}{3} \text{ fin. } \frac{s^2}{81}) \text{ etc.}$$

os sequenti
forma gene-
ralis $1 - \frac{1}{3} \text{ fin. } \Phi^2$, ob $\text{fin } \Phi^2 = 1 - \frac{1}{3} \text{ cof. } 2 \Phi$, reducitur ad
hanc: $\frac{1}{2} + \frac{1}{2} \text{ cof. } 2 \Phi$, quam hoc modo referre licet: $\frac{\text{tag. cof. } 2 \Phi + \text{cof. cof. } \Phi}{2}$.

erit

$$1 - \frac{1}{2} \text{ fin. } \Phi^2 = \frac{1}{2} \text{ cof. } (30^\circ + \Phi) \text{ cof. } (30^\circ - \Phi).$$

Quare

Quare si haec reductio ad singulos factores supra inuen-
tios applicetur, habebimus sequens productum infinitum :

$$\frac{\text{fin. } s}{s} = \frac{1}{2} \text{ cof. } (30^\circ + \frac{s}{2}) \text{ cof. } (30^\circ - \frac{s}{2}).$$

$$\frac{1}{2} \text{ cof. } (30^\circ + \frac{s}{2}) \text{ cof. } (30^\circ - \frac{s}{2})$$

$$\frac{1}{2} \text{ cof. } (30^\circ + \frac{s}{4}) \text{ cof. } (30^\circ - \frac{s}{4}) \text{ etc.}$$

quod per secantes ita exhibebitur :

$$\frac{s}{\text{fin. } s} = \frac{1}{2} \text{ sec. } (30^\circ + \frac{s}{2}) \text{ sec. } (30^\circ - \frac{s}{2}).$$

$$\frac{1}{2} \text{ sec. } (30^\circ + \frac{s}{2}) \text{ sec. } (30^\circ - \frac{s}{2})$$

$$\frac{1}{2} \text{ sec. } (30^\circ + \frac{s}{4}) \text{ sec. } (30^\circ - \frac{s}{4}) \text{ etc.}$$

qui factores, quo magis arcus s diminuitur, eo propius ad
unitatem accedunt.

§. 9. Si nunc logarithmos sumamus et singulos
terminos differentiemus, factores illi numerici $\frac{1}{2}$; vix
constantes, penitus ex calculo excedent ; et quia, vt supra vidi-
mus est $d \cdot \text{fin. } \Phi = d \Phi \text{ tag. } \Phi$, obinuitur sequens aequatio :

$$\frac{1}{2} = \text{cof. tag. } s + \frac{1}{2} \text{ tag. } (30^\circ + \frac{s}{2}) + \frac{1}{2} \text{ tag. } (30^\circ + \frac{s}{4})$$

$$- \frac{1}{2} \text{ tag. } (30^\circ - \frac{s}{2}) - \frac{1}{2} \text{ tag. } (30^\circ - \frac{s}{4})$$

id quod exemplo comprobasse iuuabit. Sit igitur $s = \frac{\pi}{2}$
eritque

$$\frac{1}{2} = \frac{1}{2} \text{ tag. } 60^\circ + \frac{1}{2} \text{ tag. } 40^\circ + \frac{1}{2} \text{ tag. } (33^\circ 20') + \frac{1}{2} \text{ tag. } (31^\circ 6')$$

$$- \frac{1}{2} \text{ tag. } 0^\circ - \frac{1}{2} \text{ tag. } 20^\circ - \frac{1}{2} \text{ tag. } (26^\circ 40') - \frac{1}{2} \text{ tag. } (28^\circ 53')$$

$$+ \frac{1}{2} \text{ tag. } \text{tag. } (30^\circ 22') \text{ etc.}$$

$$- \frac{1}{2} \text{ tag. } \text{tag. } (29^\circ 37') \text{ etc.}$$

§. 10. Haec potteremam series eo magis videtur no-
tatu digna, quod vix quisquam eius veritatem demonstrare
valuerit, nisi eadem methodo fuerit usus. Haec autem se-
ries sine dubio multo altioris est indaginis quam ea, ad
quam

X x 3

quam per evolutionem precedentis casu: finis deducti, quae erat

$\frac{1}{2} = \cot s + \frac{1}{2} \text{tag. } \frac{1}{2} s + \frac{1}{2} \text{tag. } \frac{1}{2} s + \frac{1}{2} \text{tag. } \frac{1}{2} s + \text{etc.}$
 cuius veritas ex notissima formula $a \cot. a \Phi = \cot. \Phi - \text{tag. } \Phi$ sequitur, vnde habemus $\text{tag. } \Phi = \cot. \Phi - a \cot. a \Phi$. Hinc si loco singularium tangentium valores debiti substituantur, operatio sequenti modo infiruetur:

$$i = \begin{cases} \cot. s + \frac{1}{2} \cot. \frac{1}{2} s + \frac{1}{2} \cot. \frac{1}{2} s + \frac{1}{2} \cot. \frac{1}{2} s + \dots + \frac{1}{2} \cot. \frac{1}{2} s \\ \cot. s - \frac{1}{2} \cot. \frac{1}{2} s - \frac{1}{2} \cot. \frac{1}{2} s - \frac{1}{2} \cot. \frac{1}{2} s - \dots \end{cases}$$

Vbi omnes termini manifesto se mutuo destrunt, vique ad vltimum $\frac{1}{2} \cot. \frac{1}{2} s$, qui ad hanc formam redigatur: $\frac{\text{cot. } \frac{1}{2} s}{\frac{1}{2} \sin. \frac{1}{2} s}$, denotante i numerum infinitum. Iam quia arcus $\frac{1}{2}$ est infinitus parvus, erit $\cot. \frac{1}{2} s = 1$, sinus vero ipsi arcui $\frac{1}{2}$ aequalis, ex quo vltimus iste terminus fit $= \frac{1}{2}$, qui est ipse valor huius ferrei aequalis inuentus.

§. 11. Interim tamen etiam pro casu praesenti demonstratione directa simili modo exhibetur, ex formula, qua tangens anguli triplici exprimitur, repetenda. Si enim ponatur $\text{tag. } \Phi = t$, quia habetur $\text{tag. } 3 \Phi = \frac{3t - t^3}{1 - 3t^2}$ erit $\cot. 3 \Phi = \frac{1 - 3t^2}{3t - t^3}$ et $3 \cot. 3 \Phi = \frac{3 - 9t^2}{3t - t^3}$. Hinc subtrahatur $\cot. \Phi = \frac{1}{t}$, fietque $3 \cot. 3 \Phi - \cot. \Phi = \frac{3 - 9t^2}{3t - t^3} - \frac{1}{t} = \frac{3 - 9t^2 - 3t + t^3}{3t - t^3}$; hinc loco substituitur eius valor $\frac{\sin. \Phi}{\cos. \Phi}$ et habebimus,

$3 \cot. 3 \Phi - \cot. \Phi = \frac{3 \sin. \Phi - \cos. \Phi}{\sin. \Phi}$

cuius fractionis numeratorem et denominatorem sequenti modo tractamus. Cum fit $\cot. \Phi = \frac{1}{t} + \cot. a \Phi$ et $\sin. \Phi = \frac{1}{\sqrt{1 + t^2}}$ et $\cos. a \Phi$, inducet denominator hanc formam: $1 + 2 \cot. a \Phi$, quae prop-

terti ob
 hanc rem
 Qui sum
 3 cc
 quo
 finis deducti,
 $\frac{1}{2} \text{tag. } \frac{1}{2} s + \text{etc.}$
 $a \Phi$. Hinc si loco
 tantur, operatio
 $\dots + \frac{1}{2} \cot. \frac{1}{2} s$
 destrunt, vique
 nam redigatur:
 arcus $\frac{1}{2}$ est in-
 ni $\frac{1}{2}$ aequalis, ex
 ipse valor huius
 Simi
 Eod
 et f
 randi
 erit:
 cot. s
 m sequenti modo
 $\frac{1}{2} \cot. a \Phi$, quae prop-
 circa

terea ita referri poterit: $a \cot. 60^\circ + \cot. a \Phi$, quae porro ob $\cot. a + \cot. b = a \cot. \frac{a+b}{2}$, $\cot. \frac{a-b}{2}$ reducitur ad hanc: $4 \cot. (30^\circ + \Phi) \cot. (30^\circ - \Phi)$, numerator alterem manifesto fit $-4 \sin. a \Phi$, ita vt iam habeamus

$$3 \cot. 3 \Phi - \cot. \Phi = \frac{\sin. a \Phi}{\cos. (30^\circ + \Phi) \cos. (30^\circ - \Phi)}$$

Quia nunc in genere est

$$\sin. a \Phi = \sin. (a + \Phi) \cos. (a - \Phi) - \cos. (a + \Phi) \sin. (a - \Phi)$$

sumamus $a = 30^\circ$ et habemus sequentem aequationem:
 $3 \cot. 3 \Phi - \cot. \Phi = \frac{\sin. (30^\circ + \Phi) \cos. (30^\circ - \Phi) - \cos. (30^\circ + \Phi) \sin. (30^\circ - \Phi)}{\cos. (30^\circ + \Phi) \cos. (30^\circ - \Phi)}$
 $= \frac{-\text{tag. } (30^\circ + \Phi) + \text{tag. } (30^\circ - \Phi)}{\cos. (30^\circ + \Phi) \cos. (30^\circ - \Phi)}$
 quocirca perigimus ad hanc aequationem notatu dignam:
 $\cot. 3 \Phi = \frac{1}{2} \cot. \Phi - \frac{1}{2} \text{tag. } (30^\circ + \Phi) + \frac{1}{2} \text{tag. } (30^\circ - \Phi)$

§. 12. Iam pro nostro casu loco 3Φ scribendo s statim nanciscimur

$$\cot. s = \frac{1}{2} \cot. \frac{1}{2} s - \frac{1}{2} \text{tag. } (30^\circ + \frac{1}{2} s) + \frac{1}{2} \text{tag. } (30^\circ - \frac{1}{2} s).$$

Simili vero modo vltimus erit

$$\frac{1}{2} \cot. \frac{1}{2} s = \frac{1}{2} \cot. \frac{1}{4} s - \frac{1}{2} \text{tag. } (30^\circ + \frac{1}{4} s) + \frac{1}{2} \text{tag. } (30^\circ - \frac{1}{4} s).$$

Eodem porro modo fit

$$\frac{1}{4} \cot. \frac{1}{4} s = \frac{1}{4} \cot. \frac{1}{8} s - \frac{1}{4} \text{tag. } (30^\circ + \frac{1}{8} s) + \frac{1}{4} \text{tag. } (30^\circ - \frac{1}{8} s).$$

et si hoc modo in infinitum progrediamur pervenimus tandem ad huiusmodi cotangentem:

$$\frac{1}{2} \cot. \frac{1}{2} s = \frac{\text{cot. } \frac{1}{2} s}{\frac{1}{2} \sin. \frac{1}{2} s}; \text{ quamobrem nostra aequatio perducta erit ad hanc formam:}$$

$$\cot. s = \frac{1}{2} \text{tag. } (30^\circ + \frac{1}{2} s) - \frac{1}{2} \text{tag. } (30^\circ + \frac{1}{4} s) - \frac{1}{4} \text{tag. } (30^\circ + \frac{1}{8} s) + \frac{1}{2} \text{tag. } (30^\circ - \frac{1}{2} s) + \frac{1}{4} \text{tag. } (30^\circ - \frac{1}{4} s) + \frac{1}{8} \text{tag. } (30^\circ - \frac{1}{8} s) + \dots$$

ex