# DE SERIEBVS
## IN QVIBVS
## PRODVCTA EX BINIS TERMINIS CONTIGVIS DATAM CONSTITVVNT PROGRESSIONEM.

Propofita progreffione numerorum quacunque:

A, B, C, D, E, F, etc.

quaeftio, quam hic tractare ftatui, in hoc confiftit, vt inveniatur eiusmodi feries:

$a, b, c, d, e, f,$ etc.

in qua fit:

$ab=A$; $bc=B$; $cd=C$; $de=D$; $ef=E$; $fg=F$; etc.

vbi, etfi numeri A, B, C, D, etc. fint rationales, fatisque fimplici lege procedant, plerumque fieri folet, vt numeri $a, b, c, d,$ etc. euadant adeo maxime tranfcendentes. Euidens autem eft totum negotium ad vnicum terminum huius feriei reuocari; quippe quo cognito reliqui omnes faciIlime definientur: inuento enim primo $a$ reliqui ita fe habebunt:

$b=\frac{A}{a}$; $c=\frac{B}{b}$; $d=\frac{C}{c}$; $e=\frac{D}{d}$; etc.

 Dupli-

Duplicem autem ad solutionem huius quaestionis patere viam observavi, quarum altera interpolatione certae cuiusdam seriei absolvatur, altera autem, quae magis directa videatur, ad fractiones continuas perducatur, quae duae methodi cum diverso plane modo negotium conficiant, earum collatio haud contemnendas proprietates patefaciet. Vtramque igitur methodum seorsim exponam, deinceps, quae fuerint eruta, inter se comparaturus.

## Methodus prior, interpolatione innixa.

§. 1. Consideretur series, ex quaesita hoc modo formanda:

1, 2, 3, 4, 5, 6, 7

$a,\ ab,\ abc,\ abcd,\ abcde,\ abcdef,\ abcdefg$, etc.

quae ob

$ab = A,\ bc = B,\ cd = C,\ de = D,\ ef = E$, etc.

in hanc abibit formam:

1, 2, 3, 4, 5, 6, 7

$a;\ A;\ aB;\ AC;\ aBD;\ ACE;\ aBDF$; etc.

cuius ergo termini, locis paribus constituti, ob progressionem A, B, C, D, etc. datam, per se innotescunt.

§. 2. Cum ergo progressio terminorum alternorum

$A;\ AC;\ ACE;\ ACEG$; etc.

sit cognita, eius interpolatio ad verum valorem termini quaesiti



quaesiti $a$ manuducet. At ista progressio semper ita est comparata, vt in infinitum continuata cum eiusmodi progressione simplici confundatur, cuius interpolatio nulli amplius difficultati sit obnoxia. Plerumque autem illa progressio in infinitum producta in geometricam abire solet, ita vt interpolandi sint medii proportionales inter binos contiguos.

§. 3. Si ergo seriem

$a,\ A,\ aB,\ AC,\ aBD,\ ACE$, etc.

totam vt geometricam spectemus, indeque terminos medios definiamus, ab initio multum fortasse a veritate aberrabimus; sed quo longius progrediamur, eo propius ad veritatem accedemus, quam tandem in infinito plane assequemur. Hinc sequentes determinationes ad verum continuo magis appropinquabunt:

$aa = \frac{AA}{B}$ $\qquad aa = \frac{AAC}{BB}$

$aa = \frac{AACC}{BBD}$ $\qquad aa = \frac{AACCE}{BBDD}$

$aa = \frac{AACCEE}{BBDDF}$ $\qquad aa = \frac{AACCEEG}{BBDDFF}$

etc. $\qquad$ etc.

Sicque revera in infinitum progrediendo erit

$aa = A \cdot \frac{AC}{BB} \cdot \frac{CE}{DD} \cdot \frac{EG}{FF} \cdot \frac{GI}{HH} \cdot \frac{IL}{KK} \cdot$ etc.

§. 4. Expressio haec infinita verum valorem ipsius $a$ exhibet, quoties progressio numerorum A, B, C, D, etc. ita est comparata, vt termini infinitesimi inter se rationem aequalitatis teneant, illiusque expressionis factores tandem in vnitatem abeant. Veluti si pro A, B, C, D, etc. series numerorum naturalium accipiatur, vt sit

$$ab = 1,\ bc = 2,\ cd = 3,\ de = 4,\ ef = 5,\ fg = 6, \text{ etc.}$$

erit

$$aa = \frac{1.3}{2.2}\cdot\frac{3.5}{4.4}\cdot\frac{5.7}{6.6}\cdot\frac{7.9}{8.8}\cdot\frac{9.11}{10.10}\cdot\frac{11.13}{12.12} \text{ etc.}$$

Conſtat autem hoc productum infinitum, poſita ratione diametri ad peripheriam $= 1 : \pi$, eſſe $= \frac{2}{\pi}$, ita vt ſit $a = \sqrt{\frac{2}{\pi}}$, hincque $b\sqrt{\frac{\pi}{2}}$; $c = \frac{2\sqrt{2}}{\sqrt{\pi}}$; $d = \frac{3\sqrt{\pi}}{2\sqrt{2}}$; etc.

§. 5. Haec ergo ſeries numerorum tranſcendentalium lege quadam vniformitatis procedit, quos numeros per approximationem euoluiſſe eorumque differentias notaſſe iuuabit.

| | diff. 1. | diff. 2. | diff. 3. |
|---|---|---|---|
| $a = 0,7978845$ | | | |
| $b = 1,2533140$ | 4554295 | | |
| $c = 1,5957690$ | 3424550 | 1129745 | |
| $d = 1,8799710$ | 2842020 | 582530 | 547215 |
| $e = 2,1276920$ | 2477210 | 364810 | 217720 |
| $f = 2,3499637$ | 2222717 | 254493 | 110317 |
| $g = 2,5532304$ | 2032667 | 190050 | 64443 |
| $h = 2,7416243$ | 1883929 | 148738 | 41312 |

Si enim pro $a$ alius quicunque numerus aſſumeretur, ex eoque ſequentes definirentur, in differentiis ingentes ſaltus eſſent apparituri.

§. 6. Eodem modo negotium procedit, ſi pro numeris A, B, C, D, etc. quaecunque capiatur progreſſio arithmetica. Quaerenda enim ſit ſeries $a, b, c, d, e$, etc. ita vt ſit

$a\,b$



$$ab = p;\ bc = p+q;\ cd = p+2q;\ de = p+3q; \text{ etc.}$$

& quia termini infiniteſimi ad rationem aequalitatis accedunt, erit

$$aa = p\cdot\frac{p(p+2q)}{(p+q)(p+q)}\cdot\frac{(p+2q)(p+4q)}{(p+3q)(p+3q)}\cdot\frac{(p+4q)(p+6q)}{(p+5q)(p+5q)} \text{ etc.}$$

cuius expreſſionis valor ita per formulas integrales exhiberi poteſt, vt ſit

$$aa = p\cdot\frac{\int z^{p+q-1}\,dz : \sqrt{(1-z^{2q})}}{\int z^{p-1}\,dz : \sqrt{(1-z^{2q})}}$$

poſito poſt vtramque integrationem $z = 1$.

§. 7. Si pro A, B, C, D, etc. ſumatur progreſſio mixta ex arithmetica & harmonica, vt talis ſeries numerorum $a, b, c, d, e$, etc. ſit inueſtiganda:

$$ab = \frac{p}{r};\ bc = \frac{p+q}{r+s};\ cd = \frac{p+2q}{r+2s};\ de = \frac{p+3q}{r+3s}; \text{ etc.}$$

quia & hic numeri A, B, C, D, etc. ad rationem aequalitatis conuergunt, erit

$$aa = \frac{p}{r}\cdot\frac{p(r+s)(p+2q)(r+s)}{r(p+q)(r+2s)(p+q)}\cdot\frac{(p+2q)(r+3s)(p+4q)(r+3s)}{(r+2s)(p+3q)(r+4s)(p+3q)} \text{ etc.}$$

cuius valor vt ſupra colligitur:

$$aa = \frac{p}{r}\cdot\frac{\int z^{p+q-1}\,dz : \sqrt{(1-z^{2q})}\,.\int z^{r-1}\,dz : \sqrt{(1-z^{2s})}}{\int z^{p-1}\,dz : \sqrt{(1-z^{2q})}\,.\int z^{r+s-1}\,dz : \sqrt{(1-z^{2s})}}$$

vbi iterum poſt integrationem poni oportet $z = 1$.

§. 8. Si fuerit $s = q$, numeri A, B, C, D, etc. continuo propius ad vnitatem accedunt, eique tandem fient aequales. Vnde cum ſeriei

$$a,\ A,\ aB,\ AC,\ aBD,\ ACE, \text{ etc.}$$

termini

termini infinitesimi inter se aequales sint confendi, inde concludetur

$$a = \frac{p}{r} \cdot \frac{(p+2q)(r+q)}{(p+q)(r+2q)} \cdot \frac{(p+4q)(r+3q)}{(p+3q)(r+4q)} \cdot \frac{(p+6q)(r+5q)}{(p+5q)(r+6q)} \text{ etc.}$$

quae expressio etiam ita referri potest:

$$a = \frac{q(r+q)}{r(p+q)} \cdot \frac{(p+2q)(r+3q)}{(r+2q)(p+3q)} \cdot \frac{(p+4q)(r+5q)}{(r+4q)(p+5q)} \text{ etc.}$$

cuius valor per formulas integrales est:

$$a = \frac{\int z^{r-1}\,dz : \sqrt{(1-z^{2q})}}{\int z^{p-1}\,dz : \sqrt{(1-z^{2q})}}.$$

§. 9. Hinc etiam casus, quo $s$ & $q$ sunt inaequales, facilius expediri potest. Sit enim $s = nnq$, ac ponatur $r = nnt$; tum vero statuatur:

$$a = \frac{\alpha}{n};\; b = \frac{\beta}{n};\; c = \frac{\gamma}{n};\; d = \frac{\delta}{n};\; e = \frac{\varepsilon}{n};\; \text{etc.}$$

eritque per conditionem praescriptam:

$$\alpha\beta = \frac{p}{t};\; \beta\gamma = \frac{p+q}{t+q};\; \gamma\delta = \frac{p+2q}{t+2q};\; \delta\varepsilon = \frac{p+3q}{t+3q};\; \text{etc.}$$

ex cuius convenientia cum praecedenti est

$$\alpha = \frac{p(t+q)}{t(p+q)} \cdot \frac{(p+2q)(t+3q)}{(t+2q)(p+3q)} \cdot \frac{(p+4q)(t+5q)}{(t+4q)(p+5q)} \text{ etc.}$$

ideoque

$$\alpha = \frac{\int z^{t-1}\,dz : \sqrt{(1-z^{2q})}}{\int z^{p-1}\,dz : \sqrt{(1-z^{2q})}}.$$

§. 10. Cum igitur sit

$$n = \sqrt{\frac{s}{q}};\; t = \frac{qr}{s} \text{ et } a = \frac{\alpha\sqrt{q}}{\sqrt{s}},$$

erit pro casu in §. 7. exposito:

$$a = \frac{\sqrt{q}}{\sqrt{s}} \cdot \frac{p(r+s)}{r(p+q)} \cdot \frac{(p+2q)(r+3s)}{(r+2s)(p+3q)} \cdot \frac{(p+4q)(r+5s)}{(r+4s)(p+5q)} \text{ etc.}$$





ac per formulas integrales:

$$a = \frac{\sqrt{q}}{\sqrt{s}} \cdot \frac{\int z^{\frac{qr}{s}-1}\,dz : \sqrt{(1-z^{2q})}}{\int z^{p-1}\,dz : \sqrt{(1-z^{2q})}},$$

vbi si in numeratore pro $z^q$ scribatur $z^s$, fiet

$$a = \frac{\sqrt{s}}{\sqrt{q}} \cdot \frac{\int z^{r-1}\,dz : \sqrt{(1-z^{2s})}}{\int z^{p-1}\,dz : \sqrt{(1-z^{2q})}},$$

cuius ergo quadratum aequetur necesse est formulae supra inuentae, ita vt sit

$$\frac{s}{q} \cdot \frac{\int z^{r-1}\,dz : \sqrt{(1-z^{2s})}}{\int z^{p-1}\,dz : \sqrt{(1-z^{2q})}} = \frac{p}{r} \cdot \frac{\int z^{p+q-1}\,dz : \sqrt{(1-z^{2q})}}{\int z^{r+s-1}\,dz : \sqrt{(1-z^{2s})}}.$$

§. 11. Harum ergo formularum consensus casu, quo post integrationem statuitur $z = 1$, sequens nobis suppeditat Theorema:

$$pq \int \frac{z^{p-1}\,dz}{\sqrt{(1-z^{2q})}} \cdot \int \frac{z^{p+q-1}\,dz}{\sqrt{(1-z^{2q})}} = rs \int \frac{z^{r-1}\,dz}{\sqrt{(1-z^{2s})}} \cdot \int \frac{z^{r+s-1}\,dz}{\sqrt{(1-z^{2s})}}.$$

cuius veritatem quidem iam alibi ex aliis principiis demonstratam dedi. Hinc ergo sequitur, sumendo $r = s = 1$ fore

$$pq \int \frac{z^{p-1}\,dz}{\sqrt{(1-z^{2q})}} \cdot \int \frac{z^{p+q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\pi}{2}, \text{ ob}$$

$$\int \frac{dz}{\sqrt{(1-zz)}} = \frac{\pi}{2} \text{ et } \int \frac{z\,dz}{\sqrt{(1-zz)}} = 1.$$

§. 12. Contemplemur igitur aliquot exempla.

I. Si esse debeat

$$ab = 1;\; bc = 2;\; cd = 3;\; de = 4;\; ef = 5;\; \text{etc.}$$

erit

$$aa = 1 \cdot \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdot \frac{7 \cdot 9}{8 \cdot 8} \cdot \text{ etc. et}$$
$$aa \frac{\int z\,dz : \sqrt{(1-zz)}}{\int dz : \sqrt{(1-zz)}} = \frac{2}{\pi}.$$

## II. Si esse debeat

$$ab = 1,\ bc = 3,\ cd = 5,\ de = 7,\ ef = 9, \text{ etc.}$$

erit

$$aa = \frac{1 \cdot 5}{3 \cdot 3} \cdot \frac{5 \cdot 9}{7 \cdot 7} \cdot \frac{9 \cdot 13}{11 \cdot 11} \cdot \frac{13 \cdot 17}{15 \cdot 15} \cdot \text{ etc.}$$

seu

$$aa = \frac{\int zz\,dz : \sqrt{(1-z^4)}}{\int dz : \sqrt{(1-z^4)}}.$$

Cum vero sit ex theoremate modo exposito

$$\frac{\pi}{4} = \int \frac{dz}{\sqrt{(1-z^4)}} \cdot \int \frac{zz\,dz}{\sqrt{(1-z^4)}}, \text{ colligitur}$$
$$a = \frac{2}{\sqrt{\pi}} \int \frac{zz\,dz}{\sqrt{(1-z^4)}}.$$

## III. Si esse debeat

$$ab = 1,\ bc = 4,\ cd = 7,\ de = 10,\ ef = 13, \text{ etc.}$$

erit

$$aa = \frac{1 \cdot 7}{4 \cdot 4} \cdot \frac{7 \cdot 13}{10 \cdot 10} \cdot \frac{13 \cdot 19}{16 \cdot 16} \cdot \text{ etc.}$$

seu

$$aa = \frac{\int z^4\,dz : \sqrt{(1-z^6)}}{\int dz : \sqrt{(1-z^6)}}, \text{ hincque}$$
$$a = \frac{\sqrt{6}}{\sqrt{\pi}} \int \frac{z^3\,dz}{\sqrt{(1-z^6)}}.$$

## IV. Si generalius esse debeat

$$ab = p;\ bc = p+q;\ cd = p+2q;\ de = p+3q;$$
$$ef = p+4q; \text{ etc.}$$

per reductionem, ope Theorematis superioris instituendam, colli-



colligemus.

$$a = p \frac{\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p-1}\,dz}{\sqrt{(1-z^{2q})}}.$$

§. 13. Haec exempla ex progressione arithmetica sunt desumta, quibus adiungamus aliquot, in quibus numerorum A, B, C, D, etc. progressio est mixta ex arithmetica et harmonica.

## I. Si esse debeat

$$ab = \tfrac{1}{2};\ bc = \tfrac{2}{3};\ cd = \tfrac{3}{4};\ de = \tfrac{4}{5};\ ef = \tfrac{5}{6}; \text{ etc.}$$

ob

$$p = 1,\ q = 1,\ r = 2,\ s = 1, \text{ erit}$$
$$a = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdot \frac{7 \cdot 9}{8 \cdot 8} \text{ etc.}$$

seu

$$a = \frac{\int z\,dz : \sqrt{(1-zz)}}{\int dz : \sqrt{(1-zz)}} = \frac{2}{\pi}.$$

## II. Si esse debeat

$$ab = \tfrac{1}{2};\ bc = \tfrac{3}{4};\ cd = \tfrac{5}{6};\ de = \tfrac{7}{8};\ ef = \tfrac{9}{10}; \text{ etc.}$$

ob

$$p = 1;\ q = 2;\ r = 2;\ s = 2; \text{ erit}$$
$$a = \frac{1 \cdot 4}{2 \cdot 3} \cdot \frac{5 \cdot 8}{6 \cdot 7} \cdot \frac{9 \cdot 12}{10 \cdot 11} \cdot \frac{13 \cdot 16}{14 \cdot 15} \cdot \frac{17 \cdot 20}{18 \cdot 19} \cdot \text{ etc.}$$

seu

$$a = \frac{\int zz\,dz : \sqrt{(1-z^4)}}{\int dz : \sqrt{(1-z^4)}} = \frac{\pi}{4} : \int \frac{dz}{\sqrt{(1-z^4)}} = \int \frac{zz\,dz}{\sqrt{(1-z^4)}}.$$

## III. Si esse debeat

$$ab = \tfrac{1}{1};\ bc = \tfrac{2}{3};\ cd = \tfrac{3}{5};\ de = \tfrac{4}{7};\ ef = \tfrac{5}{9}; \text{ etc.}$$

ob

$$p = 1;\ q = 1;\ r = 1;\ s = 2; \text{ erit}$$



a =

$$a = \frac{1}{\sqrt{2}} \cdot \frac{1 \cdot 3}{1 \cdot 2} \cdot \frac{3 \cdot 7}{5 \cdot 4} \cdot \frac{5 \cdot 11}{9 \cdot 6} \cdot \frac{7 \cdot 15}{13 \cdot 8} \cdot \frac{9 \cdot 19}{17 \cdot 10} \cdot \text{ etc.}$$

vel

$$a = \frac{\sqrt{2}}{1} \cdot \frac{\int dz : \sqrt{(1-z^4)}}{\int dz : \sqrt{(1-zz)}} = \frac{2\sqrt{2}}{\pi} \cdot \int \frac{dz}{\sqrt{(1-z^4)}} = \frac{1}{\sqrt{2}} : \int \frac{zz\,dz}{\sqrt{(1-z^4)}}.$$

Productum autem ex hoc valore et praecedente manifesto est $= \frac{1}{\sqrt{2}}$.

## Methodus altera,
### per fractiones continuas.

§. 14. Seriem inueniendam ita cum indicibus repraesentemus

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | | | | $n$ | $n+1$ | . |
| $a$, | $b$, | $c$, | $d$, | $e$, | etc. | . | . | $x$, | $y$, | |

ac primo inuestigemus eam seriem, in qua sit

$$ab = p;\ bc = p+q;\ cd = p+2q;\ de = p+3q; \text{ etc.}$$

vt sit per methodum praecedentem

$$aa = p \cdot \frac{p(p+2q)}{(p+q)(p+q)} \cdot \frac{(p+2q)(p+4q)}{(p+3q)(p+3q)} \cdot \frac{(p+4q)(p+6q)}{(p+5q)(p+5q)} \cdot \text{ etc.}$$

et

$$aa = p \cdot \frac{\int z^{p+q-1}\,dz : \sqrt{(1-z^{2q})}}{\int z^{p-1}\,dz : \sqrt{(1-z^{2q})}},$$

seu

$$a = p\frac{\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p-1}\,dz}{\sqrt{(1-z^{2q})}}, \text{ hinc}$$

$$b = \frac{(p+q)\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+2q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p+q-1}\,dz}{\sqrt{(1-z^{2q})}} \text{ et}$$

$$c = \frac{(p+2q)\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+3q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p+2q-1}\,dz}{\sqrt{(1-z^{2q})}}$$

ideoque





$$x = \frac{(p+nq)\sqrt{2q}}{\sqrt{\pi}} \cdot \int \frac{z^{p+(n+1)q-1}\,dz}{\sqrt{(1-z^{2q})}} = \frac{\sqrt{\pi}}{\sqrt{2q}} : \int \frac{z^{p+nq-1}\,dz}{\sqrt{(1-z^{2q})}}.$$

§. 15. Cum ergo pro hac serie in genere sit $xy = p + nq$, quantitas $x$ eiusmodi functio indicis $n$ esse debet, vt posito in ea $n+1$ loco $n$ prodeat $y$, fiatque productum $xy = p + nq$: quod cum rationalitati aduersetur, quaeri conuenit valores quadratorum $xx$ et $yy$, ex aequatione

$$xxyy = pp + 2npq + nnqq;$$

quandoquidem ratio illa functionum etiam ad quadrata patet. Haec igitur inuestigatio commode latius extendetur ad resolutionem huius aequationis:

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma;$$

vnde valor ipsius $xx$ pluribus modis ad fractiones continuas reduci potest, qui sequentibus lemmatibus innituntur.

## Lemma I.

§. 16. Proposita hac aequatione:

$$(X + \lambda n + \mu)(Y + \lambda n + \nu) = \zeta\zeta nn + 2\zeta\eta n + \theta,$$

in qua Y perinde ex $n+1$ atque X ex $n$ definitur: ponatur

$$X + \lambda n + \mu = \zeta n + f + \frac{h}{X'} \text{ et}$$
$$Y + \lambda n + \nu = \zeta n + g + \frac{h}{Y'},$$

vt sit

$$X = (\zeta - \lambda) n + f - \mu + \frac{h}{X'} \text{ et}$$
$$Y = (\zeta - \lambda) n + g - \nu + \frac{h}{Y'},$$

ubi jam X' et Y' sint novae functiones similes ipsarum $n$ et $n+1$; atque necesse est sit

$$g-\nu=\zeta-\lambda+f-\mu, \text{ seu } g=\zeta-\lambda-\mu+\nu+f.$$

§. 17. Hoc posito aequatio praescripta abit in hanc:

$$\zeta\zeta nn+\zeta(f+g)n+fg+\frac{k(\zeta n+f)}{Y'}+\frac{k(\zeta n+g)}{X'}+\frac{kk}{X'Y'}$$
$$=\zeta\zeta nn+2\zeta\eta n+\theta.$$

Statuatur $f+g=2\eta$ et $k=fg-\theta$, vt prodeat

$$X'Y'+(\zeta n+f)X'+(\zeta n+g)Y'+fg-\theta=0,$$

seu

$$(X'+\zeta n+g)(Y'+\zeta n+f)=\zeta\zeta nn+\zeta(f+g)n+\theta,$$

quae similis est formae propositae. At ob $f+g=2\eta$, habebitur

$$\zeta-\lambda-\mu+\nu+2f=2\eta;\ f=\eta+\frac{\lambda-\zeta+\mu-\nu}{2}, \text{ et}$$
$$g=\eta-\frac{\lambda+\zeta-\mu+\nu}{2},$$

hincque

$$k=fg-\theta=\eta\eta-\tfrac{1}{4}(\lambda-\zeta+\mu-\nu)^2-\theta.$$

§. 18. Quocirca aequatio proposita

$$(X+\lambda n+\mu)(Y+\lambda n+\nu)=\zeta\zeta nn+2\zeta\eta n+\theta,$$

ope huius substitutionis:

$$X=(\zeta-\lambda)n+\eta+\frac{\lambda-\zeta-\mu-\nu}{2}+\frac{\eta\eta-\frac{1}{4}(\lambda-\zeta+\mu-\nu)^2-\theta}{X'}$$

$$Y=(\zeta-\lambda)n+\eta-\frac{\lambda+\zeta-\mu-\nu}{2}+\frac{\eta\eta-\frac{1}{4}(\lambda-\zeta+\mu-\nu)^2-\theta}{Y'}$$

reducitur ad hanc aequationem ipsi propositae similem:

$$(X'+\zeta n+\eta-\tfrac{\lambda+\zeta-\mu+\nu}{2})(Y'+\zeta n+\eta+\tfrac{\lambda-\zeta+\mu-\nu}{2})$$
$$=\zeta\zeta nn+2\zeta\eta n+\theta.$$

§. 19.

ipsarum $n$
$-\nu+f.$
it in hanc:
at
$=0,$
$+g)n+\theta,$
$+g=2\eta,$
et
$\zeta\eta n+\theta,$
similem:
§. 19.



§. 19. Simili modo eadem aequatio proposita

$$(X+\lambda n+\mu)(Y+\lambda n+\nu)=\zeta\zeta nn+2\zeta\eta n+\theta$$

factis his substitutionibus:

$$X=(\zeta-\lambda)n+\eta+\frac{\lambda-\zeta-\mu-\nu}{2}+\frac{\frac{1}{4}(\lambda-\zeta+\mu-\nu)^2-\eta\eta+\theta}{X'}$$

$$Y=(\zeta-\lambda)n+\eta-\frac{\lambda+\zeta-\mu-\nu}{2}+\frac{\frac{1}{4}(\lambda-\zeta+\mu-\nu)^2-\eta\eta+\theta}{Y'}$$

reducitur ad hanc sui similem:

$$(X'-\zeta n-\eta+\tfrac{\lambda-\zeta+\mu-\nu}{2})(Y'-\zeta n-\eta-\tfrac{\lambda-\zeta-\mu+\nu}{2})$$
$$=\zeta\zeta nn+2\zeta\eta n+\theta.$$

## Lemma II.

§. 20. Proposita hac aequatione:

$$(X+\lambda n+\mu)(Y+\lambda n+\nu)=\zeta\zeta nn+2\zeta\eta n+\theta$$

in qua Y perinde ab $n+1$ atque X ab $n$ pendet, ponatur

$$X+\lambda n+\mu=\zeta n+f+\frac{hn+k}{X'}$$
$$Y-\lambda n+\nu=\zeta n+g+\frac{hn+h+k}{Y'},$$

ubi ob similitudinem functionum esse debet vt ante

$$g=\zeta-\lambda-\mu+\nu+f.$$

§. 21. Porro substitutione horum valorum facta habebimus:

$$\zeta\zeta nn+\zeta(f+g)n+fg+\frac{(\zeta n+f)(hn+h+k)}{Y'}+\frac{(\zeta n+g)(hn+k)}{X'}$$
$$+\frac{(hn+k)(hn+h+k)}{X'Y'}=\zeta\zeta nn+2\zeta\eta n+\theta,$$

unde fit:

ζζ

$$(\zeta(f+g-2\eta)n+fg-\theta)X'Y'+(\zeta n+f)(bn+b+k)X'$$
$$+(\zeta n+g)(bn+k)Y'+(bn+k)(bn+b+k)=0,$$

quae vt ſimilis ſit formae propoſitae, diviſibilis eſſe debet per $\zeta(f+g-2\eta)n+fg-\theta$; cui quantitati ergo vel $bn+k$, vel $bn+b+k$ aequale vel multiplum ſtatui oportet.

§. 22. Sit primo

$$bn+k=\alpha\zeta(f+g-2\eta)n+\alpha(fg-\theta),$$

et $\zeta n+f$ ſubmultiplum ipſius $\zeta(f+g-2\eta)n+fg-\theta$ eſſe oportet; vnde fit $f(f+g-2\eta)=fg-\theta$, ſeu $ff=2\eta f-\theta$, hincque

$$f=\eta+\sqrt{(\eta\eta-\theta)} \text{ et } g=\zeta-\lambda-\mu+\nu+\eta+\sqrt{(\eta\eta-\theta)}:$$

quare porro

$$b=\alpha\zeta(f+g-2\eta) \text{ et } k=\alpha(fg-\theta),$$

et aequatio reſultans euadet:

$$X'Y'+\frac{\alpha\zeta(f+g-2\eta)n+\alpha\zeta(f+g-2\eta)+\alpha(fg-\theta)}{f+g-2\eta}X'$$
$$+\alpha(\zeta n+g)Y'+\alpha\alpha(\zeta(f+g-2\eta)n$$
$$+\zeta(f+g-2\eta)+fg-\theta)=0.$$

§. 23. Vt fractiones tollamus ponamus:

$$\alpha=f+g-2\eta=\zeta-\lambda-\mu+\nu+2\sqrt{(\eta\eta-\theta)}$$

ſicque fiet

$$X'Y'+(\zeta(f+g-2\eta)n+\zeta(f+g-2\eta)+fg-\theta)X'$$
$$+(\zeta(f+g-2\eta)n+g(f+g-2\eta))Y'$$
$$+(f+g-2\eta)(\zeta(f+g-2\eta)n+\zeta(f+g-2\eta)+fg-\theta)=0.$$

Verum ſi fractiones non curemus, habebimus:

X'



$$X'Y'+\alpha\left(\zeta n+\zeta+\frac{fg-\theta}{f+g-2\eta}\right)X'+\alpha(\zeta n+g)Y'$$
$$+\alpha\alpha(f+g-2\eta)\left(\zeta n+\zeta+\frac{fg-\theta}{f+g-2\eta}\right)=0,$$

quae aequatio, poſito breuitatis gratia $\frac{fg-\theta}{f+g-2\eta}=\varepsilon$, reducitur ad hanc propoſitae ſimilem:

$$(X'+\alpha(\zeta n+g))(Y'+\alpha(\zeta n+\zeta+\varepsilon))$$
$$=\alpha\alpha(\zeta\zeta nn+\zeta(\zeta+\varepsilon-f+2\eta)+(\zeta+\varepsilon)(2\eta-f))$$
$$=\alpha\alpha(\zeta n+\zeta+\varepsilon)(\zeta n+2\eta-f)$$

§. 24. Propoſita ergo aequatione

$$(X+\lambda n+\mu)(Y+\lambda n+\nu)=\zeta\zeta nn+2\zeta\eta n+\theta,$$

ſi breuitatis gratia ponatur

$$f=\eta+\sqrt{(\eta\eta-\theta)};\ g=\zeta-\lambda-\mu+\nu+\eta+\sqrt{(\eta\eta-\theta)}$$

atque

$$\frac{fg-\theta}{f+g-2\eta}=\varepsilon,$$

ſequens ſubſtitutio:

$$X=(\zeta-\lambda)n+f-\mu+\frac{\zeta(f+g-2\eta)n+fg-\theta}{X'},$$
$$Y=(\zeta-\lambda)n+g-\nu+\frac{\zeta(f+g-2\eta)(n+1)+fg-\theta}{Y'},$$

ſuppeditabit ſequentem aequationem propoſitae ſimilem:

$$(X'+\zeta n+g)(Y'+\zeta(n+1)+\varepsilon)$$
$$=\zeta\zeta nn+\zeta(\zeta+\varepsilon-f+2\eta)n+(\zeta+\varepsilon)(2\eta-f).$$

§. 25. Quemadmodum hic ſumſimus $\alpha=1$, ita poſitio $\alpha=-1$, manentibus iisdem abbreuiationibus, dabit hanc ſubſtitutionem:

$$X=(\zeta-\lambda)n+f-\mu+\frac{\zeta(2\eta-f-g)n-fg+\theta}{X'},$$
$$Y=(\zeta-\lambda)n+g-\nu+\frac{\zeta(2\eta-f-g)(n+1)-fg+\theta}{Y'},$$



vnde

n+b+k)X'
b+k)=0,
s eſſe debet
i ergo vel
m ſtatui o-
−θ),
n+fg−θ
ff=2ηf−θ,
+√(ηη−θ):
),
g−θ) X'
−2η)n
.
us:
η−θ)
fg−θ)X'
η)) Y'
η)+fg−θ)=0.
:
X'

vnde oritur haec aequatio ſimilis propoſitae:

$$(X' - \zeta n - g)(Y' - \zeta(n+1) - \varepsilon)$$
$$= \zeta\zeta nn + \zeta(\zeta + \varepsilon - f + 2\eta)n + (\zeta + \varepsilon)(2\eta - f).$$

§. 26. Ponamus porro eſſe

$$hn + h + k = \zeta(f + g - 2\eta)n + fg - \theta,$$

vt ſit

$$h = \zeta(f + g - 2\eta) \text{ et } k = fg - \theta - \zeta(f + g - 2\eta)$$

ac neceſſe eſt, vt fiat

$$\zeta(f + g - 2\eta)n + fg - \theta = (f + g - 2\eta)(\zeta n + g)$$

ideoque

$$g(f + g - 2\eta) = fg - \theta, \text{ ſeu } g = \eta + \sqrt{(\eta\eta - \theta)},$$

hincque

$$f = \lambda - \zeta + \mu - \nu + \eta + \sqrt{(\eta\eta - \theta)}.$$

Aequatio autem reſultans erit

$$X'Y' + (\zeta n + f)X' + \left(\zeta n - \zeta + \tfrac{fg - \theta}{f + g - 2\eta}\right)Y'$$
$$+ (f + g - 2\eta)\left(\zeta(n - 1) + \tfrac{fg - \theta}{f + g - 2\eta}\right) = 0,$$

quae, poſito $\frac{fg - \theta}{f + g - 2\eta} = \varepsilon$, abit in hanc:

$$(X' + \zeta n - \zeta + \varepsilon)(Y' + \zeta n + f) = \zeta\zeta nn + \zeta n(2\eta - g - \zeta + \varepsilon)$$
$$+ (\varepsilon - \zeta)(2\eta - g) = (\zeta n - \zeta + \varepsilon)(\zeta n + 2\eta - g).$$

§. 27. Propoſita ergo aequatione

$$(X + \lambda n + \mu)(Y + \lambda n + \nu) = \zeta\zeta nn + 2\zeta\eta n + \theta,$$

ſi ponatur breuitatis gratia,

$f =$



$$f = \lambda - \zeta + \mu - \nu + \eta + \sqrt{(\eta\eta - \theta)};$$
$$g = \eta + \sqrt{(\eta\eta - \theta)} \text{ atque } \varepsilon = \tfrac{fg - \theta}{f + g - 2\eta},$$

ſequens ſubſtitutio:

$$X = (\zeta - \lambda)n + f - \mu + \tfrac{\zeta(f + g - 2\eta)(n - 1) + fg - \theta}{X'},$$
$$Y = (\zeta - \lambda)n + g - \nu + \tfrac{\zeta(f + g - 2\eta)n + fg - \theta}{Y'},$$

praebebit hanc aequationem propoſitae ſimilem:

$$(X' + \zeta n - \zeta + \varepsilon)(Y' + \zeta n + f)$$
$$= \zeta\zeta nn + \zeta(2\eta - g - \zeta + \varepsilon)n + (\varepsilon - \zeta)(2\eta - g).$$

§. 28. Simili modo, manentibus iisdem abbreuiaturis, eadem aequatio propoſita ope harum ſubſtitutionum:

$$X = (\zeta - \lambda)n + f - \mu + \tfrac{\zeta(2\eta - f - g)(n - 1) - fg + \theta}{X'},$$
$$Y = (\zeta - \lambda)n + g - \nu + \tfrac{\zeta(2\eta - f - g)n - fg + \theta}{Y'},$$

reducetur ad hanc aequationem propoſitae ſimilem:

$$(X' - \zeta n + \zeta - \varepsilon)(Y' - \zeta n - f)$$
$$= \zeta\zeta nn + \zeta(2\eta - g - \zeta + \varepsilon)n + (\varepsilon - \zeta)(2\eta - g).$$

Ope ergo harum ſenarum reductionum in §§. 18, 19, 24, 25, 27, 28, traditarum omnes huiusmodi aequationes infinitis modis per fractiones continuas reſolui poterunt.

## Reſolutio aequationis

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

per §. 18.

§. 29. Cum hic ſit

$X = xx$; $Y = yy$; $\lambda = 0$; $\mu = 0$; $\nu = 0$; $\zeta = \alpha$; $\eta = \beta$ et $\theta = \gamma$, prodibit haec ſubſtitutio:



xx

$$xx = \alpha n + \beta - \tfrac{1}{2}\alpha + \frac{\beta\beta - \frac{1}{4}\alpha\alpha - \gamma}{X'},$$

$$yy = \alpha n + \beta + \tfrac{1}{2}\alpha + \frac{\beta\beta - \frac{1}{4}\alpha\alpha - \gamma}{Y'},$$

quae deducit ad hanc secundam aequationem:

$$(X' + \alpha n + \beta + \tfrac{1}{2}\alpha)(Y' + \alpha n + \beta - \tfrac{1}{2}\alpha) = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

§. 30. Ad hanc simili modo resoluendam, ob

$$\lambda = \alpha,\ \mu = \beta + \tfrac{1}{2}\alpha;\ \nu = \beta - \tfrac{1}{2}\alpha;\ \zeta = \alpha,\ \eta = \beta,\ \theta = \gamma,$$

consequemur hanc substitutionem:

$$X' = 0 + \frac{\beta\beta - \frac{1}{4}\alpha\alpha - \gamma}{X''};\quad Y' = 0 + \frac{\beta\beta - \frac{1}{4}\alpha\alpha - \gamma}{Y''},$$

quae deducit ad hanc tertiam aequationem:

$$(X'' + \alpha n + \beta - \tfrac{1}{2}\alpha)(Y'' + \alpha n + \beta + \tfrac{1}{2}\alpha) = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

Haec autem porro istas substitutiones praebet:

$$X'' = 0 + \frac{\beta\beta - \frac{1}{4}\alpha\alpha - \gamma}{X'''},\ \text{et}\ Y'' = 0 + \frac{\beta\beta - \frac{1}{4}\alpha\alpha - \gamma}{Y'''},$$

vnde ob $X''' = X'$ et $Y''' = Y'$ nihil vltra concludi potest.

## Resolutio aequationis

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

per §. 19.

§. 31. Factis his substitutionibus:

$$xx = \alpha n - \tfrac{1}{2}\alpha + \beta + \frac{\frac{1}{4}\alpha\alpha - \beta\beta + \gamma}{X},$$

$$yy = \alpha n + \tfrac{1}{2}\alpha + \beta + \frac{\frac{1}{4}\alpha\alpha - \beta\beta + \gamma}{Y},$$

perue-



peruenitur ad hanc aequationem:

$$(X - \alpha n - \tfrac{1}{2}\alpha - \beta)(Y - \alpha n + \tfrac{1}{2}\alpha - \beta) = \alpha\alpha nn + 2\alpha\beta n + \gamma,$$

quae secundum §. 19. reducta, ob

$$\lambda = -\alpha;\ \mu = -\tfrac{1}{2}\alpha - \beta;\ \nu = \tfrac{1}{2}\alpha - \beta;\ \zeta = \alpha;$$
$$\eta = \beta;\ \theta = \gamma;$$

dat has substitutiones:

$$X = 2\alpha n - \alpha + 2\beta + \frac{\frac{1}{4}\alpha\alpha - \beta\beta + \gamma}{X'},$$

$$Y = 2\alpha n + \alpha + 2\beta + \frac{\frac{1}{4}\alpha\alpha - \beta\beta + \gamma}{Y'},$$

vnde nascitur haec noua aequatio:

$$(X' - \alpha n - \tfrac{1}{2}\alpha - \beta)(Y' - \alpha n + \tfrac{1}{2}\alpha - \beta) = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

§ 32. Haec aequatio vlterius reducatur, et ob

$$\lambda = -\alpha;\ \mu = -\tfrac{1}{2}\alpha - \beta;\ \nu = \tfrac{1}{2}\alpha - \beta;\ \zeta = \alpha,\ \eta = \beta,\ \theta = \gamma,$$

habebimus has substitutiones:

$$X' = 2\alpha n - \alpha + 2\beta + \frac{\frac{1}{4}\alpha\alpha - \beta\beta + \gamma}{X''},$$

$$Y' = 2\alpha n + \alpha + 2\beta + \frac{\frac{1}{4}\alpha\alpha - \beta\beta + \gamma}{Y''},$$

hincque hanc aequationem nouam:

$$(X'' - \alpha n - \tfrac{1}{2}\alpha - \beta)(Y'' - \alpha n + \tfrac{1}{2}\alpha - \beta) = \alpha\alpha nn + 2\alpha\beta n + \gamma,$$

vnde sequentes substitutiones facile colliguntur.

§. 33. Quodsi ergo ad abbreviandum ponatur:

$$\alpha n - \tfrac{1}{2}\alpha + \beta = N\ \text{et}\ \beta\beta - \gamma = B,$$

valor ipsius $xx$ sequenti fractione continua exprimetur:



$xx =$

$$xx = N + \cfrac{\frac{1}{4}\alpha\alpha - B}{2N + \cfrac{\frac{9}{4}\alpha\alpha - B}{2N + \cfrac{\frac{25}{4}\alpha\alpha - B}{2N + \cfrac{\frac{49}{4}\alpha\alpha - B}{2N + \cfrac{\frac{81}{4}\alpha\alpha - B}{2N - \text{etc.}}}}}}$$

qui conuenit aequationi propositae

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma.$$

## Resolutio aequationis

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

per §. 24.

§. 34. Cum hic sit

$$\lambda = 0;\ \mu = 0;\ \nu = 0;\ \zeta = \alpha;\ \eta = \beta;\ \theta = \gamma;\ \text{erit}$$
$$f = \beta + \sqrt{(\beta\beta - \gamma)},\ g = \alpha + \beta + \sqrt{(\beta\beta - \gamma)}$$

hinc

$$fg - \theta = \alpha\beta + 2\beta\beta - 2\gamma + (\alpha + 2\beta)\sqrt{(\beta\beta - \gamma)}\ \text{et}$$
$$f + g - 2\eta = \alpha + 2\sqrt{(\beta\beta - \gamma)}.$$

Ponatur ergo

$$\frac{fg - \theta}{f + g - 2\eta} = \beta + \sqrt{(\beta\beta - \gamma)} = \varepsilon,\ \text{ita vt sit}$$
$$\varepsilon = f\ \text{et}\ g = \alpha + f,$$

vnde oriuntur hae substitutiones:

$$xx = \alpha n + f + \frac{(f + g - 2\eta)(\zeta n + f)}{X} = (\alpha n + f)\left(1 + \frac{\alpha + 2\sqrt{(\beta\beta - \gamma)}}{X}\right)$$
$$yy = \alpha n + g + \frac{(f + g - 2\eta)(\alpha n + g)}{Y} = (\alpha n + g)\left(1 + \frac{\alpha + 2\sqrt{(\beta\beta - \gamma)}}{Y}\right),$$

indeque haec aequatio noua:

$$(X + \alpha n + g)(Y + \alpha n + \alpha + f) = \alpha\alpha nn + \alpha(\alpha + 2\beta)n$$
$$+ (\alpha + f)(2\beta - f).$$



§. 35. Ponamus $\beta + \sqrt{(\beta\beta - \gamma)} = \delta$, vt sit $f = \delta$;

$$g = \alpha + \delta\ \text{et}\ f + g - 2\eta = \alpha - 2\beta + 2\delta,$$

sicque substitutiones

$$xx = \alpha n + \delta + \frac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{X}$$
$$yy = \alpha(n + 1) + \delta + \frac{(\alpha - 2\beta + 2\delta)(\alpha(n + 1) + \delta)}{Y}$$

dabunt hanc aequationem:

$$(X + \alpha n + \alpha + \delta)(Y + \alpha n + \alpha + \delta)$$
$$= \alpha\alpha nn + \alpha(\alpha + 2\beta)n + (\alpha + \delta)(2\beta - \delta).$$

§. 36. Pro huius aequationis reductione est

$$\lambda = \alpha;\ \mu = \alpha + \delta;\ \nu = \alpha + \delta;\ \zeta = \alpha;\ \eta = \beta + \tfrac{1}{2}\alpha;$$
$$\theta = (\alpha + \delta)(2\beta - \delta),$$

vnde ob

$$ff - (2\beta + \alpha)f + (\alpha + \delta)(2\beta - \delta) = 0,$$

erit vel $f = \alpha + \delta$, vel $f = 2\beta - \delta$; at prior positio non vlterius deducit, vnde posteriori vtendo erit

$$f = 2\beta - \delta;\ g = 2\beta - \delta\ \text{et}\ \varepsilon = 2\beta - \delta,$$

sicque hae obtinentur substitutiones:

$$X = 2\beta - \alpha - 2\delta + \frac{(2\beta - \alpha - 2\delta)(\alpha n + 2\beta - \delta)}{X'},$$
$$Y = 2\beta - \alpha - 2\delta + \frac{(2\beta - \alpha - 2\delta)(\alpha(n + 1) + 2\beta - \delta)}{Y'},$$

quae hanc praebent aequationem:

$$(X' + \alpha n + 2\beta - \delta)(Y' + \alpha n + \alpha + 2\beta - \delta)$$
$$= \alpha\alpha nn + 2\alpha(\alpha + \beta)n + (\alpha + \delta)(\alpha + 2\beta - \delta).$$

§. 37. Nunc igitur est

$$\lambda = \alpha;\ \mu = 2\beta - \delta;\ \nu = \alpha + 2\beta - \delta,\ \zeta = \alpha;\ \eta = \alpha + \beta;$$
$$\theta = (\alpha + \delta)(\alpha + 2\beta - \delta),$$

vnde ob

$ff -$

$$ff - 2(\alpha+\beta)f + (\alpha+\delta)(\alpha+2\beta-\delta) = 0,$$

sumatur valor

$$f = \alpha + 2\beta - \delta, \text{ erit } g = 2\alpha + 2\beta - \delta,$$

atque

$$\varepsilon = \frac{(\alpha+2\beta-\delta)(\alpha+\beta-2\delta)}{\alpha+2\beta-2\delta} = \alpha + 2\beta - \delta.$$

Quare haec substitutio:

$$X' = \alpha + \frac{(\alpha+2\beta-2\delta)(\alpha n+\alpha+2\beta-\delta)}{X''}$$

$$Y' = \alpha + \frac{(\alpha+2\beta-2\delta)(\alpha n+2\alpha+2\beta-\delta)}{Y''}$$

dabit hanc aequationem:

$$(X''+\alpha n+2\alpha+2\beta-\delta)(Y''+\alpha n+2\alpha+2\beta-\delta)$$
$$= \alpha\alpha nn + \alpha(3\alpha+2\beta)n + (2\alpha+2\beta-\delta)(\alpha+\delta).$$

§. 38. Si vtamur altero valore $f = \alpha + \delta$, fit $g = 2\alpha + \delta$ et $\varepsilon = \alpha + \delta$, et facta substitutione

$$X' = \alpha - 2\beta + 2\delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+\alpha+\delta)}{X''},$$

$$Y' = \alpha - 2\beta + 2\delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{Y''},$$

nanciscimur hanc aequationem:

$$(X''+\alpha n+2\alpha+\delta)(Y''+\alpha n+2\alpha+\delta)$$
$$= \alpha\alpha nn + (3\alpha+2\beta)n + (2\alpha+\delta)(\alpha+2\beta-\delta).$$

§. 39. Prosequamur hanc posteriorem aequationem, quia magis similis est secundae, cum ex ea nascatur ponendo $\delta+\alpha$ pro $\delta$ et $\beta+\alpha$ pro $\beta$, vnde prodit haec substitutio:

$$X'' = 2\beta - \alpha - 2\delta + \frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+\alpha-\delta)}{X'''},$$

$$Y'' = 2\beta - \alpha - 2\delta + \frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+2\alpha-\delta)}{Y'''},$$

quae ducit ad hanc aequationem:

$$(X'''+\alpha n+2\beta+\alpha-\delta)(Y'''+\alpha n+2\alpha+2\beta-\delta)$$
$$= \alpha\alpha nn + 2\alpha(2\alpha+\beta)n + (2\alpha+\delta)(2\alpha+2\beta-\delta).$$

§. 40.



§. 40. Haec aequatio porro vti in §. 38. tractata ope harum substitutionum:

$$X''' = \alpha - 2\beta + 2\delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+2\alpha+\delta)}{X''''}$$

$$Y''' = \alpha - 2\beta + 2\delta + \frac{(\alpha-2\beta+2\delta)(\alpha n+3\alpha+\delta)}{Y''''},$$

reducitur ad hanc:

$$(X''''+\alpha n+3\alpha+\delta)(Y''''+\alpha n+3\alpha+\delta)$$
$$= \alpha\alpha nn + \alpha(5\alpha+2\beta)n + (3\alpha+\delta)(2\alpha+2\beta-\delta)$$

haecque vlterius per has substitutiones:

$$X'''' = 2\beta - \alpha - 2\delta + \frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+2\alpha-\delta)}{X'''''}$$

$$Y'''' = 2\beta - \alpha - 2\delta + \frac{(2\beta-\alpha-2\delta)(\alpha n+2\beta+3\alpha-\delta)}{Y'''''},$$

ad istam reducitur:

$$(X'''''+\alpha n+2\beta+2\alpha-\delta)(Y'''''+\alpha n+3\alpha+2\beta-\delta)$$
$$= \alpha\alpha nn + 2\alpha(3\alpha+\beta)n + (3\alpha+\delta)(3\alpha+2\beta-\delta).$$

§. 41. Hinc ergo valor ipsius $xx$ ex hac aequatione:

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma,$$

posito breuitatis causa

$$\beta + \surd(\beta\beta - \gamma) = \delta \text{ et } \alpha - 2\beta + 2\delta = A, \text{ erit}$$

$$xx = \alpha n + \delta + \cfrac{A(\alpha n+\delta)}{-A - \cfrac{A(\alpha n+2\beta-\delta)}{A + \cfrac{A(\alpha n+\alpha+\delta)}{-A - \cfrac{A(\alpha n+\alpha+2\beta-\delta)}{A + \cfrac{A(\alpha n+2\alpha+\delta)}{-A - \cfrac{A(\alpha n+2\alpha+2\beta-\delta)}{A + \cfrac{A(\alpha n+3\alpha+\delta)}{-A - \cfrac{A(\alpha n+3\alpha+2\beta-\delta)}{A + \text{etc.}}}}}}}}}$$



Haec

Haec autem expreſſio euoluta praebet pro $xx$ ipſum illud productum ex infinitis factoribus conſtans, quod per methodum priorem elicitur.

§. 42. Iſta fractio continua ſimplicius hoc modo exprimi poteſt:

$$xx = \alpha n + \delta - \cfrac{(\alpha n + \delta)}{1 + \alpha n + 2\beta - \delta + \cfrac{A - (\alpha n + \alpha + \delta)}{1 + \alpha n + \alpha + 2\beta - \delta + \cfrac{A - (\alpha n + 2\alpha + \delta)}{1 + \alpha n + 2\alpha + 2\beta - \delta + \cfrac{A - (\alpha n + 3\alpha + \delta)}{1 + \alpha n + 3\alpha + 2\beta - \delta + \cfrac{A - (\alpha n + 4\alpha + \delta)}{1 + \text{etc.}}}}}}$$

Sin autem formulae §. 37 hoc modo vlterius reducentur, inuenitur haec expreſſio ab initio irregularis:

$$xx = \alpha n + \delta - \cfrac{(\alpha n + \delta)}{1 + \alpha n + 2\beta - \delta + \cfrac{\alpha - (\alpha n + \alpha + 2\beta - \delta)}{1 + \alpha n + \alpha + \delta + \cfrac{A - (\alpha n + 2\alpha + 2\beta - \delta)}{1 + \alpha n + 2\alpha + \delta + \cfrac{A - (\alpha n + 3\alpha + 2\beta - \delta)}{1 + \alpha n + 3\alpha + \delta + \cfrac{}{A - \text{etc.}}}}}}$$

§. 43.



§. 43. Si vtraque expreſſio capite communi truncetur, pro $2\beta$ valor aſſumtus $\alpha + 2\delta - A$ ſubſtituatur, inſuperque pro $\alpha n + \alpha + \delta$ ſcribatur N, habebitur haec aequalitas:

$$A - \cfrac{N}{1 + N + \alpha - A + \cfrac{A - N - \alpha}{1 + N + 2\alpha - A + \cfrac{A - N - 2\alpha}{1 + N + 3\alpha - A + \cfrac{}{A - \text{etc.}}}}}$$

$$= \alpha - \cfrac{N - \alpha + A}{1 + N + \cfrac{A - N - 2\alpha + A}{1 + N + \alpha + \cfrac{A - N - 3\alpha + A}{1 + N + 2\alpha + \cfrac{}{A - \text{etc.}}}}}$$

vbi pro A, $\alpha$ et N numeri quicunque aſſumi poſſunt.

## Reſolutio aequationis

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma,$$

ope §. 25.

§. 44. Prima ſubſtitutio, ex reſolutione praecedente, ſumendis X et Y negatiuis, petita

$$xx = \alpha n + \delta + \frac{(2\beta - \alpha - 2\delta)(\alpha n + \delta)}{X}$$

$$yy = \alpha(n + 1) + \delta + \frac{(2\beta - \alpha - 2\delta)(\alpha n + \alpha + \delta)}{Y};$$

 poſito

posito $\delta = \beta + \sqrt{(\beta\beta - \gamma)}$ deducit ad hanc aequationem:

$$(X - \alpha n - \alpha - \delta)(Y - \alpha n - \alpha - \delta)$$
$$= \alpha\alpha nn + \alpha(\alpha + 2\beta)n + (\alpha + \delta)(2\beta - \delta),$$

quae cum §. 25. comparata praebet

$$\lambda = -\alpha;\ \mu = -\alpha - \delta;\ \nu = -\alpha - \delta;\ \zeta = \alpha;$$
$$\eta = \tfrac{1}{2}\alpha + \beta;\ \theta = (\alpha + \delta)(2\beta - \delta),$$

vnde colligitur

$$ff - (\alpha + 2\beta)f + (\alpha + \delta)(2\beta - \delta) = 0.$$

Sit $f = \alpha + \delta$, erit $g = 3\alpha + \delta$ et $\varepsilon = \alpha + \delta$; hincque nascitur haec substitutio:

$$X = 2\alpha n + 2\alpha + 2\delta - \frac{(3\alpha - 2\beta + 2\delta)(\alpha n + \alpha + \delta)}{X'}$$
$$Y = 2\alpha n + 4\alpha + 2\delta - \frac{(3\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{Y'}$$

quae ducit ad sequentem aequationem:

$$(X' - \alpha n - 3\alpha - \delta)(Y' - \alpha n - 2\alpha - \delta)$$
$$= \alpha\alpha nn + \alpha(2\alpha + 2\beta)n + (2\alpha + \delta)(2\beta - \delta).$$

§. 45. Tractetur haec aequatio simili modo secundum §. 25. et ob valores

$$\lambda = -\alpha;\ \mu = -3\alpha - \delta;\ \nu = -2\alpha - \delta;\ \zeta = \alpha;$$
$$\eta = \alpha + \beta;\ \theta = (2\alpha + \delta)(2\beta - \delta).$$

erit

$$ff - (2\alpha + 2\beta)f + (2\alpha + \delta)(2\beta - \delta) = 0,$$

vnde sumatur $f = 2\alpha + \delta$, fietque $g = 5\alpha + \delta$ et $\varepsilon = 2\alpha + \delta$. Habebitur ergo ista substitutio:

$$X' = 2\alpha n + 5\alpha + 2\delta - \frac{(5\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{X''}$$
$$Y' = 2\alpha n + 7\alpha + 2\delta - \frac{(5\alpha - 2\beta + 2\delta)(\alpha n + 3\alpha + \delta)}{Y''},$$

quae



quae praebet hanc aequationem:

$$(X'' - \alpha n - 5\alpha - \delta)(Y'' - \alpha n - 3\alpha - \delta)$$
$$= \alpha\alpha nn + \alpha(3\alpha + 2\beta)n + (3\alpha + \delta)(2\beta - \delta).$$

§. 46. Nunc igitur eodem modo erit

$$\lambda = -\alpha;\ \mu = -5\alpha - \delta;\ \nu = -3\alpha - \delta;$$
$$\zeta = \alpha;\ \eta = \tfrac{3}{2}\alpha + \beta;\ \theta = (3\alpha + \delta)(2\beta - \delta);$$

vnde ob $f = 3\alpha + \delta$ colligitur $g = 7\alpha + \delta$ et $\varepsilon = 3\alpha + \delta$. Substitutio ergo

$$X'' = 2\alpha n + 8\alpha + 2\delta - \frac{(7\alpha - 2\beta + 2\delta)(\alpha n + 3\alpha + \delta)}{X'''},$$
$$Y'' = 2\alpha n + 10\alpha + 2\delta - \frac{(7\alpha - 2\beta + 2\delta)(\alpha n + 4\alpha + \delta)}{Y'''},$$

istam dabit aequationem:

$$(X''' - \alpha n - 7\alpha - \delta)(Y''' - \alpha n - 4\alpha - \delta)$$
$$= \alpha\alpha nn + \alpha(4\alpha + 2\beta) + (4\alpha + \delta)(2\beta - \delta).$$

§. 47. Cum lex progressionis hic sit satis manifesta, facile concluditur fore:

$$xx = \alpha n + \delta - \cfrac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{2\alpha n + 2\alpha + 2\delta - \cfrac{(3\alpha - 2\beta + 2\delta)(\alpha n + \alpha + \delta)}{2\alpha n + 5\alpha + 2\delta - \cfrac{(5\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{2\alpha n + 8\alpha + 2\delta - \cfrac{(7\alpha - 2\beta + 2\delta)(\alpha n + 3\alpha + \delta)}{2\alpha n + 11\alpha + 2\delta - \text{etc.}}}}}$$

vbi notandum est, ex aequatione proposita

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

duplici modo dari $\delta$, cum sit $\delta = \beta \pm \sqrt{(\beta\beta - \gamma)}$, sicque binae eiusmodi series obtinentur, quarum altera proditura fuisset, si vbique pro $f$ alteros valores assumsissemus.

 Alia

## Alia resolutio,
binos valores ipsius $f$ alternando.

§. 48. Sumamus in resolutione §. 44. $f = 2\beta - \delta$, vt sit $g = 2\alpha + 2\beta - \delta$ et $\varepsilon = 2\beta - \delta$, erit substitutio:

$$X = 2\alpha n + \alpha + 2\beta - \frac{(\alpha + 2\beta - 2\delta)(\alpha n + 2\beta - \delta)}{X'},$$

$$Y = 2\alpha n + 3\alpha + 2\beta - \frac{(\alpha + 2\beta - 2\delta)(\alpha n + \alpha + 2\beta - \delta)}{Y'}$$

vnde resultat haec aequatio:

$$(X' - \alpha n - 2\alpha - 2\beta + \delta)(Y' - \alpha n - \alpha - 2\beta + \delta)$$
$$= \alpha\alpha nn + \alpha(2\alpha + 2\beta)n + (\alpha + \delta)(\alpha + 2\beta - \delta),$$

quae ex superiori oritur, si ibi pro $\delta$ scribatur $-\alpha + 2\beta - \delta$, quo valore in sequentibus retento fiet:

$$xx = \alpha n + \delta \cdot \cfrac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{2\alpha n + \alpha + 2\beta - \cfrac{(\alpha + 2\beta - 2\delta)(\alpha n + 2\beta - \delta)}{2\alpha n + 3\alpha + 4\beta - 2\delta - \cfrac{(3\alpha + 2\beta - 2\delta)(\alpha n + \alpha + 2\beta - \delta)}{2\alpha n + 6\alpha + 4\beta - 2\delta - \cfrac{(5\alpha + 2\beta - 2\delta)(\alpha n + 2\alpha + 2\beta - \delta)}{2\alpha n + 9\alpha + 4\beta - 2\delta - \text{etc.}}}}}$$

§ 49. At aequatio modo eruta cum §. 25. collata dat

$$\lambda = -\alpha;\ \mu = -2\alpha - 2\beta + \delta;\ \nu = -\alpha - 2\beta + \delta;$$
$$\zeta = \alpha;\ \eta = \alpha + \beta;\ \theta = (\alpha + \delta)(\alpha + 2\beta - \delta) \text{ et}$$
$$ff - 2(\alpha + \beta)f + (\alpha + \delta)(\alpha + 2\beta - \delta) = 0.$$

Si hic sumeremus $f = \alpha + 2\beta - \delta$, haberemus formulam modo inuentam. Sit ergo $f = \alpha + \delta$, erit $g = 4\alpha + \delta$, et $\varepsilon = \alpha + \delta$, ideoque:

$$X' = 2\alpha n + 3\alpha + 2\beta - \frac{(3\alpha - 2\beta + 2\delta)(\alpha n + \alpha + \delta)}{X''},$$

$$Y' = 2\alpha n + 5\alpha + 2\beta - \frac{(3\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{Y''},$$

hinc-



hincque ista nascitur aequatio:

$$(X'' - \alpha n - 4\alpha - \delta)(Y'' - \alpha n - 2\alpha - \delta)$$
$$= \alpha\alpha nn + \alpha(3\alpha + 2\beta)n + (2\alpha + \delta)(\alpha + 2\beta - \delta),$$

quae ex praecedente oritur, si ibi pro $\delta$ scribatur $-\alpha + \delta$, sicque erit:

$$xx = \alpha n + \delta \cdot \cfrac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{2\alpha n + \alpha + 2\beta - \cfrac{(\alpha + 2\beta - 2\delta)(\alpha n + 2\beta - \delta)}{2\alpha n + 3\alpha + 2\beta - \cfrac{(3\alpha - 2\beta + 2\delta)(\alpha n + 2\alpha + \delta)}{2\alpha n + 6\alpha + 2\beta - \text{etc.}}}}$$

§. 50. Verum illa aequatio altero modo resoluta, ob valores

$$\lambda = -\alpha;\ \mu = -4\alpha - \delta;\ \nu = -2\alpha - \delta;$$
$$\zeta = \alpha;\ \eta = \tfrac{3}{2}\alpha + \beta;\ \theta = (2\alpha + \delta)(\alpha + 2\beta - \delta); \text{ dat}$$
$$ff - (3\alpha + 2\beta)f + (2\alpha + \delta)(\alpha + 2\beta - \delta) = 0;$$

vnde nunc sumamus

$f = \alpha + 2\beta - \delta$, vt sit $g = 5\alpha + 2\beta - \delta$ et $\varepsilon = \alpha + 2\beta - \delta$ prodibitque haec substitutio:

$$X'' = 2\alpha n + 5\alpha + 2\beta - \frac{(3\alpha + 2\beta - 2\delta)(\alpha n + \alpha + 2\beta - \delta)}{X'''},$$

$$Y'' = 2\alpha n + 7\alpha + 2\beta - \frac{(3\alpha + 2\beta - 2\delta)(\alpha n + 2\alpha + 2\beta - \delta)}{Y'''},$$

quae ducit ad hanc aequationem:

$$(X''' - \alpha n - 5\alpha - 2\beta - \delta)(Y''' - \alpha n - 2\alpha - 2\beta + \delta)$$
$$= \alpha\alpha nn + \alpha(4\alpha + 2\beta)n + (2\alpha + \delta)(2\alpha + 2\beta - \delta).$$

§. 51. Cum lex progressionis hinc iam colligi possit, habebimus:

$$xx = \alpha n + \delta - \cfrac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{2\alpha n + \alpha + 2\beta - \cfrac{(\alpha + 2\beta - 2\delta)(\alpha n + 2\beta - \delta)}{2\alpha n + 3\alpha + 2\beta - \cfrac{(3\alpha - 2\beta + 2\delta)(\alpha n + \alpha + \delta)}{2\alpha n + 5\alpha + 2\beta - \cfrac{(3\alpha + 2\beta - 2\delta)(\alpha n + \alpha + 2\beta - \delta)}{2\alpha n + 7\alpha + 2\beta - \text{etc.}}}}}$$

quae

quae fractio continua ob satis concinnam progressionis legem est notatu digna.

## Resolutio aequationis
$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma.$
Per §. 28.

§. 52. Posito $\delta = \beta + \nu(\beta\beta - \gamma)$, ob $\lambda = 0$, $\mu = 0$, $\nu = 0$, $\zeta = \alpha$, $\eta = \beta$, $\theta = \gamma$, erit $g = \delta$, $f = -\alpha + \delta$ et $\varepsilon = \delta$; vnde substitutio

$$xx = \alpha n - \alpha + \delta + \frac{(-\alpha - 2\beta + 2\delta)(\alpha n - \alpha + \delta)}{X},$$
$$yy = \alpha n + \delta + \frac{(-\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{Y},$$

dabit hanc aequationem ex §. 27.

$$(X + \alpha n - \alpha + \delta)(Y + \alpha n + \delta)$$
$$= \alpha\alpha nn + \alpha(2\beta - \alpha)n + (\delta - \alpha)(2\beta - \delta).$$

Sumtis autem X et Y negatiuis, vt fit ex §. 28.

$$xx = \alpha n - \alpha + \delta + \frac{(\alpha + 2\beta - 2\delta)(\alpha n - \alpha + \delta)}{X},$$
$$yy = \alpha n + \delta + \frac{(\alpha + 2\beta - 2\delta)(\alpha n + \delta)}{Y},$$

habebitur:

$$(X - \alpha n + \alpha - \delta)(Y - \alpha n - \delta)$$
$$= \alpha\alpha nn - \alpha(\alpha - 2\beta)n - (\alpha - \delta)(2\beta - \delta).$$

§. 53. Hacc aequatio porro secundum easdem formulas tractata praebet:

$\lambda = -\alpha$; $\mu = \alpha - \delta$; $\nu = -\delta$; $\zeta = \alpha$;
$\eta = -\frac{1}{2}\alpha + \beta$; $\theta = (\delta - \alpha)(2\beta - \delta)$, vnde fit
$gg - (2\beta - \alpha)g + (\delta - \alpha)(2\beta - \delta) = 0$,

ergo

onis le-
, μ=0,
+ δ et
),
- δ).
).
lem for-
fit
ergo



ergo vel $g = \delta - \alpha$, vel $g = 2\beta - \delta$, et $f = -\alpha + g$ atque $\varepsilon = g$. Quare substitutio erit:

$$X = 2\alpha n - 2\alpha + g + \delta + \frac{(2\beta - 2g)(\alpha n - \alpha + g)}{X'},$$
$$Y = 2\alpha n + g + \delta + \frac{(2\beta - 2g)(\alpha n + g)}{Y'},$$

quae ducit ad hanc aequationem:

$$(X' - \alpha n + \alpha - g)(Y' - \alpha n + \alpha - g)$$
$$= \alpha\alpha nn + \alpha(2\beta - 2\alpha)n + (g - \alpha)(2\beta - \alpha - g).$$

§. 54. Retineamus hanc litteram $g$ geminum valorem inuoluentem, et sequentes per $g'$, $g''$ indicemus. Cum ergo hic fit $\lambda = -\alpha$; $\mu = \alpha - g$; $\nu = \alpha - g$; $\zeta = \alpha$; $\eta = -\alpha + \beta$; $\theta = (g - \alpha)(2\beta - \alpha - g)$; erit vel $g' = g - \alpha$, vel $g' = 2\beta - \alpha - g$; hincque $f = -2\alpha + g'$ et $\varepsilon = g'$, ideoque

$$X' = 2\alpha n - 3\alpha + g + g' + \frac{(2\beta - 2g')(\alpha n - \alpha + g')}{X''},$$
$$Y' = 2\alpha n - \alpha + g + g' + \frac{(2\beta - 2g')(\alpha n + g')}{Y''},$$

vnde prodit haec aequatio:

$$(X'' - \alpha n + \alpha - g')(Y'' - \alpha n + 2\alpha - g')$$
$$= \alpha\alpha nn + \alpha(2\beta - 3\alpha)n + (g' - \alpha)(2\beta - 2\alpha - g').$$

§. 55. Nunc igitur porro erit:

$\lambda = -\alpha$; $\mu = \alpha - g'$; $\nu = 2\alpha - g'$; $\zeta = \alpha$;
$\eta = \beta - \frac{1}{2}\alpha$; $\theta = (g' - \alpha)(2\beta - 2\alpha - g')$;

hincque vel $g'' = g' - \alpha$, vel $g'' = 2\beta - 2\alpha - g'$ et
$f = -3\alpha + g''$ atque $\varepsilon = g''$.

Quare substitutio

$$X'' = 2\alpha n - 4\alpha + g' + g'' + \frac{(2\beta - 2g'')(\alpha n - \alpha + g'')}{X'''},$$

$$Y'' = 2\alpha n - 2\alpha + g' + g'' + \frac{(2\beta - 2g'')(\alpha n + g'')}{Y'''},$$

dabit hanc aequationem:

$$(X''' - \alpha n + \alpha - g'')(Y''' - \alpha n + 3\alpha - g'')$$
$$= \alpha\alpha nn + \alpha(2\beta - 4\alpha)n + (g'' - \alpha)(2\beta - 3\alpha - g'').$$

§. 56. Iam pro huius aequationis resolutione est:

$$\lambda = -\alpha;\ \mu = \alpha - g'';\ \nu = 3\alpha - g'';\ \zeta = \alpha;$$
$$\eta = \beta - 2\alpha;\ \theta = (g'' - \alpha)(2\beta - 3\alpha - g'')$$

vnde vel

$$g''' = g'' - \alpha, \text{ vel } g''' = 2\beta - 3\alpha - g'',$$
$$f = -4\alpha + g''' \text{ atque } \varepsilon = g'''.$$

Quare ex substitutione

$$X''' = 2\alpha n - 5\alpha + g'' + g''' + \frac{(2\beta - 2g''')(\alpha n - \alpha + g''')}{X''''},$$

$$Y''' = 2\alpha n - 3\alpha + g'' + g''' + \frac{(2\beta - 2g''')(\alpha n + g''')}{Y''''},$$

orietur haec aequatio:

$$(X'''' - \alpha n + \alpha - g''')(Y'''' - \alpha n + 4\alpha - g''')$$
$$= \alpha\alpha nn + \alpha(2\beta - 5\alpha)n + (g''' - \alpha)(2\beta - 4\alpha - g''').$$

§. 57. His igitur colligendis ex aequatione proposita

$$xxyy = \alpha\alpha nn + 2\alpha\beta n + \gamma$$

posito $\delta = \beta + \sqrt{(\beta\beta - \gamma)}$, si pro litteris $g$, $g'$, $g''$, $g'''$ etc. sequentes valores gemini assumantur:

$$g = \left\{\begin{matrix}\delta - \alpha \\ 2\beta - \delta\end{matrix}\right);\ g' = \left\{\begin{matrix}g - \alpha \\ 2\beta - \alpha - g\end{matrix}\right);\ g'' = \left\{\begin{matrix}g' - \alpha \\ 2\beta - 2\alpha - g'\end{matrix}\right);\ g''' = \left\{\begin{matrix}g'' - \alpha \\ 2\beta - 3\alpha - g''\end{matrix}\right);\text{etc.}$$

colligetur pro $xx$ sequens valor:

$xx =$



$$xx = \alpha n - \alpha + \delta + \cfrac{(\alpha + 2\beta - 2\delta)(\alpha n - \alpha + \delta)}{2\alpha n - 2\alpha + \delta + g + \cfrac{(2\beta - 2g)(\alpha n - \alpha + g)}{2\alpha n - 3\alpha + g + g' + \cfrac{(2\beta - 2g')(\alpha n - \alpha + g')}{2\alpha n - 4\alpha + g' + g'' + \text{etc.}}}}$$

qui ergo ob geminos valores singularum $\delta$, $g$, $g'$, $g''$, $g'''$ etc. in infinitum variari potest.

§. 58. Si eadem aequatio simili modo, retentis valoribus omnibus ambiguis, secundum §. 25. resoluatur, ac sumto $\delta = \beta \pm \sqrt{(\beta\beta - \gamma)}$ ponatur

$$f = \left\{\begin{matrix}\alpha + \delta \\ 2\beta - \delta\end{matrix}\right);\ f' = \left\{\begin{matrix}\alpha + f \\ \alpha + 2\beta - f\end{matrix}\right);\ f'' = \left\{\begin{matrix}\alpha + f' \\ 2\alpha + 2\beta - f'\end{matrix}\right);\ f''' = \left\{\begin{matrix}\alpha + f'' \\ 3\alpha + 2\beta - f''\end{matrix}\right);\text{etc.}$$

erit

$$xx = \alpha n + \delta - \cfrac{(\alpha - 2\beta + 2\delta)(\alpha n + \delta)}{2\alpha n + \alpha + \delta + f - \cfrac{(\alpha - 2\beta + 2f)(\alpha n + f)}{2\alpha n + 2\alpha + f + f' - \cfrac{(\alpha - 2\beta + 2f')(\alpha n + f')}{2\alpha n + 3\alpha + f' + f'' - \cfrac{(\alpha - 2\beta + 2f'')(\alpha n + f'')}{2\alpha n + 4\alpha + f'' + f''' - \text{etc.}}}}}$$

§. 59. Simili modo aequationem propositam secundum §. 24. tractando, si posito $\delta = \beta \pm \sqrt{(\beta\beta - \gamma)}$ statuatur vt ante:

$$f = \left\{\begin{matrix}\alpha + \delta \\ 2\beta - \delta\end{matrix}\right);\ f' = \left\{\begin{matrix}\alpha + f \\ \alpha + 2\beta - f\end{matrix}\right);\ f'' = \left\{\begin{matrix}\alpha + f' \\ 2\alpha + 2\beta - f'\end{matrix}\right);\ f''' = \left\{\begin{matrix}\alpha + f'' \\ 3\alpha + 2\beta - f''\end{matrix}\right);\text{etc.}$$

erit

$$xx = \alpha n + \delta - \cfrac{(2\beta\alpha - \alpha - 2\delta)(\alpha n + \delta)}{-\alpha - \delta + f - \cfrac{(2\beta + \alpha - 2f)(\alpha n + f)}{-f + f' - \cfrac{(2\beta + \alpha - 2f')(\alpha n + f')}{-\alpha - f' + f'' - \cfrac{(2\beta + 3\alpha - 2f'')(\alpha n + f'')}{-f'' + f''' - \cfrac{(2\beta + 3\alpha - 2f''')(\alpha n + f''')}{-\alpha - f''' + f'''' - \cfrac{(2\beta + 5\alpha - 2f'''')(\alpha n + f'''')}{-f'''' + f''''' - \text{etc.}}}}}}}$$



§. 60.

§. 60. Porro ex §. 27. si post
$\delta = \beta \pm \sqrt{(\beta\beta - \gamma)}$, statuatur

$$g = \left\{\begin{matrix}\delta - \alpha\\ 2\beta - \delta\end{matrix}\right\};\; g' = \left\{\begin{matrix}g - \alpha\\ 2\beta - \alpha - g\end{matrix}\right\};\; g'' = \left\{\begin{matrix}g' - \alpha\\ 2\beta - 2\alpha - g'\end{matrix}\right\};\; g''' = \left\{\begin{matrix}g'' - \alpha\\ 2\beta - 3\alpha - g''\end{matrix}\right\};$$

erit

$$xx = an - \alpha + \delta - \cfrac{(2\beta - 2\delta + \alpha)(an - \alpha + \delta)}{g - \delta - \cfrac{(2\beta - 2g)(an - \alpha + g)}{g' - g + \alpha - \cfrac{(2\beta - 2g' - 2\alpha)(an - \alpha + g')}{g'' - g' - \cfrac{(2\beta - 2g'' - 2\alpha)(an - \alpha + g'')}{g''' - g'' + \alpha - \cfrac{(2\beta - 2g''' - 4\alpha)(an - \alpha + g''')}{g'''' - g''' - \cfrac{(2\beta - 2g'''' - 4\alpha)(an - \alpha + g'''')}{g'''' - g''' + \alpha - \text{etc.}}}}}}}$$

§. 61. Possent autem permiscendis his reductionibus innumerabiles aliae fractiones continuae elici, quae omnes valorem ipsius $xx$ exprimerent; verum his quaternis formis generalibus, quibus prima §. 33. exhibita addi potest, acquiescamus, easque ad casum quempiam determinatum accommodemus. Sit scilicet $xxyy = nn$, seu quaeratur eiusmodi series $a, b, c, d, e, f$, etc. vt sit $ab = 1$; $bc = 2$; $cd = 3$; $de = 4$; $ef = 5$; etc. . . . $xy = n$, atque iam notauimus (12) fore

$$aa = \tfrac{2}{\pi};\; bb = \tfrac{\pi}{2};\; cc = \tfrac{4}{1}\cdot\tfrac{2}{\pi};\; dd = \tfrac{9}{4}\cdot\tfrac{\pi}{2};\; ee = \tfrac{4\cdot 16}{1\cdot 9}\cdot\tfrac{2}{\pi};\; \text{etc.}$$

Deinde vero ex §. 6. colligitur

$$xx = n.\frac{\int z^n\, dz : \sqrt{(1 - zz)}}{\int z^{n-1}\, dz : \sqrt{(1 - zz)}}$$

seu per productum infinitum:

$$xx = n.\frac{n(n+2)}{(n+1)^2}.\frac{(n+2)(n+4)}{(n+3)^2}.\frac{(n+4)(n+6)}{(n+5)^2}.\frac{(n+6)(n+8)}{(n+7)^2}\;\text{etc.}$$

Hunc igitur valorem ipsius $xx$ quemadmodum per fractiones continuas exprimi possit, videamus.

§. 62.

$g'' - \alpha$ ; $2\beta - 3\alpha - g''$ ; $g''''$) ($an - \alpha + g''''$) — etc. reductioni- elici, quae n his qua- 3. exhibita quempiam $yy = nn$, etc. vt sit etc. . . . etc. etc. per frac- §. 62.



§. 62. Cum igitur pro aequatione $xxyy = nn$ sit $\alpha = 1$; $\beta = 0$; $\gamma = 0$; erit secundum §. 33. $N = n - \tfrac{1}{2}$, et $B = 0$, vnde fit:

$$xx = n - \tfrac{1}{2} + \cfrac{1:4}{2n - 1 + \cfrac{9:4}{2n - 1 + \cfrac{25:4}{2n - 1 + \cfrac{49:4}{2n - 1 + \text{etc.}}}}}$$

siue

$$2xx = 2n - 1 + \cfrac{1}{2(2n - 1) + \cfrac{9}{2(2n - 1) + \cfrac{25}{2(2n - 1) + \cfrac{49}{2(2n - 1) + \cfrac{81}{\text{etc.}}}}}}$$

§. 63. Porro ex §. 59. ob $\beta = 0$; $\gamma = 0$ et $\delta = 0$; si sumatur

$$f = \left\{\begin{matrix}1\\0\end{matrix}\right\};\; f' = \left\{\begin{matrix}1 + f\\ 1 - f\end{matrix}\right\};\; f'' = \left\{\begin{matrix}1 + f'\\ 2 - f'\end{matrix}\right\};\; f''' = \left\{\begin{matrix}1 + f''\\ 3 - f''\end{matrix}\right\};\; f'''' = \left\{\begin{matrix}1 + f'''\\ 4 - f'''\end{matrix}\right\}$$

erit

$$xx = n + \cfrac{n}{f - 1 - \cfrac{(1 - 2f)(n + f)}{f' - f - \cfrac{(1 - 2f')(n + f')}{f'' - f' - 1 - \cfrac{(3 - 2f'')(n + f'')}{f''' - f'' - \cfrac{(3 - 2f''')(n + f''')}{f'''' - f''' - 1 - \cfrac{(5 - 2f'''')(n + f'''')}{f''''' - f'''' - \cfrac{(5 - 2f''''')(n + f''''')}{\text{etc.}}}}}}}}$$

 vel

vel ex §. 58. sub iisdem denominationibus:

$$xx = n - \cfrac{n}{2n+1+f - \cfrac{(1+2f)(n+f)}{2n+2+f+f' - \cfrac{(1+2f')(n+f')}{2n+3+f'+f'' - \cfrac{(1+2f'')(n+f'')}{2n+4+f''+f''' - \cfrac{(1+2f''')(n+f''')}{2n+5+f'''+f'''' - \text{etc.}}}}}}$$

§. 64. Deinde posito

$$g = \begin{Bmatrix} -1 \\ 0 \end{Bmatrix};\ g' = \begin{Bmatrix} g-1 \\ -1-g \end{Bmatrix};\ g'' = \begin{Bmatrix} g'-1 \\ -2-g' \end{Bmatrix};\ g''' = \begin{Bmatrix} g''-1 \\ -3-g'' \end{Bmatrix};\ \text{etc.}$$

erit ex §. 60.

$$xx = n-1 - \cfrac{(n-1)}{g + \cfrac{2g(n-1+g)}{g'-g+1 + \cfrac{2(1+g')(n-1+g')}{g''-g'+2 + \cfrac{(1+g'')(n-1+g'')}{g'''-g''+1 + \cfrac{2(2+g''')(n-1+g''')}{g''''-g'''+\text{etc.}}}}}}$$

atque ex §. 57:

$$xx = n-1 + \cfrac{(n-1)}{2n-2+g - \cfrac{2g(n-1+g)}{2n-3+g+g' - \cfrac{2g'(n-1+g')}{2n-4+g'+g'' - \cfrac{2g''(n-1+g'')}{2n-5+g''+g''' - \text{etc.}}}}}$$

§. 65. Generaliter ergo pro serie $a$, $b$, $c$, $d$, etc. in qua sit

$$ab = p;\ bc = p+q;\ cd = p+2q;\ de = p+3q;\ \text{etc.}$$

$xy = p + nq$; ex superioribus constat esse.

$$xx = (p+nq) \cdot \frac{(p+nq)(p+(n+2)q)}{(p+(n+1)q)(p+(n+1)q)} \cdot \frac{(p+(n+2)q)(p+(n+4)q)}{(p+(n+3)q)(p+(n+3)q)}\ \text{etc.}$$

et



et per formulas integrales:

$$xx = (p+nq) \cdot \frac{\int z^{p+(n+1)q-1}\, dz : \sqrt{(1-z^{2q})}}{\int z^{p+nq-1}\, dz : \sqrt{(1-z^{2q})}}$$

posito $z = 1$. Iam ob $xxyy = qqnn + 2pqn + pp$ habebimus $\alpha = q$; $\beta = p$ et $\gamma = pp$; hinc $\delta = p$. Quare ex §. 33. erit $N = nq - \frac{q}{2} + p$ et $B = 0$; ideoque

$$xx = p + q(n-\tfrac{1}{2}) + \cfrac{\tfrac{1}{4}qq}{2p+q(2n-1) + \cfrac{\tfrac{9}{4}qq}{2p+q(2n-1) + \cfrac{\tfrac{25}{4}qq}{2p+q(2n-1) + \cfrac{\tfrac{49}{4}qq}{\text{etc.}}}}}$$

§. 66. At per reliquas formulas, si ponamus primo:

$$f = \begin{Bmatrix} q+p \\ p \end{Bmatrix};\ f' = \begin{Bmatrix} q+f \\ q+2p-f \end{Bmatrix};\ f'' = \begin{Bmatrix} q+f' \\ 2q+2p-f' \end{Bmatrix};\ f''' = \begin{Bmatrix} q+f'' \\ 3q+2p-f'' \end{Bmatrix};$$
etc.

habebimus ex §. 59.

$$xx = qn+p + \cfrac{q(qn+p)}{f-p-q - \cfrac{(q+2p-2f)(qn+f)}{f'-f - \cfrac{(q+2p-2f')(qn+f')}{f''-f'-q - \cfrac{(3q+2p-2f'')(qn+f'')}{f'''-f'' - \cfrac{(3q+2p-2f''')(qn+f''')}{\text{etc.}}}}}}$$

et ex §. 58.

$$xx = qn+p - \cfrac{q(qn+p)}{2qn+q+p+f - \cfrac{(q-2p+2f)(qn+f)}{2qn+2q+f+f' - \cfrac{(q-2p+2f')(qn+f')}{2qn+3q+f'+f''\ \text{etc.}}}}$$

vbi

vbi ex tribus numeris datis $p$, $q$, $n$, bini quicunque negatiui assumi possunt, quandoquidem aequatio resoluenda hinc nullam mutationem patitur.

§. 67. Deinde si ponamus

$$g = \left\{ \begin{matrix} p-q \\ p \end{matrix} \right\}; \; g' = \left\{ \begin{matrix} g-q \\ 2p-q-g \end{matrix} \right\}; \; g'' = \left\{ \begin{matrix} g'-q \\ 2p-2q-g' \end{matrix} \right\}; \; g''' = \left\{ \begin{matrix} g''-q \\ 2p-3q-g'' \end{matrix} \right\} \text{etc}$$

erit per §. 60.

$$xx = qn-q+p - \cfrac{q(qn-q+p)}{g-p - \cfrac{2(p-g)(qn-q+g)}{g'-g+q - \cfrac{2(p-g'-q)(qn-q+g')}{g''-g' - \cfrac{2(p-g''-q)(qn-q+g'')}{g'''-g''+q - \text{etc.}}}}}$$

per §. 57.

$$xx = qn-q+p + \cfrac{q(qn-q+p)}{2qn-2q+p+g + \cfrac{2(p-g)(qn-q+g)}{2qn-3q+g+g' + \cfrac{2(p-g')(qn-q+g')}{2qn-4q+g'+g'' - \text{etc.}}}}$$

§. 68. Verum de his expressionibus in infinitum excurrentibus tenendum est, eas saepenumero seriebus diuergentibus aequiualere, ita vt quo vlterius earum valores colligamus, eo magis a veritate aberremus: quod incommodum tamen in expressione prima non vsu venit. Quamobrem his casibus de iis eadem locum habent, quae de serierum diuergentium natura iam annotaui, scilicet eas spectandas esse tanquam formulas infinitas, ex euolutione cuiuspiam formulae finitae natas, quae nihilominus pro earum summa sit habenda, etiamsi vbicunque in collec-

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collectione partium subsisterimus, verum nunquam attingamus.

§. 69. Examinemus etiam seriem $a$, $b$, $c$, $d$, etc. in qua sit

$$ab = \frac{\beta}{\gamma}; \; bc = \frac{\alpha+\beta}{\alpha+\gamma}; \; cd = \frac{2\alpha+\beta}{2\alpha+\gamma}; \; de = \frac{3\alpha+\beta}{3\alpha+\gamma}; \text{ etc.}$$

hincque in genere $xy = \frac{\alpha n+\beta}{\alpha n+\gamma}$; ac habebitur

$$x = \frac{\alpha n+\beta}{\alpha n+\gamma}, \; \frac{\alpha(n+1)+\gamma}{\alpha(n+1)+\beta}, \; \frac{\alpha(n+2)+\beta}{\alpha(n+2)+\gamma}, \; \frac{\alpha(n+3)+\gamma}{\alpha(n+3)+\beta} \text{ etc.}$$

siue

$$x = \frac{\int z^{\alpha n+\gamma-1}\,dz : \sqrt{(1-z^{2\alpha})}}{\int z^{\alpha n+\beta-1}\,dz : \sqrt{(1-z^{2\alpha})}},$$

Cum nunc, si esset $n=\infty$, foret $x=y=1$, valores $x$ et $y$ continuo magis ad vnitatem accedent; quare ponatur $x = 1 + \frac{A}{X}$ et $y = 1 + \frac{A}{Y}$, fietque

$$(X+A)(Y+A) = \frac{\alpha n+\beta}{\alpha n+\gamma} XY, \text{ seu}$$

$$(\beta-\gamma)XY - A(\alpha n+\gamma)X - A(\alpha n+\gamma)Y = AA(\alpha n+\gamma).$$

Sit $A = \beta-\gamma$, seu

$$x = 1 + \frac{\beta-\gamma}{X} \text{ et } y = 1 + \frac{\beta-\gamma}{Y},$$

habebiturque

$$XY - (\alpha n+\gamma)X - (\alpha n+\gamma)Y = +(\beta-\gamma)(\alpha n+\gamma)$$

siue

$$(X-\alpha n-\gamma)(Y-\alpha n-\gamma) = (\alpha n+\beta)(\alpha n+\gamma).$$

§. 70. Ex hac iam aequatione valores X et Y per formulas supra datas infinitis modis exhiberi possunt, ex quibus §. 19. maxime conuergentem suppeditat. Cum autem sit

 $\lambda=$

$$\lambda = -\alpha;\ \mu = -\gamma;\ \nu = -\gamma;\ \zeta = \alpha;\ \eta = \tfrac{\beta+\gamma}{2} \text{ et } \theta = \beta\gamma,$$

fiet

$$X = 2\alpha n - \alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma + \frac{\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{X'}$$

$$Y = 2\alpha n + \alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma + \frac{\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{Y'}$$

hincque emergit ista nova aequatio:

$$(X' - \alpha n - \alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma)(Y' - \alpha n + \alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma) = (\alpha n + \beta)(\alpha n + \gamma).$$

§. 70. Quodsi haec aequatio denuo simili modo euoluatur, ob

$$\lambda = -\alpha;\ \mu = -\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma;\ \nu = \alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma;\ \zeta = \alpha;$$
$$\eta = \tfrac{\beta+\gamma}{2} \text{ et } \theta = \beta\gamma,$$

vt sit $\eta\eta - \theta = \frac{1}{4}(\beta-\gamma)^2$, orietur haec substitutio:

$$X' = 2\alpha n - \alpha + \beta + \gamma + \frac{4\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{X''},$$

$$Y' = 2\alpha n + \alpha + \beta + \gamma + \frac{4\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{Y''},$$

quae praebet istam aequationem:

$$(X'' - \alpha n - 2\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma)(Y'' - \alpha n + 2\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma) = (\alpha n + \beta)(\alpha n + \gamma).$$

§. 72. Iam cum hic sit

$$\lambda = -\alpha;\ \mu = -2\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma;\ \nu = 2\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma;\ \zeta = \alpha;$$
$$\eta = \tfrac{\beta+\gamma}{2};\ \theta = \beta\gamma;$$

orietur haec substitutio:

X =

βγ,
· ½γ)
modo
= α;
½γ)
= α;
X =



$$X'' = 2\alpha n - \alpha + \beta + \gamma + \frac{9\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{X'''},$$

$$Y'' = 2\alpha n + \alpha + \beta + \gamma + \frac{9\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{Y'''},$$

hincque ista aequalitas:

$$(X''' - \alpha n - 3\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma)(Y''' - \alpha n + 3\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma) = (\alpha n + \beta)(\alpha n + \gamma).$$

§. 73. Hoc modo progrediendo consequemur tandem aequationis propositae $xy = \frac{\alpha n + \beta}{\alpha n + \gamma}$ hanc resolutionem:

$$x = 1 + \cfrac{\beta - \gamma}{2\alpha n - \alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma + \cfrac{\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{2\alpha n - \alpha + \beta + \gamma + \cfrac{4\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{2\alpha n - \alpha + \beta + \gamma + \cfrac{9\alpha\alpha - \frac{1}{4}(\beta-\gamma)^2}{2\alpha n - \alpha + \beta + \gamma \text{ etc.}}}}}$$

ita vt sit

$$x = \frac{\int z^{2\alpha n + \gamma - 1}\, dz : \sqrt{(1 - z^{2\alpha})}}{\int z^{2\alpha n + \beta - 1}\, dz : \sqrt{(1 - z^{2\alpha})}},$$

§. 74. Percurramus quaedam exempla, sitque primo $\alpha = 1$, $\beta = 2$ et $\gamma = 0$, erit

$$x = \frac{\int z^{n-1}\, dz : \sqrt{(1 - zz)}}{\int z^{n+1}\, dz : \sqrt{(1 - zz)}} = \frac{n+1}{n},$$

ideoque

$$x = 1 + \cfrac{2}{2n + \cfrac{1 \cdot 1}{2n + 1 \text{ etc.}}} = 1 + \tfrac{1}{n},$$

 vti

vti patet. Atque in genere si $\alpha = 1$, quoties $\beta - \gamma$ est numerus par, valor ipsius $x$ rationaliter exprimitur.

§. 75. Manente $\alpha = 1$, sit $\beta = 1$ et $\gamma = 0$, erit

$$x = \frac{\int z^{n-1}\,dz : \sqrt{(1-zz)}}{\int z^{n}\,dz : \sqrt{(1-zz)}},$$

ex aequatione $xy = \frac{n+1}{n}$; at per fractionem continuam:

$$x = 1 + \cfrac{1}{2n - \frac{1}{2} + \cfrac{1 - \frac{1}{4}}{2n + \cfrac{4 - \frac{1}{4}}{2n + \cfrac{9 - \frac{1}{4}}{2n + \text{etc.}}}}}$$

$$= 1 + \cfrac{2}{4n - 1 + \cfrac{1.3}{4n + \cfrac{3.5}{4n + \cfrac{5.7}{4n + \text{etc.}}}}}$$

Sumto autem $\beta = 0$ et $\gamma = 1$, prodit valor reciprocus

$$\frac{1}{x} = 1 - \cfrac{1}{2n + \frac{1}{2} + \cfrac{1 - \frac{1}{4}}{2n + \cfrac{4 - \frac{1}{4}}{2n + \text{etc.}}}}$$

$$= 1 - \cfrac{2}{4n + 1 + \cfrac{1.3}{4n + \cfrac{3.5}{4n + \cfrac{5.7}{4n + \text{etc.}}}}}$$

cuius consensus cum praecedente facile perspicitur.

§. 76.

$\beta - \gamma$ est
itur.
$= 0$, erit
atinuam:
x.
c.
iprocus
§. 76.



§. 76. Quodsi iam pro $n$ successiue numeri 1, 2, 3, 4, etc. substituantur, reperientur sequentes fractiones continuae:

$$\frac{\pi}{2} = 1 + \cfrac{2}{3 + \cfrac{1.3}{4 + \cfrac{3.5}{4 + \cfrac{5.7}{4 + \cfrac{7.9}{4 + \text{etc.}}}}}}$$

$$\frac{2}{1}\cdot\frac{2}{\pi} = 1 + \cfrac{2}{7 + \cfrac{1.3}{8 + \cfrac{3.5}{8 + \cfrac{5.7}{8 + \cfrac{7.9}{8 + \text{etc.}}}}}}$$

$$\frac{1.3}{2.2}\cdot\frac{\pi}{2} = 1 + \cfrac{2}{11 + \cfrac{1.3}{12 + \cfrac{3.5}{12 + \cfrac{5.7}{12 + \cfrac{7.9}{12 + \text{etc.}}}}}}$$

$$\frac{2.2.4}{1.3.3}\cdot\frac{2}{\pi} = 1 + \cfrac{2}{15 + \cfrac{1.3}{16 + \cfrac{3.5}{16 + \cfrac{5.7}{16 + \cfrac{7.9}{16 + \text{etc.}}}}}}$$



§. 77.

§. 77. Hinc etiam vicissim huiusmodi fractionum continuarum valores inuestigari poterunt. Sit enim proposita haec fractio:

$$s = \alpha\alpha - \delta\delta \cfrac{}{m + 4\alpha\alpha - \delta\delta \cfrac{}{m + 9\alpha\alpha - \delta\delta \cfrac{}{m + 16\alpha\alpha - \delta\delta \cfrac{}{m + \text{etc.}}}}}$$

erit $\beta - \gamma = 2\delta$ et $2\alpha n + \beta + \gamma = m$; vnde $\beta = 2\delta + \gamma$ et $2\alpha n = m - 2\delta - 2\gamma$, sicque

$$x = 1 + \frac{2\delta}{m - \alpha - \delta + 1} = \frac{m - \alpha + \delta + 1}{m - \alpha - \delta + 1},$$

ergo

$$s = \frac{2\delta}{x - 1} - m + \alpha + \delta.$$

Verum est

$$x = \frac{\int z^{\frac{1}{2}m - \delta - 1} dz : \sqrt{(1 - z^{2\alpha})}}{\int z^{\frac{1}{2}m + \delta - 1} dz : \sqrt{(1 - z^{2\alpha})}},$$

vnde si ponatur

$$\int \frac{z^{\frac{1}{2}m - \delta - 1} dz}{\sqrt{(1 - z^{2\alpha})}} = P \text{ et } \int \frac{z^{\frac{1}{2}m + \delta - 1}}{\sqrt{(1 - z^{2\alpha})}} = Q,$$

erit

$$s = \frac{(m - \alpha + \delta) Q - (m - \alpha - \delta) P}{P - Q},$$

§. 78. Hinc si ponatur $\delta = \varepsilon\sqrt{-1}$, vt fit



i fractionum
it enim pro-
: $\beta = 2\delta + \gamma$
$= Q,$
vt fit
s =



$$s = \alpha\alpha + \varepsilon\varepsilon \cfrac{}{m + 4\alpha\alpha + \varepsilon\varepsilon \cfrac{}{m + 9\alpha\alpha + \varepsilon\varepsilon \cfrac{}{m + 16\alpha\alpha + \varepsilon\varepsilon \cfrac{}{m + \text{etc.}}}}}$$

ob $z^{-\delta} = z^{-\varepsilon\sqrt{-1}} = e^{-\varepsilon\sqrt{-1}.lz} = \text{cof.}\, \varepsilon lz - \sqrt{-1}.\ \text{fin.}\, \varepsilon lz$ et $z^{\delta} = \text{cof.}\, \varepsilon lz + \sqrt{-1}.\ \text{fin.}\, \varepsilon lz$, statuatur

$$\int \frac{z^{\frac{1}{2}m - 1} dz\, \text{cof.}\, \varepsilon lz}{\sqrt{(1 - z^{2\alpha})}} = R \text{ et } \int \frac{z^{\frac{1}{2}m - 1} dz\, \text{fin.}\, \varepsilon lz}{\sqrt{(1 - z^{2\alpha})}} = S,$$

erit $P = R - S\sqrt{-1}$; et $Q = R + S\sqrt{-1}$; ideoque

$$s = \frac{2(m - \alpha) S\sqrt{-1} + 2\varepsilon R\sqrt{-1}}{-2S\sqrt{-1}} = \alpha - m - \varepsilon\frac{R}{S},$$

vbi notandum est integralia R et S ita sumi debere, vt posito $z = 0$ euanescant, tum vero poni $z = 1$.