DE
INTEGRATIONE FORMVLAE
$\int \frac{dx\, lx}{\sqrt{(1-xx)}}$,
AB $x=0$ AD $x=1$ EXTENSA.

Auctore

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§. 1.

Methodus maxime naturalis huiusmodi formulas $\int p\, dx\, lx$ tractandi in hoc consistit, vt eae ad alias huiusmodi formas $\int q\, dx$ reducantur, in quibus littera q sit functio algebraica ipsius x; quandoquidem regulae integrandi potissimum ad tales formulas sunt accommodatae. Huiusmodi autem reductio nulla prorsus laborat difficultate, quando functio p ita est comparata, vt integrale $\int p\, dx$ algebraice exhiberi queat. Si enim fuerit $\int p\, dx = P$, ita vt formula proposita sit $\int dP\, lx$, ea sponte reducitur ad hanc expressionem: $P\, lx - \int \frac{P\, dx}{x}$, sicque iam totum negotium ad integrationem huius formulae $\int \frac{P\, dx}{x}$ est perductum. Quando vero formula $\int p\, dx$ integrationem algebraicam non admittit, quemadmodum euenit in nostra formula proposita $\int \frac{dx\, lx}{\sqrt{1-xx}}$, talis reductio successu penitus caret. Cum enim ſit $\int \frac{dx}{\sqrt{1-xx}} = $ A. ſin. x, iſta reductio daret

$$\int \frac{dx\, lx}{\sqrt{1-xx}} = \text{A ſin.}\, x\, l\, x - \int \frac{dx}{x}\, \text{A ſin.}\, x$$

ſicque poſt ſignum integrationis noua quantitas tranſcendens A ſin. x occurreret, cuius integratio aeque eſt abſcondita ac ipſius propoſitae. Quare cum nuper ſingulari methodo inueniſſem eſſe

$$\int \frac{dx\, lx}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} ab & x = 0 \\ ad & x = 1 \end{smallmatrix} \right] = -\tfrac{1}{2}\pi\, l\, 2$$

expreſſio integralis eo maiori attentione digna eſt cenſenda, quod eius inueſtigatio neutiquam eſt obuia; vnde operae pretium eſſe duxi eius veritatem etiam ex aliis fontibus oſtendiſſe, ante quam ipſam methodum, quae me eo perduxit, exponerem.

Prima demonſtratio integrationis propoſitae.

§. 2. Quoniam hic potiſſimum ad ſeries infinitas eſt recurrendum, formula autem $l\,x$ talem reſolutionem ſimplicem reſpuit, adhibeamus ſubſtitutionem $\sqrt{(1-xx)} = y$ vnde fit $x = \sqrt{1-yy}$, hincque porro

$$l\,x = -\frac{yy}{2} - \frac{y^4}{4} - \frac{y^6}{6} - \frac{y^8}{8} - \text{etc.}$$

hoc igitur modo formula integralis propoſita $\int \frac{dx\, lx}{\sqrt{1-xx}}$ transformatur in ſequentem formam:

$$\int \frac{dy}{\sqrt{1-yy}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.} \right)$$

vbi, cum ſit $y = \sqrt{(1-xx)}$, notetur integrationem extendi debere ab $y = 1$ vsque ad $y = 0$; quare ſi hos terminos integrationis permutare velimus, ſignum totius formae mutari oportet.

§. 3. Quo autem minus tali ſignorum mutatione confundamur, deſignemus valorem quaeſitum littera S, vt ſit

$$S = \int \frac{dx\, lx}{\sqrt{1-xx}} \left[\begin{smallmatrix} ab & x = 0 \\ ad & x = 1 \end{smallmatrix} \right]$$

atque

quae facta substitutione $y = \sqrt{1 - xx}$, habebimus, vti modo monuimus

$$S = -\int \frac{dy}{\sqrt{1-yy}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \text{etc.} \right) \left[\begin{smallmatrix} ab\ y = 0 \\ ad\ y = 1 \end{smallmatrix} \right].$$

sub his autem integrationis terminis, scilicet ab $y = 0$ ad $y = 1$, iam satis notum est, singulas partes, quae hic occurrunt, ad sequentes valores reduci:

$$\int \frac{yy\,dy}{\sqrt{(1-yy)}} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int \frac{y^4\,dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2}$$

$$\int \frac{y^6\,dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2}$$

$$\int \frac{y^8\,dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{\pi}{2}$$

$$\int \frac{y^{10}\,dy}{\sqrt{(1-yy)}} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot \frac{\pi}{2}$$

vbi nimirum est $\frac{\pi}{2} = \int \frac{dy}{\sqrt{(1-yy)}}$, ita vt $1 : \pi$ exprimat rationem diametri ad peripheriam circuli.

§. 4. Quodsi ergo singulos istos valores introducamus, pro valore quaesito S impetrabimus sequentem seriem infinitam:

$$S = \frac{\pi}{2} \left(\frac{1}{2^2} + \frac{1 \cdot 3}{2 \cdot 4^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6^2} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8^2} + \text{etc.} \right)$$

sicque nunc totum negotium eo est reductum, vt istius seriei infinitae summa inuestigetur; qui labor fortasse haud minus operosus videri potest, quam id ipsum, quod nobis exsequi est propositum. Interim tamen ad cognitionem summae huius seriei haud difficulter sequenti modo nobis pertingere licebit.

§. 5. Cum sit

$$\frac{1}{\sqrt{(1-zz)}} = 1 + \frac{1}{2} zz + \frac{1 \cdot 3}{2 \cdot 4} z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} z^6 + \text{etc.}$$

si vtrinque per $\frac{dz}{z}$ multiplicemus et integremus, obtinebimus

$$\int \frac{dz}{z\sqrt{1-zz}} = lz + \frac{1}{2^2} zz + \frac{1 \cdot 3}{2 \cdot 4^2} z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6^2} z^6 + \text{etc.}$$

ficque

ſicque ad ipſam ſeriem noſtram ſumus perducti, cuius ergo valor quaeri debet ex hac expreſſione: $\int \frac{dz}{z\sqrt{1-zz}} - lz$, integrali ſcilicet ita ſumto, vt euaneſcat poſito $z=0$, quo facto ſtatuatur $z=1$, ac prodibit ipſa ſeries

$$\frac{1}{2^2} + \frac{1\cdot 3}{2\cdot 4^2} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6^2} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8^2} + \text{etc.}$$

Hoc igitur modo totum negotium perductum eſt ad iſtam formulam integralem $\int\frac{dz}{z\sqrt{1-zz}}$, quae poſito $\sqrt{1-zz}=v$ tranſit in hanc formam: $\frac{-dv}{1-vv}$ cuius integrale conſtat eſſe $-\frac{1}{2}l\frac{1+v}{1-v} = -l\frac{1+v}{\sqrt{(1-vv)}}$. Quodſi loco v reſtituatur valor $\sqrt{1-zz}$, tota expreſſio, qua indigemus, ita ſe habebit: $\int\frac{dz}{(z\sqrt{1-zz})} - lz = -l\frac{(1+\sqrt{1-zz})}{z} - lz + C = C - l(1+\sqrt{1-zz})$, vbi conſtans ita accipi debet, vt valor euaneſcat, poſito $z=0$, ideoque erit $C=l2$. Quamobrem, poſito $z=1$, ſumma ſeriei quaeſita erit $l2$, hincque valor ipſius formulae integralis propoſitae erit

$$\int\frac{dx\, lx}{\sqrt{(1-xx)}} = S = -\frac{\pi}{2}\, l2$$

prorſus vti longe alia methodo inueneram, ex quo iam ſatis intelligitur, iſtam veritatem vtique altioris eſſe indaginis, ideoque attentione Geometrarum maxime dignam.

Alia demonſtratio integrationis propoſitae.

§. 6. Cum ſit $\frac{dx}{\sqrt{1-xx}}$ elementum arcus circuli cuius ſinus $=x$, ponamus iſtum angulum $=\Phi$, ita vt ſit $x=\text{ſin.}\,\Phi$ et $\frac{dx}{\sqrt{1-xx}}=d\Phi$, atque facta hac ſubſtitutione valor quantitatis S, in quem inquirimus, ita repraeſentabitur: $S=\int d\Phi\, l\,\text{ſin.}\,\Phi \left[\begin{smallmatrix} a & \Phi=0 \\ ad & \Phi=90^\circ \end{smallmatrix}\right]$. Cum enim ante termini fuiſſent $x=0$ et $x=1$, iis nunc reſpondent $\Phi=0$ et $\Phi=90^\circ$ ſiue $\Phi=\frac{\pi}{2}$. Hic igitur totum negotium eo redit, vt formula $l\,\text{ſin.}\,\Phi$ commode in ſeriem infinitam con-

verta-

vertatur. Hunc in finem ponamus $l\,\text{fin.}\,\Phi = s$ eritque $ds = \frac{d\Phi \,\text{cof.}\,\Phi}{\text{fin.}\,\Phi}$. Nouimus autem esse

$$\frac{\text{cof.}\,\Phi}{\text{fin.}\,\Phi} = 2\,\text{fin.}\,2\Phi + 2\,\text{fin.}\,4\Phi + 2\,\text{fin.}\,6\Phi + 2\,\text{fin.}\,8\Phi + \text{etc.}$$

Si enim vtrinque per fin. Φ multiplicemus, ob

$$2\,\text{fin.}\,n\Phi\,\text{fin.}\,\Phi = \text{cof.}\,(n-1)\Phi - \text{cof.}\,(n+1)\Phi,$$

vtrinque prodit

$$\text{cof.}\,\Phi = \text{cof.}\,\Phi + \text{cof.}\,3\Phi + \text{cof.}\,5\Phi + \text{cof.}\,7\Phi + \text{cof.}\,9\Phi + \text{etc.}$$
$$- \text{cof.}\,3\Phi - \text{cof.}\,5\Phi - \text{cof.}\,7\Phi - \text{cof.}\,9\Phi - \text{etc.}$$

Hac igitur ferie pro $\frac{\text{cof.}\,\Phi}{\text{fin.}\,\Phi}$ in vfum vocata erit

$$s = C - \text{cof.}\,2\Phi - \tfrac{1}{2}\text{cof.}\,4\Phi - \tfrac{1}{3}\text{cof.}\,6\Phi - \tfrac{1}{4}\text{cof.}\,8\Phi - \tfrac{1}{5}\text{cof.}\,10\Phi - \text{etc.}$$

vbi cum fit $s = l\,\text{fin.}\,\Phi$ ideoque $s = 0$, quando fin. $\Phi = 1$ ideoque $\Phi = \frac{\pi}{2}$, conftantem C ita definire oportet, vt pofito $\Phi = \frac{\pi}{2} = 90^\circ$ euadat $s = 0$, ex quo colligitur fore

$$C = -1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \text{etc.} = -l\,2.$$

§. 7. Cum igitur fit

$$l\,\text{fin.}\,\Phi = -l\,2 - \text{cof.}\,2\Phi - \tfrac{1}{2}\text{cof.}\,4\Phi - \tfrac{1}{3}\text{cof.}\,6\Phi - \tfrac{1}{4}\text{cof.}\,8\Phi - \text{etc.}$$

erit valor formulae propofitae

$$\int d\Phi\, l\,\text{fin.}\,\Phi = C - \Phi\, l\,2 - \tfrac{1}{2}\text{fin.}\,2\Phi - \tfrac{1}{8}\text{fin.}\,4\Phi - \tfrac{1}{18}\text{fin.}\,6\Phi$$
$$- \tfrac{1}{32}\text{fin.}\,8\Phi - \tfrac{1}{50}\text{fin.}\,10\Phi - \text{etc.}$$

quae expreffio cum euanefcere debeat pofito $\Phi = 0$, conftans hic ingreffa erit $C = 0$, ita vt iam in genere fit

$$\int d\Phi\, l\,\text{fin.}\,\Phi = -\Phi\, l\,2 - \frac{2\,\text{fin.}\,2\Phi}{2^2} - \frac{2\,\text{fin.}\,4\Phi}{4^2} - \frac{2\,\text{fin.}\,6\Phi}{6^2} - \frac{2\,\text{fin.}\,8\Phi}{8^2}$$
$$- \frac{2\,\text{fin.}\,10\Phi}{10^2} - \frac{2\,\text{fin.}\,12\Phi}{12^2} - \text{etc.}$$

Quodfi iam hic capiatur $\Phi = 90^\circ = \frac{\pi}{2}$, omnium angulorum 2Φ; 4Φ; 6Φ; 8Φ etc. qui hic occurrunt finus euanefcunt, ideoque valor quaefitus erit

$$S = \int d\Phi\, l\,\text{fin.}\,\Phi \left[\begin{smallmatrix} a & \Phi = 0 \\ ad & \Phi = 90^\circ \end{smallmatrix} \right] = -\tfrac{\pi}{2}\, l\,2$$

quemadmodum etiam in priore demonftratione oftendimus.

§. 8.

§. 8. Ista autem demonstratio praecedenti ideo longe antecellit, quod nobis non solum valorem formulae propositae exhibeat casu quo $\phi = 90°$, sed etiam verum eius valorem ostendat, quicunque angulus pro ϕ accipiatur, id quod ad ipsam formulam propositam $\int \frac{dx\, lx}{\sqrt{1-xx}}$ transferri poterit, cuius adeo valorem pro quolibet valore ipsius x assignare poterimus. Quodsi enim istius formulae valorem desideremus ab $x = 0$ vsque ad $x = a$, quaeratur angulus α cuius sinus sit aequalis ipsi a atque semper habebitur

$$\int \frac{dx\, lx}{\sqrt{1-xx}} \left[\begin{matrix} ab & x = 0 \\ ad & x = a \end{matrix} \right] = -\alpha\, l\, 2 - \frac{2\,\text{fin.}\, 2\alpha}{2^2} - \frac{2\,\text{fin.}\, 4\alpha}{4^2} - \frac{2\,\text{fin.}\, 6\alpha}{6^2} - \frac{2\,\text{fin.}\, 8\alpha}{8^2} - \text{etc.}$$

Vnde patet quoties fuerit $\alpha = \frac{i\pi}{2}$, denotante i numerum integrum quemcunque, quoniam omnes finus euanescunt, valor formulae his casibus finite exprimi per $-\frac{i\pi}{2}\, l\, 2$; aliis vero casibus valor nostrae formulae per seriem infinitam satis concinnam exprimetur. Ita si capiatur $a = \frac{1}{\sqrt{2}}$, vt fit $\alpha = \frac{\pi}{4}$ valor nostrae formulae erit

$$-\frac{\pi}{4}\, l\, 2 - \frac{2}{2^2} + \frac{2}{6^2} - \frac{2}{10^2} + \frac{2}{14^2} - \frac{2}{18^2} + \frac{2}{22^2} - \text{etc.}$$

quae feries elegantius ita exprimitur:

$$-\frac{\pi}{4}\, l\, 2 - \frac{1}{2}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \text{etc.}\right)$$

ficque hic occurrit feries fatis memorabilis

$$1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \frac{1}{121} + \text{etc.}$$

cuius fummam nullo adhuc modo ad menfuras cognitas reuocare licuit.

§. 9. Quoniam tam egregia feries hic fe quafi praeter exfpectationem obtulit, etiam alios cafus euoluamus notabiliores, fumamusque $a = \frac{1}{2}$, vt fit $\alpha = 30° = \frac{\pi}{6}$ atque noftrae formulae hoc cafu valor erit

$$-\frac{\pi}{6}\, l\, 2 - \frac{\sqrt{3}}{2^2} - \frac{\sqrt{3}}{4^2} + \frac{\sqrt{3}}{8^2} + \frac{\sqrt{3}}{10^2} - \frac{\sqrt{3}}{14^2} - \frac{\sqrt{3}}{16^2} + \text{etc.}$$

quae expreffio ita exhiberi poteft

$$-\frac{\pi}{6}\, l\, 2 - \frac{\sqrt{3}}{4}\left(1 + \frac{1}{2^2} - \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - \frac{1}{10^2} - \frac{1}{11^2} + \text{etc.}\right)$$

in

in qua ferie quadrata multiplorum ternarii deficiunt. Sumamus nunc fimili modo $a = \frac{\sqrt{3}}{2}$, vt fit $\alpha = 60^\circ = \frac{\pi}{3}$, ac valor noftrae formulae hoc cafu prodibit

$$S = \frac{\pi}{3} l 2 - \frac{\sqrt{3}}{2^2} + \frac{\sqrt{3}}{4^2} - \frac{\sqrt{3}}{8^2} + \frac{\sqrt{3}}{10^2} - \frac{\sqrt{3}}{14^2} + \frac{\sqrt{3}}{16^2} - \text{etc.}$$

fiue hoc modo exprimetur:

$$S = \frac{\pi}{3} l 2 - \frac{\sqrt{3}}{4} \left(1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{10^2} - \frac{1}{11^2} + \text{etc.} \right)$$

Adhuc alia demonftratio integrationis propofitae.

§. 10. Introducatur in formulam noftram angulus Φ, cuius cofinus fit $= x$, fiue fit $x = \text{cof.}\,\Phi$ et formula noftra induet hanc formam: $-\int d\Phi\, l\, \text{cof.}\,\Phi$, quod integrale a $\Phi = 90^\circ$ vsque ad $\Phi = 0$ erit extendendum. Quodfi autem hos terminos permutemus, valor S, quem quaerimus, ita exprimetur:

$$S = \int d\Phi\, l\, \text{cof.}\,\Phi \left[{a \atop ad} \; {\Phi = 0 \atop \Phi = 90^\circ} \right].$$

Vt hic $l\,\text{cof.}\,\Phi$ in feriem idoneam conuertamus, ftatuamus vt ante $s = l\,\text{cof.}\,\Phi$, eritque $ds = -\frac{d\Phi\, \text{fin.}\,\Phi}{\text{cof.}\,\Phi}$. Conftat autem per feriem effe

$$\frac{\text{fin.}\,\Phi}{\text{cof.}\,\Phi} = 2\,\text{fin.}\,2\Phi - 2\,\text{fin.}\,4\Phi + 2\,\text{fin.}\,6\Phi - 2\,\text{fin.}\,8\Phi + \text{etc.}$$

Cum enim in genere fit

$$2\,\text{fin.}\,n\Phi\,\text{cof.}\,\Phi = \text{fin.}\,(n+1)\Phi + \text{fin.}\,(n-1)\Phi$$

fi vtrinque per cof. Φ multiplicemus orietur

$$\begin{aligned} \text{fin.}\,\Phi = {} & \text{fin.}\,3\Phi - \text{fin.}\,5\Phi + \text{fin.}\,7\Phi - \text{fin.}\,9\Phi \\ & + \text{fin.}\,\Phi - \text{fin.}\,3\Phi + \text{fin.}\,5\Phi - \text{fin.}\,7\Phi + \text{fin.}\,9\Phi \end{aligned} \;\text{etc.}$$

quare cum fit $ds = -\frac{d\Phi\,\text{fin.}\,\Phi}{\text{cof.}\,\Phi}$, erit nunc

$$S = C + \frac{\text{cof.}\,2\Phi}{1} - \frac{\text{cof.}\,4\Phi}{2} + \frac{\text{cof.}\,6\Phi}{3} - \frac{\text{cof.}\,8\Phi}{4} + \frac{\text{cof.}\,10\Phi}{5} - \text{etc.}$$

Quia igitur eft $s = l\,\text{cof.}\,\Phi$, euidens eft pofito $\Phi = 0$ fieri debere $s = 0$, vnde colligitur

$$C = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \text{etc.} = -l\,2$$

sicque erit $l\cos\Phi = -l2 + \frac{\cos 2\Phi}{1} - \frac{\cos 4\Phi}{2} + \frac{\cos 6\Phi}{3} - \frac{\cos 8\Phi}{4} +$ etc. quae series ducta in $d\Phi$ et integrata praebet

$$S = \int d\Phi l\cos\Phi = C - \Phi l 2 + \frac{\sin 2\Phi}{2} - \frac{\sin 4\Phi}{8} + \frac{\sin 6\Phi}{18} - \frac{\sin 8\Phi}{32} + \frac{\sin 10\Phi}{50} - \text{etc.}$$

quae expressio quia sponte euanescit posito $\Phi = 0$, inde patet fore $C = 0$, sicque habebimus

$$\int d\Phi l\cos\Phi = -\Phi l 2 + \frac{1}{2}\left(\frac{\sin 2\Phi}{1} - \frac{\sin 4\Phi}{2^2} + \frac{\sin 6\Phi}{3^2} - \frac{\sin 8\Phi}{4^2} + \frac{\sin 10\Phi}{5^2} - \text{etc.}\right)$$

Sumto igitur $\Phi = \frac{\pi}{2} = 90°$, oritur vt ante $S = -\frac{\pi}{2} l 2$. Praeterea vero etiam hinc integrale ad quemuis terminum vsque extendere licet.

§. 11. Quodsi formulam posteriorem a praecedente subtrahamus, adipiscemur in genere hanc integrationem: $\int d\Phi\, l\,\text{tang.}\,\Phi = -\sin. 2\Phi - \frac{1}{3^2}\sin. 6\Phi - \frac{1}{5^2}\sin. 10\Phi - \text{etc.}$ vnde patet hoc integrale euanescere casibus $\Phi = 90°$ et in genere $\Phi = i\frac{\pi}{2}$. Postquam igitur istam integrationem triplici modo demonstrauimus, ipsam Analysin, quae me primum huc perduxit, hic delucide sum expositurus.

Analysis ad integrationem formulae $\int \frac{dx\, lx}{\sqrt{1-xx}}$ aliarumque similium perducens.

§. 12. Tota haec Analysis innititur sequenti lemmati a me iam olim demonstrato: Posito breuitatis gratia $(1-x^n)^{\frac{m-n}{n}} = X$, si hinc duae formulae integrales formentur $\int X x^{p-1} dx$ et $\int X x^{q-1} dx$, quae a termino $x = 0$ vsque ad terminum $x = 1$ extendantur, ratio horum valorum sequenti modo ad productum ex infinitis factoribus conflatum reduci potest

$$\frac{\int X x^{p-1} dx}{\int X x^{q-1} dx} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

vbi

vbi scilicet singuli factores tam numeratoris, quam denominatoris continuo eadem quantitate n augentur. Hic autem probe tendendum est, veritatem istius lemmatis subsistere non posse, nisi singulae litterae m, n, p et q denotent numeros positiuos, quos tamen semper tanquam integros spectare licet.

§. 13. Circa has duas formulas integrales, a termino $x = 0$ vsque ad $x = 1$ extensas, duo casus imprimis seorsim notari merentur, quibus integratio actu succedit, verusque valor absolute assignari potest. Prior casus locum habet, si fuerit $p = n$, ita vt formula sit $\int X x^{n-1} dx$. Posito enim $x^n = y$ fiet $X = (1-y)^{\frac{m-n}{n}}$ et $x^{n-1} dx = \frac{1}{n} dy$ sicque ista formula euadet $\frac{1}{n} \int dy (1-y)^{\frac{m-n}{n}}$, pariter a termino $y = 0$ vsque ad $y = 1$ extendenda, quae porro posito $1 - y = z$, abit in hanc formulam $-\frac{1}{n} \int z^{\frac{m-n}{n}} dz$ a termino $z = 1$ vsque ad $z = 0$ extendendam; eius ergo integrale manifesto est $= -\frac{1}{m} z^{\frac{m}{n}} + \frac{1}{m}$, vnde facto $z = 0$ valor erit $= \frac{1}{m}$. Consequenter pro casu $p = n$ habebimus

$$\int X x^{n-1} dx \left[\begin{smallmatrix} ab & x = 0 \\ ad & x = 1 \end{smallmatrix}\right] = \frac{1}{m}$$

sicque si fuerit vel $p = n$ vel $q = n$, integrale absolute innotescit.

§. 14. Alter casus notatu dignus est quo $p = n - m$, ita vt formula integranda sit $\int X x^{n-m-1} dx$; tum enim, si ponatur $x (1 - x^n)^{\frac{-1}{n}}$ siue $\frac{x}{(1-x^n)^{\frac{1}{n}}} = y$, posito $x = 0$ fiet $y = 0$, at posito $x = 1$ fiet $y = \infty$; tum autem erit

$$y^{n-m} = \frac{x^{n-m}}{(1-x^n)^{\frac{n-m}{n}}} = X x^{n-m}$$

 vnde

vnde formula integranda erit $\int y^{n-m} \frac{dx}{x}$. Cum igitur sit $\frac{x}{(1-x^n)^{\frac{1}{n}}} = y$, erit $\frac{x^n}{1-x^n} = y^n$, vnde colligitur $x^n = \frac{y^n}{1+y^n}$, ideoque $n\,l\,x = n\,l\,y - l\,(1+y^n)$, cuius differentiatio praebet

$$\frac{dx}{x} = \frac{dy}{y(1+y^n)}$$

quo valore substituto formula nostra integranda erit $\int \frac{y^{n-m-1}\,dy}{1+y^n}$ a termino $y = 0$ vsque ad $y = \infty$ extendenda, quae formula ideo est notatu digna, quod ab omni irrationalitate est liberata.

§. 15. Quoniam igitur hoc casu ad formulam rationalem sumus perducti, ex elementis calculi integralis constat, eius integrationem semper per logarithmos et arcus circulares absolui posse, tum vero pro hoc casu non ita pridem ostendi, huius formulae $\int \frac{x^{m-1}\,dx}{1+x^n}$ integrale ab $x = 0$ vsque ad $x = \infty$ extensum reduci ad valorem $\frac{\pi}{n\,\text{sin.}\,\frac{m\pi}{n}}$; facta igitur applicatione pro nostro casu habebimus

$$\int \frac{y^{n-m-1}\,dy}{1+y^n} = \frac{\pi}{n\,\text{sin.}\,\frac{(n-m)\pi}{n}} = \frac{\pi}{n\,\text{sin.}\,\frac{m\pi}{n}}$$

quamobrem pro casu $p = n - m$ valor integralis sequenti modo absolute exprimi potest, eritque

$$\int X\,x^{n-m-1}\,dx \left[\begin{matrix}\text{ab } x = 0\\ \text{ad } x = 1\end{matrix}\right] = \frac{\pi}{n\,\text{sin.}\,\frac{m\pi}{n}}$$

quod idem manifesto tenendum est, si fuerit $q = n - m$.

§. 16. His praemissis ponamus porro breuitatis gratia

$$\int X\,x^{p-1}\,dx \left[\begin{matrix}ab\ x = 0\\ ad\ x = 1\end{matrix}\right] = P \text{ et}$$

$$\int X\,x^{q-1}\,dx \left[\begin{matrix}ab\ x = 0\\ ad\ x = 1\end{matrix}\right] = Q$$

atque

atque lemma allatum nobis praebet hanc aequationem

$$\frac{P}{Q} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

Hinc igitur ſumendis logarithmis deducimus

$$lP - lQ = l(m+p) - lp + l(m+p+n) - l(p+n) + l(m+p+2n) - l(p+2n) \text{ etc.}$$
$$+ lq - l(m+q) + l(q+n) - l(m+q+n) + l(q+2n) - l(m+q+2n) \text{ etc.}$$

haecque aequalitas ſemper locum habebit, quicunque valores litteris m, n, p et q tribuantur, dummodo fuerint poſitiui.

§. 17. Cum igitur haec aequalitas in genere ſubſiſtat, etiam veritati erit conſentanea, quando quaepiam harum litterarum m, n, p et q infinite parum immutantur, ſiue tanquam variabiles ſpectantur. Hanc ob rem conſideremus ſolam quantitatem p tanquam variabilem, ita vt reliquae litterae m, n et q maneant conſtantes, ideoque etiam quantitas Q erit conſtans, dum altera P variabitur; ex quo differentiando nanciſcemur hanc aequationem:

$$\frac{dP}{P} = \frac{dp}{m+p} - \frac{dp}{p} + \frac{dp}{m+p+n} - \frac{dp}{p+n} + \frac{dp}{m+p+2n} - \frac{dp}{p+2n}$$
$$+ \frac{dp}{m+p+2n} - \frac{dp}{p+3n} + \text{etc.}$$

vbi totum negotium eo redit, quemadmodum differentiale formulae P, quae eſt integralis, exprimi oporteat.

§. 18. Cum igitur P ſit formula integralis ſolam quantitatem x tanquam variabilem inuoluens, quandoquidem in eius integratione exponens p vt conſtans tractari debet, demum poſt integrationem ipſam quaentitatem P tanquam functionem duarum variabilium x et p ſpectare licebit, vnde quaeſtio huc redit, quomodo valorem, hoc charactere $\left(\frac{dP}{dp}\right)$ exprimi ſolitum inueſtigari oporteat, qui ſi indicetur littera Π, aequatio ante inuenta hanc induet formam:

$$\frac{\Pi}{P} = \frac{1}{m+p} - \frac{1}{p} + \frac{1}{m+p+n} - \frac{1}{p+n} + \frac{1}{m+p+2n} - \frac{1}{p+2n} + \text{etc.}$$

 Hanc

Hanc vero feriem infinitam haud difficulter ad expreffionem finitam reuocare licebit hoc modo: Ponatur

$$s = \frac{v^{m+p}}{m+p} - \frac{v^p}{p} + \frac{v^{m+p+n}}{m+p+n} - \frac{v^{p+n}}{p+n} + \frac{v^{m+p+2n}}{m+p+2n} - \frac{v^{p+2n}}{p+2n} + \text{etc.}$$

ita vt facto $v = 1$ littera s nobis exhibeat valorem quaefitum $\frac{\Pi}{P}$; at vero differentiatio nobis dabit

$$\frac{ds}{dv} = v^{m+p-1} - v^{p-1} + v^{m+p+n-1} - v^{p+n-1} + v^{m+p+2n-1} - v^{p+2n-1} + \text{etc.}$$

cuius feriei infinitae fumma manifefto eft

$$\frac{v^{m+p-1} - v^{p-1}}{1 - v^n} = \frac{v^{p-1}(v^m - 1)}{1 - v^n}.$$

Hinc igitur viciffim concludimus fore

$$S = \int \frac{v^{p-1}(v^m - 1)\,dv}{1 - v^n}$$

quae formula integralis a $v = 0$ vsque ad $v = 1$ eft extendenda; ficque habebimus

$$\frac{\Pi}{P} = \int \frac{v^{p-1}(v^m - 1)\,dv}{1 - v^n} \left[\begin{matrix} \text{a} & v = 0 \\ \text{ad} & v = 1 \end{matrix} \right].$$

§. 19. Ad valorem autem $\left(\frac{dP}{dp}\right)$, quem hic littera Π indicauimus, inueftigandum, ex principiis calculi integralis ad functiones duarum variabilium applicati iam fatis notum eft differentiale formulae integralis $P = \int X x^{p-1} dx$ ex fola variabilitate ipfius p oriundum obtineri, fi formula poft fignum integrationis pofita $X x^{p-1}$ ex fola variabilitate ipfius p differentietur atque elementum dp figno integrationis praefigatur; at vero quia X non continet p, hic vt conftans tractari debet: poteftatis vero x^{p-1} differentiale hinc natum erit $x^{p-1}\,dp\,lx$; quam ob rem ex hac differentiatione orietur $dP = dp \int X x^{p-1}\,dx\,lx$, ita vt tantum poft fignum integrationis factor lx acceſſerit, ex quo manifeftum manifestum est, fore

$$\Pi = \int X x^{p-1} dx\, lx \left[\begin{smallmatrix} ab & x = 0 \\ ad & x = 1 \end{smallmatrix} \right].$$

hinc igitur sequens theorema generale constituere licebit.

Theorema generale.

§. 20. Posito breuitatis gratia $X = (1 - x^n)^{\frac{m-n}{n}}$ si sequentes formulae integrales omnes a termino $x = 0$ ad terminum $x = 1$ extendantur, sequens aequalitas semper erit veritati consentanea:

$$\int \frac{X x^{p-1} dx\, lx}{\int X x^{p-1} dx} = \int \frac{x^{p-1}(x^m - 1)\, dx}{1 - x^n}$$

nihil enim obstabat, quo minus loco v scriberemus x, quandoquidem isti valores tantum a terminis integrationis pendent.

§. 21. Hoc igitur modo deducti sumus ad integrationem huiusmodi formularum $\int X x^{p-1} dx\, lx$ in quibus quantitas logarithmica lx post signum integrationis tanquam factor inest, quarum valorem exprimere licuit per binas formulas integrales ordinarias, cum sit

$$\int X x^{p-1} dx\, lx = \int X x^{p-1} dx . \int \frac{x^{p-1}(x^m - 1)\, dx}{1 - x^n}$$

integralibus scilicet ab $x = 0$ ad $x = 1$ extensis, vbi breuitatis gratia posuimus $(1 - x^n)^{\frac{m-n}{n}} = X$. Hinc igitur pro binis casibus memorabilibus supra expositis bina theoremata particularia deriuemus.

Theorema particulare I, quo $p = n$.

§. 22. Quoniam supra vidimus casu $p = n$ fieri $\int X x^{n-1} dx = \frac{1}{m}$, hoc valore substituto habebimus istam aequationem satis elegantem:

$$\int X x^{n-1} dx\, lx = \frac{1}{m} \int \frac{x^{n-1}(x^m - 1)\, dx}{1 - x^n}$$

dum

dum scilicet ambo integralia ab $x=0$ ad $x=1$ extenduntur.

Theorema particulare II, quo $p=n-m$.

§. 23. Quoniam pro hoc casu, quo $p=n-m$ supra ostendimus esse

$$\int X x^{n-m-1} dx = \frac{\pi}{n \sin. \frac{m\pi}{n}}$$

nunc deducimur ad sequentem integrationem maxime notata dignam:

$$\int X x^{n-m-1} dx\, lx = \frac{\pi}{n \sin. \frac{m\pi}{n}} \int \frac{x^{n-m-1}(x^m-1)dx}{1-x^n}$$

si quidem haec ambo integralia ab $x=0$ vsque ad $x=1$ extendantur; vbi meminisse oportet esse

$$X=(1-x^n)^{\frac{m-n}{n}}.$$

§. 24. Hic probe notetur theorema generale latissime patere, propterea quod in eo insunt tres exponentes indefiniti, scilicet m, n et p, qui penitus arbitrio nostro relinquuntur, quos ergo infinitis modis pro lubitu definire licet, dummodo singulis valores positiui tribuantur, ita vt semper valor huius formulae integralis $\int X x^{p-1} dx\, lx$, quam ob factorem lx tanquam transcendentem spectari oportet, per formulas integrales ordinarias exprimi queat, quae cum sint generalissima, operae pretium erit non nullos casus speciales euoluere.

I. Euolutio casus quo $m=1$ et $n=2$.

§. 25. Hoc igitur casu erit $X=\frac{1}{\sqrt{1-xx}}$, vnde pro hoc casu theorema generale ita se habebit

$$\int \frac{x^{p-1} dx\, lx}{\sqrt{1-xx}} = -\int \frac{x^{p-1}dx}{\sqrt{1-xx}} \cdot \int \frac{x^{p-1}dx}{1+x}$$ si-

siquidem singula haec integralia ab $x=0$ ad $x=1$ extendantur. Quoniam igitur hic tantum exponens p arbitrio nostro relinquitur, hinc sequentia exempla perlustremus.

Exemplum I. quo $p=1$.

§. 26. Hoc igitur casu aequatio superior hanc induet formam:

$$\int \frac{dx\, lx}{\sqrt{1-xx}} = -\int \frac{dx}{\sqrt{1-xx}} \cdot \int \frac{dx}{1+x}$$

ubi, integralibus ab $x=0$ ad $x=1$ extensis, notum est fieri

$$\int \frac{dx}{\sqrt{1-xx}} = \frac{\pi}{2} \text{ et } \int \frac{dx}{1+x} = l\,2$$

ita vt iam habeamus

$$\int \frac{dx\, lx}{\sqrt{1-xx}} \left[\begin{smallmatrix} ab\ x=0 \\ ad\ x=1 \end{smallmatrix}\right] = -\frac{\pi}{2}\, l\,2,$$

quae est ea ipsa formula, quam initio huius dissertationis tractauimus et cuius veritatem iam triplici demonstratione corroborauimus.

§. 27. Eundem valorem elicere licet ex theoremate particulari secundo, quo erat $p=n-m$, siquidem nunc ob $n=2$ et $m=1$ erit $p=1$; inde enim ob $X=\frac{1}{\sqrt{1-xx}}$ istud theorema praebet

$$\int \frac{dx\, lx}{\sqrt{1-xx}} = -\frac{\pi}{2\,\text{fin.}\frac{\pi}{2}} \int \frac{dx}{1+x} = -\frac{\pi}{2}\, l\,2.$$

Exemplum II. quo $p=2$.

§. 28. Hoc igitur casu aequatio superior hanc induet formam:

$$\int \frac{x\,dx\, lx}{\sqrt{1-xx}} = -\int \frac{x\,dx}{\sqrt{1-xx}} \int \frac{x\,dx}{1+x}.$$

Iam vero integralibus ab $x=0$ ad $x=1$ extensis notum est fore

$$\int \frac{x\,dx}{\sqrt{1-xx}} = 1 \text{ et } \int \frac{x\,dx}{1+x} = 1 - l\,2$$

ita vt habeamus

$$\int\frac{x\,dx\,lx}{\sqrt{1-xx}}\left[\begin{smallmatrix}ab\ x=0\\ ad\ x=1\end{smallmatrix}\right]=l\,2-1.$$

§. 29. Quoniam in hac formula integrale $\int\frac{x\,dx}{\sqrt{1-xx}}$ algebraice exhiberi potest, cum sit $=1-\sqrt{1-xx}$, valor quaesitus etiam per reductiones consuetas erui potest, cum sit

$$\int\frac{x\,dx\,lx}{\sqrt{1-xx}}=(1-\sqrt{1-xx})\,lx-\int\frac{dx}{x}(1-\sqrt{1-xx})$$

positoque $x=1$ erit

$$\int\frac{x\,dx\,lx}{\sqrt{1-xx}}=-\int\frac{dx}{x}(1-\sqrt{1-xx})$$

ad quam formam integrandam fiat $1-\sqrt{1-xx}=z$, vnde colligitur $xx=2z-zz$, ergo $2\,lx=lz+l(2-z)$ sicque fiet $\frac{dx}{x}=\frac{dz(1-z)}{z(2-z)}$, quibus valoribus substitutis erit

$$+\int\frac{dx}{x}(1-\sqrt{1-xx})=+\int\frac{dz(1-z)}{2-z}$$

qui ergo valor erit $=C-z-l(2-z)$. Quia igitur posito $x=0$ fit $z=0$, constans erit $C=+l\,2$; facto igitur $x=1$, quia tum fit $z=1$, iste valor integralis erit $l\,2-1$, prorsus vt ante.

§. 30. Eundem valorem suppeditat theorema prius supra allatum, quo erat $p=n=2$; inde enim statim fit $\int\frac{x\,dx\,lx}{\sqrt{1-xx}}=\int-\frac{x\,dx}{1+x}$. Ante autem vidimus esse $\int\frac{x\,dx}{1+x}=1-l\,2$ ita vt etiam hinc prodeat valor quaesitus $l\,2-1$.

Exemplum III. quo $p=3$.

§. 31. Hoc igitur casu aequatio in theoremate generali allata hanc induet formam:

$$\int\frac{xx\,dx\,lx}{\sqrt{1-xx}}=-\int\frac{xx\,dx}{\sqrt{1-xx}}\cdot\int\frac{xx\,dx}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int\frac{xx\,dx}{\sqrt{1-xx}}\left[\begin{smallmatrix}ab\ x=0\\ ad\ x=1\end{smallmatrix}\right]=\frac{1}{2}\cdot\frac{\pi}{2}$$

at

tum vero fractio ſpuria $\frac{xx}{1+x}$ reſoluitur in has partes $x-1+\frac{1}{1+x}$, vnde erit $\int\frac{xx\,dx}{1+x}=\frac{1}{2}xx-x+l(1+x)$, quod integrale iam euaneſcit poſito $x=0$; facto ergo $x=1$ eius valor erit $=-\frac{1}{2}+l2$; quamobrem integrale quod quaerimus erit

$$\int\frac{xx\,dx\,lx}{\sqrt{1-xx}}\left[\begin{smallmatrix}ab & x=0\\ ad & x=1\end{smallmatrix}\right]=-\frac{\pi}{4}\left(l2-\frac{1}{2}\right).$$

Exemplum IV. quo $p=4$.

§. 32. Hoc igitur caſu aequatio ſuperior hanc induet formam:

$$\int\frac{x^3\,dx\,lx}{\sqrt{1-xx}}=-\int\frac{x^3\,dx}{\sqrt{1-xx}}\cdot\int\frac{x^3\,dx}{1+x}.$$

Per reductiones autem notiſſimas conſtat eſſe

$$\int\frac{x^3\,dx}{\sqrt{1-xx}}\left[\begin{smallmatrix}ab & x=0\\ ad & x=1\end{smallmatrix}\right]=\frac{2}{3}$$

tum vero fractio ſpuria $\frac{x^3}{1+x}$ reſoluitur in has partes: $xx-x+1-\frac{1}{x+1}$, vnde integrando fit

$$\int\frac{x^3\,dx}{1+x}=\frac{1}{3}x^3-\frac{1}{2}xx+x-l(1+x)$$

ex quo valor formulae erit $=\frac{5}{6}-l2$. His ergo valoribus ſubſtitutis adipiſcimur hanc integrationem:

$$\int\frac{x^3\,dx\,lx}{\sqrt{1-xx}}\left[\begin{smallmatrix}ab & x=0\\ ad & x=1\end{smallmatrix}\right]=-\frac{2}{3}\left(\frac{5}{6}-l2\right).$$

Exemplum V. quo $p=5$.

§. 33. Hoc igitur caſu aequatio ſuperior hanc induet formam:

$$\int\frac{x^4\,dx\,lx}{\sqrt{1-xx}}=-\int\frac{x^4\,dx}{\sqrt{1-xx}}\cdot\int\frac{x^4\,dx}{1+x}.$$

Conſtat autem eſſe

$$\int\frac{x^4\,dx}{\sqrt{1-xx}}\left[\begin{smallmatrix}ab & x=0\\ ad & x=1\end{smallmatrix}\right]=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi}{2}$$

tum vero fractio ſpuria $\frac{x^4}{1+x}$ manifeſto reſoluitur in has partes: $x^3-xx+x-1+\frac{1}{x+1}$, vnde integrando fit

$$\int\frac{x^4\,dx}{1+x}=\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}xx-x+l(1+x)$$

 ex

ex quo valor formulae erit $= -\frac{7}{12} + l\,2$. His igitur valoribus substitutis prodibit ista integratio:

$$\int \frac{x^4\,dx\,lx}{\sqrt{1-xx}} \left[\begin{smallmatrix} ab & x=0 \\ ad & x=1 \end{smallmatrix}\right] = -\frac{1.3}{2.4}\cdot\frac{\pi}{2}\left(l\,2 - \frac{7}{12}\right).$$

Exemplum VI. quo $p = 6$.

§. 34. Hoc igitur casu aequatio superior induet hanc formam:

$$\int \frac{x^5\,dx\,lx}{\sqrt{1-xx}} = -\int \frac{x^5\,dx}{\sqrt{1-xx}} \cdot \int \frac{x^5\,dx}{1+x}.$$

Constat autem per reductiones notas esse

$$\int \frac{x^5\,dx}{\sqrt{1-xx}} \left[\begin{smallmatrix} ab & x=0 \\ ad & x=1 \end{smallmatrix}\right] = \frac{2.4}{3.5}$$

tum vero fractio spuria $\frac{x^5}{1+x}$ resoluitur in has partes:

$$x^4 - x^3 + xx - x + 1 - \frac{1}{x+1}$$

vnde integrando nanciscimur

$$\int \frac{x^5\,dx}{1+x} = \frac{1}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}xx + x - l(1+x)$$

ex quo valor huius formulae erit $= \frac{47}{60} - l\,2$; quibus valoribus substitutis prodibit ista integratio:

$$\int \frac{x^5\,dx\,lx}{\sqrt{1-xx}} \left[\begin{smallmatrix} ab & x=0 \\ ad & x=1 \end{smallmatrix}\right] = -\frac{2.4}{3.5}\left(\frac{47}{60} - l\,2\right).$$

II. Euolutio casus quo $m = 3$ et $n = 2$.

§. 35. Hic ergo erit $X = \sqrt{(1-xx)}$, vnde theorema nostrum generale nobis praebebit hanc aequationem:

$$\int x^{p-1}\,dx\,lx.\sqrt{1-xx} = \int x^{p-1}\,dx\sqrt{1-xx}.\int \frac{x^{p-1}(x^3-1)\,dx}{1-xx}$$

vbi cum sit

$$\frac{x^3-1}{1-xx} = \frac{-xx-x-1}{x+1} = -x - \frac{1}{x+1}.$$

erit postrema formula integralis

$$-\int x^p\,dx - \int \frac{x^{p-1}\,dx}{1+x}$$

quae

quae integrata ab $x = 0$ ad $x = 1$ dat

$$\frac{1}{p+1} + \int \frac{x^{p-1}\, dx}{1+x}$$

quam ob rem habebimus

$$\int x^{p-1} dx\, lx\, \sqrt{1-xx} = -\int x^{p-1} dx \sqrt{1-xx} \left(\frac{1}{p+1} + \int \frac{x^{p-1}\, dx}{1+x}\right).$$

Hinc igitur ſequentia exempla notaſſe iuuabit.

Exemplum I. quo $p = 1$.

§. 36. Pro hoc igitur caſu poſtremus factor euadet, $\frac{1}{2} + l\,2$, ita vt ſit

$$\int dx\, lx.\sqrt{1-xx} = -\left(\tfrac{1}{2} + l\,2\right) \int dx \sqrt{1-xx}.$$

Pro formula autem $\int dx \sqrt{1-xx}$ ſtatuatur $\sqrt{1-xx} = 1 - vx$ fietque $x = \frac{2v}{1+vv}$ et $\sqrt{1-xx} = \frac{1-vv}{1+vv}$ atque $dx = \frac{2dv(1-vv)}{(1+vv)^2}$, vnde fiet $dx \sqrt{1-xx} = \frac{2dv(1-vv)^2}{(1+vv)^3}$, cuius integrale reſoluitur in has partes: $\frac{2v}{(1+vv)^2} - \frac{v}{1+vv} + A\,\mathrm{tang.}\,v$; quae expreſſio, cum extendi debeat ab $x = 0$ vsque ad $x = 1$, prior terminus erit $v = 0$, alter vero terminus eſt $v = 1$; ita vt integrale illud a $v = 0$ vsque ad $v = 1$ extendi debeat. At vero illa expreſſio ſponte euaneſcit poſito $v = 0$, facto autem $v = 1$, valor integralis erit $= \frac{\pi}{4}$, quam ob rem habebimus

$$\int dx\, lx.\sqrt{1-xx} \left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = 1 \end{smallmatrix}\right] = -\frac{\pi}{4}\left(\tfrac{1}{2} + l\,2\right).$$

§. 37. Hic quidem calculum per longas ambages euoluimus, prouti reductio ad rationalitatem formulae $\sqrt{1-xx}$ manuduxit; at vero ſolus aſpectus formulae $\int dx \sqrt{1-xx}$ ſtatim declarat, eam exprimere aream quadrantis circuli, cuius radius $= 1$, quem nouimus eſſe $= \frac{\pi}{4}$. Caeterum adhiberi potuiſſet iſta reductio:

$$\int dx \sqrt{1-xx} = \tfrac{1}{2} x \sqrt{1-xx} + \tfrac{1}{2} \int \frac{dx}{\sqrt{1-xx}}$$

cuius valor ab $x = 0$ ad $x = 1$ extenſus manifeſto dat $\frac{\pi}{4}$.

Exem-

Exemplum II. quo $p = 2$.

§. 38. Hoc ergo caſu poſtremus factor fit

$$\tfrac{1}{3} + \int \frac{x\,dx}{1+x} = \tfrac{4}{3} - l\,2$$

ſicque habebimus

$$\int x\,dx\,lx.\sqrt{1-xx} = -(\tfrac{4}{3} - l\,2)\int x\,dx\sqrt{1-xx}$$

perſpicuum autem eſt eſſe

$$\int x\,dx\sqrt{1-xx} = C - \tfrac{1}{3}(1-xx)^{\frac{3}{2}}$$

qui valor ab $x = 0$ ad $x = 1$ extenſus praebet $\frac{1}{3}$, ita vt habeamus

$$\int x\,dx\,lx.\sqrt{1-xx}\left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = 1 \end{smallmatrix}\right] = -\tfrac{1}{3}(\tfrac{4}{3} - l\,2).$$

III. Euolutio caſus quo $m = 1$ et $n = 3$.

§. 39. Hoc igitur caſu erit $X = \frac{1}{\sqrt[3]{(1-x^3)^2}}$, vnde theorema generale nobis praebet hanc aequationem:

$$\int \frac{x^{p-1}\,dx\,lx}{\sqrt[3]{(1-x^3)^2}} = \int \frac{x^{p-1}\,dx}{\sqrt[3]{(1-x^3)^2}} - \int \frac{x^{p-1}(x-1)\,dx}{1-x^3}$$

vbi poſtrema formula reducitur ad hanc: $-\int \frac{x^{p-1}\,dx}{xx+x+1}$, ita vt habeamus

$$\int \frac{x^{p-1}\,dx\,lx}{\sqrt[3]{(1-x^3)^2}} = -\int \frac{x^{p-1}\,dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x^{p-1}\,dx}{xx+x+1}$$

ſequentia igitur exempla adiungamus.

Exemplum I. quo $p = 1$.

§. 40. Hoc igitur caſu poſtremus factor euadit $\int \frac{dx}{xx+x+1}$, cuius integrale indefinitum reperitur $\frac{2}{\sqrt{3}}$ A tg. $\frac{x\sqrt{3}}{2+x}$ qui valor poſito $x = 1$ abit in $\frac{\pi}{3\sqrt{3}}$; quocirca hoc caſu habe-

habebimus

$$\int \frac{d x\, x}{\sqrt[3]{(1-x^3)^2}} = -\frac{\pi}{3\sqrt{3}} \int \frac{d x\, l x}{\sqrt[3]{(1-x^3)^2}}$$

at vero formula integralis $\int \frac{dx}{\sqrt[3]{(1-x^3)^2}}$ peculiarem quantitatem transcendentem inuoluit, quam neque per logarithmos, neque per arcus circulares explicare licet.

Exemplum II. quo $p = 2$.

§. 41. Hoc igitur casu postremus factor erit $\int \frac{x dx}{1+x+xx}$ qui in has partes resoluatur:

$$\frac{1}{2}\int \frac{2x\,dx + dx}{1+x+xx} - \frac{1}{2}\int \frac{dx}{1+x+xx}$$

vbi partis prioris integrale est

$$\frac{1}{2} l(1+x+xx) = \frac{1}{2} l\, 3 \text{ (posito scilicet } x = 1)$$

alterius vero partis integrale est $-\frac{1}{2} \cdot \frac{\pi}{3\sqrt{3}}$, quo valore substituto habebimus

$$\int \frac{x d x\, l x}{\sqrt[3]{(1-x^3)^2}} = -\frac{1}{2}\left(l\, 3 - \frac{\pi}{3\sqrt{3}}\right)\int \frac{x\, d x}{\sqrt[3]{(1-x^3)^2}}.$$

Nunc vero istam formulam integralem commode assignare licet per reductionem supra initio indicatam; cum enim hic sit $m = 1$ et $n = 3$, tum vero sumserimus $p = 2$, erit $p = n - m$. Supra autem §. 15. inuenimus, hoc casu integrale fore $= \frac{\pi}{n \sin \frac{m\pi}{n}}$, qui valor nostro casu abit in $\frac{\pi}{3 \sin \frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}$. Hoc igitur valore substituto nostram formulam per meras quantitates cognitas exprimere poterimus, hoc modo:

$$\int \frac{x d x\, l x}{\sqrt[3]{(1-x^2)^2}} \left[\begin{smallmatrix} ab\ x = 0 \\ ad\ x = 1 \end{smallmatrix}\right] = -\frac{\pi}{3\sqrt{3}}\left(l\, 3 - \frac{\pi}{3\sqrt{3}}\right).$$

IV.

IV. Euolutio caſus quo $m = 2$ et $n = 3$.

§. 42. Hoc igitur caſu erit $X = \frac{1}{\sqrt[3]{(1 - x^3)}}$ vnde theorema generale praebet iſtam aequationem:

$$\int \frac{x^{p-1}\, dx\, lx}{\sqrt[3]{(1 - x^3)}} = \int \frac{x^{p-1}\, dx}{\sqrt[3]{(1 - x^3)}} \cdot \int \frac{x^{p-1}(xx - 1)\, dx}{1 - x^3}.$$

vbi forma poſtrema transmutatur in hanc: $-\int \frac{x^{p-1} dx (1 + x)}{1 + x + xx}$;
vnde fiet

$$\int \frac{x^{p-1}\, dx\, lx}{\sqrt[3]{(1 - x^3)}} = -\int \frac{x^{p-1}\, dx}{\sqrt[3]{(1 - x^3)}} \cdot \int \frac{x^{p-1}\, dx\, (1 + x)}{1 + x + xx}$$

vnde ſequentia exempla expediamus.

Exemplum I. quo $p = 1$.

§. 43. Hoc ergo caſu membrum poſtremum erit $\int \frac{dx(1 + x)}{1 + x + xx}$, cuius integrale in has partes diſtribuatur:

$$\frac{1}{2}\int \frac{2x\, dx + dx}{1 + x + xx} + \frac{1}{2}\int \frac{dx}{1 + x + xx}$$

vnde manifeſto pro caſu $x = 1$ prodit $\frac{1}{2}(l3 + \frac{\pi}{3\sqrt{3}})$; quamobrem noſtra aequatio erit

$$\int \frac{dx\, lx}{\sqrt[3]{1 - x^3}} = -\frac{1}{2}(l3 + \frac{\pi}{3\sqrt{3}}) \int \frac{dx}{\sqrt[3]{(1 - 1x)^3}}.$$

In hac autem formula integrali, ob $m = 2$ et $n = 3$, quia ſumſimus $p = 1$, erit $p = n - m$; pro hoc ergo caſu per §. 15. valor iſtius formulae abſolute exprimi poterit, eritque $\int \frac{dx}{\sqrt[3]{(1 - x^3)}} = \frac{2\pi}{3\sqrt{3}}$; conſequenter etiam hoc caſu per quantitates abſolutas conſequimur hanc formam:

$$\frac{dx\, lx}{\sqrt[3]{(1 - x^3)}} \left[\begin{matrix} ab\ x = 0 \\ ad\ x = 1 \end{matrix}\right] = -\frac{\pi}{3\sqrt{3}}(l3 + \frac{\pi}{3\sqrt{3}}).$$

§. 44. Quodſi hanc formam cum poſtrema caſus praecedentis, quae itidem abſolute prodiit expreſſa, combinemus,

nemus, earum summa primo dabit

$$\int \frac{x\,dx\,lx}{\sqrt[3]{(1-x^3)^2}} + \int \frac{dx\,lx}{\sqrt[3]{(1-x^3)}} = -\frac{2\pi\, l3}{3\sqrt{3}}$$

sin autem posterior a priore subtrahatur, orietur ista aequatio:

$$\int \frac{xx\,dx\,lx}{\sqrt[3]{(1-x^3)^2}} - \int \frac{dx\,lx}{\sqrt[3]{(1-x)^3}} = \frac{2\pi\pi}{27}.$$

Quoniam hoc modo ad expressiones satis simplices sumus perducti, operae pretium erit ambas aequationes sub alia forma repraesentare, qua binae partes integrales commode in vnam coniungi queant; statuamus scilicet $\frac{x}{\sqrt[3]{(1-x^3)}} = z$, vnde fit $\frac{xx}{\sqrt[3]{(1-x^3)^2}} = zz$, sicque prior formula induet hanc speciem $\int \frac{zz\,dx\,lx}{x}$, posterior vero istam: $\int \frac{z\,dx\,lx}{x}$; tum vero habebimus $\frac{x^3}{1-x^3} = z^3$, vnde fit $x^3 = \frac{z^3}{1+z^3}$, ideoque

$$lx = lz - \tfrac{1}{3} l(1+z^3) = l\frac{z}{\sqrt[3]{(1+z^3)}}$$

hincque porro

$$\frac{dx}{x} = \frac{dz}{z} - \frac{zz\,dz}{1+z^3} = \frac{dz}{z(1+z^3)}.$$

quare his valoribus adhibitis prior formula integralis euadit $\int \frac{z\,dz}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}}$; altera vero formula erit $\int \frac{dz}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}}$.

§. 45. Quoniam autem integralia ab $x=0$ ad $x=1$ extendi debent, notandum est, casu $x=0$ fieri $z=0$, at vero casu $x=1$ prodire $z=\infty$, ita vt nouas istas formas a $z=0$ ad $z=\infty$ extendi oporteat. Quo obseruato prior harum formularum dabit

$$\int \frac{z\,dz}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z)^3}} \left[\begin{smallmatrix} a & z = 0 \\ ad & z = \infty \end{smallmatrix}\right] = -\frac{\pi\, l3}{3\sqrt{3}} + \frac{\pi\pi}{27}$$

posterior vero

$$\int \frac{dz}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}} \left[\begin{smallmatrix} a & z = 0 \\ ad & z = \infty \end{smallmatrix}\right] = -\frac{\pi\, l3}{3\sqrt{3}} - \frac{\pi\pi}{27}.$$

Hinc igitur summa harum formularum erit

$$\int \frac{dz(1+z)}{1+z^3} \cdot l\frac{z}{\sqrt[3]{(1+z^3)}} = -\frac{2\pi\, l3}{3\sqrt{3}}$$

at vero differentia
$$\int\frac{dz(z-1)}{1+z^2}.\,l\frac{z}{\sqrt[3]{(1+z^3)}}=\frac{2\pi\pi}{27}.$$

§. 46. Hic non inutile erit obseruasse, istum logarithmum $l\frac{z}{\sqrt[3]{(1+z^3)}}$ commode in seriem infinitam satis simplicem conuerti posse; cum enim sit
$$l\frac{z}{\sqrt[3]{(1+z^3)}}=\tfrac{1}{3}\,l\,\frac{z^3}{1+z^3}=-\tfrac{1}{3}\,l\,\frac{1+z^3}{z^3}$$
erit per seriem
$$l\frac{z}{\sqrt[3]{1+z^3}}=-\tfrac{1}{3}\left(\frac{1}{z^3}-\frac{1}{2z^6}+\frac{1}{3z^9}-\frac{1}{4z^{12}}+\frac{1}{5z^{15}}\text{ etc.}\right)$$
verum ista resolutio nullum vsum praestare potest ad integralia haec per series euoluenda, propterea quod potestates ipsius z in denominatoribus occurrunt, ideoque singulae partes non ita integrari possunt, vt euanescant posito $z=0$.

Exemplum II. quo $p=2$.

§. 47. Hoc igitur casu factor postremus euadit $\int\frac{xdx(1+x)}{1+x+xx}$, qui in has duas partes discerpitur: $\int dx-\int\frac{dx}{1+x+xx}$, cuius ergo integrale ab $x=0$ ad $x=1$ extensum est $=1-\frac{\pi}{3\sqrt{3}}$.
Hinc igitur deducimur ad hanc aequationem:
$$\int\frac{xdx\,lx}{\sqrt[3]{(1-x^3)}}=-\left(1-\frac{\pi}{3\sqrt{3}}\right)\int\frac{xdx}{\sqrt[3]{(1-x^3)}}.$$
Hic autem notandum istam formulam integralem nullo modo absolute exhiberi posse, sed peculiarem quandam quantitatem transcendentem inuoluere.

V. Euolutio casus, quo $m=2$ et $n=4$.

§. 48. Hoc igitur casu erit $X=\frac{1}{\sqrt{(1-x^4)}}$, vnde theorema nostrum generale nobis dabit hanc aequationem
$$\int\frac{x^{p-1}dx\,lx}{\sqrt{(1-x^4)}}=-\int\frac{x^{p-1}dx}{\sqrt{(1-x^4)}}\cdot\int\frac{x^{p-1}dx}{1+xx}$$

[illegible] vero problema particulare prius pro hoc casu praebet $\int\frac{x^3\,dx\,lx}{\sqrt{(1-x^4)}} = -\frac{1}{2}\int\frac{x^3\,dx}{1+xx}$. Cum autem sit $\int\frac{x^3\,dx}{1+xx} = \frac{1}{2} - \frac{1}{2}l\,2$, erit absolute

$$\int\frac{x^3\,dx\,lx}{\sqrt{(1-x^4)}}\left[\begin{matrix}ab\ x=0\\ ad\ x=1\end{matrix}\right] = -\tfrac{1}{4}(1-l\,2)$$

at vero hic casus congruit cum supra §. 28. tractato. Si enim hic ponamus $xx=y$, quo facto termini integrationis manent $y=0$ et $y=1$, erit $lx=\frac{1}{2}ly$ et $x\,dx=\frac{1}{2}dy$; quibus valoribus substitutis nostra aequatio abibit in hanc formam: $\frac{1}{4}\int\frac{y\,dy\,ly}{\sqrt{(1-yy)}} = -\frac{1}{4}(1-l\,2)$ siue $\int\frac{y\,dy\,ly}{\sqrt{(1-yy)}} = l\,2-1$, prorsus vt supra.

§. 49. Alterum vero theorema particulare ad praesentem casum accommodatum dabit $\int\frac{x\,dx\,lx}{\sqrt{(1-x^4)}} = -\frac{\pi}{4}\int\frac{x\,dx}{1+xx}$; est vero

$$\int\frac{x\,dx}{1+xx} = l\sqrt{1+xx} = \tfrac{1}{2}l\,2$$

ita vt habeamus

$$\int\frac{x\,dx\,lx}{\sqrt{(1-x^4)}}\left[\begin{matrix}ab\ x=0\\ ad\ x=1\end{matrix}\right] = -\tfrac{\pi}{8}l\,2.$$

Quodsi vero hic vt ante statuamus $xx=y$, obtinebitur $\int\frac{dy\,ly}{\sqrt{(1-yy)}} = -\frac{\pi}{2}l\,2$, qui est casus supra §. 26. tractatus. His duobus casibus exponens p erat numerus par, vnde casus impares euolui conueniet.

Exemplum I. quo $p=1$.

§. 50. Hoc igitur casu formula integralis postrema fiet $\int\frac{dx}{1+xx} = A\ tang.\,x$, ita vt posito $x=1$ prodeat $\frac{\pi}{4}$; tum vero aequatio nostra erit, $\int\frac{dx\,lx}{\sqrt{(1-x^4)}} = -\frac{\pi}{4}\int\frac{dx}{\sqrt{(1-x^4)}}$, integralibus scilicet ab $x=0$ ad $x=1$ extensis; vbi formula $\int\frac{dx}{\sqrt{(1-x^4)}}$ arcum curuae elasticae rectangulae exprimit, ideoque absolute exhiberi nequit.

Exemplum II. quo $p = 3$.

§. 51. Hoc ergo casu formula integralis postrema erit $\int \frac{xxdx}{1+xx} = \int dx - \int \frac{dx}{1+xx}$, cuius integrale posito $x = 1$ fit $= 1 - \frac{\pi}{4}$, ita vt nunc aequatio nostra euadat

$$\int \frac{xxdxlx}{\sqrt{(1-x^4)}} = -\left(1 - \frac{\pi}{4}\right) \int \frac{xxdx}{\sqrt{(1-x^4)}}$$

quae formula integralis pariter absolute exhiberi nequit; exprimit enim applicatam curuae elasticae rectangulae.

§. 52. Quanquam autem haec duo exempla ad formulas inextricabiles perduxerunt, tamen iam pridem demonstraui, productum horum duorum integralium $\int \frac{dx}{\sqrt{(1-x^4)}} \cdot \int \frac{xxdx}{\sqrt{(1-x^4)}}$ aequari areae circuli, cuius diameter $= 1$, siue esse $= \frac{\pi}{4}$; quam ob rem, binis exemplis coniungendis, hoc insigne theorema adipiscimur $\int \frac{dxlx}{\sqrt{(1-x^4)}} \cdot \int \frac{xxdxlx}{\sqrt{(1-x^4)}} = \frac{\pi^2}{16}\left(1 - \frac{\pi}{4}\right)$. Facile autem patet, innumera alia huiusmodi theoremata ex hoc fonte hauriri posse, quae, per se spectata, profundissimae indaginis sunt censenda.

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