minis poteſtatibus obtinebuntur, eaque ipſa, quae iam ſupra ſunt recenſita. Similisque eſt ratio omnium reliquorum numerorum primorum.

§. 53. Quod autem ad multitudinem horum numerorum a attinet, obſeruo eam quouis caſu $p = 2q + 1$ aequalem eſſe multitudini eorum numerorum ipſo p minorum, qui ſint ad $2q$ primi: atque alio loco oſtendi, ad hanc multitudinem inueniendam numerum $2q$ in factores ſuos primos reſolui debere, ita vt ſi fuerit $2q = f^{\zeta} g^{\eta} h^{\theta} k^{\kappa}$, ſit iſta multitudo

$$= (f-1) f^{\zeta-1} \cdot (g-1) g^{\eta-1} \cdot (h-1) h^{\theta-1} \cdot (k-1) k^{\kappa-1}.$$

Definito autem pro quouis numero $p = 2q + 1$ hac multitudine, ſint ipſi numeri ad $2q$ primi $1, \alpha, \beta, \gamma, \delta,$ etc. atque ſi datus fuerit vnus numerus a quicunque, reliqui ideoque omnes erunt:

$$a,\ a^{\alpha} - np;\ a^{\beta} - np;\ a^{\gamma} - np;\ a^{\delta} - np;\ \text{etc.}$$

ſumendo n ita, vt omnes iſti numeri infra p deprimantur. Haec fortaſſe conſideratio viam aperiet pro quouis caſu hos numeros inueſtigandi.

iam ſu-
reliquo-
ME
rum nu-
$= 2q + 1$
ſo p mi-
oſtendi,
in fac-
it $2q =$
In n
va
dati
quoq
eoden
quae
ergo
curua
vani,
curua
ſempe
aeque
cum
curua
maxir
ſtanti
innuit
primis
interp
nerali
$-1) k^{\kappa-1}.$
hac mul-
$\gamma, \delta,$ etc.
, reliqui
etc.
depriman-
ro quouis
DE

DE EXIMIO VSV
METHODI INTERPOLATIONVM
IN SERIERVM DOCTRINA.

In methodo interpolationum eiusmodi relatio inter binas variabiles x et y quaeritur, vt ſi alteri x ſucceſſiue dati valores $a, b, c, d,$ etc. tribuantur, altera y inde quoque datos valores $p, q, r, s,$ etc. ſortiatur; ſeu quod eodem redit, aequatio pro eiusmodi linea curua quaeritur, quae per quotcunque puncta data tranſeat. Quo maior ergo fuerit horum punctorum numerus, eo magis linea curua limitatur: interim tamen iam alia occaſione obſeruaui, etiamſi punctorum numerus in infinitum augeatur, curuam per ea tranſeuntem non prorſus determinari, ſed ſemper infinitas adhuc lineas curuas exhiberi poſſe, quae aeque per cuncta eadem puncta ſint tranſiturae. Quare cum methodus interpolationum pro quouis caſu lineam curuam ſuppeditet determinatam, ſolutio haec ſemper pro maxime particulari erit habenda: verum haec ipſa circumſtantia ſingularem quandam indolem ſolutionis inuentae innuit, quae accuratiorem conſiderationem meretur. Imprimis autem iſta ſolutionis indoles pendet a ratione, qua interpolatio inſtituitur, ſeu a forma, quae aequationi generali tribuitur, in qua aequationem quaeſitam contineri oportet. Quae forma cum infinitis modis conſtitui poſſit, inueſtigationes meas ad hanc formam reſtringam:

$$y = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 + \varepsilon x^9 + \text{etc.}$$

quae ſcilicet tantum poteſtates impares ipſius x contineat, ita vt, qui ipſius y valores quibuscunque valoribus poſitiuis ipſius x conueniant, iidem negatiue ſumti valoribus iisdem negatiuis ipſius x reſpondeant: quo ipſo innumerabiles aliae lineae curuae excluduntur, quae per eadem puncta eſſent tranſiturae.

Problema I.

§. 1. Inuenire aequationem inter binas variabiles x et y huius formae: $y = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 +$ etc. vt, ſi ipſi x dati valores a, b, c, d, etc. tribuantur, altera variabilis y itidem datos conſequatur valores p, q, r, s, etc.

Solutio.

Quo aequatio generalis aſſumta facilius ad hunc caſum accommodari poſſit, ea hac forma exhibeatur:

$$\begin{aligned} y = Ax &+ Bx(xx-aa) + Cx(xx-aa)(xx-bb) \\ &+ Dx(xx-aa)(xx-bb)(xx-cc) \\ &+ Ex(xx-aa)(xx-bb)(xx-cc)(xx-dd) \end{aligned}$$

quae, etſi forte in infinitum progrediatur, ſi ſcilicet conditionum numerus ſit infinitus, tamen pro ſingulis conditionibus propoſitis aequationes ſuppeditat finitas ſequentes:

I. $p = Aa$

II. $q = Ab + Bb(bb-aa)$

III. $r = Ac + Bc(cc-aa) + Cc(cc-aa)(cc-bb)$

IV. $s = Ad + Bd(dd-aa) + Cd(dd-aa)(dd-bb) + Dd(dd-aa)(dd-bb)(dd-cc)$

etc. etc.

quae

quae ita repraeſententur:

I. $\frac{p}{a} = A$

II. $\frac{q}{b} = A + B(bb-aa)$

III. $\frac{r}{c} = A + B(cc-aa) + C(cc-aa)(cc-bb)$

IV. $\frac{s}{d} = A + B(dd-aa) + C(dd-aa)(dd-bb) + D(dd-aa)(dd-bb)(dd-cc)$

etc. etc.

Iam prima a ſingulis ſequentibus ſubtrahatur, et differentiae per coefficientes ipſius B diuidantur, vt prodeant hae aequationes:

$$\frac{aq-bp}{ab(bb-aa)} = q' = B$$

$$\frac{ar-cp}{ac(cc-aa)} = r' = B + C(cc-bb)$$

$$\frac{as-dp}{ad(dd-aa)} = s' = B + C(dd-bb) + D(dd-bb)(dd-cc)$$

etc.

Nunc ſimili modo primam a ſequentibus ſubtrahentes, et reſidua per coefficientes ipſius C diuidentes, perueniemus ad has aequationes:

$$\frac{r'-q'}{cc-bb} = r'' = C$$

$$\frac{s'-q'}{dd-bb} = s'' = C + D(dd-cc)$$

etc.

porroque ad hanc $\frac{s''-r''}{dd-cc} = D$.

Quamobrem ex quantitatibus datis a, b, c, d, etc. et p, q, r, s, etc. coefficientes A, B, C, D, etc. ita commodiſſime definientur. Deriuentur primo ex quantitatibus datis iſtae:

$$P = \frac{p}{a};\ Q = \frac{q}{b};\ R = \frac{r}{c};\ S = \frac{s}{d};\ \text{etc.}$$

hinc-

c
Ia
ti
ac
Nu
rei
ad
poi
Qu
s,
def
itui poſſit,
n:
c.
contineat,
ibus poſiti-
i valoribus
nnumerabi-
idem punc-
s variabiles
+ etc. vt, ſi
altera va-
, r, s, etc.
us ad hunc
eatur:
-cc)
-cc)(xx-dd)
ſcilicet con-
ulis condi-
s ſequentes:
d · cc)
quae

hincque formentur hae:

$$Q' = \frac{Q-P}{bb-aa};\ R' = \frac{R-P}{cc-aa};\ S' = \frac{S-P}{dd-aa};\ T' = \frac{T-P}{ee-aa};\ \text{etc.}$$

$$R'' = \frac{R'-Q'}{cc-bb};\ S'' = \frac{S'-Q'}{dd-bb};\ T'' = \frac{T'-Q'}{ee-bb};\ \text{etc.}$$

$$S''' = \frac{S''-R''}{dd-cc};\ T''' = \frac{T''-R''}{ee-cc};\ \text{etc.}$$

$$T'''' = \frac{T'''-S'''}{ee-dd};\ \text{etc.}$$

Quibus valoribus inuentis habebimus:

$$A = P;\ B = Q';\ C = R'';\ D = S''';\ E = T'''';\ \text{etc.}$$

Corollarium 1.

§. 2. Cum sit $P = \frac{p}{a}$, erit primus coefficiens $A = \frac{p}{a}$; pro sequentibus vero, ob:

$$Q' = \frac{aq-bp}{ab(bb-aa)};\ R' = \frac{ar-cp}{ac(cc-aa)};\ S' = \frac{as-dp}{ad(dd-aa)};$$

$$T' = \frac{at-ep}{ae(ee-aa)};\ \text{etc.}$$

erit secundus coefficiens

$$B = \frac{aq-bp}{ab(bb-aa)}\ \text{seu}\ B = \frac{p}{a(aa-bb)} + \frac{q}{b(bb-aa)}.$$

Corollarium 2.

§. 3. Cum porro sit

$$R'' = \frac{ar-cp}{ac(cc-aa)(cc-bb)} - \frac{aq+bp}{ab(bb-aa)(cc-bb)},\ \text{fiet}$$

$$C = \frac{r}{c(cc-aa)(cc-bb)} + \frac{q}{b(aa-bb)(cc-bb)} + \frac{p}{a(cc-aa)(bb-aa)},$$

quae forma ita concinnius exhiberi potest, vt sit:

$$C = \frac{p}{a(aa-bb)(aa-cc)} + \frac{q}{b(bb-aa)(bb-cc)} + \frac{r}{c(cc-aa)(cc-bb)}.$$

Corol-

$= \frac{T-P}{ee-aa}$; etc.
etc.
$= T''''$; etc.
mus coefficiens
$= \frac{as-dp}{ad(dd-aa)}$;
$\frac{q}{bb-aa)}$.
$-bb)$, fiet
$c-bb)$
vt sit:
$-cc)$
Corol-

Corollarium 3.

§. 4. Simili modo calculum vlterius prosequendo reperietur:

$$D = \frac{p}{a(aa-bb)(aa-cc)(aa-dd)} + \frac{q}{b(bb-aa)(bb-cc)(bb-dd)} + \frac{r}{c(cc-aa)(cc-bb)(cc-dd)} + \frac{s}{d(dd-aa)(dd-bb)(dd-cc)}$$

vnde formam sequentium quantitatum E, F, G, etc. iam tuto coniectura assequi licet.

Scholion.

§. 5. Plerumque autem expeditius singularum coefficientium A, B, C, D, E, etc. valores ex praecedentibus definiuntur: Ex aequationibus enim fundamentalibus deducuntur formulae sequentes:

$$A = \frac{p}{a};$$

$$B = \frac{q-bA}{b(bb-aa)};$$

$$C = \frac{r-cA}{c(cc-aa)(cc-bb)} - \frac{B}{cc-bb};$$

$$D = \frac{s-dA}{d(dd-aa)(dd-bb)(dd-cc)} - \frac{B}{(dd-bb)(dd-cc)} - \frac{C}{dd-cc};$$

$$E = \frac{t-eA}{e(ee-aa)(ee-bb)(ee-cc)(ee-dd)} - \frac{B}{(ee-bb)(ee-cc)(ee-dd)} - \frac{C}{(ee-cc)(ee-dd)} - \frac{D}{ee-dd};$$

etc.

vbi plerumque mox eiusmodi ordo obseruatur, vnde sequentes facile deriuari possunt, quemadmodum ex sequentibus problematibus, in quibus hanc methodum ad quosdam casus particulares sum accommodaturus, patebit.

 §. 6.

Scholion 2.

§. 6. Antequam autem huiusmodi casus euoluam, in genere obseruasse iuuabit, quod si pro quocunque casu oblato aequatio satisfaciens inter binas variabiles x et y fuerit inuenta, quam hoc modo designabo: $y = X$, ita vt sit $X = \alpha x + \beta x^3 + \gamma x^5 + \delta x^7 +$ etc. tum inde facile aequationem multo latius patentem et pariter satisfacientem formari posse. Statuatur enim:

$$Q = x \cdot \frac{xx - aa}{aa} \cdot \frac{xx - bb}{bb} \cdot \frac{xx - cc}{cc} \cdot \frac{xx - dd}{dd} \cdot \text{ etc.}$$

quae quantitas euanescit pro omnibus valoribus ipsius x propositis:

$$x = 0, \; x = \pm a; \; x = \pm b; \; x = \pm c, \text{ etc.}$$

idemque praestabunt omnes functiones ipsius Q vna cum ipsa Q euanescentes: ex quo manifestum est, si statuatur $y = X + Q$, vel $y = X + f:Q$, omnibus conditionibus aeque satisfieri. Quoniam igitur haec functio $f:Q$ omnino est arbitraria, dummodo euanescat facto $Q = 0$, haec aequatio: $y = X + f:Q$ solutionem generalissimam exhibere est censenda.

Problema 2.

§. 7. Sint a, b, c, d, etc. quotcunque arcus circulares, existente radio $= 1$, valores autem p, q, r, s, etc. sint sinus eorundem arcuum, quandoquidem hoc casu ea proprietas locum habet, vt arcubus negatiuis iidem sinus negatiue sumti respondeant; hinc proxime definire rationem inter diametrum et peripheriam.

Solutio.

Cum hic sit $p = \text{fin.}\, a$, $q = \text{fin.}\, b$, $r = \text{fin.}\, c$, etc. aequatio inter x et y ita erit comparata, vt sumto pro x arcu

casus euoluam, quocunque casu variabiles x et y $y = X$, ita vt tum inde facile iter satisfacien- $\frac{dd}{d}$. etc. oribus ipsius x $\pm c$, etc. us Q vna cum est, si statuatur conditionibus io $f:Q$ omni- $Q = 0$, haec ssimam exhibe- nque arcus cir- p, q, r, s, etc. m hoc casu ea uis iidem sinus efinire rationem $r = \text{fin.}\, c$, etc. t sumto pro x arcu

arcu circuli, quantitas y proxime eius sinum sit expressura, fiatque $y = \text{fin.}\, x$. Definitis ergo per praecedens problema coefficientibus A, B, C, D, etc. habebitur haec aequatio:

$$\text{fin.}\, x = Ax + Bx(xx - aa) + Cx(xx - aa)(xx - bb) + \text{etc.}$$

quae adeo veritati est consentanea, quoties fuerit vel

$$x = 0, \text{ vel } x = \pm a, \text{ vel } x = \pm b, \text{ vel } x = \pm c \text{ etc.}$$

Statuamus nunc arcum x infinite paruum, et quia tum eius sinus, fin. x ipsi arcui x aequatur, orietur haec aequatio:

$$1 = A - Baa + Caabb - Daabbcc + Eaabbccdd - \text{etc.}$$

Substituamus hic pro litteris A, B, C, D, etc. valores supra inuentos, ac perueniemus ad hanc aequationem:

$$\begin{aligned} 1 = &\frac{p}{a}\left(1 - \frac{aa}{aa - bb} + \frac{aabb}{(aa - bb)(aa - cc)} - \frac{aabbcc}{(aa - bb)(aa - cc)(aa - dd)} + \text{etc.}\right) \\ &- \frac{q}{b}\left(\frac{aa}{bb - aa} - \frac{aabb}{(bb - aa)(bb - cc)} + \frac{aabbcc}{(bb - aa)(bb - cc)(bb - dd)} - \text{etc.}\right) \\ &+ \frac{r}{c}\left(\frac{aabb}{(cc - aa)(cc - bb)} - \frac{aabbcc}{(cc - aa)(cc - bb)(cc - dd)} + \text{etc.}\right) \\ &- \frac{s}{d}\left(\frac{aabbcc}{(dd - aa)(dd - bb)(dd - cc)} - \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

quae reducitur ad hanc, in qua omnes series sibi sunt similes:

$$\begin{aligned} 1 = &\frac{p}{a}\left(1 + \frac{aa}{bb - aa} + \frac{aabb}{(bb - aa)(cc - aa)} + \frac{aabbcc}{(bb - aa)(cc - aa)(dd - aa)} + \text{etc.}\right) \\ &- \frac{aaq}{b(bb - aa)}\left(1 + \frac{bb}{cc - bb} + \frac{bbcc}{(cc - bb)(dd - bb)} + \frac{bbccdd}{(cc - bb)(dd - bb)(ee - bb)} + \text{etc.}\right) \\ &+ \frac{aabbr}{c(cc - aa)(cc - bb)}\left(1 + \frac{cc}{dd - cc} + \frac{ccdd}{(dd - cc)(ee - cc)} + \text{etc.}\right) \\ &- \frac{aabbccs}{d(dd - aa)(dd - bb)(dd - cc)}\left(1 + \frac{dd}{ee - dd} + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

Quaelibet autem harum serierum sponte est summabilis; primae enim duo termini primi coniuncti dant $\frac{bb}{bb - aa}$, cui si tertius addatur, prodit $\frac{bbcc}{(bb-aa)(cc-aa)}$, hincque porro quartus adiunctus praebet $\frac{bbccdd}{(bb-aa)(cc-aa)(dd-aa)}$, sicque ulterius, ita vt nostrae aequationis prima series euadat:

$$\frac{p}{a}\cdot\frac{bb}{bb-aa}\cdot\frac{cc}{cc-aa}\cdot\frac{dd}{dd-aa}\cdot\frac{ee}{ee-aa}\cdot \text{ etc.}$$

Simili vero modo pro secunda reperitur:

$$-\frac{q}{b}\cdot\frac{aa}{bb-aa}\cdot\frac{cc}{cc-bb}\cdot\frac{dd}{dd-bb}\cdot\frac{ee}{ee-bb}\cdot \text{ etc.}$$

sicque tandem nostra aequatio reducitur ad hanc formam:

$$\begin{aligned}1 = &\frac{p}{a}\cdot\frac{bb}{bb-aa}\cdot\frac{cc}{cc-aa}\cdot\frac{dd}{dd-aa}\cdot\frac{ee}{ee-aa}\cdot \text{ etc.}\\ &+\frac{q}{b}\cdot\frac{aa}{aa-bb}\cdot\frac{cc}{cc-bb}\cdot\frac{dd}{dd-bb}\cdot\frac{ee}{ee-bb}\cdot \text{ etc.}\\ &+\frac{r}{c}\cdot\frac{aa}{aa-cc}\cdot\frac{bb}{bb-cc}\cdot\frac{dd}{dd-cc}\cdot\frac{ee}{ee-cc}\cdot \text{ etc.}\\ &+\frac{s}{d}\cdot\frac{aa}{aa-dd}\cdot\frac{bb}{bb-dd}\cdot\frac{cc}{cc-dd}\cdot\frac{ee}{ee-dd}\cdot \text{ etc.}\\ &+\frac{t}{e}\cdot\frac{aa}{aa-ee}\cdot\frac{bb}{bb-ee}\cdot\frac{cc}{cc-ee}\cdot\frac{dd}{dd-ee}\cdot \text{ etc.}\\ &\text{etc.}\end{aligned}$$

vnde si arcus dati a, b, c, d, etc. cognitam rationem ad semiperipheriam π teneant, huius quantitatis valor π definietur.

Corollarium 1.

§. 8. Si numerus horum arcuum a, b, c, d, etc. fuerit finitus, tum circuli peripheria eo accuratius definietur, quo maior sit ille numerus, et simul quo minores arcus inter eos occurrant. Aucto autem arcuum propositorum numero in infinitum, vera ratio peripheriae ad diametrum inde deriuabitur.

Corollarium 2.

§. 9. Simili modo in genere sinus arcus indefiniti x definiri potest. Substitutis enim loco coefficientium A, B,

ãque porro quar-
ã sicque vlterius,
adat:
l hanc formam:
c.
am rationem ad
ntitatis valor π
. a, b, c, d, etc.
ccuratius definie-
quo minores ar-
rcuum proposito-
ripheriae ad dia-
inus arcus indefi-
oco coefficientium
A, B,

A, B, C, D, etc. valoribus inuentis, aequatio ad hanc formam redigetur:

$$\begin{aligned}\frac{\sin x}{x} = &\frac{p}{a}\cdot\frac{bb-xx}{bb-aa}\cdot\frac{cc-xx}{cc-aa}\cdot\frac{dd-xx}{dd-aa}\cdot \text{ etc.}\\ &+\frac{q}{b}\cdot\frac{aa-xx}{aa-bb}\cdot\frac{cc-xx}{cc-bb}\cdot\frac{dd-xx}{dd-bb}\cdot \text{ etc.}\\ &+\frac{r}{c}\cdot\frac{aa-xx}{aa-cc}\cdot\frac{bb-xx}{bb-cc}\cdot\frac{dd-xx}{dd-cc}\cdot \text{ etc.}\\ &+\frac{s}{d}\cdot\frac{aa-xx}{aa-dd}\cdot\frac{bb-xx}{bb-dd}\cdot\frac{cc-xx}{cc-dd}\cdot \text{ etc.}\\ &\text{etc.}\end{aligned}$$

quae aequatio, sumto arcu x euanescente, in illam abit.

Corollarium 3.

§. 10. Haec autem reductio multo latius patet, nulla arcuum ratione habita. Si enim quaeratur eiusmodi aequatio inter binas variabiles x et y, vt

sumto $x = 0$, a, b, c, d, e, etc.

fiat $y = 0$, p, q, r, s, t, etc.

aequatio haec in genere ita repraesentari poterit:

$$\begin{aligned}\frac{y}{x} = &\frac{p}{a}\cdot\frac{bb-xx}{bb-aa}\cdot\frac{cc-xx}{cc-aa}\cdot\frac{dd-xx}{dd-aa}\cdot\frac{ee-xx}{ee-aa}\cdot \text{ etc.}\\ &+\frac{q}{b}\cdot\frac{aa-xx}{aa-bb}\cdot\frac{cc-xx}{cc-bb}\cdot\frac{dd-xx}{dd-bb}\cdot\frac{ee-xx}{ee-bb}\cdot \text{ etc.}\\ &+\frac{r}{c}\cdot\frac{aa-xx}{aa-cc}\cdot\frac{bb-xx}{bb-cc}\cdot\frac{dd-xx}{dd-cc}\cdot\frac{ee-xx}{ee-cc}\cdot \text{ etc.}\\ &+\frac{s}{d}\cdot\frac{aa-xx}{aa-dd}\cdot\frac{bb-xx}{bb-dd}\cdot\frac{cc-xx}{cc-dd}\cdot\frac{ee-xx}{ee-dd}\cdot \text{ etc.}\\ &\text{etc.}\end{aligned}$$

ex qua forma simul manifestum est, quomodo singulis conditionibus satisfiat.

Scholion.

§. 11. Casibus hic non immorabor, quibus numerus conditionum praescriptarum a, b, c, d, etc. assumi-

 tur

tur finitus, quoniam inde tantum approximationes pro mensura circuli suppeditantur. Interim tamen obseruasse haud pigebit, si tantum quatuor arcus accipiantur, qui sint

$$a = \Phi,\ b = 2\Phi,\ c = 3\Phi,\ d = 4\Phi,$$

fore ex solutione problematis

$$\Phi = \frac{\sin.\Phi}{1}.\frac{2.2}{1.3}.\frac{3.3}{2.4}.\frac{4.4}{3.5} = \frac{8}{5}\sin.\Phi - \frac{2}{3}\sin.2\Phi + \frac{8}{105}\sin.3\Phi - \frac{1}{140}\sin.4\Phi$$
$$-\frac{\sin.2\Phi}{2}.\frac{1.1}{1.3}.\frac{3.3}{1.5}.\frac{4.4}{2.6}$$
$$+\frac{\sin.3\Phi}{3}.\frac{1.1}{2.4}.\frac{2.2}{1.5}.\frac{4.4}{2.7}$$
$$-\frac{\sin.4\Phi}{4}.\frac{1.1}{3.5}.\frac{2.2}{2.6}.\frac{3.3}{1.7}$$

quae expressio eo propius ad veritatem accedit, quo minor capiatur arcus Φ: interim tamen etsi ad quadrantem vsque augeatur, vt sit $\Phi = \frac{\pi}{2}$, error non fit enormis; prodit enim $\frac{\pi}{2} = \frac{8}{5} - \frac{8}{105} = \frac{32}{21}$, sicque $\pi = 3\frac{1}{21}$. At si sumamus $\Phi = 30° = \frac{\pi}{6}$, fit

$$\frac{\pi}{6} = \frac{8}{5}.\frac{1}{2} - \frac{2}{3}.\frac{\sqrt{3}}{2} + \frac{8}{105} - \frac{1}{140}.\frac{\sqrt{3}}{2},\ \text{seu}$$
$$\pi = \frac{184}{35} - \frac{171\sqrt{3}}{140},$$

qui valor tantum aliquot particulis centies millesimis vnitatis a vero differt. Verum missa hac speculatione casus aliquot, vbi arcuum propositorum a, b, c, d, etc. certa lege progredientium numerus est infinitus, percurram.

Exemplum I.

§. 12. *Progrediantur arcus* a, b, c, d, *etc. secundum seriem numerorum naturalium, sitque*

$$a = \Phi,\ b = 2\Phi,\ c = 3\Phi,\ d = 4\Phi,\ \text{etc.}$$

in infinitum, ex quorum finibus p, q, r, *etc. veram longitudinem arcus* Φ *determinari oporteat.*

Solutio

imationes pro
men obseruasse
iantur, qui sint
3Φ − 1/140 sin.4Φ
edit, quo mi-
ad quadrantem
enormis; pro-
At si sumamus
millesimis vni-
eculatione casus
, d, etc. certa
percurram.
, d, etc. secun-
Φ, etc.
tc. veram longi-
Solutio

Solutio ergo problematis pro hoc casu suppeditat hanc aequationem:

$$\Phi = \frac{\sin.\Phi}{1}.\frac{2.2}{1.3}.\frac{3.3}{2.4}.\frac{4.4}{3.5}.\frac{5.5}{4.6}.\ \text{etc.}$$
$$-\frac{\sin.2\Phi}{2}.\frac{1.1}{1.3}.\frac{3.3}{1.5}.\frac{4.4}{2.6}.\frac{5.5}{3.7}.\ \text{etc.}$$
$$+\frac{\sin.3\Phi}{3}.\frac{1.1}{2.4}.\frac{2.2}{1.5}.\frac{4.4}{1.7}.\frac{5.5}{2.8}.\ \text{etc.}$$
$$-\frac{\sin.4\Phi}{4}.\frac{1.1}{3.5}.\frac{2.2}{2.6}.\frac{3.3}{1.7}.\frac{5.5}{1.9}.\ \text{etc.}$$
$$+\frac{\sin.5\Phi}{5}.\frac{1.1}{4.6}.\frac{2.2}{3.7}.\frac{3.3}{2.8}.\frac{4.4}{1.9}.\ \text{etc.}$$

omnia autem haec producta eundem reperiuntur habere valorem $= 2$, ita vt sit:

$$\tfrac{1}{2}\Phi = \sin.\Phi - \tfrac{1}{2}\sin.2\Phi + \tfrac{1}{3}\sin.3\Phi - \tfrac{1}{4}\sin.4\Phi + \tfrac{1}{5}\sin.5\Phi - \text{etc.}$$

cuius seriei veritas casu, quo angulus Φ est infinite paruus, per se est manifesta. Euoluamus ergo casus sequentes.

1°. Sit $\Phi = 90° = \frac{\pi}{2}$, ac prodit series *Leibniziana*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.}$$

2°. Sit $\Phi = 45° = \frac{\pi}{4}$, orieturque haec series:

$$\frac{\pi}{8} = \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{3\sqrt{2}} * - \frac{1}{5\sqrt{2}} + \frac{1}{6} - \frac{1}{7\sqrt{2}} * + \frac{1}{9\sqrt{2}} - \frac{1}{10} + \frac{1}{11\sqrt{2}} - \text{etc.}$$

quae resoluitur in has duas:

$$\frac{\pi}{8} = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.}\right)$$
$$- \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}\right)$$

ita vt sit

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \text{etc.} = \frac{\pi}{2\sqrt{2}}.$$

3°. Sit $\Phi = 60° = \frac{\pi}{3}$, eritque

$$\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2}.\frac{\sqrt{3}}{2} * + \frac{1}{4}.\frac{\sqrt{3}}{2} - \frac{1}{5}.\frac{\sqrt{3}}{2} + \text{etc.}$$

seu

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \frac{1}{11} + \text{etc.}$$

4°.

4°. Sit $\Phi = 30° = \frac{\pi}{6}$, ac fit

$$\frac{\pi}{12} = \frac{1}{1}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{3} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{1}{5}\cdot\frac{1}{2} * - \frac{1}{7}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{\sqrt{3}}{2} - \text{etc.}$$

seu

$$\frac{\pi}{12} = \frac{1}{2}\left(1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{19} - \frac{1}{23} + \text{etc.}\right)$$
$$- \frac{\sqrt{3}}{4}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \text{etc.}\right)$$
$$+ \frac{1}{3}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}\right)$$

quarum ferierum vltima fit $= \frac{\pi}{12}$, hinc concluditur

$$1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \text{etc.} = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \text{etc.}\right)$$

Vtraque autem feries aequatur arcui $\frac{\pi}{3}$, quod quidem in priori ex *Leibniziana* eft manifeftum.

Corollarium 1.

§. 13. Ex aequatione hic inuenta:

$$\frac{1}{2}\Phi = \text{fin.}\,\Phi - \frac{1}{2}\,\text{fin.}\,2\Phi + \frac{1}{3}\,\text{fin.}\,3\Phi - \frac{1}{4}\,\text{fin.}\,4\Phi + \text{etc.}$$

plures aliae non minus notatu dignae deriuari poffunt. Veluti inftituta differentiatione prodit

$$\frac{1}{2} = \text{cof.}\,\Phi - \text{cof.}\,2\Phi + \text{cof.}\,3\Phi - \text{cof.}\,4\Phi + \text{etc.}$$

cuius ratio inde eft manifefta, quod multiplicando vtrinque per $2\,\text{cof.}\,\frac{1}{2}\Phi$ prodit aequatio identica $\text{cof.}\,\frac{1}{2}\Phi = \text{cof.}\,\frac{1}{2}\Phi$.

Corollarium 2.

§. 14. At fi illam aequationem per $-d\Phi$ multiplicatam integremus, prouenit

$$C - \frac{1}{4}\Phi\Phi = \text{cof.}\,\Phi - \frac{1}{4}\,\text{cof.}\,2\Phi + \frac{1}{9}\,\text{cof.}\,3\Phi - \frac{1}{16}\,\text{cof.}\,4\Phi + \text{etc.}$$

vbi ex cafu $\Phi = 0$ conftans per integrationem ingreffa determinatur, fcilicet:

$$C = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \text{etc.} = \frac{\pi\pi}{12},$$

ita

$\frac{1}{8}\cdot\frac{\sqrt{3}}{2}$ − etc.
$\frac{1}{23}$ + etc.)
+ etc.)
ditur
$+\frac{1}{7}+\frac{1}{8}$ − etc.)
quidem in
in. 4Φ + etc.
uari poffunt.
. 4Φ + etc.
ndo vtrinque
= cof. ½Φ.
− dΦ mul-
cof. 4Φ + etc.
nem ingreffa
ita

ita vt fit

$$\frac{\pi\pi}{12} - \frac{\Phi\Phi}{4} = \text{cof.}\,\Phi - \frac{1}{4}\,\text{cof.}\,2\Phi + \frac{1}{9}\,\text{cof.}\,3\Phi - \frac{1}{16}\,\text{cof.}\,4\Phi + \text{etc.}$$

quae ergo feries, fumto $\Phi = \frac{\pi}{\sqrt{3}}$, fit $= 0$. Eft autem proxime

$$\frac{\pi}{\sqrt{3}} = 103°, 55', 23'' \text{ et cof.}\,\frac{\pi}{\sqrt{3}} = 0,2406186.$$

Corollarium 3.

§. 15. Si hanc aequationem denuo per $d\Phi$ multiplicatam integremus, orietur haec noua fummatio:

$$\frac{1}{12}\pi\pi\Phi - \frac{1}{12}\Phi^3 = \text{fin.}\,\Phi - \frac{1}{8}\,\text{fin.}\,2\Phi + \frac{1}{27}\,\text{fin.}\,3\Phi - \frac{1}{64}\,\text{fin.}\,4\Phi + \text{etc.}$$

vnde fumto arcu $\Phi = 90° = \frac{\pi}{2}$ obtinetur

$$\frac{1}{32}\pi^3 = 1 - \frac{1}{27} + \frac{1}{125} - \frac{1}{343} + \text{etc.}$$

vti iam aliunde eft notum.

Scholion.

§. 16. Circa feriem inuentam

$$\frac{1}{2}\Phi = \text{fin.}\,\Phi - \frac{1}{2}\,\text{fin.}\,2\Phi + \frac{1}{3}\,\text{fin.}\,3\Phi - \text{etc.}$$

dubium oriri poteft, quod fumto arcu $\Phi = 180° = \pi$, finguli feriei termini euanefcant, ideoque fumma nequeat ipfi $\frac{1}{2}\pi$ aequari. Verum ad hoc dubium foluendum ftatuatur prime $\Phi = \pi - \omega$, et refultabit haec aequatio:

$$\frac{\pi - \omega}{2} = \text{fin.}\,\omega + \frac{1}{2}\,\text{fin.}\,2\omega + \frac{1}{3}\,\text{fin.}\,3\omega + \frac{1}{4}\,\text{fin.}\,4\omega + \text{etc.}$$

nunc vero arcus ω infinite paruus fumatur, vnde adipifcimur hanc:

$$\frac{\pi - \omega}{2} = \omega + \omega + \omega + \omega + \omega + \text{etc.}$$

quae nihil amplius continet abfurdi. Quod idem tenendum eft, fi velimus accipere $\Phi = 2\pi$ vel $\Phi = 3\pi$ etc.

Exemplum II.

§. 17. *Si arcus* a, b, c, d, *etc. progressionem arithmeticam quamcunque constituant, vt sit*

$$a = n\Phi,\ b = (n+1)\Phi,\ c = (n+2)\Phi,\ d = (n+3)\Phi, \text{ etc.}$$

ex eorum sinibus longitudinem arcus Φ *definire.*

Solutio generalis ante exhibita pro hoc casu dat

$$\Phi = \frac{\text{sin.}\,n\Phi}{n} \cdot \frac{(n+1)^2}{1(1+2n)} \cdot \frac{(n+2)^2}{2(2+2n)} \cdot \frac{(n+3)^2}{3(3+2n)} \cdot \frac{(n+4)^2}{4(4+2n)} \cdot \frac{(n+5)^2}{5(5+2n)} \text{ etc.}$$
$$- \frac{\text{sin.}\,(n+1)\Phi}{n+1} \cdot \frac{n^2}{1(1+2n)} \cdot \frac{(n+2)^2}{1(3+2n)} \cdot \frac{(n+3)^2}{2(4+2n)} \cdot \frac{(n+4)^2}{3(5+2n)} \cdot \frac{(n+5)^2}{4(6+2n)} \text{ etc.}$$
$$+ \frac{\text{sin.}\,(n+2)\Phi}{n+2} \cdot \frac{n^2}{2(2+2n)} \cdot \frac{(n+1)^2}{1(3+2n)} \cdot \frac{(n+3)^2}{1(5+2n)} \cdot \frac{(n+4)^2}{2(6+2n)} \cdot \frac{(n+5)^2}{3(7+2n)} \text{ etc.}$$
$$- \frac{\text{sin.}\,(n+3)\Phi}{n+3} \cdot \frac{n^2}{3(3+2n)} \cdot \frac{(n+1)^2}{2(4+2n)} \cdot \frac{(n+2)^2}{1(5+2n)} \cdot \frac{(n+4)^2}{1(7+2n)} \cdot \frac{(n+5)^2}{2(8+2n)} \text{ etc.}$$
$$+ \frac{\text{sin.}\,(n+4)\Phi}{n+4} \cdot \frac{n^2}{4(4+2n)} \cdot \frac{(n+1)^2}{3(5+2n)} \cdot \frac{(n+2)^2}{2(6+2n)} \cdot \frac{(n+3)^2}{1(7+2n)} \cdot \frac{(n+5)^2}{1(9+2n)} \text{ etc.}$$

Ad valores horum productorum in infinitum excurrentium inuestigandos ponamus breuitatis gratia

$$\Phi = \mathfrak{A} \cdot \frac{\text{sin.}\,n\Phi}{n} - \mathfrak{B}\frac{\text{sin.}\,(n+1)\Phi}{n+1} + \mathfrak{C}\frac{\text{sin.}\,(n+2)\Phi}{n+2} - \mathfrak{D}\frac{\text{sin.}\,(n+3)\Phi}{n+3} + \text{etc.}$$

et hi coefficientes sequenti modo inter se comparentur:

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{nn}{(n+1)^2} \cdot \frac{2(2+2n)}{1(3+2n)} \cdot \frac{3(3+2n)}{2(4+2n)} \cdot \frac{4(4+2n)}{3(5+2n)} \text{ etc.}$$

qui valor reducitur ad $\frac{nn}{(n+1)^2} \cdot \frac{(i-1)(2+2n)}{1(i+2n)}$, denotante i numerum infinitum, sicque erit $\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{2nn}{n+1}$. Simili modo colligitur:

$$\frac{\mathfrak{C}}{\mathfrak{B}} = \frac{1(1+2n)}{2(2+2n)} \cdot \frac{(n+1)^2}{(n+2)^2} \cdot \frac{(i-3)(4+2n)}{1(i+2n)} = \frac{(n+1)(2n+1)}{2(n+2)};$$

tum vero porro

$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(n+2)(2n+2)}{3(n+3)};\ \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(n+3)(2n+3)}{4(n+4)},$$

et sic deinceps; vnde sequitur fore

$$\mathfrak{B} = \frac{2nn}{(n+1)}\mathfrak{A}$$

$\mathfrak{C} =$

c. progressionem a-
d=(n+3)Φ, etc.
efinire.
casu dat
itum excurrentium
−𝔇 sin.(n+3)Φ/(n+3) + etc.
se comparentur:
denotante i nu-
. Simili modo col-
𝔈 =

$$\mathfrak{C} = \frac{2nn(2n+1)}{1 \cdot 2(n+2)}\mathfrak{A}$$
$$\mathfrak{D} = \frac{2nn(2n+1)(2n+2)}{1 \cdot 2 \cdot 3(n+3)}\mathfrak{A}$$
$$\mathfrak{E} = \frac{2nn(2n+1)(2n+2)(2n+3)}{1 \cdot 2 \cdot 3 \cdot 4(n+4)}\mathfrak{A}$$
etc.

sicque totum negotium redit ad inuentionem primae litterae:

$$\mathfrak{A} = \frac{(n+1)^2}{1(2n+1)} \cdot \frac{(n+2)^2}{2(2n+2)} \cdot \frac{(n+3)^2}{3(2n+3)} \cdot \frac{(n+4)^2}{4(2n+4)} \text{ etc.}$$

Iam dudum autem demonstraui, huius producti generalis:

$$\frac{a(b+c)}{b(a+c)} \cdot \frac{(a+d)(b+c+d)}{(b+d)(a+c+d)} \cdot \frac{(a+2d)(b+c+2d)}{(b+2d)(a+c+2d)} \text{ etc.}$$

valorem ita exprimi, vt sit

$$= \frac{\int x^{b-1}\,dx\,(1-x^d)^{\frac{c-d}{d}}}{\int x^{a-1}\,dx\,(1-x^d)^{\frac{c-d}{d}}}$$

vtraque scilicet integratione a termino $x = 0$ ad $x = 1$ extensa. Quare cum pro nostro casu sumi oporteat

$$a = n+1;\ b+c = n+1,\ b = 1,\ c = n, \text{ et } d = 1,$$

habebimus:

$$\mathfrak{A} = \frac{\int dx\,(1-x)^{n-1}}{\int x^n\,dx\,(1-x)^{n-1}} = \frac{1}{n\int x^n\,dx\,(1-x)^{n-1}}$$

hincque sequentem pro arcu Φ expressionem:

$$\Phi\int x^n\,dx\,(1-x)^{n-1} = \frac{1}{nn}\,\text{sin.}\,n\Phi - \frac{2n}{1(n+1)^2}\,\text{sin.}\,(n+1)\Phi$$
$$+ \frac{2n(2n+1)}{1 \cdot 2(n+2)^2}\,\text{sin.}\,(n+2)\Phi - \frac{2n(2n+1)(2n+2)}{1 \cdot 2 \cdot 3 \cdot (n+3)^2}\,\text{sin.}\,(n+3)\Phi$$
$$+ \frac{2n(2n+1)(2n+2)(2n+3)}{1 \cdot 2 \cdot 3 \cdot 4\,(n+4)^2}\,\text{sin.}\,(n+4)\Phi + \text{etc.}$$

Quae series eo maiorem attentionem meretur, quod formulam integralem $\int x^n\,dx\,(1-x)^{n-1}$ inuoluit.

 Corol-

Corollarium 1.

§. 18. De hac formula integrali $\int x^n dx(1-x)^{n-1}$ primum obseruasse iuuabit, si casu $n=\lambda$ eius valor fuerit $=\Delta$, tum eum casu $n=\lambda+1$ fore $=\frac{\lambda}{2(2\lambda+1)}\Delta$. Ita cum casu $n=1$ sit $\int x\,dx=\frac{1}{2}$, erit

$$\int x^2 dx(1-x)=\frac{1}{2}\cdot\frac{1}{2\cdot 3};\ \int x^3 dx(1-x)^2=\frac{1}{2}\cdot\frac{1}{2\cdot 3}\cdot\frac{2}{2\cdot 5}\ \text{etc.}$$

Corollarium 2.

§. 19. Si ergo in genere ponatur $\int x^n dx(1-x)^{n-1}=f:n$, quandoquidem eius valor vt functio ipsius n spectari potest: erit

$$f:1=\frac{1}{2};\ f:2=\frac{1}{2}\cdot\frac{1}{6};\ f:3=\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{2}{10};\ f:4=\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{2}{10}\cdot\frac{3}{14},$$

atque in genere $f:(n+1)=\frac{n}{2(2n+1)}f:n$. Vnde quoties n est numerus integer, valor istius formulae $f:n$ facile assignatur.

Corollarium 3.

§. 20. Sit nunc $n=\frac{1}{2}$, eritque

$$f:\tfrac{1}{2}=\int\frac{dx\sqrt{x}}{\sqrt{(1-x)}}=2\int\frac{yy\,dy}{\sqrt{(1-yy)}},$$

posito $x=yy$; at

$$\int\frac{yy\,dy}{\sqrt{(1-yy)}}=\frac{1}{2}\int\frac{dy}{\sqrt{(1-yy)}}=\frac{\pi}{4};$$

vnde fit $f:\frac{1}{2}=\frac{\pi}{2}$, hincque porro:

$$f:\tfrac{3}{2}=\tfrac{1}{8}\cdot\tfrac{\pi}{2};\ f:\tfrac{5}{2}=\tfrac{1}{8}\cdot\tfrac{3}{16}\cdot\tfrac{\pi}{2};\ f:\tfrac{7}{2}=\tfrac{1}{8}\cdot\tfrac{3}{16}\cdot\tfrac{5}{24}\cdot\tfrac{\pi}{2};\ \text{etc.}$$

At si in genere sit $n=\frac{\mu}{\nu}$, reperitur:

$$f:\tfrac{\mu}{\nu}=\int x^{\frac{\mu}{\nu}}dx(1-x)^{\frac{\mu}{\nu}-1}=\nu\int y^{\mu+\nu-1}dy(1-y^\nu)^{\frac{\mu}{\nu}-1}$$

posito $x=y^\nu$, hincque facta reductione

f:

li $\int x^n dx(1-x)^{n-1}$ λ eius valor fuerit $\frac{\lambda}{(2\lambda+1)}\Delta$. Ita cum
$x)^2=\frac{1}{2}\cdot\frac{1}{2\cdot 3}\cdot\frac{2}{2\cdot 5}$ etc.
ur $\int x^n dx(1-x)^{n-1}$ io ipsius n spectari
$f:4=\frac{1}{2}\cdot\frac{1}{6}\cdot\frac{2}{10}\cdot\frac{3}{14}$, :n. Vnde quoties ormulae $f:n$ facile
$=\frac{1}{8}\cdot\frac{3}{16}\cdot\frac{5}{24}\cdot\frac{\pi}{2}$; etc.
$dy(1-y^\nu)^{\frac{\mu}{\nu}-1}$
f:

$$f:\tfrac{\mu}{\nu}=\tfrac{\nu}{\mu}\int y^{\mu-1}dy(1-y^\nu)^{\frac{\mu}{\nu}-1},$$

quae forma omnis generis quantitates transcendentes inuoluit.

Corollarium 4.

§. 21. Huius ipsius autem formulae integralis $\int x^n dx(1-x)^{n-1}$ valor casu $x=1$ vicissim ex serie inuenta satis eleganter determinatur: facta enim differentiatione, solum arcum Φ vt variabilem spectando, prodit

$$\int x^n dx(1-x)^{n-1}=\frac{1}{n}\cos. n\Phi-\frac{2n}{1(n+1)}\cos.(n+1)\Phi$$
$$+\frac{2n(2n+1)}{1\cdot 2(n+2)}\cos.(n+2)\Phi$$
$$-\frac{2n(2n+1)(2n+2)}{1\cdot 2\cdot 3(n+3)}\cos.(n+3)\Phi+\text{etc.}$$

quae ergo series aequalis est huic ex ipsa euolutione solita ortae:

$$\int x^n dx(1-x)^{n-1}=\frac{1}{n+1}-\frac{(n-1)}{1(n+2)}+\frac{(n-1)(n-2)}{1\cdot 2(n+3)}-\frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3(n+4)}\ \text{etc.}$$

Scholion 1.

§. 22. Quoniam casum $n=1$ in praecedente exemplo euoluimus, consideremus hic potissimum casum $n=\frac{1}{2}$, quo vidimus esse $\int x^n dx(1-x)^{n-1}=\frac{\pi}{2}$; eritque propterea:

$$\tfrac{\pi\Phi}{2}=\tfrac{4}{1}\sin.\tfrac{1}{2}\Phi-\tfrac{4}{9}\sin.\tfrac{3}{2}\Phi+\tfrac{4}{25}\sin.\tfrac{5}{2}\Phi-\tfrac{4}{49}\sin.\tfrac{7}{2}\Phi+\text{etc.}$$

Ponamus $\Phi=2\omega$, prodibitque haec series concinnior:

$$\tfrac{\pi\omega}{4}=\tfrac{1}{1}\sin.\omega-\tfrac{1}{9}\sin 3\omega+\tfrac{1}{25}\sin.5\omega-\tfrac{1}{49}\sin.7\omega+\text{etc.}$$

quae primo, si arcus ω sumatur euanescens, dat

$$\tfrac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ \text{etc.}$$

Sit autem $\omega = \frac{\pi}{5}$, oriturque series etiam cognita

$$\frac{\pi\pi}{8} = \frac{1}{1} + \frac{1}{9} + \frac{1}{37} + \frac{1}{49} + \frac{1}{81} + \text{etc.}$$

At sumto arcu $\omega = 45^\circ = \frac{\pi}{4}$, prouenit

$$\frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{9} - \frac{1}{25} + \frac{1}{49} + \frac{1}{81} - \frac{1}{121} - \frac{1}{169} + \text{etc.}$$

Sit $\omega = 30^\circ = \frac{\pi}{6}$, erit

$$\frac{\pi\pi}{24} = \frac{1}{2}\left(1 + \frac{1}{7^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{25^2} + \text{etc.}\right)$$
$$- 1\left(\frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \text{etc.}\right)$$
$$+ \frac{1}{2}\left(\frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{17^2} + \frac{1}{23^2} + \text{etc.}\right)$$

vbi media est $= \frac{\pi\pi}{72}$, reliquarumque ratio perspicua. Deinde differentiatio nostrae seriei suppeditat hanc formam notatu dignam:

$$\frac{\pi}{4} = \frac{1}{1}\cos\omega - \frac{1}{3}\cos 3\omega + \frac{1}{5}\cos 5\omega - \frac{1}{7}\cos 7\omega + \text{etc.}$$

quoniam omnes plane arcus pro ω assumti eandem praebent summam. Tum vero iterata differentiatio praebet:

$$0 = \sin\omega - \sin 3\omega + \sin 5\omega - \sin 7\omega + \text{etc.}$$

Per integrationem autem elicimus:

$$C - \frac{\pi\omega^2}{8} = \frac{1}{1}\cos\omega - \frac{1}{3^3}\cos 3\omega + \frac{1}{5^3}\cos 5\omega - \frac{1}{7^3}\cos 7\omega + \text{etc.}$$

vbi cum sumto $\omega = 0$ sit $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.} = \frac{\pi^3}{32}$, erit $C = \frac{\pi^3}{32}$, ita vt sit

$$\frac{\pi}{8}\left(\frac{\pi\pi}{4} - \omega\omega\right) = \frac{1}{1}\cos\omega - \frac{1}{3^3}\cos 3\omega + \frac{1}{5^3}\cos 5\omega - \frac{1}{7^3}\cos 7\omega + \text{etc.}$$

Scholion 2.

§. 23. Ponamus nunc in genere $\Phi = \pi$, et cum sit $\sin.(n+1)\pi = -\sin.n\pi$; $\sin.(n+2)\pi = +\sin.n\pi$; etc. aequatio nostra per $\sin.n\pi$ diuisa induet hanc formam:

$$\frac{\pi}{\sin.n\pi}\int x^n dx(1-x)^{n-1} = \frac{1}{n^2} + \frac{2n}{1(n+1)^2} + \frac{2n(2n+1)}{1.2(n+2)^2} + \frac{2n(2n+1)(2n+2)}{1.2.3(n+3)^2} + \text{etc.}$$

sumto

cognita
$+ \text{etc.}$
etc.)
perspicua. Dein-
at hanc formam
$- \frac{1}{7}\cos 7\omega + \text{etc.}$
mti eandem prae-
entiatio praebet:
$7\omega + \text{etc.}$
$- \frac{1}{7^3}\cos 7\omega + \text{etc.}$
$\frac{1}{7^3} + \text{etc.} = \frac{\pi^3}{32}$, erit
$- \frac{1}{7^3}\cos 7\omega + \text{etc.}$
$\Phi = \pi$, et cum
$2)\pi = +\sin.n\pi$;
uet hanc formam:
sumto

sumto autem $\Phi = 2\pi$, erit simili modo:

$$\frac{2\pi}{\sin.2n\pi}\int x^n dx(1-x)^{n-1} = \frac{1}{n^2} - \frac{2n}{1(n+1)^2} + \frac{2n(2n+1)}{1.2(n+2)^2} - \frac{2n(2n+1)(2n+2)}{1.2.3(n+3)^2} + \text{etc.}$$

quorum serierum ergo illa per hanc diuisa quotum praebet $= \cos.n\pi$, quod incongruum videtur, cum quotus sit vnitate maior. At similem difficultatem iam supra resoluimus, quae ex positione $\Phi = 2\pi$ est nata: si enim poneremus $\Phi = 3\pi$, ipsa series prior emergeret, summam habitura $= \frac{3\pi}{\sin.3n\pi}\int x^n dx(1-x)^{n-1}$, quae illi non est aequalis, nisi n sit ratio euanescens. Quare prima tantum series locum habere est censenda, cuius summam vt ex ipsa eius natura inuestigemus, statuamus:

$$s = \frac{1}{n^2}t^n + \frac{2n}{(n+1)^2}t^{n+1} + \frac{2n(2n+1)}{1.2(n+2)^2}t^{n+2} + \text{etc.}$$

eritque hinc

$$\frac{d.t\,ds}{dt^2} = 1\,t^{n-1} + \frac{2n}{1}t^n + \frac{2n(2n+1)}{1.2}t^{n+1} + \text{etc.}$$

cuius seriei summa manifesto est $= t^{n-1}(1-t)^{-2n}$, ita vt sit

$$\frac{t\,ds}{dt} = \int t^{n-1}dt(1-t)^{-2n} \text{ et}$$

$$s = \int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t)^{2n}},$$

sicque posito post integrationem $t = 1$ habebitur:

$$\frac{\pi}{\sin.n\pi}\int x^n dx(1-x)^{n-1} = \int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t)^{2n}}.$$

Quae binarum formularum integralium comparatio eo magis est memorabilis, quod inter plurimas alias, quae adhuc sunt erutae, huius generis non reperiantur.

Scho-

Scholion 3.

§. 24. Ponamus in genere $\phi = \frac{\pi}{2}$, eritque

$\sin. n\phi = \sin. \frac{n\pi}{2}$; $\sin. (n+1)\phi = \cos. \frac{n\pi}{2}$;

$\sin. (n+2)\phi = -\sin. \frac{n\pi}{2}$; $\sin. (n+3)\phi = -\cos. \frac{n\pi}{2}$; etc.

vnde resultat haec aequatio:

$$\tfrac{\pi}{2}\int x^n dx(1-x)^{n-1} = \sin. \tfrac{n\pi}{2}\left(\tfrac{1}{2n} - \tfrac{2n(2n+1)}{1.2(n+2)^2} + \tfrac{2n(2n+1)(2n+2)(2n+3)}{1.2.3.4(n+4)^2} - \text{etc.}\right)$$
$$- \cos. \tfrac{n\pi}{2}\left(\tfrac{2n}{1(n+1)^2} - \tfrac{2n(2n+1)(2n+2)}{1.2.3(n+3)^2} + \text{etc.}\right)$$

At ex superiori reductione manifestum est fore:

$$1 - \tfrac{2n(2n+1)}{1.2}t^2 + \tfrac{2n(2n+1)(2n+2)(2n+3)}{1.2.3.4}t^4 - \text{etc.}$$
$$= \frac{(1+t\sqrt{-1})^{-2n} + (1-t\sqrt{-1})^{-2n}}{2},$$
$$\tfrac{2n}{1}t - \tfrac{2n(2n+1)(2n+2)}{1.2.3}t^3 + \text{etc.}$$
$$= \frac{(1+t\sqrt{-1})^{-2n} - (1-t\sqrt{-1})^{-2n}}{2\sqrt{-1}},$$

hincque colligitur:

$$\tfrac{\pi}{2}\int x^n dx(1-x)^{n-1} = \tfrac{1}{2}\sin. \tfrac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1+t\sqrt{-1})^{2n}}$$
$$+ \tfrac{1}{2}\sin. \tfrac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t\sqrt{-1})^{2n}},$$
$$- \tfrac{1}{2\sqrt{-1}}\cos. \tfrac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1+t\sqrt{-1})^{2n}}$$
$$+ \tfrac{1}{2\sqrt{-1}}\cos. \tfrac{n\pi}{2}\int\frac{dt}{t}\int\frac{t^{n-1}dt}{(1-t\sqrt{-1})^{2n}},$$

vbi quidem post integrationem poni oportet $t = 1$. Vt autem hanc expressionem ab imaginariis liberemus ponamus:

$t =$

eritque
$\frac{n\pi}{2}$;
$= -\cos. \frac{n\pi}{2}$; etc.
— etc.)
+ etc.)
re:
t^4 — etc.
$\sqrt{-1})^{-2n}$,
$\sqrt{-1})^{-2n}$,
tet $t = 1$. Vt
liberemus po-
$t =$

$t = \text{tang.}\,\omega = \frac{\sin. \omega}{\cos. \omega}$; erit $dt = \frac{d\omega}{\cos. \omega^2}$; $\frac{dt}{t} = \frac{d\omega}{\sin. \omega \cos. \omega}$;

$$t^{n-1}dt = \frac{d\omega \sin. \omega^{n-1}}{\cos. \omega^{n+1}},$$

tum vero

$$(1+t\sqrt{-1})^{-2n} = \cos. \omega^{2n}(\cos. \omega + \sqrt{-1}.\sin. \omega)^{-2n}$$
$$= \cos. \omega^{2n}(\cos. 2n\omega - \sqrt{-1}.\sin. 2n\omega)$$
$$(1-t\sqrt{-1})^{-2n} = \cos. \omega^{2n}(\cos. \omega - \sqrt{-1}.\sin. \omega)^{-2n}$$
$$= \cos. \omega^{2n}(\cos. 2n\omega + \sqrt{-1}.\sin. 2n\omega).$$

Quibus valoribus substitutis imaginaria se mutuo tollent, prodibitque haec aequatio:

$$\tfrac{\pi}{2}\int x^n dx(1-x)^{n-1} = \sin. \tfrac{n\pi}{2}\int\frac{d\omega}{\sin. \omega \cos. \omega}\int d\omega \sin. \omega^{n-1}\cos. \omega^{n-1}\cos. 2n\omega$$
$$+ \cos. \tfrac{n\pi}{2}\int\frac{d\omega}{\sin. \omega \cos. \omega}\int d\omega \sin. \omega^{n-1}\cos. \omega^{n-1}\sin. 2n\omega$$

quae in hanc simpliciorem contrahitur:

$$\tfrac{\pi}{2}\int x^n dx(1-x)^{n-1} = \int\frac{d\omega}{\sin. \omega \cos. \omega}\int d\omega \sin. \omega^{n-1}\cos. \omega^{n-1}\sin. (\tfrac{n\pi}{2} + 2n\omega)$$

vel, ob $\sin. \omega \cos. \omega = \frac{1}{2}\sin. 2\omega$, in hanc:

$$\tfrac{\pi}{2}\int x^n dx(1-x)^{n-1} = \frac{1}{2^n}\int\frac{2d\omega}{\sin. 2\omega}\int 2d\omega \sin. 2\omega^{n-1}\sin. (\tfrac{n\pi}{2} + 2n\omega).$$

Sit nunc angulus $2\omega = \theta$, vt fiat concinnius:

$$\tfrac{\pi}{2}\int x^n dx(1-x)^{n-1} = \frac{1}{2^n}\int\frac{d\theta}{\sin. \theta}\int d\theta \sin. \theta^{n-1}\sin. n(\tfrac{\pi}{2} + \theta)$$

vbi post integrationem statui oportet $\theta = 90^\circ = \frac{\pi}{2}$, vt tum fiat $\omega = 45^\circ$ et $t = \cos. \omega = 1$.

Exemplum 3.

§. 25. *Si arcus a, b, c, d, etc. constituant progressionem arithmeticam interruptam, vt sit:*

 $a =$

$a = m\Phi$; $b = n\Phi$; $c = (1+m)\Phi$;
$d = (1+n)\Phi$; $e = (2+m)\Phi$; $f = (2+n)\Phi$; etc.
ex eorum finibus longitudinem arcus Φ definire.

Solutio generalis supra data (7) suppeditat hanc aequationem:

$$\Phi = \frac{\sin. m\Phi}{m} \cdot \frac{nn}{(n-m)(n+m)} \cdot \frac{(1+m)^2}{1(1+2m)} \cdot \frac{(1+n)^2}{(1+n-m)(1+n+m)} \cdot \frac{(2+m)^2}{2(2+2m)} \cdot \frac{(2+n)^2}{(2+n-m)(2+n+m)} \text{ etc.}$$
$$- \frac{\sin. n\Phi}{n} \cdot \frac{mm}{(m-n)(m+n)} \cdot \frac{(1+m)^2}{(1+m-n)(1+m+n)} \cdot \frac{(1+n)^2}{1(1+2n)} \cdot \frac{(2+m)^2}{(2+m-n)(2+m+n)} \cdot \frac{(2+n)^2}{2(2+2n)} \text{ etc.}$$
$$+ \frac{\sin.(1+m)\Phi}{1+m} \cdot \frac{mm}{1(1+2m)} \cdot \frac{nn}{(1+m-n)(1+m+n)} \cdot \frac{(1+n)^2}{(n-m)(2+m+n)} \cdot \frac{(2+m)^2}{2(3+2m)} \cdot \frac{(2+n)^2}{(1+n-m)(3+n+m)} \text{ etc.}$$
$$- \frac{\sin.(1+n)\Phi}{1+n} \cdot \frac{mm}{(1+n-m)(1+n+m)} \cdot \frac{nn}{1(1+2n)} \cdot \frac{(1+m)^2}{(m-n)(2+n+m)} \cdot \frac{(2+m)^2}{(1+m-n)(3+m+n)} \cdot \frac{(2+n)^2}{2(3+2n)} \text{ etc.}$$
$$+ \frac{\sin.(2+m)\Phi}{2+m} \cdot \frac{mm}{2(2+2m)} \cdot \frac{nn}{(2+m-n)(2+m+n)} \cdot \frac{(1+m)^2}{1(3+2m)} \cdot \frac{(1+n)^2}{(1+m-n)(3+m+n)} \cdot \frac{(2+n)^2}{(n-m)(4+m+n)} \text{ etc.}$$

etc.

Hinc autem in genere nihil attentione dignum concludere licet, vnde casum praecipue memorabilem euoluam, quo est $n = 1 - m$, pro quo statuo breuitatis gratia:

$$\Phi = \frac{\mathfrak{A} \sin. m\Phi}{m} - \frac{\mathfrak{B} \sin.(1-m)\Phi}{1-m} + \frac{\mathfrak{C} \sin.(1+m)\Phi}{1+m} - \frac{\mathfrak{D} \sin.(2-m)\Phi}{2-m} + \text{etc.}$$

ita vt sit:

$$\mathfrak{A} = \frac{(1-m)^2}{1(1-2m)} \cdot \frac{(1+m)^2}{1(1+2m)} \cdot \frac{(2-m)^2}{2(2-2m)} \cdot \frac{(2+m)^2}{2(2+2m)} \cdot \frac{(3-m)^2}{3(3-2m)} \text{ etc.}$$
$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{mm}{(1-m)^2} \cdot \frac{1(1+2m)}{2 \cdot 2m} \cdot \frac{2(2-2m)}{1(3-2m)} \cdot \frac{2(2+2m)}{3(1+2m)} \cdot \frac{3(3-2m)}{2(4-2m)} \text{ etc.}$$
$$\frac{\mathfrak{C}}{\mathfrak{B}} = \frac{1(1-2m)}{1(1+2m)} \cdot \frac{(1-m)^2}{(1+m)^2} \cdot \frac{2(2-2m)}{4(1-2m)} \cdot \frac{2(1+2m)}{3(3+2m)} \cdot \frac{3(4-2m)}{4(2-2m)} \text{ etc.}$$
$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{1(1+2m)}{2(2-2m)} \cdot \frac{2 \cdot 2m}{1(3-2m)} \cdot \frac{(1+m)^2}{(2-m)^2} \cdot \frac{2(3+2m)}{4 \cdot 2m} \cdot \frac{4(2-2m)}{3(5-2m)} \text{ etc.}$$
$$\frac{\mathfrak{E}}{\mathfrak{D}} = \frac{2(2-2m)}{3(2+2m)} \cdot \frac{1(3-2m)}{2(1+2m)} \cdot \frac{3(1-2m)}{1(5+2m)} \cdot \frac{(2-m)^2}{(2+m)^2} \cdot \frac{1(5-2m)}{5(1-2m)} \text{ etc.}$$

etc.

$2+n)\Phi$; etc.
ire.
at hanc aequa-
etc.
etc.
etc.
etc.
etc.
num concludere
euoluam, quo
gratia:
$\frac{\mathfrak{D}\sin.(2-m)\Phi}{2-m}$ + etc.
etc.
etc.
etc.
etc.
etc.
At-

Atque ex superiori reductione reperitur

$$\mathfrak{A} = \frac{\int x^{m-1} dx (1-x)^{-2m}}{m \int x^m dx (1-x)^{m-1} \cdot \int x^{m-1} dx (1-x)^{-m}};$$

tum vero pro reliquis colligitur ex ipsa forma productorum:

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{m}{1-m}; \quad \frac{\mathfrak{C}}{\mathfrak{B}} = \frac{1-m}{1+m}; \quad \frac{\mathfrak{D}}{\mathfrak{C}} = \frac{1+m}{2-m}; \quad \frac{\mathfrak{E}}{\mathfrak{D}} = \frac{2-m}{2+m}; \text{ etc.}$$

ita vt sit

$$\mathfrak{B} = \frac{m}{1-m}\mathfrak{A}; \quad \mathfrak{C} = \frac{m}{1+m}\mathfrak{A}; \quad \mathfrak{D} = \frac{m}{2-m}\mathfrak{A}; \quad \mathfrak{E} = \frac{m}{2+m}\mathfrak{A}; \text{ etc.}$$

Ponamus ergo breuitatis gratia

$$\int x^m dx (1-x)^{m-1} \cdot \frac{\int x^{m-1} dx (1-x)^{-m}}{\int x^{m-1} dx (1-x)^{-2m}} = \mathfrak{M}$$

eritque vt sequitur:

$$M\Phi = \frac{\sin. m\Phi}{m^2} - \frac{\sin.(1-m)\Phi}{(1-m)^2} + \frac{\sin.(1+m)\Phi}{(1+m)^2} - \frac{\sin.(2-m)\Phi}{(2-m)^2} + \frac{\sin.(2+m)\Phi}{(2+m)^2} - \text{etc.}$$

vnde differentiando concludimus fore

$$M = \frac{\cos. m\Phi}{m} - \frac{\cos.(1-m)\Phi}{1-m} + \frac{\cos.(1+m)\Phi}{1+m} - \frac{\cos.(2-m)\Phi}{2-m} + \frac{\cos.(2+m)\Phi}{2+m} - \text{etc.}$$

quae series ob insignem simplicitatem imprimis est notatu digna: quandoquidem inde, ponendo $\Phi = 0$, deducimus

$$M = \frac{1}{m} - \frac{1}{1-m} + \frac{1}{1+m} - \frac{1}{2-m} + \frac{1}{2+m} - \frac{1}{3-m} + \frac{1}{3+m} - \text{etc.}$$

cuius seriei summam iam olim ostendi esse

$$M = \frac{\pi \cos. m\pi}{\sin. m\pi},$$

vnde colligimus hanc elegantem comparationem:

$$\int x^m dx (1-x)^{m-1} = \frac{\pi \cos. m\pi}{\sin. m\pi} \cdot \frac{\int x^{m-1} dx (1-x)^{-2m}}{\int x^{m-1} dx (1-x)^{-m}}$$

quae redigitur porro ad hanc:

$$\int x^m dx(1-x)^{m-1} = \frac{(1-m)\pi \cos. m\pi}{\sin. m\pi} \cdot \frac{\int x^m dx(1-x)^{-2m}}{\int x^m dx(1-x)^{-m}},$$

vel ad hanc adhuc concinniorem:

$$\int x^{m-1} dx(1-x)^{m-1} = \frac{2\pi \cos. m\pi}{\sin. m\pi} \cdot \frac{\int x^{m-1} dx(1-x)^{-2m}}{\int x^{m-1} dx(1-x)^{-m}}.$$

Corollarium 1.

§. 26. En ergo aliquot insignia Theoremata, quae huius exempli euolutio nobis suppeditat, quorum primum est: Si Φ denotet angulum quemcunque, erit

$$\frac{\pi \cos. m\pi}{\sin. m\pi} = \frac{\cos. m\Phi}{m} - \frac{\cos.(1-m)\Phi}{1-m} + \frac{\cos.(1+m)\Phi}{1+m} - \frac{\cos.(2-m)\Phi}{2-m} + \text{etc.}$$

quae aequalitas etiam ita exhiberi potest, vt sit:

$$\frac{\pi \cos. m\pi}{\sin. m\pi} = \cos. m\Phi\left(\frac{1}{m} - \frac{2m \cos. \Phi}{1-mm} - \frac{2m \cos. 2\Phi}{4-mm} - \frac{2m \cos. 3\Phi}{9-mm} - \text{etc.}\right)$$
$$- 2\sin. m\Phi\left(\frac{\sin. \Phi}{1-mm} + \frac{2\sin. 2\Phi}{4-mm} + \frac{3\sin. 3\Phi}{9-mm} + \frac{4\sin. 4\Phi}{16-mm} - \text{etc.}\right)$$

vnde si $m\Phi = 90° = \frac{\pi}{2}$, ideoque $\Phi = \frac{\pi}{2m}$, erit

$$\frac{\pi \cos. m\pi}{\sin. m\pi} = \frac{\sin. \frac{\pi}{2m}}{1-mm} + \frac{2\sin. \frac{2\pi}{2m}}{4-mm} + \frac{3\sin. \frac{3\pi}{2m}}{9-mm} + \frac{4\sin. \frac{4\pi}{2m}}{16-mm} + \text{etc.}$$

Corollarium 2.

§. 27. Secundum Theorema ita enunciatur: Si Φ denotet angulum quemcunque, erit:

$$\frac{\pi\Phi \cos. m\pi}{\sin. m\pi} = \frac{\sin. m\Phi}{mm} - \frac{\sin.(1-m)\Phi}{(1-m)^2} + \frac{\sin.(1+m)\Phi}{(1+m)^2} - \frac{\sin.(2-m)\Phi}{(2-m)^2} + \text{etc.}$$

Quare sumto $\Phi = \pi$ erit

$$\frac{\pi\pi \cos. m\pi}{\sin. m\pi} = \frac{\sin. m\pi}{mm} - \frac{\sin. m\pi}{(1-m)^2} - \frac{\sin. m\pi}{(1+m)^2} + \frac{\sin. m\pi}{(2-m)^2} + \frac{\sin. m\pi}{(2+m)^2} - \text{etc.}$$

siue

$^m dx(1-x)^{-2m}$ / $dx(1-x)^{-m}$,
$x(1-x)^{-2m}$ / $x(1-x)^{-m}$.
Theoremata, ditat, quorum anque, erit $\frac{\cos.(2-m)\Phi}{2-m}$ + etc.
t sit: $\frac{2m\cos.3\Phi}{9-mm}$ − etc.) $\frac{\Phi}{m}$ + $\frac{4\sin.4\Phi}{16-mm}$ − etc.
erit $\frac{4\sin.\frac{4\pi}{2m}}{16-mm}$ + etc.
unciatur: Si Φ $\frac{\sin.(2-m)\Phi}{(2-m)^2}$ + etc.
$\frac{\sin. m\pi}{(2+m)^2}$ − etc.
siue

siue

$$\frac{\pi\pi}{\sin. \pi \tan g. m\pi} = \frac{1}{m^2} - \frac{1}{(1-m)^2} - \frac{1}{(1+m)^2} + \frac{1}{(2-m)^2} + \frac{1}{(2+m)^2} - \text{etc.}$$

At posito $m\Phi = \pi$ habebitur

$$\frac{\pi\pi \cos. m\pi}{mm \sin. m\pi} = \frac{\sin. \frac{\pi}{m}}{(1-m)^2} - \frac{\sin. \frac{\pi}{m}}{(1+m)^2} + \frac{\sin. \frac{2\pi}{m}}{(2-m)^2} - \frac{\sin. \frac{2\pi}{m}}{(2+m)^2} + \text{etc.}$$

siue hoc modo:

$$\frac{\pi\pi \cos. m\pi}{4mm \sin. m\pi} = \frac{1\sin. \frac{\pi}{m}}{(1-mm)^2} + \frac{2\sin. \frac{2\pi}{m}}{(4-mm)^2} + \frac{3\sin. \frac{3\pi}{m}}{(9-mm)^2} + \text{etc.}$$

Corollarium 3.

§. 28. Tertium Theorema spectat ad comparationem formularum integralium, et ita enunciatur: Si sequentium formularum integratio a termino $x = 0$ vsque ad terminum $x = 1$ extendatur, erit semper:

$$\int x^{m-1} dx(1-x)^{m-1} \cdot \int x^{m-1} dx(1-x)^{-m}$$
$$= \frac{2\pi \cos. m\pi}{\sin. m\pi} \int x^{m-1} dx(1-x)^{-2m}.$$

Seu si ponatur $m = \frac{\lambda}{n}$ et $x = y^n$, erit

$$\int \frac{y^{\lambda-1} dy}{\sqrt[n]{(1-y^n)^{n-\lambda}}} \cdot \int \frac{y^{\lambda-1} dy}{\sqrt[n]{(1-y^n)^{\lambda}}} = \frac{2\pi \cos. \frac{\lambda\pi}{n}}{n \sin. \frac{\lambda\pi}{n}} \int \frac{y^{\lambda-1} dy}{\sqrt[n]{(1-y^n)^{2\lambda}}}.$$

Scholion.

§. 29. Demonstratio huius postremi Theorematis non parum ardua videtur: interim tamen per ea, quae olim de huiusmodi formulis integralibus sum commentatus, eius veritas sequenti modo ostendi potest. Indicemus enim, vti ibi feci, hanc formulam integralem:

$$\int \frac{y^{p-1}\,dy}{\sqrt[n]{(1-y^n)^{n-q}}}$$

hoc charactere: $\left(\frac{p}{q}\right)$, ac demonstrandum est esse:

$$\left(\frac{\lambda}{\lambda}\right)\left(\frac{\lambda}{n-\lambda}\right)=\frac{2\,\pi\,\mathrm{cos.}\frac{\lambda\pi}{n}}{n\,\mathrm{sin.}\frac{\lambda\pi}{n}}\left(\frac{\lambda}{n-2\lambda}\right).$$

Iam primum demonstravi, si fuerit $q+r=n$, fore

$$\left(\frac{q}{r}\right)=\frac{\pi}{n\,\mathrm{sin.}\frac{q\pi}{n}},$$

vnde statim sequitur

$$\left(\frac{\lambda}{n-\lambda}\right)=\int \frac{y^{\lambda-1}\,dy}{\sqrt[n]{(1-y^n)^{\lambda}}}=\frac{\pi}{n\,\mathrm{sin.}\frac{\lambda\pi}{n}},$$

ita vt demonstrandum restet esse:

$$\left(\frac{\lambda}{\lambda}\right)=2\,\mathrm{cos.}\frac{\lambda\pi}{n}\left(\frac{\lambda}{n-2\lambda}\right).$$

Verum ibidem ostendi, si fuerit $p+q+r=n$, fore

$$\frac{1}{\mathrm{sin.}\frac{r\pi}{n}}\left(\frac{p}{q}\right)=\frac{1}{\mathrm{sin.}\frac{q\pi}{n}}\left(\frac{p}{r}\right)=\frac{1}{\mathrm{sin.}\frac{p\pi}{n}}\left(\frac{q}{r}\right).$$

Sumamus igitur $p=\lambda$, $q=\lambda$, eritque $r=n-2\lambda$, quo, ob $\mathrm{sin.}\frac{(n-2\lambda)\pi}{n}=\mathrm{sin.}\frac{2\lambda\pi}{n}$, colligimus,

$$\frac{1}{\mathrm{sin.}\frac{2\lambda\pi}{n}}\left(\frac{\lambda}{\lambda}\right)=\frac{1}{\mathrm{sin.}\frac{\lambda\pi}{n}}\left(\frac{\lambda}{n-2\lambda}\right),$$

ita vt, ob

$$\mathrm{sin.}\frac{2\lambda\pi}{n}=2\,\mathrm{sin.}\frac{\lambda\pi}{n}\,\mathrm{cos.}\frac{\lambda\pi}{n},\ \text{fit revera}$$

$$\left(\frac{\lambda}{\lambda}\right)=2\,\mathrm{cos.}\frac{\lambda\pi}{n}\left(\frac{\lambda}{n-2\lambda}\right).$$

Multo magis autem abstrusum est Theorema supra (23) erutum,

esse:
).
= n, fore
= n, fore
$\left(\frac{q}{r}\right)$.
= n − 2 λ, quo,
ema supra (23) erutum,

erutum, quod sub iisdem integrationum terminis fit:

$$\frac{\pi}{\mathrm{sin.}\,n\,\pi}\int x^n\,dx\,(1-x)^{n-1}=\int\frac{dx}{x}\int\frac{x^{n-1}\,dx}{(1-x)^{2n}}$$

seu

$$\frac{\pi}{2\,\mathrm{sin.}\,n\pi}\int x^{n-1}\,dx\,(1-x)^{n-1}=\int\frac{dx}{x}\int\frac{x^{n-1}\,dx}{(1-x)^{2n}},$$

quae aequatio vt ad illam formam reducatur, loco n scribamus $\frac{\lambda}{n}$, sitque $x=y^n$; vnde fit

$$\frac{\pi}{2\,n\,\mathrm{sin.}\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}\,dy}{\sqrt[n]{(1-y^n)^{n-\lambda}}}=\int\frac{dy}{y}\int\frac{y^{\lambda-1}\,dy}{\sqrt[n]{(1-y^n)^{2\lambda}}}.$$

Modo autem vidimus esse

$$\int\frac{y^{\lambda-1}\,dy}{\sqrt[n]{(1-y^n)^{n-\lambda}}}=2\,\mathrm{cos.}\frac{\lambda\pi}{n}\int\frac{y^{\lambda-1}\,dy}{\sqrt[n]{(1-y^n)^{2\lambda}}},$$

sicque vi huius Theorematis colligimus esse:

$$\frac{\pi}{n\,\mathrm{tang.}\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}\,dy}{\sqrt[n]{(1-y^n)^{2\lambda}}}=\int\frac{dy}{y}\int\frac{y^{\lambda-1}\,dy}{\sqrt[n]{(1-y^n)^{2\lambda}}},$$

hincque porro istud Theorema non minus notatu dignum:

$$\frac{\pi}{n\,\mathrm{tang.}\frac{\lambda\pi}{n}}\int\frac{y^{\lambda-1}\,dy}{\sqrt[n]{(1-y^n)^{2\lambda}}}=-\int\frac{y^{\lambda-1}\,dy\,.\,ly}{\sqrt[n]{(1-y^n)^{2\lambda}}},$$

vnde sumto $\lambda=1$ sequentem proportionem elicimus:

$$\frac{\pi}{n}:\mathrm{tang.}\frac{\pi}{n}=\int\frac{dy\,l\frac{1}{y}}{\sqrt[n]{(1-y^n)^2}}:\int\frac{dy}{\sqrt[n]{(1-y^n)^2}},$$

Proble-

Problema 3.

§. 30. Inuenire eiusmodi aequationem pro linea curua inter binas variabiles, abſciſſam x et applicatam y, vt abſciſſis, in arithmetica progreſſione ſumtis, datae conueniant applicatae, ſcilicet:

Si ſit $x = n\theta,\ (n+1)\theta,\ (n+2)\theta,\ (n+3)\theta,\ (n+4)\theta,$ etc.
vt fiat $y = p,\quad q\quad,\quad r\quad,\quad s\quad,\quad t\quad,$ etc.

Solutio.

Ponamus in genere $x = \theta\omega$, atque ex ſolutione generali §. 10. data conſequimur hanc aequationem:

$$\frac{y}{\omega} = \frac{p}{n}\cdot\frac{(n+1-\omega)(n+1+\omega)}{1(2n+1)}\cdot\frac{(n+2-\omega)(n+2+\omega)}{2(2n+2)}\cdot\frac{(n+3-\omega)(n+3+\omega)}{3(2n+3)}\text{ etc.}$$
$$-\frac{q}{n+1}\cdot\frac{(n-\omega)(n+\omega)}{1(2n+1)}\cdot\frac{(n+2-\omega)(n+2+\omega)}{1(2n+3)}\cdot\frac{(n+3-\omega)(n+3+\omega)}{2(2n+4)}\text{ etc.}$$
$$+\frac{r}{n+2}\cdot\frac{(n-\omega)(n+\omega)}{2(2n+2)}\cdot\frac{(n+1-\omega)(n+1+\omega)}{1(2n+3)}\cdot\frac{(n+3-\omega)(n+3+\omega)}{1(2n+5)}\text{ etc.}$$
etc.

quam aequationem breuitatis gratia ita repraeſentemus:

$$\frac{y}{\omega} = \mathfrak{A}.\frac{p}{n} - \mathfrak{B}.\frac{q}{n+1} + \mathfrak{C}.\frac{r}{n+2} - \mathfrak{D}.\frac{s}{n+3} + \text{etc.}$$

ac primo quidem pro valore ipſius $\mathfrak{A}$ eliciendo ex forma generali §. 17 allata pro hoc caſu habebimus

$$a = n + 1 - \omega;\ b = 1,\ c = n - \omega \text{ et } d = 1,$$

vnde per formulas integrales, a termino $z = 0$ ad $z = 1$, extendendas, colligimus

$$\mathfrak{A} = \frac{\int dz\,(1-z)^{n-\omega-1}}{\int z^{n-\omega}\,dz\,(1-z)^{n-\omega-1}} = \frac{1}{(n-\omega)\int z^{n-\omega}\,dz\,(1-z)^{n-\omega-1}},$$

ſeu

$$\mathfrak{A} = \frac{2}{(n-\omega)\int z^{n-\omega-1}\,dz\,(1-z)^{n-\omega-1}};$$

qua

ionem pro linea et applicatam y, ntis, datae conue- 3)θ, (n+4)θ, etc. , t. , etc. ſolutione generali m: etc. etc. etc. epraeſentemus: D. s/(n+3) + etc. iciendo ex forma bimus ω et d = 1, z = 0 ad z = 1, qua

qua integratione conceſſa reliqua facile expediuntur. Erit enim vt ſupra §. 17

$$\frac{\mathfrak{B}}{\mathfrak{A}} = \frac{(n-\omega)(n+\omega)}{(n+1-\omega)(n+1+\omega)}\cdot(2+2n) = \frac{2(n+1)(n-\omega)(n+\omega)}{(n+1-\omega)(n+1+\omega)}$$
$$\frac{\mathfrak{C}}{\mathfrak{B}} = \frac{(n+1-\omega)(n+1+\omega)}{(n+2-\omega)(n+2+\omega)}\cdot\frac{(1+2n)(2+n)}{2(n+1)},$$
$$\frac{\mathfrak{D}}{\mathfrak{C}} = \frac{(n+2-\omega)(n+2+\omega)}{(n+3-\omega)(n+3+\omega)}\cdot\frac{(2+2n)(3+n)}{3(n+2)},$$
$$\frac{\mathfrak{E}}{\mathfrak{D}} = \frac{(n+3-\omega)(n+3+\omega)}{(n+4-\omega)(n+4+\omega)}\cdot\frac{(3+2n)(4+n)}{4(n+3)},$$
etc.

Statuamus igitur formulam integralem:

$$\int z^{n-\omega-1}\,dz\,(1-z)^{n-\omega-1} = \Delta,$$

vt ſit $\mathfrak{A} = \frac{2}{(n-\omega)\Delta}$, et reliqui coefficientes ita per $\mathfrak{A}$ definientur:

$$\mathfrak{B} = \frac{2(n+1)}{1}\,\frac{nn-\omega\omega}{(n+1)^2-\omega\omega}\,\mathfrak{A}$$
$$\mathfrak{C} = \frac{2(n+2)(2n+1)}{1.\ 2}\cdot\frac{nn-\omega\omega}{(n+2)^2-\omega\omega}\,\mathfrak{A}$$
$$\mathfrak{D} = \frac{2(n+3)(2n+1)(2n+2)}{1.\ 2.\ 3}\cdot\frac{nn-\omega\omega}{(n+3)^2-\omega\omega}\,\mathfrak{A}$$
$$\mathfrak{E} = \frac{2(n+4)(2n+1)(2n+2)(2n+3)}{1.\ 2.\ 3.\ 4}\cdot\frac{nn-\omega\omega}{(n+4)^2-\omega\omega}\,\mathfrak{A}$$
etc.

Quamobrem aequatio quaeſita inter y et $x = \theta\omega$ ita erit comparata:

$$\frac{n\Delta y}{2(n+\omega)\omega} = \frac{p}{nn-\omega\omega} - \frac{2n}{1}\cdot\frac{q}{(n+1)^2-\omega\omega} + \frac{2n(2n+1)}{1.\ 2}\cdot\frac{r}{(n+2)^2-\omega\omega}$$
$$-\frac{2n(2n+1)(2n+2)}{1.\ 2.\ 3}\cdot\frac{s}{(n+3)^2-\omega\omega} + \text{etc.}$$

vnde pro quouis valore ipſius $x = \theta\omega$ conueniens ipſius y valor definitur; idque per applicatas p, q, r, etc. quae abſciſſis $n\theta$, $(n+1)\theta$, $(n+2)\theta$, etc. conuenientes aſſumuntur. Vbi quidem notari oportet, ſi ω capiatur aequalis cuipiam termino progreſſionis n, $n+1$, $n+2$, etc. tum deno-

denominatorem applicatae respondentis datae euanescere, ita vt prae eo termino, quippe infinito, reliqui euanescant. Verum tum simul quoque valor Δ prodit infinitus, et praecise eiusmodi, vt tum fiat vel $y = p$, vel $y = q$, vel $y = r$, etc. quemadmodum rei natura postulat.

Corollarium 1.

§. 31. Si abscissae propositae denotent arcus circulares, applicatae vero eorundem sinus, vt sit

$$p = \sin. n\theta,\; q = \sin.(n+1)\theta,\; r = \sin.(n+2)\theta; \text{ etc.}$$

erit $y = \sin. \omega\theta$, vnde ista resultat aequatio generalis:

$$\frac{n\Delta \sin.\omega\theta}{2(n+\omega)\omega} = \frac{\sin. n\theta}{nn-\omega\omega} - \frac{2n}{1}\cdot\frac{\sin.(n+1)\theta}{(n+1)^2-\omega^2} + \frac{2n(2n+1)}{1.\,2}\cdot\frac{\sin.(n+2)\theta}{(n+2)^2-\omega^2}$$
$$- \frac{2n(2n+1)(2n+2)}{1.\;\;2.\;\;3.}\cdot\frac{\sin.(n+3)\theta}{(n+3)^2-\omega^2} \text{ etc.}$$

vbi imprimis notatu est dignum, quod tres litterae, n, θ et ω, pro lubitu accipi queant.

Corollarium 2.

§. 32. Si ergo sumamus $\theta = \pi$, vt omnes seriei sinus ad eundem $\sin. n\theta$ reducantur, erit

$$\frac{n\Delta \sin.\omega\pi}{2(n+\omega)\omega \sin. n\pi} = \frac{1}{nn-\omega\omega} + \frac{2n}{1}\cdot\frac{1}{(n+1)^2-\omega^2} + \frac{2n(2n+1)}{1.\;\;2}\cdot\frac{1}{(n+2)^2-\omega^2}$$
$$+ \frac{2n(2n+1)(2n+2)}{1.\;\;2.\;\;3.}\cdot\frac{1}{(n+3)^2-\omega^2} \text{ etc.}$$

Hinc si fit

$$n = \tfrac{1}{2} \text{ et } \Delta = \int z^{-\omega-\frac{1}{2}} dz\,(1-z)^{-\omega-\frac{1}{2}}, \text{ seu}$$

$$\Delta = 2\int \frac{z^{\frac{1}{2}-\omega}\, dz}{(1-z)^{\frac{1}{2}+\omega}}, \text{ habebitur:}$$

$$\frac{\Delta \sin.\omega\pi}{2(1+2\omega)\omega} = \frac{1}{1-4\omega^2} + \frac{1}{9-4\omega^2} + \frac{1}{25-4\omega^2} + \frac{1}{49-4\omega^2} + \text{etc.}$$

cuius seriei summam ostendi esse $= \frac{\pi}{8\omega}$ tang. $\omega\pi$, ita vt sit

$$\frac{\Delta \sin.\omega\pi}{2(1+2\omega)\omega} = \frac{\pi}{8\omega} \text{ tang. } \omega\pi, \text{ ideoque } \Delta = \frac{(1+2\omega)\pi}{4\cos.\omega\pi}.$$

Scholion 1.

§. 33. His autem conclusionibus nimium confidere non licet, ob rationem iam supra allegatam: Positis enim applicatis

$$p = \sin. n\theta,\; q = \sin.(n+1)\theta,\; r = \sin.(n+2)\theta, \text{ etc.}$$

dum arcus $n\theta$, $(n+1)\theta$ $(n+2)\theta$ etc. vt abscissae spectantur, aequatio inuenta eiusmodi lineam curuam praebet, quae per omnia haec puncta transit; neque vero hinc sequitur, hanc curuam ipsam esse lineam sinuum, cum infinitae aliae dentur lineae curuae per eadem illa puncta infinita data transeuntes. Quare seruata littera y pro applicata, abscissae $x = \theta\omega$ respondente, significanda, solutio nostra pro curua quaesita hanc quidem suppeditat aequationem:

$$\frac{n\Delta y}{2(n+\omega)\omega} = \frac{\sin. n\theta}{n^2-\omega^2} - \frac{2n}{1}\cdot\frac{\sin.(n+1)\theta}{(n+1)^2-\omega^2} + \frac{2n(2n+1)}{1.\;\;2}\cdot\frac{\sin.(n+2)\theta}{(n+2)^2-\omega^2}$$
$$- \frac{2n(2n+1)(2n+2)}{1.\;\;2.\;\;3}\cdot\frac{\sin.(n+3)\theta}{(n+3)^2-\omega^2} + \text{etc.}$$

ita vt abscissae $x = (n \pm i)\theta$ respondeat haec applicata: $y = \sin.(n \pm i)\theta$, si modo i sit numerus integer quicunque. Fieri vero posset, vt pro aliis abscissis, vbi i non est numerus integer, ideoque generatim si $x = \omega\theta$, non foret applicata $y = \sin.\omega\theta$. Quo hoc clarius perspiciatur, inuestigemus aequationem generalem pro omnibus plane lineis per puncta data transeuntibus, sitque valor hactenus repertus $y = \Theta$, et quaeratur functio pro omnibus abscissis datis euanescens, cuiusmodi est

Aa 2 $\omega(nn$

$$\omega(nn-\omega\omega)\frac{((n+1)^2-\omega^2)}{1(2n+1)}\cdot\frac{((n+2)^2-\omega^2)}{2(2n+2)}\cdot\frac{((n+3)^2-\omega^2)}{3(2n+3)} \text{ etc.}$$

quae per superiora est $= \omega(nn-\omega\omega)\mathfrak{A} = \frac{2\omega(n+\omega)}{\Delta}$. Vocetur haec quantitas $= \Omega$, sitque $f:\Omega$ eiusmodi functio ipsius Ω, quae euanescat si $\Omega = 0$, eritque aequatio generalis pro omnibus curuis satisfacientibus:

$$y = \Theta + f:\Omega = \Theta + f:\frac{\omega(n+\omega)}{\Delta}.$$

Ac iam sine vllo dubio certum est, in hac aequatione contineri aequationem $y = \sin.\omega\theta$, posito $x = \omega\theta$, quandoquidem haec aequatio conditionibus praescriptis satisfacit. Ex quo euenire prorsus posset, vt aequatio $y = \Theta$ ab ista: $y = \sin.\omega\theta$, esset diuersa; quod imprimis pendere potest a valoribus, litteris θ et n tributis, ita vt aliis casibus aequatio inuenta $y = \Theta$ cum hac: $y = \sin.\omega\theta$, conueniat, aliis vero ab eadem discrepet.

Scholion 2.

§. 34. Accommodemus haec ad casum, quo $\theta = \pi$ et $n = \frac{1}{2}$ atque $\Delta = 2\int \frac{z^{\frac{1}{2}-\omega}\,dz}{(1-z)^{\frac{1}{2}+\omega}}$; et quoniam seriei inuentae summa est $= \frac{\pi}{4\omega}$ tang. $\omega\pi$, habebitur haec aequatio generalis:

$$\frac{\Delta y}{4(1+2\omega)\omega} = \frac{\pi}{4\omega}\text{ tang. }\omega\pi + \frac{\Delta}{4(1+2\omega)\omega} f:\frac{\omega(1+2\omega)}{2\Delta},\text{ seu}$$

$$y = \frac{\pi(1+2\omega)}{\Delta}\text{ tang. }\omega\pi + f\frac{\omega(1+2\omega)}{2\Delta},$$

vbi functio adiecta in genere ita est comparata, vt casibus

$$\omega = 0,\ \omega = \pm\tfrac{1}{2},\ \omega = \pm\tfrac{3}{2},\ \omega = \pm\tfrac{5}{2},\text{ etc.}$$

euanescat, cuiusmodi formulae sunt: $\sin. 2\omega\pi$, $\omega\cos.\omega\pi$; item $\sin. 2i\omega\pi$ et $\omega\cos.(2i-1)\omega\pi$, denotante i numerum integrum quemcunque; vnde quotcunque huiusmodi formulas pro lubitu combinare licet. Huiusmodi ergo dabitur certa quaedam functio, quae sit Φ, vt fiat $y = \sin.\omega\pi$, hincque

$$\sin.\omega\pi = \frac{\pi(1+2\omega)}{\Delta}\text{ tang. }\omega\pi + \Phi,\text{ seu}$$

$$\Delta = \frac{\pi(1+2\omega)\text{ tang. }\omega\pi}{\sin.\omega\pi - \Phi} = \frac{2\int z^{\frac{1}{2}-\omega}\,dz}{(1-z)^{\frac{1}{2}+\omega}}.$$

Cum igitur casu $\omega = 0$ functio Φ certe euanescat, erit vtique $\Delta = \pi$, quod indicio est, functionem Φ factorem continere ω^λ, cuius exponens λ sit vnitate maior, quia alioquin, sumto $\omega = 0$, quantitas Φ prae $\sin.\omega\pi$ non euanesceret. Atque ob hanc rationem conclusiones praecedentis problematis secundi pro veris sunt habendae.

Problema 4.

§. 35. Inuenire eiusmodi aequationem pro linea curua, inter abscissam x et applicatam y, vt abscissis, in arithmetica progressione interrupta progredientibus, datae conueniant applicatae, scilicet:

Si sit $x = n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta,\ (3-n)\theta$, etc.
vt fiat $y = p,\quad q,\quad r,\quad s,\quad t,\quad u$, etc.

Solutio.

Ponamus in genere abscissam $x = \theta\omega$, et pro aequatione inter x et y statuamus hanc aequationem:

$$\frac{y}{\omega} = \mathfrak{A}.\frac{p}{n} - \mathfrak{B}.\frac{q}{1-n} + \mathfrak{C}.\frac{r}{1+n} - \mathfrak{D}.\frac{s}{2-n} + \mathfrak{E}.\frac{t}{2+n} - \mathfrak{F}.\frac{u}{3-n} + \text{etc.}$$

 atque

atque ex §. 25. ad hunc casum generalem extense habebitur:

$$\mathfrak{A}=\frac{(1-n-\omega)(1-n+\omega)}{1(1-2n)}\cdot\frac{(1+n-\omega)(1+n+\omega)}{1(1+2n)}\cdot\frac{(2-n-\omega)(2-n+\omega)}{2(2-2n)}\cdot\frac{(2+n-\omega)(2+n+\omega)}{2(2+2n)}\text{ etc.}$$

$$\frac{\mathfrak{B}}{\mathfrak{A}}=\frac{(n-\omega)(n+\omega)}{(1-n-\omega)(1-n+\omega)}\cdot\frac{1-n}{n};\quad \frac{\mathfrak{C}}{\mathfrak{B}}=\frac{(1-n-\omega)(1-n+\omega)}{(1+n-\omega)(1+n+\omega)}\cdot\frac{1+n}{1-n};$$

$$\frac{\mathfrak{D}}{\mathfrak{C}}=\frac{(1+n-\omega)(1+n+\omega)}{(2-n-\omega)(2-n+\omega)}\cdot\frac{2-n}{1+n};\quad \frac{\mathfrak{E}}{\mathfrak{D}}=\frac{(2-n-\omega)(2-n+\omega)}{(2+n-\omega)(2+n+\omega)}\cdot\frac{2+n}{2-n};$$

etc.

Euoluamus valorem ipsius $\mathfrak{A}$ in duo producta:

$$\mathfrak{P}=\frac{(1-n-\omega)(1-n+\omega)}{1(1-2n)}\cdot\frac{(2-n-\omega)(2-n+\omega)}{2(2-2n)}\cdot\frac{(3-n-\omega)(3-n+\omega)}{3(3-2n)}\text{ etc.}$$

$$\mathfrak{Q}=\frac{(1+n-\omega)(1+n+\omega)}{1(1+2n)}\cdot\frac{(2+n-\omega)(2+n+\omega)}{2(2+2n)}\cdot\frac{(3+n-\omega)(3+n+\omega)}{3(3+2n)}\text{ etc.}$$

vt sit $\mathfrak{A}=\mathfrak{P}\,\mathfrak{Q}$, et vtriusque valorem per formulas integrales secundum praecepta §. 17 definiamus. Ac primo quidem pro infinito producto $\mathfrak{P}$ faciamus $a=1-n-\omega$, $b=1$, $c=-n+\omega$ et $d=1$, eritque

$$\mathfrak{P}=\frac{\int dx(1-x)^{-1-n+\omega}}{\int x^{-n-\omega}dx(1-x)^{-1-n+\omega}}=\frac{1}{\omega-n}\cdot\frac{1}{\int x^{-n-\omega}dx(1-x)^{-1-n+\omega}}$$

siquidem sit $\omega-n>0$. Pro altero producto infinito, sumendo tantum n negatiue, fiet

$$\mathfrak{Q}=\frac{1}{\omega+n}\cdot\frac{1}{\int x^{n-\omega}dx(1-x)^{n+\omega-1}}.$$

Ne autem conditione $\omega-n>0$ opus sit, alia distributione vtamur, sitque

$$\mathfrak{P}=\frac{(1+n+\omega)(1-n-\omega)}{1\cdot 1}\cdot\frac{(2+n+\omega)(2-n-\omega)}{2\cdot 2}\cdot\frac{(3+n+\omega)(3-n-\omega)}{3\cdot 3}\text{ etc.}$$

$$\mathfrak{Q}=\frac{(1+n-\omega)(1-n+\omega)}{(1-2n)(1+2n)}\cdot\frac{(2+n-\omega)(2-n+\omega)}{(2-2n)(2+2n)}\cdot\frac{(3+n-\omega)(3-n+\omega)}{(3-2n)(3+2n)}\text{ etc.}$$

ac pro $\mathfrak{P}$ statuamus: $a=1-n-\omega$, $b=1$, $c=+n+\omega$ $d=1$; pro $\mathfrak{Q}$ vero: $a=1+n-\omega$, $b=1-2n$; $c=n+\omega$ et $d=1$, eritque

$\mathfrak{P}=$

alem extense habebitur:

roducta:

a per formulas integrales ... niamus. Ac primo ... us $a=1-n-\omega$,

oducto infinito, su-

sit, alia distributio-

$=1$, $c=+n+\omega$
$=1-2n$; $c=n+\omega$

$\mathfrak{P}=$

$$\mathfrak{P}=\frac{\int dx(1-x)^{-1+n+\omega}}{\int x^{-n-\omega}dx(1-x)^{-1+n+\omega}}=\frac{1}{n+\omega}\cdot\frac{1}{\int x^{-n-\omega}dx(1-x)^{-1+n+\omega}},$$

$$\mathfrak{Q}=\frac{\int x^{-2n}dx(1-x)^{-1+n+\omega}}{\int x^{n-\omega}dx(1-x)^{-1+n+\omega}}.$$

Est vero in genere

$$\int x^m dx(1-x)^{k-1}=\frac{m+k+1}{k}\int x^m dx(1-x)^k,$$

ergo

$$\int x^{-n-\omega}dx(1-x)^{-1+n+\omega}=\frac{1}{n+\omega}\int x^{-n-\omega}dx(1-x)^{n+\omega}$$
$$=\frac{1}{n+\omega}\int y^{n+\omega}dy(1-y)^{-n-\omega}$$

$$\int x^{-2n}dx(1-x)^{-1+n+\omega}=\frac{(1-n+\omega)}{n+\omega}\int x^{-2n}dx(1-x)^{n+\omega}$$
$$=\frac{(1-n+\omega)}{n+\omega}\int y^{n+\omega}dy(1-y)^{-2n}$$

$$\int x^{n-\omega}dx(1-x)^{-1+n+\omega}=\frac{(1+2n)}{n+\omega}\int x^{n-\omega}dx(1-x)^{n+\omega}$$
$$=\frac{(1+2n)}{n+\omega}\int y^{n+\omega}dy(1-y)^{n-\omega};$$

vnde concluditur:

$$\mathfrak{A}=\mathfrak{P}\,\mathfrak{Q}=\frac{(1-n+\omega)\int y^{n+\omega}dy(1-y)^{-2n}}{\int y^{n+\omega}dy(1-y)^{-n+\omega}\cdot\int y^{n+\omega}dy(1-y)^{n-\omega}}$$

vel

$$\mathfrak{A}=\frac{\int y^{n+\omega-1}dy(1-y)^{-2n}}{\int y^{n+\omega}dy(1-y)^{-n-\omega}\cdot\int y^{n+\omega-1}dy(1-y)^{n-\omega}},$$

seu

$$\mathfrak{A}=\frac{\int y^{n+\omega-1}dy(1-y)^{-2n}}{(n+\omega)\int y^{n+\omega-1}dy(1-y)^{-n-\omega}\cdot\int y^{n+\omega-1}dy(1-y)^{n-\omega}}.$$

Cum igitur sit

$$\mathfrak{B}=\frac{1-n}{n}\cdot\frac{nn-\omega\omega}{(1-n)^2-\omega\omega}\mathfrak{A};\quad \mathfrak{C}=\frac{1+n}{n}\cdot\frac{nn-\omega\omega}{(1+n)^2-\omega\omega}\mathfrak{A};$$

$$\mathfrak{D}=\frac{2-n}{n}\cdot\frac{nn-\omega\omega}{(2-n)^2-\omega\omega}\mathfrak{A};\quad \mathfrak{E}=\frac{2+n}{n}\cdot\frac{nn-\omega\omega}{(2+n)^2-\omega\omega}\mathfrak{A};\text{ etc.}$$

erit

erit per feriem fatis concinnam:

$$\frac{y}{\mathfrak{A}\omega}=\frac{p}{n}-\frac{(nn-\omega\omega)q}{n((1-n)^2-\omega^2)}+\frac{(nn-\omega\omega)r}{n((1+n)^2-\omega^2)}-\frac{(nn-\omega\omega)s}{n((2-n)^2-\omega^2)}+\text{etc.}$$

fiue

$$\frac{ny}{\mathfrak{A}\omega(nn-\omega\omega)}=\frac{p}{n^2-\omega^2}-\frac{q}{(1-n)^2-\omega^2}+\frac{r}{(1+n)^2-\omega^2}-\text{etc.}$$

Loco $\mathfrak{A}$ autem formam integralem reftituendo, vbi quidem nouam variabilem diftinctionis caufa littera z defignabo, haec eadem feries aequalis eft huic expreffioni:

$$\frac{ny}{(n-\omega)\omega}\cdot\frac{\int z^{n+\omega-1}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}}{\int z^{n+\omega-1}dz(1-z)^{-2n}}$$

quarum formularum integratio a termino $z=0$ ad $z=1$ extenfa eft intelligenda.

Corollarium 1.

§. 36. Si ergo breuitatis gratia hanc integralem formam ponamus:

$$\frac{\int z^{n+\omega-1}dz(1-z)^{-n-\omega}\cdot\int z^{n+\omega-1}dz(1-z)^{n-\omega}}{\int z^{n+\omega-1}dz(1-z)^{-2n}}=\Delta$$

et fingulos ferici terminos in binos refoluamus, habebimus:

$$\frac{2n\Delta y}{n-\omega}=+\frac{p}{n-\omega}-\frac{q}{1-n-\omega}+\frac{r}{1+n-\omega}-\frac{s}{2-n-\omega}+\frac{t}{2+n-\omega}-\text{etc.}$$
$$-\frac{p}{n+\omega}+\frac{q}{1-n+\omega}-\frac{r}{1+n+\omega}+\frac{s}{2-n+\omega}-\frac{t}{2+n+\omega}+\text{etc.}$$

Corollarium 2.

§. 37. Haec ergo aequatio eiusmodi definit lineam curuam, in qua abfciffis

$$x=0,\ n\theta,\ (1-n)\theta,\ (1+n)\theta,\ (2-n)\theta,\ (2+n)\theta,$$

refpondent applicatae

$$y=0,\quad p,\quad q,\quad r,\quad s,\quad t,\ \text{etc.}$$

iisdem

iisdem vero abfciffis negatiue fumtis refpondeant eaedem applicatae negatiue fumtae. In genere autem hic pofita eft abfciffa $x=\theta\omega$.

Corollarium 3.

§. 38. Quoniam littera θ ex calculo exceffit, eius loco vnitatem fcribere licuiffet, vt littera ω ipfam abfciffam denotaret. Verum fi applicationem ad arcus eorumque finus facere velimus, commodum eft litteram θ in calculo retinere.

Scholion.

§. 39. Vfus huius problematis imprimis cernitur, fi vt fupra abfciffae tanquam arcus circulares fpectentur, et abfciffae datae ita accipiantur, vt applicatae p, q, r, s, t etc. fiant inter fe aequales, fiue pofitiue fiue negatiue. Quo igitur his cafibus appareat, an feries inuenta aliunde de fummari poffit, in fubfidium vocentur, quae olim de fimilibus feriebus fum commentatus; vnde quidem fequentium duarum ferierum fummae colliguntur:

$$\frac{1}{\alpha}-\frac{1}{\beta-\alpha}+\frac{1}{\beta+\alpha}-\frac{1}{2\beta-\alpha}+\frac{1}{2\beta+\alpha}-\text{etc.}=\frac{\pi}{\beta\,\text{tang.}\frac{\alpha\pi}{\beta}};$$

$$\frac{1}{\alpha}+\frac{1}{\beta-\alpha}-\frac{1}{\beta+\alpha}-\frac{1}{2\beta-\alpha}+\frac{1}{2\beta+\alpha}+\text{etc.}=\frac{\pi}{\beta\,\text{fin.}\frac{\alpha\pi}{\beta}}.$$

Hinc ergo pro noftro problemate deducimus fequentes fummationes quatuor:

I. $\frac{1}{n-\omega}-\frac{1}{1-n+\omega}+\frac{1}{1+n-\omega}-\frac{1}{2-n+\omega}+\frac{1}{2+n-\omega}-\text{etc.}=\frac{\pi}{\text{tang.}(n-\omega)\pi};$

II. $\frac{1}{n-\omega}+\frac{1}{1-n+\omega}-\frac{1}{1+n-\omega}-\frac{1}{2-n+\omega}+\frac{1}{2+n-\omega}+\text{etc.}=\frac{\pi}{\text{fin.}(n-\omega)\pi};$

III.

III. $\frac{1}{n+\omega}-\frac{1}{1-n-\omega}+\frac{1}{1+n+\omega}-\frac{1}{2-n-\omega}+\frac{1}{2+n+\omega}-\text{etc.}=\frac{\pi}{\sin.(n+\omega)\pi}$;

IV. $\frac{1}{n+\omega}+\frac{1}{1-n-\omega}-\frac{1}{1+n+\omega}-\frac{1}{2-n-\omega}+\frac{1}{2+n+\omega}+\text{etc.}=\frac{\pi}{\sin.(n+\omega)\pi}$.

His obſeruatis euoluamus caſus, quos harum ſummationum ope ad expreſſiones finitas reducere licet.

Exemplum I.

§. 40. Sint applicatae, quae abſciſſis $n\theta$, $(1-n)\theta$, $(1+n)\theta$, $(2-n)\theta$, $(2+n)\theta$ etc. reſpondent:

$p=f$; $q=f$; $r=-f$; $s=-f$; $t=+f$; $u=+f$; etc.

et per aequationem finitam relatio inter applicatam y et abſciſſam $x=\theta\,\omega$ inueſtigetur.

Solutio.

Corollarium primum pro hoc caſu hanc praebet aequationem:

$$\frac{2n\,\Delta y}{f(n-\omega)}=+\frac{1}{n-\omega}-\frac{1}{1-n-\omega}-\frac{1}{1+n-\omega}+\frac{1}{2-n-\omega}+\frac{1}{2+n-\omega}-\text{etc.}$$
$$-\frac{1}{n+\omega}+\frac{1}{1-n+\omega}+\frac{1}{1+n+\omega}-\frac{1}{2-n+\omega}-\frac{1}{2+n+\omega}+\text{etc.}$$

quae binae ſeries reducuntur ope quatuor ſupra allatarum, quarum ſummatio conſtat, ad II – IV, ideoque aequatio quaeſita in forma finita ita ſe habebit:

$$\frac{2n\,\Delta y}{f(n-\omega)}=\frac{\pi}{\sin.(n-\omega)\pi}-\frac{\pi}{\sin.(n+\omega)\pi},$$

quae expreſſio redigitur ad hanc:

$$\frac{2\pi\cos.n\pi\sin.\omega\pi}{\sin.(n-\omega)\pi.\sin.(n+\omega)\pi}=\frac{4\pi\cos.n\pi.\sin.\omega\pi}{\cos.2\omega\pi-\cos.2n\pi}$$

ita vt pro noſtra curua haec habeatur aequatio:

$$\frac{n\,\Delta y}{f(n-\omega)}=\frac{\pi\cos.n\pi\sin.\omega\pi}{\sin.(n-\omega)\pi.\sin.(n+\omega)\pi}.$$

Valorem ipſius Δ ante per formulas integrales expreſſum dedimus; cum autem ex ſuperioribus ſit $\Delta=\frac{1}{\pi(n+\omega)}$, habebimus per productum infinitum:

$$\Delta=\frac{1}{n+\omega}\cdot\frac{1(1-2n)}{(1-n)^2-\omega^2}\cdot\frac{1(1+2n)}{(1+n)^2-\omega^2}\cdot\frac{2(2-2n)}{(2-n)^2-\omega^2}\cdot\frac{2(2+2n)}{(2+n)^2-\omega^2}\cdot\text{etc.}$$

vnde

- etc. = π/sin.(n+ω)π;
etc. = π/sin.(n+ω)π.
arum ſummationum
et.
bſciſſis nθ, (1−n)θ,
nt:
-f; u=+f; etc.
er applicatam y et
caſu hanc praebet
ω + 1/(2+n−ω) − etc.
+ω − 1/(2+n+ω) + etc.
or ſupra allatarum,
, ideoque aequatio
aequatio:
ntegrales expreſſum
: Δ = 1/π(n+ω), ha-
etc.
vnde

vnde clarius conſtat, quam ex formulis integralibus, valorem Δ fieri infinitum, quoties fuerit $\omega=\pm(i\pm n)$, denotante i numerum integrum quemcunque: eundem vero valorem Δ euaneſcere, caſibus quibus eſt $n=\pm\frac{i}{2}$. Tum vero etiam notaſſe iuuabit, ſi abeunte ω in $1+\omega$ valor ipſius Δ notetur Δ', fore $\Delta'=-\frac{(1-n+\omega)\Delta}{n-\omega}$. Ac ſi ſimili modo Δ'' conueniat valori $2+\omega$ loco ω aſſumto, erit

$$\Delta''=\frac{-(2-n+\omega)\Delta'}{-(1-n+\omega)}=\frac{-(2-n+\omega)\Delta}{n-\omega}.$$

Corollarium 1.

§. 41. Quatenus quantitas Δ ab ω pendet, conſideretur vt eius functio, hocque modo deſignetur: $\Delta=f:\omega$; tum igitur erit

$$f:(1+\omega)=\frac{n-1-\omega}{n-\omega}f:\omega,\ \text{et}$$
$$f:(2+\omega)=\frac{n-2-\omega}{n-\omega}f:\omega\ \text{etc.}$$

Quare ſi ω denotet numerum integrum quemcunque, habebitur hoc Theorema: $\frac{f:(i+\omega)}{n-i-\omega}=\frac{f:\omega}{n-\omega}$.

Corollarium 2.

§. 42. Cum deinde, ſumto ω negatiuo, ſit

$f:(-\omega)=\frac{n+\omega}{n-\omega}f:\omega$, erit $\frac{f:-\omega}{n+\omega}=\frac{f:\omega}{n-\omega}$,

hinc etiam in genere $\frac{f:(i-\omega)}{n-i+\omega}=\frac{f:\omega}{n-\omega}$.

Scholion.

§. 43. Hic caſus reſpondet illi, quem ſupra §. 25. euoluimus, vbi applicatae datae quoque erant abſciſſarum ſinus; ac pro praeſenti quidem caſu ſtatui oportet $\theta=\pi$,

ve

vt fit $f = \text{fin.}\, n\pi$, et puncta omnia data in linea finuum fint fita. Hinc autem non fequitur ipfam curuam, quam aequatio inuenta exhibet, effe lineam finuum, cum innumerabiles aliae curuae per eadem puncta data tranfire queant. Quare neutiquam etiamnunc certum eft, valorem ipfius y, abfciffae $x = \omega\pi$ conuenientem, et hac aequatione definitum:

$$\frac{n\Delta y}{(n-\omega)\,\text{fin.}\, n\pi} = \frac{\pi\,\text{cof.}\, n\pi.\,\text{fin.}\,\omega\pi}{\text{fin.}\,(n-\omega)\pi.\,\text{fin.}\,(n+\omega)\pi},$$

aequari finui arcus $\pi\omega$, vt fit $y = \text{fin.}\,\pi\omega$, etiamfi hoc verum fit cafibus $\omega = \pm(i \pm n)$ et $\omega = 0$. Supra quidem vidimus, cafu etiam, quo ω eft quantitas minima, aequationem fore veritati confentaneam, fumendo $y = \text{fin.}\,\pi\omega$, ita vt fit $\Delta = \frac{\pi\,\text{cof.}\, n\pi}{\text{fin.}\, n\pi}$, exiftente

$$\Delta = \frac{\int z^{n-1} dz (1-z)^{-n} \int z^{n-1} dz (1-z)^{n}}{\int z^{n-1} dz (1-z)^{-2n}},$$

quemadmodum etiam ibi demonftraui. Quo autem haec res facilius in genere explorari poffit, pro valore Δ commodius exprimendo obferuo effe:

$$\frac{\int z^{n+\omega-1} dz (1-z)^{-n-\omega}}{\int z^{n+\omega-1} dz (1-z)^{-2n}} = \frac{\int z^{\omega-n} dz (1-z)^{-n-\omega}}{\int dz (1-z)^{-2n}}$$
$$= (1-2n)\int z^{\omega-n} dz (1-z)^{-n-\omega},$$

dum fit $n < \frac{1}{2}$, vnde erit:

$$\Delta = (1-2n)\int z^{\omega-n} dz (1-z)^{-n-\omega} \int z^{n+\omega-1} dz (1-z)^{n-\omega}.$$

Verum fi effet in genere $y = \text{fin.}\,\omega\pi$, foret

$$\Delta = \frac{(n-\omega)\pi\,\text{fin.}\, n\pi\,\text{cof.}\, n\pi}{n\,\text{fin.}\,(n-\omega)\pi.\,\text{fin.}\,(n+\omega)\pi}.$$

Quaeftio ergo huc redit: vtrum haec aequatio:

$$(1-2n)\int z^{\omega-n} dz (1-z)^{-n-\omega}.\int z^{n+\omega-1} dz (1-z)^{n-\omega}$$
$$= \frac{(n-\omega)\pi\,\text{fin.}\, n\pi\,\text{cof.}\, n\pi}{n\,\text{fin.}\,(n-\omega)\pi.\,\text{fin.}\,(n+\omega)\pi},$$

etiam

in linea finuum etia m curuam, quam ne ? uum, cum innu- ω = cta data tranfire tum eft, valorem et hac aequatio- prio τω, etiamfi hoc = o. Supra qui- titas minima, ae- quae nendo y = fin. πω, (1 − z)ⁿ, cuius Quo autem haec ftrau valore Δ com- figne quen (1 − z)⁻ⁿ⁻ω 1 − z)⁻²ⁿ −ω, cipia d z(1 − z)ⁿ⁻ω. (1 − ct fi qu ad te atio: d z(1 − z)ⁿ⁻ω tracta etiam

etiam aliis cafibus, praeter fupra memoratos, fit vera, nec ne? Hunc in finem confideremus cafum, quo $n = \frac{1}{4}$ et $\omega = \frac{1}{2}$, vbi pofterior quidem pars fit

$$= \frac{-\frac{1}{4}\cdot\pi\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}{-\frac{1}{4}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}} = \pi;$$

prior vero pars erit

$$= \tfrac{1}{2}\int \frac{z^{\frac{1}{4}} dz}{(1-z)^{\frac{3}{4}}}.\int \frac{z^{-\frac{1}{4}} dz}{(1-z)^{\frac{1}{4}}}$$

quae pofito $z = v^4$ abit in hanc formam:

$$8\int \frac{v^4 dv}{\sqrt[4]{(1-v^4)^3}}.\int \frac{v^2 dv}{\sqrt[4]{(1-v^4)}} = 4\int \frac{dv}{\sqrt[4]{(1-v^4)^3}}.\int \frac{vv\,dv}{\sqrt[4]{(1-v^4)}}$$

cuius valor per ea, quae circa huiusmodi formulas demonftraui, reuera fit $= \pi$, quod ergo iam eft documentum infigne pro veritate noftrae aequationis, quam autem fequenti modo perfecte demonftrare licet.

Theorema.

§. 44. Quomodocunque bini numeri n et ω accipiantur, haec aequatio veritati erit confentanea:

$$(1-2n)\int \frac{z^{\omega-n} dz}{(1-z)^{n+\omega}}.\int \frac{z^{n+\omega-1} dz}{(1-z)^{\omega-n}} = \frac{(n-\omega)\pi\,\text{fin.}\, n\pi.\,\text{cof.}\, n\pi}{n\,\text{fin.}\,(n-\omega)\pi.\,\text{fin}\,(n+\omega)\pi},$$

fi quidem illarum formularum integratio a termino $z = 0$ ad terminum $z = 1$ extendatur.

Demonftratio.

Quo has formulas integrales ad formam, quam tractaui, reducamus, ponamus $n + \omega = \frac{\mu}{\lambda}$ et $\omega - n = \frac{\nu}{\lambda}$,

vt

vt sit $2n = \frac{\mu-\nu}{\lambda}$, ac demonstrari oportet hanc aequationem:

$$\frac{\lambda-\mu+\nu}{\lambda}\int\frac{z^{\frac{\nu}{\lambda}}dz}{\sqrt[\lambda]{(1-z)^\mu}}\cdot\int\frac{z^{\frac{\mu-\lambda}{\lambda}}dz}{\sqrt[\lambda]{(1-z)^\nu}} = \frac{\nu}{\mu-\nu}\cdot\frac{\pi \sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\nu\pi}{\lambda}.\sin.\frac{\mu\pi}{\lambda}}.$$

Ponatur nunc $z = v^\lambda$, et habebitur:

$$\lambda(\lambda-\mu+\nu)\int\frac{v^{\lambda+\nu-1}dv}{\sqrt[\lambda]{(1-v^\lambda)^\mu}}\cdot\int\frac{v^{\mu-1}dv}{\sqrt[\lambda]{(1-v^\lambda)^\nu}} = \frac{\nu}{\mu-\nu}\cdot\frac{\pi \sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\nu\pi}{\lambda}.\sin.\frac{\mu\pi}{\lambda}};$$

more autem has formulas integrales exprimendi ibi recepto membrum prius ita repraesentabitur:

$$\lambda(\lambda-\mu+\nu)\left(\frac{\lambda+\nu}{\lambda-\mu}\right)\cdot\left(\frac{\mu}{\lambda-\nu}\right)$$

quod per reductionem primam

$$\left(\frac{p}{q}\right) = \frac{p-\lambda}{p+q-\lambda}\left(\frac{p-\lambda}{q}\right) \text{ abit in}$$

$$\lambda\nu\left(\frac{\nu}{\lambda-\mu}\right)\cdot\left(\frac{\mu}{\lambda-\nu}\right) = \lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right).$$

Haec vero reductio:

$$\left(\frac{\lambda-q}{p}\right)\left(\frac{\lambda+p-q}{q}\right) = \frac{\pi}{\lambda p \sin.\frac{q\pi}{\lambda}}$$

sumto $p = \mu-\nu$ et $q = \mu$ dat

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right) = \frac{\pi}{\lambda(\mu-\nu)\sin.\frac{\mu\pi}{\lambda}}.$$

Est vero etiam

$$\left(\frac{\lambda-\nu}{\nu}\right) = \frac{\pi}{\lambda\sin.\frac{\nu\pi}{\lambda}},$$

quarum productum est

$$\left(\frac{\lambda-\nu}{\mu}\right)\left(\frac{\lambda-\nu}{\nu}\right)\left(\frac{\lambda-\mu}{\mu-\nu}\right) = \frac{\pi\pi}{\lambda\lambda(\mu-\nu)\sin.\frac{\mu\pi}{\lambda}.\sin.\frac{\nu\pi}{\lambda}}.$$

Porro

Porr … anc aequatio- sume … in. $\frac{\mu-\nu}{\lambda}\pi$ … $\frac{\nu\pi}{\lambda}.\sin.\frac{\mu\pi}{\lambda}$. et c … sin. $\frac{\mu-\nu}{\lambda}\pi$ … n. $\frac{\nu\pi}{\lambda}.\sin.\frac{\mu\pi}{\lambda}$; … ndi ibi recepto ideoq ex q quae integr $\left(\frac{p}{q}\right)$, ti, qu … sin. $\frac{\nu\pi}{\lambda}$. Porro

Porro cum in genere sit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right) = \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right),$$

sumendo $p = \lambda-\mu$; $q = \mu-\nu$ et $r = \nu$, erit

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\nu}\right) = \left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\mu+\nu}{\mu-\nu}\right)$$

et ob $\left(\frac{\lambda-p}{p}\right) = \frac{\pi}{\lambda\sin.\frac{p\pi}{\lambda}}$; sumto $p = \mu-\nu$ erit

$$\left(\frac{\lambda-\mu}{\mu-\nu}\right)\left(\frac{\lambda-\nu}{\nu}\right) = \left(\frac{\lambda-\mu}{\nu}\right)\cdot\frac{\pi}{\lambda\sin.\frac{\mu-\nu}{\lambda}\pi}$$

ideoque

$$\left(\frac{\lambda-\nu}{\mu}\right)\left(\frac{\lambda-\mu}{\nu}\right)\cdot\frac{\pi}{\lambda\sin.\frac{\mu-\nu}{\lambda}\pi} = \frac{\pi\pi}{\lambda\lambda(\mu-\nu)\sin.\frac{\mu\pi}{\lambda}.\sin.\frac{\nu\pi}{\lambda}}$$

ex quibus prius membrum redigitur ad hanc formam:

$$\lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right) = \frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\mu\pi}{\lambda}.\sin.\frac{\nu\pi}{\lambda}}$$

quae est ipsa aequatio demonstranda.

Corollarium 1.

§. 45. In doctrina ergo de huiusmodi formulis integralibus $\int\frac{v^{p-1}dv}{\sqrt[\lambda]{(1-v^\lambda)^{\lambda-q}}}$, quas hoc charactere designo: $\left(\frac{p}{q}\right)$, cui aequiualet $\left(\frac{q}{p}\right)$, haec reductio est grauis momenti, qua demonstraui esse

$$\lambda\nu\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right) = \frac{\nu}{\mu-\nu}\cdot\frac{\pi\sin.\frac{\mu-\nu}{\lambda}\pi}{\sin.\frac{\mu\pi}{\lambda}.\sin.\frac{\nu\pi}{\lambda}}$$

ita

ita vt productum binarum talium formularum integralium $\left(\frac{\lambda-\mu}{\nu}\right)\left(\frac{\lambda-\nu}{\mu}\right)$ per folos angulos exhiberi poffit.

Corollarium 2.

§. 46. Si in valore pro Δ primum inuento pariter ponatur $n+\omega=\frac{\mu}{\lambda}$ et $\omega-n=\frac{\nu}{\lambda}$, tum vero $z=v^\lambda$, erit:

$$\Delta=\lambda\int\frac{v^{\mu-1}\,dv}{\sqrt[\lambda]{(1-v^\lambda)^\mu}}\cdot\int\frac{v^{\mu-1}\,dv}{\sqrt[\lambda]{(1-v^\lambda)^\nu}}:\int\frac{v^{\mu-1}\,dv}{\sqrt[\lambda]{(1-v^\lambda)^{\mu-\nu}}}$$

ideoque hoc fignandi modo:

$$\Delta=\frac{\lambda\left(\frac{\mu}{\lambda-\mu}\right)\cdot\left(\frac{\mu}{\lambda-\nu}\right)}{\left(\frac{\mu}{\lambda-\mu+\nu}\right)},\ \text{feu}$$

$$\Delta=\frac{\lambda\left(\frac{\lambda-\mu}{\mu}\right)\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}.$$

Idem vero valor eft quoque

$$\Delta=\frac{\nu\pi}{\mu-\nu}\cdot\frac{\text{fin.}\frac{\mu-\nu}{\lambda}\pi}{\text{fin.}\frac{\mu\pi}{\lambda}\cdot\text{fin.}\frac{\nu\pi}{\lambda}}.$$

Corollarium 3.

§. 47. Cum igitur pro hac poftrema ftatim fit

$\left(\frac{\lambda-\mu}{\mu}\right)=\frac{\pi}{\lambda\,\text{fin.}\frac{\mu\pi}{\lambda}}$; erit

$$\frac{\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}=\frac{\nu}{\mu-\nu}\cdot\frac{\text{fin.}\frac{\mu-\nu}{\lambda}\pi}{\text{fin.}\frac{\nu\pi}{\lambda}},$$

cuius veritas ex hoc theoremate generali oftenditur:

$$\frac{\left(\frac{q}{p}\right)}{\left(\frac{r}{p}\right)}=\frac{\left(\frac{p+r}{q}\right)}{\left(\frac{p+q}{r}\right)};$$

erit

erit enim

$$\frac{\left(\frac{\lambda-\nu}{\mu}\right)}{\left(\frac{\lambda-\mu+\nu}{\mu}\right)}=\frac{\left(\frac{\lambda+\nu}{\lambda-\nu}\right)}{\left(\frac{\lambda+\mu-\nu}{\lambda-\mu+\nu}\right)}=\frac{\nu}{\mu-\nu}\cdot\frac{\left(\frac{\nu}{\lambda-\nu}\right)}{\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right)},\ \text{ob}$$

$$\left(\frac{\lambda+\nu}{\lambda-\nu}\right)=\frac{\nu}{\lambda}\left(\frac{\nu}{\lambda-\nu}\right)\ \text{et}\ \left(\frac{\lambda+\mu-\nu}{\lambda-\mu+\nu}\right)=\frac{\mu-\nu}{\lambda}\left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right);$$

tum vero eft

$$\left(\frac{\nu}{\lambda-\nu}\right)=\frac{\pi}{\lambda\,\text{fin.}\frac{\nu\pi}{\lambda}}\ \text{et}\ \left(\frac{\mu-\nu}{\lambda-\mu+\nu}\right)=\frac{\pi}{\lambda\,\text{fin.}\frac{\mu-\nu}{\lambda}\pi}.$$

Exemplum 2.

§. 48. Sint applicatae, quae abfciffis $n\theta$, $(1-n)\theta$, $(1+n)\theta$, $(2-n)\theta$, $(2+n)\theta$, etc. refpondent:

$p=f$; $q=-f$; $r=+f$; $s=-f$; $t=+f$; $u=-f$; etc.

et per aequationem finitam inueftigetur relatio in genere inter abfciffam $x=\theta\omega$ et applicatam $=y$.

Aequatio generalis §. 36. ad hunc cafum accommodata praebet:

$$\frac{2n\Delta y}{f(n-\omega)}=\frac{1}{n-\omega}+\frac{1}{1-n-\omega}+\frac{1}{1+n-\omega}+\frac{1}{2-n-\omega}+\frac{1}{2+n-\omega}+\text{etc.}$$
$$-\frac{1}{n+\omega}-\frac{1}{1-n+\omega}-\frac{1}{1+n+\omega}-\frac{1}{2-n+\omega}-\frac{1}{2+n+\omega}-\text{etc.}$$

vbi nunc quidem nouimus effe

$$\Delta=\frac{(n-\omega)\pi\,\text{fin.}\,2n\pi}{2n\,\text{fin.}(n-\omega)\pi.\,\text{fin.}(n+\omega)\pi}.$$

Illa autem feries ex §. 39. fit

$$\text{I}-\text{III}=\frac{\pi}{\text{tang.}(n-\omega)\pi}-\frac{\pi}{\text{tang.}(n+\omega)\pi}=\frac{\pi\,\text{fin.}\,2\omega\pi}{\text{fin.}(n-\omega)\pi.\,\text{fin.}(n+\omega)\pi}$$

qua fumma fubftituta prodit

$$\frac{y}{f}\cdot\frac{\pi\,\text{fin.}\,2n\pi}{\text{fin.}(n-\omega)\pi.\,\text{fin.}(n+\omega)\pi}=\frac{\pi\,\text{fin.}\,2\omega\pi}{\text{fin.}(n-\omega)\pi.\,\text{fin.}(n+\omega)\pi},\ \text{feu}$$

$$y=\frac{f\,\text{fin.}\,2\omega\pi}{\text{fin.}\,2n\pi}=\frac{f\,\text{fin.}\frac{2x\pi}{\theta}}{\text{fin.}\,2n\pi}.$$

Haec

Haec ergo curua denuo est linea finuum, ac si sumatur $\theta = 2\pi$, vt sit $f = \text{fin.}\, 2n\pi$, erit applicata $y = \text{fin.}\, x$.

Corollarium 1.

§. 49. Si sumatur $\theta = \pi$ et $f = \text{tang.}\, n\theta = \text{tang.}\, n\pi$, puncta data erunt in linea tangentium; neque tamen ipsa curua inuenta erit linea tangentium: sed eius natura hac exprimetur aequatione:

$$y = \frac{\text{tang.}\, n\pi\, \text{fin.}\, 2x}{\text{fin.}\, 2n\pi} = \frac{\text{fin.}\, 2x}{2\,\text{cof.}\, n\pi^2} = \frac{\text{fin.}\, 2x}{1 + \text{cof.}\, 2n\pi}$$

eritque hic $y = \text{tang.}\, x$, quoties fuerit $x = \pm (i \pm n)\pi$.

Corollarium 2.

§. 50. Si in solutione prioris exempli, quo erat $p = f$, $q = f$, $r = -f$, $s = -f$, $t = f$, $u = f$, statim loco Δ valorem inuentum posuissemus, prodiisset haec aequatio: $y = \frac{f\,\text{fin.}\, \omega\pi}{\text{fin.}\, n\pi}$. Vnde perspicuum fuisset, sumto $\theta = \pi$ et $f = \text{fin.}\, n\pi$, curuam illam ipsam fore lineam finuum.

Scholion.

§. 51. Omnino notari meretur, quod in problemate 4. vbi abscissae datae progressionem arithmeticam interruptam constituunt, valor quantitatis Δ absolute per angulos exhiberi potuerit, cum tamen in problemate 3. vbi abscissae datae veram progressionem arithmeticam constituebant, formula integralis Δ in genere neutiquam per angulos exprimi queat. Cum enim ibi esset

$$\Delta = \int z^{n-\omega-1}\, dz\, (1 - z)^{n-\omega-1},$$

haec formula, posito $n - \omega = \frac{\nu}{\lambda}$ et $z = v^\lambda$, abit in

: si sumatur
= fin. x.
qua re dat etc. seu
θ = tang. n π, tamen ipsa natura hac
seu (i ± n) π.
vnd mar prio casu gulo
li, quo erat tim loco Δ c aequatio: θ = π et finuum.
con his hae
l in proble- rithmeticam absolute per oblemate 3. ieticam con- itiquam per
tio y
it in
Δ =

$$\Delta = \lambda \int \frac{v^{\nu-1}\, dv}{\sqrt[\lambda]{(1 - v^\lambda)^{\lambda-\nu}}}, \text{ feu } \Delta = \lambda\left(\frac{\nu}{\lambda}\right),$$

quae formula quadraturas maxime transcendentes implicare potest. Ac si in illo problemate statuantur applicatae datae $p = f$, $q = -f$, $r = f$, $s = -f$, $t = f$, $u = -f$, etc. et $n = \frac{1}{2}$, aequatio pro curua per ista puncta transeunte erit:

$$\frac{\Delta y}{2(1 + 2\omega)\omega f} = \frac{4}{1 - 4\omega\omega} + \frac{4}{9 - 4\omega\omega} + \frac{4}{25 - 4\omega\omega} + \text{etc.}$$

seu $\frac{\Delta y}{2 f \omega (1 + 2\omega)} = \frac{\pi}{2\omega}\, \text{tang.}\, \omega\pi$, ita vt sit $y = \frac{\pi f (1 + 2\omega)\, \text{tang.}\, \omega\pi}{\Delta}$ vnde, etiamsi sumatur $\theta = \pi$ et $f = \text{fin.}\, n\theta = \text{fin.}\, \frac{1}{2}\pi = 1$, manifesto non sequitur fore $y = \text{fin.}\, \theta\omega = \text{fin.}\, \omega\pi$. Cum priori exemplo iam certum sit esse $y = \frac{f\,\text{fin.}\, \omega\pi}{\text{fin.}\, n\pi}$, eundem casum ex problemate primo ita euoluamus, vt valores singulorum coefficientium A, B, C, D, etc. inuestigemus.

Problema 5.

§. 52. Aequationem generalem supra Problem. I. constitutam ita determinare, vt respondeant

his abscissis: $x = n\theta$, $(1-n)\theta$, $(1+n)\theta$, $(2-n)\theta$, $(2+n)\theta$, etc.
hae applicatae: $y = f$, $+f$, $-f$, $-f$, $+f$, etc.

Solutio.

Statuatur vt ante $x = \theta\omega$, et consideretur aequatio quaesita sub hac forma:

$$y = A\omega + B\omega(\omega\omega - nn) + C\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)$$
$$+ D\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)(\omega\omega - (1+n)^2)$$
$$+ E\omega(\omega\omega - nn)(\omega\omega - (1-n)^2)(\omega\omega - (1+n)^2)(\omega\omega - (2-n)^2)$$
etc.

 vnde

vnde deducuntur hae aequationes:

$$\frac{f}{n} = A$$

$$\frac{f}{1-n} = A + B.1(1-2n)$$

$$\frac{-f}{1+n} = A + B.1(1+2n) + C.1(1+2n).2.2n$$

$$\frac{-f}{2-n} = A + B.2(2-2n) + C.2(2-2n).1(3-2n)$$
$$+ D.2(2-2n).1(3-2n).3(1-2n)$$

etc.

hincque coefficientium valores sequentes:

$$A = \frac{f}{n};\ B = \frac{-f}{n(1-n)};\ C = \frac{f}{2n(1-n)(1+n)};$$
$$D = \frac{-f}{6n(1-n)(1+n)(2-n)};\ E = \frac{f}{24n(1-n)(1+n)(2-n)(2+n)};\ \text{etc.}$$

quae progressio cum satis sit simplex, series nostra pro valore ipsius y, quem iam nouimus esse $= \frac{f\,\sin.\omega\pi}{\sin.n\pi}$, eo maiorem attentionem meretur; haecque est

$$\frac{\sin.\omega\pi}{\sin.n\pi} = \frac{\omega}{n} - \frac{\omega}{n}.\frac{\omega\omega-nn}{1(1-n)} + \frac{\omega}{n}.\frac{\omega\omega-nn}{1(1-n)}.\frac{\omega\omega-(1-n)^2}{2(1+n)}$$
$$- \frac{\omega}{n}.\frac{\omega\omega-nn}{1(1-n)}.\frac{\omega\omega-(1-n)^2}{2(1+n)}.\frac{\omega\omega-(1+n)^2}{3(2-n)} + \text{etc.}$$

vel si Π iugiter denotet terminum praecedentem, totum erit:

$$\frac{\sin.\omega\pi}{\sin.n\pi} = \frac{\omega}{n} - \Pi.\frac{\omega\omega-nn}{1(1-n)} + \Pi.\frac{\omega\omega-(1-n)^2}{2(1+n)} - \Pi.\frac{\omega\omega-(1+n)^2}{3(2-n)}$$
$$+ \Pi.\frac{\omega\omega-(2-n)^2}{4(2+n)} - \Pi.\frac{\omega\omega-(2+n)^2}{5(3-n)} + \text{etc.}$$

Quodsi omnes termini eodem signo affecti desiderentur, erit:

$$\frac{\sin.\omega\pi}{\sin.n\pi} = \frac{\omega}{n} + \frac{\omega}{n}.\frac{nn-\omega\omega}{1(1-n)} + \frac{\omega}{n}.\frac{nn-\omega\omega}{1(1-n)}.\frac{(1-n)^2-\omega\omega}{2(1+n)}$$
$$+ \frac{\omega}{n}.\frac{nn-\omega\omega}{1(1-n)}.\frac{(1-n)^2-\omega\omega}{2(1+n)}.\frac{(1+n)^2-\omega\omega}{3(2-n)}$$
$$+ \frac{\omega}{n}.\frac{nn-\omega\omega}{1(1-n)}.\frac{(1-n)^2-\omega\omega}{2(1+n)}.\frac{(1+n)^2-\omega\omega}{3(2-n)}.\frac{(2-n)^2-\omega\omega}{4(2+n)}$$

etc.

Hacc series eo maiori attentione digna videtur, quod a soli-

solita serierum ratione plurimum recedit, in eaque adeo duo numeri arbitrarii n et ω occurrunt.

Corollarium 1.

§. 53. Si numerus ω euanescat, vt fiat $\sin.\omega\pi = \omega\pi$, diuisione per ω instituta habebitur haec aequatio:

$$\frac{\pi}{\sin.n\pi} = \frac{1}{n} + \frac{n}{1(1-n)} + \frac{n(1-n)}{1.2(1+n)} + \frac{n(1-n)(1+n)}{1.2.3(2-n)}$$
$$+ \frac{n(1-n)(1+n)(2-n)}{1.2.3.4(2+n)} + \text{etc.}$$

vnde sumto $n = \frac{1}{2}$, ob $\sin.\frac{\pi}{2} = 1$ erit

$$\pi = 2 + 1 + \frac{1.1.2}{2.4.3} + \frac{1.1.3.2}{2.4.6.3} + \frac{1.1.3.3.2}{2.4.6.8.5} + \frac{1.1.3.3.5.2}{2.4.6.8.10.5} + \text{etc.}$$

seu
$$\pi = 2 + \frac{1}{2.2^1.3} + \frac{1.3}{2.4.2^3.5} + \frac{1.3.5}{2.4.6.2^5.7} + \frac{1.3.5.7}{2.4.6.8.2^7.9}$$
$$+ 1 + \frac{1}{2.2^3.3} + \frac{1.3}{2.4.2^5.5} + \frac{1.3.5}{2.4.6.2^7.7} + \frac{1.3.5.7}{2.4.6.8.2^9.9}$$

quarum serierum posterior cum sit semissis prioris, erit summa posterioris $= \frac{\pi}{3}$, cuius quidem ratio inde est manifesta, quod sit

$$\int \frac{dx}{\sqrt{(1-xx)}} = \text{ang.}\sin.x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7} + \text{etc.}$$

vnde illa series fit $= \frac{\text{ang.}\sin.x}{x}$, sumto $x = \frac{1}{2}$ ideoque $= 2\frac{\pi}{6} = \frac{\pi}{3}$.

Corollarium 2.

§. 54 Si alter numerus n euanescat, vt fiat $\sin.n\pi = n\pi$, et aequatio per n multiplicetur, orietur:

$$\frac{\sin.\omega\pi}{\pi} = \omega - \frac{\omega^3}{1} + \frac{\omega^3(\omega^2-1)}{1.2.1^2} - \frac{\omega^3(\omega^2-1)(\omega^2-1)}{1.2.3.1^2.2} + \frac{\omega^3(\omega^2-1)(\omega^2-1)(\omega^2-4)}{1.2.3.4.1^2.2^2}$$
$$- \frac{\omega^3(\omega^2-1)(\omega^2-1)(\omega^2-4)(\omega^2-4)}{1.2.3.4.5.1^2.2^2.3} + \text{etc.}$$

quae series per ω diuisa in binas sequentes resoluitur:

$$\frac{\sin.\omega\pi}{\omega\pi} = 1 + \frac{\omega^2(\omega^2-1)}{1.2.1^2} + \frac{\omega^2(\omega^2-1)^2(\omega^2-4)}{1.2.3.4.1^2.2} + \frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2(\omega^2-9)}{1.2.3.4.5.6.1^2.2^2.3^2} + \text{etc.}$$
$$- \frac{\omega^2}{1} - \frac{\omega^2(\omega^2-1)^2}{1.2.3.1^2.2} - \frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2}{1.2.3.4.5.1^2.2^2.3} - \frac{\omega^2(\omega^2-1)^2(\omega^2-4)^2(\omega^2-9)^2}{1.2.3.4.5.6.7.1^2.2^2.3^2.4} - \text{etc.}$$

Suma-

Sumamus hic $\omega = \frac{1}{2}$, fiet:

$$\frac{2}{\pi} = 1 - \frac{1.1.3}{1.1.1.2^3} - \frac{1.1.3.1.5.3}{1.1.1.2.2.3.2^{10}} - \frac{1.1.3.1.3.5.3.5.7}{1.1.1.2.3.1.2.3.3.3.2^{13}} - \text{etc.}$$

$$- \frac{1.1}{2^3} - \frac{1.1.1.3.1.3}{1.1.1.2.2.3.2^6} - \frac{1.1.1.3.1.3.3.5.3.5}{1.1.1.2.2.2.3.3.4.5.2^{10}} - \text{etc.}$$

quae postrema series ita referri potest:

$$- \frac{1}{2^3} - \frac{1.1.1.3}{1.1.2.2^7} - \frac{1.1.3.5.3.5}{1.1.2.2.3.5.2^{12}} - \frac{1.1.3.3.3.5.5.5.7}{1.1.2.2.2.3.3.3.4.2^{17}} - \text{etc.}$$

Corollarium 3.

§. 55. Si fuerit $n = \frac{1}{2}$, vt sit $\sin. n\pi = 1$, erunt factores, ex quibus singulos seriei terminos formari oportet:

$$\frac{2\omega}{1}, \frac{1-4\omega\omega}{1.2}, \frac{1-4\omega\omega}{3.4}, \frac{9-4\omega\omega}{3.6}, \frac{9-4\omega\omega}{5.8}, \frac{25-4\omega\omega}{5.10}, \frac{25-4\omega\omega}{7.12}, \text{etc.}$$

atque seriei summa erit $\sin. \omega\pi$, scilicet:

$$\sin. \omega\pi = 2\omega + \frac{2\omega(1-4\omega\omega)}{1.2} + \frac{2\omega(1-4\omega\omega)^2}{1.2.3.4} + \frac{2\omega(1-4\omega\omega)^2(9-4\omega\omega)}{1.2.3.4.5.6} + \text{etc.}$$

vnde, sumto $\omega = 1$, esse debet:

$$0 = 2 - 3 + \frac{3}{2^3} + \frac{5}{2^8.3} + \frac{5}{2^6.5} + \frac{7}{2^7.5} + \frac{7}{2^9.5} + \frac{9}{2^{10}.7} + \frac{5.9}{2^{14}.7} + \text{etc.}$$

cuius veritas calculum instituenti mox patebit.

Scholion.

§. 56. Pro hoc casu etiam solutio supra inuenta attentius considerari meretur, quae ex §. 36. ob

$$\Delta = \frac{(n-\omega)\pi \sin. 2n\pi}{2n \sin.(n-\omega)\pi. \sin.(n+\omega)\pi} \text{ et } y = \frac{f \sin. \omega\pi}{\sin. n\pi},$$

cum sit $p = f$, $q = f$, $r = -f$, $s = -f$, $t = f$, $u = f$, etc. in hac continetur aequatione:

$$\frac{\pi \cos. n\pi. \sin. \omega\pi}{\omega \sin.(n-\omega)\pi. \sin.(n+\omega)\pi} = \frac{1}{nn-\omega\omega} - \frac{1}{(1-n)^2-\omega^2} - \frac{1}{(1+n)^2-\omega^2}$$

$$+ \frac{1}{(2-n)^2-\omega^2} + \frac{1}{(2+n)^2-\omega^2} - \text{etc.}$$

quae series maxime discrepat ab ea, quam modo inuenimus. Circa hanc autem seriem sequentia obseruo.

I. Si

$\frac{\pi\pi \cos.}{\sin. n}$ ... $\frac{1.5.3.5.7}{1.2.3.3.3.2^{13}}$ — etc.
at ... $\frac{1}{10}$ — etc.
dum ... $\frac{1.3.5.7}{1.3.3.4.2^{17}}$ — etc.
Ad euan et si ... n. $n\pi = 1$, erunt formari oportet: $\frac{2\omega}{}$ ... $\frac{25-4\omega\omega}{7.12}$, etc.
vnde ... $\frac{-4\omega\omega)^2(9-4\omega\omega)}{1.2.3.4.5.6}$ + etc.
Est e ... $\frac{9}{2^{10}.7} + \frac{5.9}{2^{14}.7}$ + etc.
tebit.
$-\frac{1}{\omega \sin}$ ... tio supra inuenta
seu ... 36. ob
vnde ... $\frac{\sin. \omega\pi}{\sin. n\pi}$,
cuius ... $t = f$, $u = f$, etc.
sumto ...
$1 - \frac{1}{4}$ ... $\omega^2 - \frac{1}{(1+n)^2-\omega^2}$
quam modo inuentia obseruo.
I. Si

I. Si ω euanescat, fore

$$\frac{\pi\pi \cos. n\pi}{(\sin. n\pi)^2} = \frac{1}{nn} - \frac{1}{(1-n)^2} - \frac{1}{(1+n)^2} + \frac{1}{(2-n)^2} + \frac{1}{(2+n)^2} - \frac{1}{(3-n)^2} - \text{etc.}$$

at si insuper n euanescat, ob $\sin. n\pi = n\pi$ incommodum nasci sequens:

$$\frac{1}{nn} = \frac{1}{nn} - \frac{2}{1} + \frac{2}{4} - \frac{2}{9} + \frac{2}{16} - \text{etc.}$$

Ad hoc autem tollendum, numerum n vt tantum non euanescentem spectemus, et cum sit $\cos. n\pi = 1 - \frac{1}{2}nn\pi\pi$ et $\sin. n\pi = n\pi - \frac{1}{6}n^3\pi^3 = n\pi(1 - \frac{1}{6}nn\pi\pi)$, erit

$$\frac{\cos. n\pi}{(\sin. n\pi)^2} = \frac{1 - \frac{1}{2}nn\pi\pi}{nn\pi\pi(1 - \frac{1}{3}nn\pi\pi)} = \frac{1 - \frac{1}{6}nn\pi\pi}{nn\pi\pi},$$

vnde obtinetur haec vera aequatio:

$$\frac{1}{nn} - \frac{1}{6}\pi\pi = \frac{1}{nn} - \frac{2}{1} + \frac{2}{4} - \frac{2}{9} + \frac{2}{16} - \frac{2}{25} + \text{etc.}$$

Est enim

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} = \frac{1}{12}\pi\pi.$$

II. Ponamus nunc $n = 0$ et habebimus:

$$-\frac{\pi}{\omega \sin. \omega\pi} = -\frac{1}{\omega^2} - \frac{1}{1-\omega^2} - \frac{1}{1-\omega^2} + \frac{1}{4-\omega^2} + \frac{1}{4-\omega^2} - \frac{1}{9-\omega^2} - \frac{1}{9-\omega^2} - \text{etc.}$$

seu $\frac{\pi}{\omega \sin. \omega\pi} = \frac{1}{\omega^2} + \frac{2}{1-\omega^2} - \frac{2}{4-\omega^2} + \frac{2}{9-\omega^2} - \frac{2}{16-\omega^2} + \frac{2}{25-\omega^2} - \text{etc.}$

vnde consequimur hanc memorabilem summationem:

$$\frac{1}{1-\omega^2} - \frac{1}{4-\omega^2} + \frac{1}{9-\omega^2} - \frac{1}{16-\omega^2} + \text{etc.} = \frac{\pi}{2\omega \sin. \omega\pi} - \frac{1}{2\omega\omega}$$

cuius veritatem alibi demonstratam dedi. Hinc autem, sumto ω infinite paruo, ob $\sin. \omega\pi = \omega\pi(1 - \frac{1}{6}\omega^2\pi^2)$ seriei $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \text{etc.}$ summa vt ante colligitur:

$$\frac{1}{2\omega\omega(1 - \frac{1}{6}\omega^2\pi^2)} - \frac{1}{2\omega\omega} = \frac{1}{12}\pi\pi.$$

III.

III. Si capiatur $n = \frac{1}{2}$, ob cof. $n\pi = 0$, etiam ipfa feries euanefcit, dum fcilicet omnes termini fe mutuo reuera deftruunt. Quid autem eueniat fi n infinite parum a $\frac{1}{2}$ difcrepet, differentiatio inftituatur, fumto n variabili. vnde fit:

$$-\frac{\pi\pi\, \text{fin.}\, n\pi\, \text{fin.}\, \omega\pi\,(1 + \text{cof.}(n-\omega)\pi.\, \text{cof.}(n+\omega)\pi)}{\omega(\text{fin.}(n-\omega)\pi.\, \text{fin.}(n+\omega)\pi)^2} =$$

$$-\frac{2n}{(nn-\omega\omega)^2} - \frac{2(1-n)}{((1-n)^2-\omega^2)^2} + \frac{2(1+n)}{((1+n)^2-\omega^2)^2}$$

$$+\frac{2(2-n)}{((2-n)^2-\omega^2)^2} - \frac{2(2+n)}{((2+n)^2-\omega^2)^2} - \text{etc.}$$

Nunc igitur fumatur $n = \frac{1}{2}$, eritque:

$$-\frac{\pi\pi\, \text{fin.}\, \omega\pi}{\omega(\text{cof.}\, \omega\pi)^2} = -\frac{16}{(1-4\omega^2)^2} - \frac{16}{(1-4\omega^2)^2} + \frac{3.16}{(9-4\omega^2)^2} + \frac{3.16}{(9-4\omega^2)^2} - \text{etc.}$$

fiue

$$-\frac{\pi\pi\, \text{fin.}\, \omega\pi}{32\,\omega(\text{cof.}\, \omega\pi)^2} = \frac{1}{(1-4\omega^2)^2} - \frac{3}{(9-4\omega^2)^2} + \frac{5}{(25-4\omega^2)^2} - \frac{7}{(49-4\omega^2)^2} - \text{etc.}$$

vnde fumto $\omega = 0$ fequitur fore:

$$\frac{\pi^2}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.}$$

quod quidem aliunde conftat.

Verum feries in praefente problemate inuenta multo magis ardua videtur. Quin adeo cafus in corollario I euolutus, etfi maxime eft particularis, diligentiori euolutione eft dignus, quam in problemate fequente expedire conabor.

Problema 6.

§ 57. Si n fit numerus quicunque, inquirere in fummam huius feriei:

$$s = \frac{1}{n} + \frac{n}{1(1-n)} + \frac{n(1-n)}{1.2(1+n)} + \frac{n(1-n)(1+n)}{1.2.3(2-n)} + \frac{n(1-n)(1+n)(2-n)}{1.2.3.4(2+n)} + \text{etc.}$$

quam quidem ante inuenimus effe $s = \frac{\pi}{\text{fin.}\, n\pi}$.

So-

, etiam ipfa
e mutuo re-
finite parum
n variabili.
— etc.
— etc.
inuenta mul-
corollario I
ori euolutio-
ite expedire
inquirere in
+etc.
So-
cam P= Q= ita inqu na a ac p dabo. debea vnde Mult ∫d Nunc eritqu feries ν=2 Euler

Solutio.

Cum in hac ferie lex progreffionis fit interrupta, eam in duas difcerpi conueniet. Statuamus ergo

$$P = \frac{1}{n} + \frac{n(1-n)}{1.2(1+n)} + \frac{n(1-n)(1+n)(2-n)}{1.2.3.4(2+n)} + \frac{n(1-n)(1+n)(2-n)(2+n)(3-n)}{1.2.3.4.5.6(3+n)} + \text{etc.}$$

$$Q = \frac{n}{1(1-n)} + \frac{n(1-n)(1+n)}{1.2.3(2-n)} + \frac{n(1-n)(1+n)(2-n)(2+n)}{1.2.3.4.5(3-n)} + \text{etc.}$$

ita vt fit $s = P + Q$. In harum iam feríerum fummas inquifiturus, in fubfidium voco fequentes feries ex doctrina angulorum petitas:

$$\frac{\text{cof.}\, \mu\Phi}{\text{cof.}\, \Phi} = 1 + \frac{(1-\mu)(1+\mu)}{1.2}\, \text{fin.}\, \Phi^2 + \frac{(1-\mu)(1+\mu)(3-\mu)(3+\mu)}{1.2.3.4}\, \text{fin.}\, \Phi^4 + \text{etc.}$$

$$\frac{\text{fin.}\, \nu\Phi}{\text{cof.}\, \Phi} = \nu\, \text{fin.}\, \Phi + \frac{\nu(2-\nu)(2+\nu)}{1.2.3}\, \text{fin.}\, \Phi^3 + \frac{\nu(2-\nu)(2+\nu)(4-\nu)(4+\nu)}{1.2.3.4.5}\, \text{fin.}\, \Phi^5 + \text{etc.}$$

ac primo quidem illam ad formam priorem P accommodabo. Cum igitur hae fractiones:

$$\frac{(1-\mu)(1+\mu)}{n(1-n)};\ \frac{(3-\mu)(3+\mu)}{(1+n)(2-n)};\ \frac{(5-\mu)(5+\mu)}{(2+n)(3-n)};\ \text{etc.}$$

debeant effe aequales, concludo capi debere $\mu = 1 - 2n$, vnde erit

$$\frac{\text{cof.}(1-2n)\Phi}{\text{cof.}\, \Phi} = 1 + \frac{n(1-n)}{1.2}.\, 2^2\, \text{fin.}\, \Phi^2 + \frac{n(1-n)(1+n)(2-n)}{1.2.3.4}\, 2^4\, \text{fin.}\, \Phi^4 + \text{etc.}$$

Multiplicemus per $d\Phi\, \text{fin.}\, \Phi^{2n-1}\, \text{cof.}\, \Phi$ et integremus, fiet

$$\int d\Phi\, \text{fin.}\, \Phi^{2n-1}\, \text{cof.}(1-2n)\Phi = \frac{1}{2n}\, \text{fin.}\, \Phi^{2n} + \frac{n(1-n)}{1.2(n+1)}\, 2\, \text{fin.}\, \Phi^{2n+2}$$

$$+ \frac{n(1-n)(1+n)(2-n)}{1.2.3.4(n+2)}\, 2^3\, \text{fin.}\, \Phi^{2n+4} + \text{etc.}$$

Nunc poft integrationem ftatuatur fin. $\Phi = \frac{1}{2}$, feu $\Phi = 30°$, eritque

$$P = 2^{2n+1} \int d\Phi\, \text{fin.}\, \Phi^{2n-1}\, \text{cof.}(1-2n)\Phi;$$

feries vero Q facile deducetur ex altera cognita, fumendo $\nu = 2n$, vnde fit

$$\frac{\text{fin.}\, 2n\Phi}{\text{cof.}\, \Phi} = n.\, 2\, \text{fin.}\, \Phi + \frac{n(1-n)(1+n)}{1.2.3}\, 2^3\, \text{fin.}\, \Phi^3$$

$$+ \frac{n(1-n)(1+n)(2-n)(2+n)}{1.\ 2.\ 3.\ 4.\ 5.}\, 2^5\, \text{fin.}\, \Phi^5\ \text{etc.}$$

Mul-

Multiplicetur per $d\Phi\,\text{fin.}\,\Phi^{-2n}\,\text{cof.}\,\Phi$ et integretur, erit

$$\int d\Phi\,\text{fin.}\,\Phi^{-2n}\,\text{fin.}\,2n\Phi = \frac{n}{1(1-n)}\,\text{fin.}\,\Phi^{2-2n}$$
$$+ \frac{n(1-n)(1+n)}{1.2.3(2-n)}\,2^2\,\text{fin.}\,\Phi^{4-2n} + \text{etc.}$$

Statuatur pariter, integratione abfoluta, $\text{fin.}\,\Phi = \frac{1}{2}$, feu $\Phi = 30°$, ac prodibit feries $Q = 2^{1-2n}\int d\Phi\,\text{fin.}\,\Phi^{-2n}\,\text{fin.}\,2n\Phi$. Quocirca feriei propofitae fumma ita exprimetur, vt fit

$$s = 2^{2n+1}\int d\Phi\,\text{fin.}\,\Phi^{2n-1}\,\text{cof.}\,(1-2n)\Phi$$
$$+ 2^{1-2n}\int d\Phi\,\text{fin.}\,\Phi^{-2n}\,\text{fin.}\,2n\Phi$$

et quia haec fumma iam aliunde eft cognita, habebitur

$$\frac{\pi}{\text{fin.}\,n\pi} = 4\int d\Phi\,\text{cof.}\,(1-2n)\Phi\,(2\,\text{fin.}\,\Phi)^{2n-1}$$
$$+ 4\int d\Phi\,\text{fin.}\,2n\Phi\,(2\,\text{fin.}\,\Phi)^{-2n}.$$

Corollarium 1.

§. 58. Si ponatur $2n = \frac{1-\lambda}{2}$, erit $1-2n = \frac{1+\lambda}{2}$, qua pofitione noftra aequatio fit concinnior, eritque

$$\frac{\pi}{\text{fin.}\,\frac{1-\lambda}{4}\pi} = 4\int\frac{d\Phi\,\text{cof.}\,\frac{1+\lambda}{2}\Phi}{(2\,\text{fin.}\,\Phi)^{\frac{1+\lambda}{2}}} + 4\int\frac{d\Phi\,\text{fin.}\,\frac{1-\lambda}{2}\Phi}{(2\,\text{fin.}\,\Phi)^{\frac{1-\lambda}{2}}} = \frac{\pi\sqrt{2}}{\text{cof.}\,\frac{\lambda\pi}{4} - \text{fin.}\,\frac{\lambda\pi}{4}}$$

pofito poft integrationem $\Phi = 30°$.

Corollarium 2.

§. 59. Simili modo fumto λ negatiue, erit

$$\frac{\pi}{\text{fin.}\,\frac{1+\lambda}{4}\pi} = 4\int\frac{d\Phi\,\text{cof.}\,\frac{1-\lambda}{2}\Phi}{(2\,\text{fin.}\,\Phi)^{\frac{1-\lambda}{2}}} + 4\int\frac{d\Phi\,\text{fin.}\,\frac{1+\lambda}{2}\Phi}{(2\,\text{fin.}\,\Phi)^{\frac{1+\lambda}{2}}} = \frac{\pi\sqrt{2}}{\text{cof.}\,\frac{\lambda\pi}{4} + \text{fin.}\,\frac{\lambda\pi}{4}}$$

vbi quidem notaffe iuuabit, omnibus cafibus, quos euoluere licet, eundem harum formularum integralium valorem actu reperiri, quem hic exhibuimus.

DE
CRITERIIS AEQVATIONIS
$fxx + gyy = hz^2$
VTRVM EA RESOLVTIONEM ADMITTAT NEC NE?

§. 1.

Notum eft huiusmodi aequationem, pro varia relatione, quae inter numeros f, g et h intercedit, modo effe poffibilem modo impoffibilem, fiquidem pro x, y et z numeros rationales accipi oportet, atque adeo integros, quia fracti facillime ad integros reuocarentur. Ita notum eft hanc aequationem: $xx + yy = 2zz$ effe poffibilem; hanc vero: $xx + yy = 3zz$ impoffibilem. Quando autem litterae f, g et h maiores tenent valores, iudicium, vtrum aequatio fit poffibilis nec ne, difficulter inftituitur; in maximis vero numeris vix fufcipiendum videtur. Hic igitur conftitui in certa criteria inquirere, ex quibus iudicare liceat, vtrum haec aequatio fit poffibilis nec ne, quantumuis magni fuerint numeri f, g et h.

§. 2. Ante omnia autem fequentia notaffe iuuabit:

I°. Numeros f, g et h non folum integros affumo, fed etiam non-quadratos, neque etiam per quadratum diuifi-

 biles;