MEDITATIONES
CIRCA SINGVLARE
SERIERVM
GENVS

Auctore

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In commercio litterario, quod olim cum Illustrissimo *Goldbachio* coluerum, inter alias varii argumenti speculationes, circa series in hac forma generali:

$$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}$$

comprehensas sumus versati, earumque summas scrutati. Tametsi autem huiusmodi series raro occurrere solent, parumque vtilitatis polliceri videntur, inuestigationes tamen, ad quas earum consideratio nos perduxerat, eo magis dignae videntur, vt ab obliuione et interitu vindicentur, quod methodi, quibus ea occasione sumus vsi, multo latius patent, ac fortasse aliquando Analysi insignem vsum afferre possunt. Non igitur tam ipsam hanc seriem, etiamsi in se spectata neutiquam spernenda videatur, quam varias methodos, quae ad eius summationem perducunt, hic exponere constitui; quae quoniam ex commercio illo epistolico sunt desumta, lectores hic statim

tim in limine monitos velim, has inueſtigationes maximam partem acumini Illuſtriſſimi *Goldbachii* eſſe attribuendas. Tres autem potiſſimum viae ad huiusmodi ſeries deducunt, quae quoniam inter ſe maxime ſunt diuerſae, vnamquamque ſeorſim explicabo, quo, quantum quaelibet praeſtet, facilius perſpici poſſit.

Prima Methodus ad huiusmodi ſeries perueniendi.

1. Si habeantur duae ſeries quaecunque, quarum ſumma conſtet:

$$1 + a + b + c + d + e + \text{etc.} = t \quad \text{et}$$
$$1 + \alpha + \beta + \gamma + \delta + \varepsilon + \text{etc.} = u$$

tum vero inſuper ſeriei ex his conflatae ſumma ſit cognita ſcilicet:

$$1 + a\alpha + b\beta + c\gamma + d\delta + e\varepsilon + \text{etc.} = v$$

tum illis ſeriebus in ſe multiplicandis colligitur:

$$\left.\begin{array}{l} 1+a(1+\alpha)+b(1+\alpha+\beta)+c(1+\alpha+\beta+\gamma)\text{etc.} \\ +1+\alpha(1+a)+\beta(1+a+b)+\gamma(1+a+b+c)\text{etc.} \end{array}\right\} = tu+v$$

quod quidem per ſe eſt manifeſtum, quoniam in his duabus poſterioribus ſeriebus occurrunt producta ſingulorum terminorum primae ſeriei per ſingulos ſecundae; id tantum notetur, producta cuiusque termini primae ſeriei per terminum reſpondentem ſecudae, veluti $1\,1$, $a\,\alpha$, $b\,\beta$, $c\,\gamma$, $d\,\delta$ etc. bis occurrere, quae quia in producto $t\,u$ ſemel tantum repeririuntur, idcirco ad id inſuper ſeriem v adiungi oportebat.

2. Quod si iam seriei $1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+$ etc. in infinitum continuatae summam per $\int\frac{1}{z^m}$ designemus, vt sit

$$1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\frac{1}{5^m}+\text{etc.}=\int\frac{1}{z^m}\text{ et}$$

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\text{etc.}=\int\frac{1}{z^n}$$

similique modo pro serie hinc conflata:

$$1+\frac{1}{2^{m+n}}+\frac{1}{3^{m+n}}+\frac{1}{4^{m+n}}+\frac{1}{5^{m+n}}+\text{etc.}=\int\frac{1}{z^{m+n}}$$

atque ex his sequentes duas series, quae in forma proposita continebuntur, formemus:

$$1+\frac{1}{2^m}(1+\frac{1}{2^n})+\frac{1}{3^m}(1+\frac{1}{2^n}+\frac{1}{3^n})+\frac{1}{4^m}(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n})+\text{etc.}=P$$

$$1+\frac{1}{2^n}(1+\frac{1}{2^m})+\frac{1}{3^n}(1+\frac{1}{2^m}+\frac{1}{3^m})+\frac{1}{4^n}(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m})+\text{etc.}=Q$$

ex principio supra stabilito habebimus:

$$P+Q=\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}+\int\frac{1}{z^{m+n}}$$

vnde si alterius harum duarum nouarum serierum summa vndecunque constaret, hinc alterius quoque seriei summa assignari posset. Summas autem serierum

$$1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\text{etc. seu }\int\frac{1}{z^m}$$

hic vt cognitas spectamus, quandoquidem quoties exponens m est numerus par, hae summae a me per

per peripheriam circuli ſunt definitae; pro caſibus autem quibus m eſt numerus impar, ſummae vero proximae facile reperiri poſſunt.

3. Quando exponentes m et n ſumuntur aequales binae ſeries inuentae conueniunt, hocque ergo caſu ſequentem ſummationem adipiſcimur:

$$1+\frac{1}{2^n}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^n}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^n}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right) \text{ etc.}$$
$$=\tfrac{1}{2}\left(\int\frac{1}{z^n}\right)^2+\tfrac{1}{2}\int\frac{1}{z^{2n}}.$$

Quocirca ſi caſus particulares conſideraturi ponamus:

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\text{etc.}=\Delta=\int\tfrac{1}{z}$$
$$1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\tfrac{1}{5^2}+\text{etc.}=A=\int\tfrac{1}{z^2}$$
$$1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\text{etc.}=B=\int\tfrac{1}{z^3}$$
$$1+\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\tfrac{1}{5^4}+\text{etc.}=C=\int\tfrac{1}{z^4}$$
$$1+\tfrac{1}{2^5}+\tfrac{1}{3^5}+\tfrac{1}{4^5}+\tfrac{1}{5^5}+\text{etc.}=D=\int\tfrac{1}{z^5}$$
etc.

ſimilique modo porro:

$$\int\tfrac{1}{z^6}=E;\ \int\tfrac{1}{z^7}=F,\ \int\tfrac{1}{z^8}=G;\ \int\tfrac{1}{z^9}=H;\ \int\tfrac{1}{z^{10}}=I \text{ etc.}$$

hinc conſequimur ſequentes ſummationes:

$$1+\tfrac{1}{2}\left(1+\tfrac{1}{2}\right)+\tfrac{1}{3}\left(1+\tfrac{1}{2}+\tfrac{1}{3}\right)+\text{etc.}=\tfrac{1}{2}\Delta\Delta+\tfrac{1}{2}A$$
$$1+\tfrac{1}{2^2}\left(1+\tfrac{1}{2^2}\right)+\tfrac{1}{3^2}\left(1+\tfrac{1}{2^2}+\tfrac{1}{3^2}\right)+\text{etc.}=\tfrac{1}{2}AA+\tfrac{1}{2}C$$
$$1+\tfrac{1}{2^3}\left(1+\tfrac{1}{2^3}\right)+\tfrac{1}{3^3}\left(1+\tfrac{1}{2^3}+\tfrac{1}{3^3}\right)+\text{etc.}=\tfrac{1}{2}BB+\tfrac{1}{2}E$$
$$1+\tfrac{1}{2^4}\left(1+\tfrac{1}{2^4}\right)+\tfrac{1}{3^4}\left(1+\tfrac{1}{2^4}+\tfrac{1}{3^4}\right)+\text{etc.}=\tfrac{1}{2}CC+\tfrac{1}{2}G$$
$$1+\tfrac{1}{2^5}\left(1+\tfrac{1}{2^5}\right)+\tfrac{1}{3^5}\left(1+\tfrac{1}{2^5}+\tfrac{1}{3^5}\right)+\text{etc.}=\tfrac{1}{2}DD+\tfrac{1}{2}I$$
etc.

vbi

vbi quidem notari conuenit, primae seriei summam Δ esse infinitam, reliquas vero omnes finitas.

4. At si exponentes m et n sunt inaequales; hoc modo series formae, quam contemplamur obtinentur, quarum quidem neutrius summam seorsim hac methodo definire licet; verumtamen ambarum iunctim sumtarum summa exhiberi potest, vti ante ostendimus. Quod quo planius reddatur, simulque scribendi compendium in vsum vocetur, huius seriei summam

$$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right) \text{ etc.}$$

hac scriptione $\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)$ indicemus, ita vt permutatis exponentibus habeatur $\int\frac{1}{z^n}\left(\frac{1}{y^m}\right)$; His notatis inuenimus fore

$$\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)+\int\frac{1}{z^n}\left(\frac{1}{y^m}\right)=\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}+\int\frac{1}{z^{m+n}}.$$

Quodsi ergo aliunde alterius harum serierum summa innotuerit, hinc alterius seriei summa erit cognita; atque plus ex hac prima methodo concludere non licet, ex quo ad secundam euoluendam progredior.

Secunda Methodus ad huiusmodi series perueniendi.

5. Seruata praecedente notandi ratione perspicuum est, quantitatem

$$\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}-\int\frac{1}{z^{m+n}}$$

ad

ad ſequentes ſeries infinitas reduci:

$$+\frac{1}{1\cdot 2^n}+\frac{1}{2^m\cdot 3^n}+\frac{1}{3^m\cdot 4^n}+\text{etc.}+\frac{1}{1\cdot 2^m}+\frac{1}{2^n\cdot 3^m}+\frac{1}{3^n\cdot 4^m}+\text{etc.}$$

$$+\frac{1}{1\cdot 3^n}+\frac{1}{2^m\cdot 4^n}+\frac{1}{3^m\cdot 5^n}+\text{etc.}+\frac{1}{1\cdot 3^m}+\frac{1}{2^n\cdot 4^m}+\frac{1}{3^n\cdot 5^m}+\text{etc.}$$

$$+\frac{1}{1\cdot 4^n}+\frac{1}{2^m\cdot 5^n}+\frac{1}{3^m\cdot 6^n}+\text{etc.}+\frac{1}{1\cdot 4^m}+\frac{1}{2^n\cdot 5^m}+\frac{1}{3^n\cdot 6^m}+\text{etc.}$$

$$+\frac{1}{1\cdot 5^n}+\frac{1}{2^m\cdot 6^n}+\frac{1}{3^m\cdot 7^n}+\text{etc.}+\frac{1}{1\cdot 5^m}+\frac{1}{2^n\cdot 6^m}+\frac{1}{3^n\cdot 7^m}+\text{etc.}$$

etc. etc.

totumque negotium iam hinc redit, vt ſingulae hae ſeries commode ſummentur; quo ipſo vberrimus campus aperitur peculiare ſerierum genus complectens, quod ob concinnitatem per ſe omni attentione dignum videtur, etiamſi cum inſtituto noſtro non tam arcto vinculo eſſet connexum.

6. Hae autem ſummationes commodius inueſtigari nequeunt, quam ſingulos terminos, quorum forma eſt $\frac{1}{x^m(x+a)^n}$, in fractiones ſimpliciores reſoluendo. Per ea autem quae de hoc argumento in Introductione ad Analyſin tradidi, patet, hanc fractionem in ſequentes diſcerpi:

$$\frac{1}{a^n}\cdot\frac{1}{x^m}-\frac{n}{1\cdot a^{n+1}}\cdot\frac{1}{x^{m-1}}+\frac{n(n+1)}{1\cdot 2\, a^{n+2}}\cdot\frac{1}{x^{m-2}}$$
$$-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3\, a^{n+3}}\cdot\frac{1}{x^{m-3}}+\text{etc.}$$

$$\pm\frac{1}{a^m}\cdot\frac{1}{(x+a)^n}\pm\frac{m}{1.\,a^{m+1}}\cdot\frac{1}{(x+a)^{n-1}}\pm\frac{m(m+1)}{1.\,2.\,a^{m+2}}\cdot\frac{1}{(x+a)^{n-2}}$$
$$\pm\frac{m(m+1)(m+2)}{1.\,2.\,3\,a^{m+3}}\cdot\frac{1}{(x+a)^{n-3}}\pm\text{etc.}$$

vbi tenendum est, in inferiori ordine signum superius $+$ valere, si m sit numerus par, contra vero signum inferius; tum vero vterque ordo non vlterius continuari debet, quam vsque ad terminos vbi exponens potestatum ipsius x et $x+a$ ad vnitatem fuerit diminutus.

7. Hinc igitur primo summam huius seriei

$$\frac{1}{1(a+1)^n}+\frac{1}{2^m(a+2)^n}+\frac{1}{3^m(a+3)^n}+\frac{1}{4^m(a+4)^n}+\text{etc.}=s$$

definire licet, dum in forma modo exhibita loco x omnes numeri 1, 2, 3 etc. in infinitum substituuntur, et in vnam summam colliguntur: Quoniam enim omnes termini ex formula $A.\frac{1}{x^\lambda}$ nati dant seriem cuius summam per $A\int\frac{1}{z^\lambda}$ exprimimus, ex formula autem $A\frac{1}{(x+a)^\lambda}$ prodit series cuius summa est $A\int\frac{1}{z^\lambda}-A\left(1+\frac{1}{2^\lambda}+\frac{1}{3^\lambda}\ldots\ldots+\frac{1}{a^\lambda}\right)$, seriei nostrae summa ita se habebit:

$$s=\frac{1}{a^n}\int\frac{1}{z^m}-\frac{n}{1.\,a^{n+1}}\int\frac{1}{z^{m-1}}+\frac{n(n+1)}{1.\,2\,a^{n+2}}\int\frac{1}{z^{m-2}}$$
$$-\frac{n(n+1)(n+2)}{1.\,2.\,3\,a^{n+3}}\int\frac{1}{z^{m-3}}+\text{etc.}$$

$\pm$

$$\pm \frac{1}{a^m}\int\frac{1}{z^n} \pm \frac{m}{1.\,a^{m+1}}\int\frac{1}{z^{n-1}} \pm \frac{m(m+1)}{1.\,2\,a^{m+2}}\int\frac{1}{z^{n-2}}$$
$$\pm \frac{m(m+)(m+2)}{1.\,2.\,3\,a^{m+3}}\int\frac{1}{z^{n-3}} \pm \text{etc.}$$

$$\mp \frac{1}{a^m}\left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \ldots\ldots + \frac{1}{a^n}\right)$$
$$\mp \frac{m}{1\,a^{m+1}}\left(1 + \frac{1}{2^{n-1}} + \frac{1}{3^{n-1}} + \ldots\ldots + \frac{1}{a^{n-1}}\right)$$
$$\mp \frac{m(m+1)}{1.\,2\,a^{m+2}}\left(1 + \frac{1}{2^{n-2}} + \frac{1}{3^{n-2}} + \ldots\ldots + \frac{1}{a^{n-2}}\right)$$
$$\mp \frac{m(m+1)(m+2)}{1.\,2.\,3\,a^{m+3}}\left(1 + \frac{1}{2^{m-3}} + \frac{1}{3^{n-3}} + \ldots\ldots + \frac{1}{a^{n-3}}\right)$$
$$\text{etc.}$$

vbi fignis fuperioribus eft vtendum, quoties m eft numerus par, contra vero inferioribus. Expreffio autem haec femper eft finita, quoniam vtrumque terminorum ordinem tantum vsque ad $\int\frac{1}{z}$ continuari conuenit.

8. Tribuantur iam quoque litterae a omnes valores ab vnitate in infinitum, vt in vna fumma conplectamur omnes feries infinitas prioris ordinis §. 5. ad finiftram notatas; atque earum fumma ita reperietur repraefentata:

$$\int\frac{1}{z^n}.\int\frac{1}{z^m} - \frac{n}{1}\int\frac{1}{z^{n+1}}\int\frac{1}{z^{m-1}} + \frac{n(n+1)}{1.\,2}\int\frac{1}{z^{n+2}}.\int\frac{1}{z^{m-2}}$$
$$- \frac{n(n+1)(n+2)}{1.\;2.\;3.}\int\frac{1}{z^{n+3}}.\int\frac{1}{z^{m-3}} \text{ etc.}$$

$$\pm \int \frac{1}{z^m} \cdot \int \frac{1}{z^n} \mp \int \frac{1}{z^m}\left(\frac{1}{y^n}\right)$$

$$\pm \frac{m}{1}\int \frac{1}{z^{m+1}} \cdot \int \frac{1}{z^{n-1}} \mp \frac{m}{1}\int \frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right)$$

$$\pm \frac{m(m+1)}{1.\ 2}\int \frac{1}{z^{m+2}} \cdot \int \frac{1}{z^{n-2}} \mp \frac{m(m+1)}{1.\ 2}\int \frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right)$$

$$\pm \frac{m(m+1)(m+2)}{1.\ 2.\ 3.}\int \frac{1}{z^{m+3}} \cdot \int \frac{1}{z^{n-3}} \mp \frac{m(m+1)(m+2)}{1.\ 2.\ 3}\int \frac{1}{z^{m+3}}\left(\frac{1}{y^{n-3}}\right)$$

etc.

Similique modo permutandis exponentibus m et n ſumma orietur alterius ſerierum ordinis ad dextram §. 5. diſpoſiti. His igitur expreſſionibus coniunctis quantitas $\int \frac{1}{z^m} \cdot \int \frac{1}{z^n} - \int \frac{1}{z^{m+n}}$ in ſequentem formam transmutatur, quam per duas partes exhiberi conuenit.

Pars prior.

$$+(1 \pm 1)\int \frac{1}{z^m} \cdot \int \frac{1}{z^n} \mp \int \frac{1}{z^m}\left(\frac{1}{y^n}\right)$$

$$-\frac{m}{1}(1 \mp 1)\int \frac{1}{z^{m+1}} \cdot \int \frac{1}{z^{n-1}} \pm \frac{m}{1}\int \frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right)$$

$$+\frac{m(m+1)}{1.\ 2}(1 \pm 1)\int \frac{1}{z^{m+2}} \cdot \int \frac{1}{z^{n-2}} \mp \frac{m(m+1)}{1.\ 2.}\int \frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right)$$

$$-\frac{m(m+1)(m+2)}{1.\ 2.\ 3}(1 \mp 1)\int \frac{1}{z^{m+3}} \cdot \int \frac{1}{z^{n-3}} \pm \frac{m(m+1)(m+2)}{1.\ 2.\ 3}\int \frac{1}{z^{m+3}}\left(\frac{1}{y^{n-3}}\right)$$

etc.

vbi

vbi figna fuperiora valent, fi m fit numerus par, inferiora vero fi m fit numerus impar.

Pars altera.

$$+(1 \pm 1)\int\frac{1}{z^n}\cdot\int\frac{1}{z^m} \mp \int\frac{1}{z^n}\left(\frac{1}{y^m}\right)$$

$$-\frac{n}{1}(1 \mp 1)\int\frac{1}{z^{n+1}}\cdot\int\frac{1}{z^{m-1}} \mp \frac{n}{1}\int\frac{1}{z^{n+1}}\left(\frac{1}{y^{m-1}}\right)$$

$$+\frac{n(n+1)}{1.\ 2}(1 \pm 1)\int\frac{1}{z^{n+2}}\cdot\int\frac{1}{z^{m-2}} \mp \frac{n(n+1)}{1.\ 2.}\int\frac{1}{z^{n+2}}\left(\frac{1}{y^{m-2}}\right)$$

$$-\frac{n(n+1)(n+2)}{1.\ 2.\ 3}(1 \mp 1)\int\frac{1}{z^{n+3}}\cdot\int\frac{1}{z^{m-3}} \mp \frac{n(n+1)(n+2)}{1.\ 2.\ 3}\int\frac{1}{z^{n+3}}\left(\frac{1}{y^{m-3}}\right)$$

etc.

vbi figna fuperiora valent, fi n numerus par, contra vero valent inferiora.

8. Prout ergo exponentes m et n fuerint vel pares vel impares, hae expreffiones ad pauciores terminos reducuntur, erit fcilicet

Pars prima fi m *fit numerus par.*

$$+2\int\frac{1}{z^m}\cdot\int\frac{1}{z^n} - \int\frac{1}{z^m}\left(\frac{1}{y^n}\right) - \frac{m}{1}\int\frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right)$$

$$+\frac{2m(m+1)}{1.\ 2.}\int\frac{1}{z^{m+2}}\cdot\int\frac{1}{z^{n-2}} - \frac{m(m+1)}{1.\ 2}\int\frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right)$$

$$-\frac{(m+1)(m+2)}{1.\ 2.\ 3}\int\frac{1}{z^{m+3}}\left(\frac{1}{y^{n-3}}\right)$$

etc.

fiue hoc modo

$$2\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}+\frac{2m(m+1)}{1.\ 2}\int\frac{1}{z^{m+2}}\cdot\int\frac{1}{z^{n-2}}$$
$$+\frac{2m(m+1)(m+2)(m+3)}{1.\ 2.\ 3.\ 4}\int\frac{1}{z^{m+4}}\int\frac{1}{z^{n-4}}+\text{etc.}$$
$$-\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)-\frac{m}{1}\int\frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right)-\frac{m(m+1)}{1.\ 2.}\int\frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right.\ \text{etc.}$$

Pars prima si m *sit numerus impar:*

$$-\frac{2m}{1}\int\frac{1}{z^{m+1}}\cdot\int\frac{1}{z^{n-1}}-\frac{2m(m+1)(m+2)}{1.\ 2.\ 3}\int\frac{1}{z^{m+3}}\int\frac{1}{z^{n-3}}-\text{etc.}$$
$$+\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)+\frac{m}{1}\int\frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right)+\frac{m(m+1)}{1.\ 2}\int\frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right)\ \text{etc.}$$

Pro posterior si n *sit numerus par.*

$$2\int\frac{1}{z^n}\cdot\int\frac{1}{z^m}+\frac{2n(n+1)}{1.\ 2}\int\frac{1}{z^{n+2}}\cdot\int\frac{1}{z^{m-2}}$$
$$+\frac{2n(n+1)(n+2)(n+3)}{1.\ 2.\ 3.\ 4}\int\frac{1}{z^{n+4}}\cdot\int\frac{1}{z^{m-4}}\ \text{etc.}$$
$$-\int\frac{1}{z^n}\left(\frac{1}{y^m}\right)-\frac{n}{1}\int\frac{1}{z^{n+1}}\left(\frac{1}{y^{m-1}}\right)-\frac{n(n+1)}{1.\ 2}\int\frac{1}{z^{n+2}}\left(\frac{1}{y^{m-2}}\right.-\text{etc.}$$

Pars posterior si n *sit numerus impar.*

$$-\frac{2n}{1}\int\frac{1}{z^{n+1}}\cdot\int\frac{1}{z^{m-1}}-\frac{2n(n+1)(n+2)}{1.\ 2.\ 3}\int\frac{1}{z^{n+3}}\left(\frac{1}{z^{m-3}}\right.-\text{etc.}$$
$$+\int\frac{1}{z^n}\left(\frac{1}{y^m}\right)+\frac{n}{1}\int\frac{1}{z^{n+1}}\left(\frac{1}{y^{m-1}}\right)+\frac{n(n+1)}{1.\ 2}\int\frac{1}{z^{n+2}}\left(\frac{1}{y^{m-2}}\right.+\text{etc.}$$

10.

10. In his formis notasse iuuabit, series formae, quam hic consideramus, $\int \frac{1}{z^{\mu}} \left(\frac{1}{y^{\nu}}\right)$ non solum occurrere, sed etiam omnes ita esse comparatas, vt summa exponentium $\mu + \nu$ vbique sit eadem $= m + n$. Quo circa nostras inuestigationes ita in ordines distribui conueniet, vt omnes resolutiones, in quibus summa exponentium $m + n$ est eadem, ad eundem ordinem referantur: quandoquidem in iis eaedem series, quas hic euoluere constitui, occurrunt; atque si theorema in prima methodo erutum, quo est

$$\int \frac{1}{z^m} \left(\frac{1}{y^n}\right) + \int \frac{1}{z^n} \left(\frac{1}{y^m}\right) = \int \frac{1}{z^m} . \int \frac{1}{z^n} + \int \frac{1}{z^{m+n}}$$

in subsidium vocemus, hinc singulas series formae nostrae $\int \frac{1}{z^{\mu}} \left(\frac{1}{y^{\nu}}\right)$ seorsim definire poterimus. Cum autem exponentes m et n vnitate minores esse nequeant, pro primo ordine erit $m + n = 2$, pro secundo $m + n = 3$, pro tertio $m + n = 4$ et ita porro: cum autem sit $\int \frac{1}{z}$ infinita, pro seriebus quarum summa est finita hoc infinitum ex calculo egredi debet.

Ordo primus quo m + n = 2.

11. Hic ergo vnico modo est $m = 1$ et $n = 1$; expressio $\int \frac{1}{z} . \int \frac{1}{z} - \int \frac{1}{z^2}$ in sequentem resoluitur:

$$\int \frac{1}{z} \left(\frac{1}{y}\right) + \int \frac{1}{z} \left(\frac{1}{y}\right) = 2 \int \frac{1}{z} \left(\frac{1}{y}\right).$$

Prior autem methodus dat

$$2 \int \frac{1}{z} \left(\frac{1}{y}\right) = \int \frac{1}{z} . \int \frac{1}{z} + \int \frac{1}{z^2}$$

quae

quae praesenti formae repugnare videtur: verum cum $\int \frac{1}{z}$ sit infinita, eius respectu vtique pars altera $\int \frac{1}{z^2}$ pro euanescente est habenda. Quam ob causam hinc nihil ad institutum nostrum concludere licet.

Ordo secundus quo est m + n = 3.

12. Hic iteram vnico modo est $m = 2$ et $n = 1$, quandoquidem permutatio horum exponentium nullum discrimen affert. Quare haec expressio $\int \frac{1}{z^2} . \int \frac{1}{z} - \int \frac{1}{z^3}$ resoluitur in hanc:

$$2 \int \frac{1}{z^2} . \int \frac{1}{z} - \int \frac{1}{z^2} \left(\frac{1}{y} \right) - 2 \int \frac{1}{z^2} . \int \frac{1}{z} + \int \frac{1}{z} \left(\frac{1}{y^2} \right) + 1 \int \frac{1}{z^2} \left(\frac{1}{y} \right)$$

quae contrahitur in $\int \frac{1}{z} \left(\frac{1}{y^2} \right)$. Per minorem autem methodum est

$$\int \frac{1}{z^2} \left(\frac{1}{y} \right) + \int \frac{1}{z} \left(\frac{1}{y^2} \right) = \int \frac{1}{z^2} . \int \frac{1}{z} + \int \frac{1}{z^3}$$

vnde sequi videtur:

$$\int \frac{1}{z^2} \left(\frac{1}{y} \right) = 2 \int \frac{1}{z^3}$$

quae conclusio etsi est certa, vti deinceps videbimus, tamen hinc ob infinita ei satis confidere non licet. Erit ergo

$$1 + \frac{1}{2^2}\left(1 + \frac{1}{2}\right) + \frac{1}{3^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \frac{1}{4^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = 2 \int \frac{1}{z^3}$$

$$= 2 \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc}\right)$$

quae aequalitas vtique omni attentione est digna.

Ordo

Ordo tertius, quo $m + n = 4$.

13. Duo casus hic sunt considerandi, quorum primus est $m = 3$, et $n = 1$, vnde forma $\int\frac{1}{z^3}\cdot\int\frac{1}{z} - \int\frac{1}{z^4}$ in hanc resoluitur:

$$+\int\frac{1}{z^3}\left(\frac{1}{y}\right) - 2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2} + \int\frac{1}{z}\left(\frac{1}{y^3}\right) + 1\int\frac{1}{z^2}\left(\frac{1}{y^2}\right) + \int\frac{1}{z^3}\left(\frac{1}{y}\right)$$

ita vt sit

$$2\int\frac{1}{z^3}\left(\frac{1}{y}\right) + \int\frac{1}{z^2}\left(\frac{1}{y^2}\right) + \int\frac{1}{z}\left(\frac{1}{y^3}\right) = 2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2} + \int\frac{1}{z^3}\cdot\int\frac{1}{z} - \int\frac{1}{z^4}.$$

Ex prima autem methodo habetur

$$\int\frac{1}{z^3}\left(\frac{1}{y}\right) + \int\frac{1}{z}\left(\frac{1}{y^3}\right) = \int\frac{1}{z^3}\cdot\int\frac{1}{z} + \int\frac{1}{z^4}$$

quae aequalitas ab illa ablata relinquit

$$\int\frac{1}{z^3}\left(\frac{1}{y}\right) + \int\frac{1}{z^2}\left(\frac{1}{y^2}\right) = 2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2} - 2\int\frac{1}{z^4}.$$

Altero casu est $m = 2$ et $n = 2$, vnde colligitur

$$\int\frac{1}{z^2}\cdot\int\frac{1}{z^2} - \int\frac{1}{z^4}$$

$$= 2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2} - \int\frac{1}{z^2}\left(\frac{1}{y^2}\right) - 2\int\frac{1}{z^3}\left(\frac{1}{y}\right)$$

$$+ 2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2} - \int\frac{1}{z^2}\left(\frac{1}{y^2}\right) - 2\int\frac{1}{z^3}\left(\frac{1}{y}\right)$$

hincque porro

$$2\int\frac{1}{z^2}\left(\frac{1}{y^2}\right) + 4\int\frac{1}{z^3}\left(\frac{1}{y}\right) = 3\int\frac{1}{z^2}\cdot\int\frac{1}{z^2} + \int\frac{1}{z^4}.$$

At methodus prima dat

$$2\int\frac{1}{z^3}\left(\frac{1}{y^2}\right) = \int\frac{1}{z^2}\cdot\int\frac{1}{z^2} + \int\frac{1}{z^4}$$

vnde concludimus fore

$$\int\frac{1}{z^2}\left(\frac{1}{y^2}\right) = \frac{1}{2}\int\frac{1}{z^2}\cdot\int\frac{1}{z^2} + \frac{1}{2}\int\frac{1}{z^4}$$

et $\int\frac{1}{z^3}\left(\frac{1}{y}\right) = \frac{1}{2}\int\frac{1}{z^2}\cdot\int\frac{1}{z^2}$.

Superior vero conclusio suppeditat

$$\int\frac{1}{z^3}\left(\frac{1}{y}\right) = \frac{3}{2}\int\frac{1}{z^2}\cdot\int\frac{1}{z^2} - \frac{5}{2}\int\frac{1}{z^4}$$

 quae

quae etiam veritati eſt conſentanea, cum ſit $\int\frac{1}{z^2}\cdot\int\frac{1}{z^2} = \frac{\pi^4}{36}$ et $\int\frac{1}{z^4} = \frac{\pi^4}{90}$; ita vt etiam prior caſus non obſtante infinito ad veritatem perducat; quod tum ſolum a vero aberrare videtur, quando infiniti quadratum $\int\frac{1}{z}\cdot\int\frac{1}{z}$, vti in primo ordine vſu venit, in calculum ingreditur; quo ipſo concluſio ex ordine ſecundo deducta iam haud mediocriter corroboratur.

Ordo quartus, quo $m + n = 5$.

14. Sit primo $m = 4$ et $n = 1$, vnde prodit pro $\int\frac{1}{z^4}\cdot\int\frac{1}{z} - \int\frac{1}{z^5}$ haec expreſſio:

$$2\int\frac{1}{z^4}\cdot\int\frac{1}{z} - \int\frac{1}{z^4}\left(\frac{1}{y}\right)$$
$$-2\int\frac{1}{z^2}\cdot\int\frac{1}{z^3} - 2\int\frac{1}{z^4}\cdot\int\frac{1}{z} + \int\frac{1}{z}\left(\frac{1}{y^4}\right) + \int\frac{1}{z^2}\left(\frac{1}{y^3}\right) + \int\frac{1}{z^3}\left(\frac{1}{y^2}\right)$$
$$+\int\frac{1}{z^4}\left(\frac{1}{y}\right)$$

vnde colligimus

$$\int\frac{1}{z}\left(\frac{1}{y^4}\right) + \int\frac{1}{z^2}\left(\frac{1}{y^3}\right) + \int\frac{1}{z^3}\left(\frac{1}{y^2}\right) = \int\frac{1}{z^4}\cdot\int\frac{1}{z} + 2\int\frac{1}{z^2}\cdot\int\frac{1}{z^3} - \int\frac{1}{z^5}.$$

Methodus autem prima dat

$$\int\frac{1}{z}\left(\frac{1}{y^4}\right) + \int\frac{1}{z^4}\left(\frac{1}{y}\right) = \int\frac{1}{z^4}\cdot\int\frac{1}{z} + \int\frac{1}{z^5}$$

qua aequalitate ab illa ablata reſtat

$$\int\frac{1}{z^2}\left(\frac{1}{y^3}\right) + \int\frac{1}{z^3}\left(\frac{1}{y^2}\right) - \int\frac{1}{z^4}\left(\frac{1}{y}\right) = 2\int\frac{1}{z^2}\cdot\int\frac{1}{z^3} - \int\frac{1}{z^5}.$$

Secundo autem ſit $m = 3$ et $n = 2$, ac pro $\int\frac{1}{z^3}\cdot\int\frac{1}{z^2} - \int\frac{1}{z^5}$ reperitur

$$-6\int\frac{1}{z^4}\cdot\int\frac{1}{z} + \int\frac{1}{z^3}\left(\frac{1}{y^2}\right) + 3\int\frac{1}{z^4}\left(\frac{1}{y}\right)$$
$$+2\int\frac{1}{z^2}\cdot\int\frac{1}{z^3} + 6\int\frac{1}{z^4}\cdot\int\frac{1}{z} - \int\frac{1}{z^2}\left(\frac{1}{y^3}\right) - 2\int\frac{1}{z^3}\left(\frac{1}{y^2}\right) - 3\int\frac{1}{z^4}\left(\frac{1}{y}\right)$$

ideoque

$$\int\frac{1}{z^2}\left(\frac{1}{y^3}\right) + \int\frac{1}{z^3}\left(\frac{1}{y^2}\right) = \int\frac{1}{z^2}\cdot\int\frac{1}{z^3} + \int\frac{1}{z^5}$$

prorſus

prorſus vti prima methodus praebet; hinc ergo fit

$$\int \frac{1}{z^4}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^5} - \int \frac{1}{z^2} \cdot \int \frac{1}{z^3}.$$

Seorſim autem hinc ſerierum $\int \frac{1}{z^2}\left(\frac{1}{y^3}\right)$ et $\int \frac{1}{z^3}\left(\frac{1}{y^2}\right)$ ſummae non definiuntur. Infra autem oſtendemus, eſſe

$$\int \frac{1}{z^2}\left(\frac{1}{y^4}\right) = 3\int \frac{1}{z^2} \cdot \int \frac{1}{z^3} - \frac{9}{2}\int \frac{1}{z^5} \text{ et } \int \frac{1}{z^2}\left(\frac{1}{y^3}\right) = -2\int \frac{1}{z^2} \cdot \int \frac{1}{z^3} + \frac{11}{2}\int \frac{1}{z^5}.$$

Ordo quintus, *quo* $m + n = 6$.

15. Sit primo $m = 5$ et $n = 1$, eritque $\int \frac{1}{z^5} \cdot \int \frac{1}{z}$

$$-\int \frac{1}{z^6} = + \int \frac{1}{z^5}\left(\frac{1}{y}\right)$$

$$- 2\int \frac{1}{z^2} \cdot \int \frac{1}{z^4} - 2\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} + \int \frac{1}{z}\left(\frac{1}{y^5}\right) + \int \frac{1}{z^2}\left(\frac{1}{y^4}\right) + \int \frac{1}{z^3}\left(\frac{1}{y^3}\right)$$
$$+ \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + \int \frac{1}{z^5}\left(\frac{1}{y}\right)$$

vnde fit

$$\int \frac{1}{z}\left(\frac{1}{y^5}\right) + \int \frac{1}{z^2}\left(\frac{1}{y^4}\right) + \int \frac{1}{z^3}\left(\frac{1}{y^3}\right) + \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + 2\int \frac{1}{z^5}\left(\frac{1}{y}\right) =$$
$$\int \frac{1}{z^5} \cdot \int \frac{1}{z} + 4\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - \int \frac{1}{z^6}.$$

Cum autem prima methodus det

$$\int \frac{1}{z}\left(\frac{1}{y^5}\right) + \int \frac{1}{z^5}\left(\frac{1}{y}\right) = \int \frac{1}{z^5} \cdot \int \frac{1}{z} + \int \frac{1}{z^6}$$

hinc terminis infinitis elidendis fit

$$\int \frac{1}{z^2}\left(\frac{1}{y^4}\right) + \int \frac{1}{z^3}\left(\frac{1}{y^3}\right) + \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + \int \frac{1}{z^5}\left(\frac{1}{y}\right) = 4\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - 2\int \frac{1}{z^6},$$

Secundo ſumatur $m = 4$ et $n = 2$ eritque $\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - \int \frac{1}{z^6} =$

$$2\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) - 4\int \frac{1}{z^5}\left(\frac{1}{y}\right)$$
$$+ 2\int \frac{1}{z^2} \cdot \int \frac{1}{z^4} + 6\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - \int \frac{1}{z^2}\left(\frac{1}{y^4}\right) - 2\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) - 3\int \frac{1}{z^4}\left(\frac{1}{y^2}\right)$$
$$- 4\int \frac{1}{z^5}\left(\frac{1}{y}\right)$$

 ſiue

siue

$$\int \frac{1}{z^2}\left(\frac{1}{y^4}\right) + 2\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) + 4\int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + 8\int \frac{1}{z^5}\left(\frac{1}{y}\right) = 9\int \frac{1}{z^4}\cdot\int \frac{1}{z^2} + \int \frac{1}{z^6}.$$

Sit tertio $m = 3$ et $n = 3$, et quia ambae partes fiunt aequales, habitur $\int \frac{1}{z^3}\cdot\int \frac{1}{z^3} - \int \frac{1}{z^6} = -12\int \frac{1}{z^4}\cdot\int \frac{1}{z^2} + 2\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) + 6\int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + 12\int \frac{1}{z^5}\left(\frac{1}{y}\right)$ seu

$$2\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) + 6\int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + 12\int \frac{1}{z^5}\left(\frac{1}{y}\right) = 12\int \frac{1}{z^4}\cdot\int \frac{1}{z^2} + \int \frac{1}{z^3}\cdot\int \frac{1}{z^3} - \int \frac{1}{z^6}$$

cum quibus coniungantur hae duae ex prima methodo ortae:

$$\int \frac{1}{z^2}\left(\frac{1}{y^4}\right) + \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) = \int \frac{1}{z^4}\cdot\int \frac{1}{z^2} + \int \frac{1}{z^6}$$

$$\text{et } 2\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) = \int \frac{1}{z^3}\cdot\int \frac{1}{z^3} + \int \frac{1}{z^6}$$

hincque singulae series formae nostrae ita determinantur:

$$\int \frac{1}{z^5}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^4}\cdot\int \frac{1}{z^2} - \frac{1}{2}\int \frac{1}{z^3}\cdot\int \frac{1}{z^3} - \frac{7}{2}\int \frac{1}{z^6}$$

$$\int \frac{1}{z^4}\left(\frac{1}{y^2}\right) = -\frac{16}{3}\int \frac{1}{z^4}\cdot\int \frac{1}{z^2} + \int \frac{1}{z^3}\cdot\int \frac{1}{z^3} + 9\int \frac{1}{z^6}$$

$$\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) = \frac{1}{2}\int \frac{1}{z^3}\cdot\int \frac{1}{z^3} + \frac{1}{2}\int \frac{1}{z^6}$$

$$\int \frac{1}{z^2}\left(\frac{1}{y^4}\right) = +\frac{19}{3}\int \frac{1}{z^4}\cdot\int \frac{1}{z^2} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^3} - 3\int \frac{1}{z^6}$$

tum vero aequationem primo inuentam adhibendo obtinetur

$$\int \frac{1}{z^6} = \frac{4}{7}\int \frac{1}{z^4}\cdot\int \frac{1}{z^2}$$

id quod ob $\int \frac{1}{z^2} = \frac{\pi\pi}{6}$, $\int \frac{1}{z^4} = \frac{\pi^4}{90}$ et $\int \frac{1}{z^6} = \frac{\pi^6}{945}$ veritati est consentaneum.

Ordo

Ordo sextus, quo $m+n=7$.

16. Sit primo $m=6$ et $n=1$, eritque
$\int\frac{1}{z^6}\cdot\int\frac{1}{z}-\int\frac{1}{z^7}=$

$$2\int\frac{1}{z^6}\cdot\int\frac{1}{z}-\int\frac{1}{z^6}\left(\frac{1}{y}\right)$$
$$-2\int\frac{1}{z^2}\cdot\int\frac{1}{z^5}=2\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}-2\int\frac{1}{z^6}\cdot\int\frac{1}{z}$$
$$+\int\frac{1}{z}\left(\frac{1}{y^6}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)+\int\frac{1}{z^6}\left(\frac{1}{y}\right)$$

vnde colligimus hanc aequationem

$$\int\frac{1}{z}\left(\frac{1}{y^6}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)=$$
$$\int\frac{1}{z^6}\cdot\int\frac{1}{z}+2\int\frac{1}{z^2}\cdot\int\frac{1}{z^5}+2\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}-\int\frac{1}{z^7}.$$

Cum vero sit

$$\int\frac{1}{z}\left(\frac{1}{y^6}\right)+\int\frac{1}{z^6}\left(\frac{1}{y}\right)=\int\frac{1}{z^6}\cdot\int\frac{1}{z}+\int\frac{1}{z^7}\text{ erit}$$
$$\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)-\int\frac{1}{z^6}\left(\frac{1}{y}\right)=$$
$$2\int\frac{1}{z^2}\cdot\int\frac{1}{z^5}+2\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}-2\int\frac{1}{z^7}.$$

Est vero etiam

$$\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)=\int\frac{1}{z^5}\cdot\int\frac{1}{z^2}+\int\frac{1}{z^7}\text{ et}$$
$$\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)=\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}+\frac{1}{z^7}$$

vnde habebitur

$$\int\frac{1}{z^6}\left(\frac{1}{y}\right)=4\int\frac{1}{z^7}-\int\frac{1}{z^2}\cdot\int\frac{1}{z^5}-\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}$$

Sit secundo $m=5$ et $n=2$, eritque $\int\frac{1}{z^5}\int\frac{1}{z^2}-\int\frac{1}{z^7}=$

$$-10\int\frac{1}{z^6}\cdot\int\frac{1}{z}+\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)+5\int\frac{1}{z^6}\left(\frac{1}{y}\right)$$
$$+2\int\frac{1}{z^2}\int\frac{1}{z^5}+6\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}+10\int\frac{1}{z^6}\int\frac{1}{z}$$
$$-\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)-2\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)-3\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)-4\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)-5\int\frac{1}{z^6}\left(\frac{1}{y}\right)$$

vnde colligitur haec aequatio:

$$\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+2\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+3\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+3\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)=$$
$$\int\frac{1}{z^2}\cdot\int\frac{1}{z^5}+6\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}+\int\frac{1}{z^7}$$

quae per ante ex methodo prima allegatas reducitur ad hanc:

$$\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+2\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)=4\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}-2\int\frac{1}{z^7}.$$

Sit tertio $m=4$ er $n=3$, erit $\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}-\int\frac{1}{z^7}=$

$$2\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}+20\int\frac{1}{z^6}\cdot\int\frac{1}{z}-\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)-4\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)-10\int\frac{1}{z^6}\left(\frac{1}{y}\right)$$
$$-6\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}-20\int\frac{1}{z^6}\cdot\int\frac{1}{z}+\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+3\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+6\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)$$
$$+10\int\frac{1}{z^6}\left(\frac{1}{y}\right).$$

Vnde colligitur

$$\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+2\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+2\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)=5\int\frac{1}{z^4}\int\frac{1}{z^3}-\int\frac{1}{z^7}\text{ seu}$$
$$\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+2\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)=4\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}-2\int\frac{1}{z^7}$$

quae cum ante inuenta congruit, ita vt hinc nihil noui concludi possit. Hinc ergo tantum determinatur primo seriei $\int\frac{1}{z^6}\left(\frac{1}{y}\right)$ summa, tum vero hae duae coniunctim $\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+2\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)$; sin autem vtraque seorsim innotesceret, tum etiam binae reliquae $\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)$ et $\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)$ innotescerent.

Ordo septimus, quo $m+n=8$.

17. Sit primo $m=7$ et $n=1$, erit

$$\int\frac{1}{z^7}\cdot\int\frac{1}{z}-\int\frac{1}{z^8}=\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$
$$-2\int\frac{1}{z^2}\cdot\int\frac{1}{z^6}-2\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-2\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}$$
$$+\int\frac{1}{z}\left(\frac{1}{y^7}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+\int\frac{1}{z^4}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)$$
$$+\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$

quae

quae vltima linea abit in

$$\int\frac{1}{z}\cdot\int\frac{1}{z^7}+\int\frac{1}{z^2}\cdot\int\frac{1}{z^6}+\int\frac{1}{z^3}\cdot\int\frac{1}{z^5}+\frac{1}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}+\frac{7}{2}\int\frac{1}{z^8}$$

sicque erit

$$\int\frac{1}{z^7}\left(\frac{1}{y}\right)=2\int\frac{1}{z^2}\cdot\int\frac{1}{z^6}-\int\frac{1}{z^3}\cdot\int\frac{1}{z^5}+\frac{3}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-\frac{9}{2}\int\frac{1}{z^8}.$$

Secundo sit $m=6$ et $n=2$, erit $\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}-\int\frac{1}{z^8}=$

$$2\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}-\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)-6\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$

$+2\int$ [illegible] $\cdot\int\frac{1}{z^6}+6\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}+10\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}$

$-\int\frac{1}{z^2}\left(\frac{1}{y^6}\right)$ [illegible] $\int\frac{1}{z^3}\left(\frac{1}{y^5}\right)-3\int\frac{1}{z^4}\left(\frac{1}{y^4}\right)-4\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)-5\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)$

$-6\int\frac{1}{z^7}\left(\frac{1}{y}\right)$

siue

$$\int\frac{1}{z^2}\left(\frac{1}{y^6}\right)+2\int\frac{1}{z^3}\left(\frac{1}{y^5}\right)+3\int\frac{1}{z^4}\left(\frac{1}{y^4}\right)+4\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)+6\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)$$
$$+12\int\frac{1}{z^7}\left(\frac{1}{y}\right)=13\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}+\int\frac{1}{z^8}+6\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}$$

quae reducitur ad hanc:

$$2\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)+5\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+12\int\frac{1}{z^7}\left(\frac{1}{y}\right)=$$
$$+12\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}-2\int\frac{1}{z^3}\cdot\int\frac{1}{z^5}+\frac{9}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-\frac{7}{2}\int\frac{1}{z^8}.$$

Sit tertio $m=5$ et $n=3$, erit $\int\frac{1}{z^5}\cdot\int\frac{1}{z^3}-\int\frac{1}{z^8}=$

$$-10\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}+\int\frac{1}{z^3}\left(\frac{1}{y^5}\right)+5\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+15\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$
$$-6\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-20\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}$$
$$+\int\frac{1}{z^3}\left(\frac{1}{y^5}\right)+3\int\frac{1}{z^4}\left(\frac{1}{y^4}\right)+6\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)+10\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+15\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$

vnde fit

$$\int\frac{1}{z^3}\left(\frac{1}{y^5}\right)+3\int\frac{1}{z^4}\left(\frac{1}{y^4}\right)+7\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)+15\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+30\int\frac{1}{z^7}\left(\frac{1}{y}\right)=$$
$$30\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}+4\int\frac{1}{z^5}\cdot\int\frac{1}{z^3}+6\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-\int\frac{1}{z^8}$$

siue

$6\int$

$$6\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)+15\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+30\int\frac{1}{z^7}\left(\frac{1}{y}\right)=$$
$$30\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}+\frac{9}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-\frac{7}{2}\int\frac{1}{z^8}.$$

Sit denique $m=4$ et $n=4$, eritque

$$\frac{1}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-\frac{1}{2}\int\frac{1}{z^8}=2\int\frac{1}{z^4}\cdot\frac{1}{z^4}+20\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}-\int\frac{1}{z^4}\left(\frac{1}{y^4}\right)-4\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)$$
$$-10\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)-20\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$

hincque

$$4\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)+10\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+20\int\frac{1}{z^7}\left(\frac{1}{y}\right)=20\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}+\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}$$

quae aequatio cum praecedente eandem continet determinationem, ac reducitur ad hanc proprietatem:

$$6\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}=7\int\frac{1}{z^8}$$

quae valoribus a me inuentis $\int\frac{1}{z^4}=\frac{\pi^4}{90}$ et $\int\frac{1}{z^8}=\frac{\pi^8}{9450}$ egregie eſt conformis. Sin autem haec vltima aequatio cum caſu ſecundo conferatur, inde colligitur

$$4\int\frac{1}{z^7}\left(\frac{1}{y}\right)=4\int\frac{1}{z^2}\int\frac{1}{z^6}-4\int\frac{1}{z^3}\int\frac{1}{z^5}+8\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-7\int\frac{1}{z^8}\quad\text{ſeu}$$
$$\int\frac{1}{z^7}\left(\frac{1}{y}\right)=\int\frac{1}{z^2}\cdot\int\frac{1}{z^6}-\int\frac{1}{z^3}\cdot\int\frac{1}{z^5}+\frac{1}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}$$

qui valor cum caſu primo collatus praebet

$$2\int\frac{1}{z^2}\cdot\int\frac{1}{z^6}+\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}=\frac{9}{2}\int\frac{1}{z^8}$$

quae aequatio etiam veritati eſt conſentanea. Hinc praeter ſeriem $\int\frac{1}{z^7}\left(\frac{1}{y}\right)$ et determinationes primae methodi, tantum hanc vnicam nouam determinationem conſequimur:

$$2\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+5\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)=10\int\frac{1}{z^3}\cdot\int\frac{1}{z^5}-\frac{9}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}$$

neque ergo ſummas harum ſerierum $\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)$ et $\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)$ ſeorſim definire licet.

Ordo

Ordo octauus, *quo* m + n = 9.

18. Pro hoc ordine methodus prima has dat aequationes:

$$\int\tfrac{1}{z}\left(\tfrac{1}{y^8}\right)+\int\tfrac{1}{z^8}\left(\tfrac{1}{y}\right)=\int\tfrac{1}{z}\cdot\int\tfrac{1}{z^8}+\int\tfrac{1}{z^9}$$

$$\int\tfrac{1}{z^2}\left(\tfrac{1}{y^7}\right)+\int\tfrac{1}{z^7}\left(\tfrac{1}{y^2}\right)=\int\tfrac{1}{z^2}\cdot\int\tfrac{1}{z^7}+\int\tfrac{1}{z^9}$$

$$\int\tfrac{1}{z^3}\left(\tfrac{1}{y^6}\right)+\int\tfrac{1}{z^6}\left(\tfrac{1}{y^3}\right)=\int\tfrac{1}{z^3}\cdot\int\tfrac{1}{z^6}+\int\tfrac{1}{z^9}$$

$$\int\tfrac{1}{z^4}\left(\tfrac{1}{y^5}\right)+\int\tfrac{1}{z^5}\left(\tfrac{1}{y^4}\right)=\int\tfrac{1}{z^4}\cdot\int\tfrac{1}{z^5}+\int\tfrac{1}{z^9}.$$

Secunda autem methodus praeterea has ſuppeditat determinationes:

$$\int\tfrac{1}{z^8}\left(\tfrac{1}{y}\right)=5\int\tfrac{1}{z^9}-\int\tfrac{1}{z^2}\cdot\int\tfrac{1}{z^7}-\int\tfrac{1}{z^3}\cdot\int\tfrac{1}{z^6}-\int\tfrac{1}{z^4}\cdot\int\tfrac{1}{z^5}$$

$$2\int\tfrac{1}{z^5}\left(\tfrac{1}{y^4}\right)+5\int\tfrac{1}{z^6}\left(\tfrac{1}{y^3}\right)+5\int\tfrac{1}{z^7}\left(\tfrac{1}{y^2}\right)=10\int\tfrac{1}{z^3}\cdot\int\tfrac{1}{z^6}$$

$$\int\tfrac{1}{z^6}\left(\tfrac{1}{y^3}\right)+3\int\tfrac{1}{z^7}\left(\tfrac{1}{y^2}\right)=6\int\tfrac{1}{z^4}\cdot\int\tfrac{1}{z^5}+6\int\tfrac{1}{z^3}\cdot\int\tfrac{1}{z^6}-10\int\tfrac{1}{z^9}.$$

Cum igitur in hoc ordine 8 occurrant ſeries formae quam contemplamur, hae ſeptem aequationes omnibus definiendis non ſufficiunt; ſin autem aliunde praeter ſeriem $\int\frac{1}{z^8}\left(\frac{1}{y}\right)$ vnica reliquarum ſummari poſſet, omnium plane ſummae hinc innoteſcerent.

Ordo nonus, *quo* m + n = 10.

19. Ex prima methodo pro hoc ordine has conſequimur aequationes:

$$\int\tfrac{1}{z}\left(\tfrac{1}{y^9}\right)+\int\tfrac{1}{z^9}\left(\tfrac{1}{y}\right)=\int\tfrac{1}{z}\cdot\int\tfrac{1}{z^9}+\int\tfrac{1}{z^{10}}$$

$$\int\tfrac{1}{z^2}\left(\tfrac{1}{y^8}\right)+\int\tfrac{1}{z^8}\left(\tfrac{1}{y^2}\right)=\int\tfrac{1}{z^2}\cdot\int\tfrac{1}{z^8}+\int\tfrac{1}{z^{10}}$$

$$\int\tfrac{1}{z^3}\left(\tfrac{1}{y^7}\right)+\int\tfrac{1}{z^7}\left(\tfrac{1}{y^3}\right)=\int\tfrac{1}{z^3}\cdot\int\tfrac{1}{z^7}+\int\tfrac{1}{z^{10}}$$

$$\int\tfrac{1}{z^5}\left(\tfrac{1}{y^6}\right)+\int\tfrac{1}{z^6}\left(\tfrac{1}{y^4}\right)=\int\tfrac{1}{z^4}\cdot\int\tfrac{1}{z^6}+\int\tfrac{1}{z^{10}}$$

$$\int\tfrac{1}{z^5}\left(\tfrac{1}{y^5}\right)=\tfrac{1}{2}\int\tfrac{1}{z^5}\cdot\int\tfrac{1}{z^5}+\tfrac{1}{2}\int\tfrac{1}{z^{10}}$$

 quo-

quoniam igitur 9 ſeries hic occurrunt; pro earum ſummatione ſecunda methodus primo dat

$$\int\frac{1}{z^9}\left(\frac{1}{y}\right) = 3\int\frac{1}{z^2}\cdot\int\frac{1}{z^8} - \int\frac{1}{z^3}\cdot\int\frac{1}{z^7} + 3\int\frac{1}{z^4}\cdot\int\frac{1}{z^6} - \frac{1}{2}\int\frac{1}{z^5}\cdot\int\frac{1}{z^5} - \frac{11}{2}\int\frac{1}{z^{10}}.$$

reliquarum vero quatuor aequationum, quae inde deducuntur, duae nihil aliud definiunt, praeter notam relationem, qua eſt $\int\frac{1}{z^{10}} = \frac{10}{11}\int\frac{1}{z^4}\cdot\int\frac{1}{z^6}$, reliquae vero duae praebent

$$\int\frac{1}{z^6}\left(\frac{1}{y^4}\right) + \int\frac{1}{z^7}\left(\frac{1}{y^3}\right) = 6\int\frac{1}{z^4}\cdot\int\frac{1}{z^6} - \int\frac{1}{z^5}\cdot\int\frac{1}{z^5} - \frac{7}{2}\int\frac{1}{z^{10}}$$

$$2\int\frac{1}{z^7}\left(\frac{1}{y^3}\right) + 7\int\frac{1}{z^8}\left(\frac{1}{y^2}\right) = 14\int\frac{1}{z^3}\cdot\int\frac{1}{z^7} - 45\int\frac{1}{z^4}\cdot\int\frac{1}{z^6} + 8\int\frac{1}{z^5}\cdot\int\frac{1}{z^5} + 33\int\frac{1}{z^{10}}$$

ita vt vna determinatio adhuc deſit omnes ſeries huius ordinis ſummandas.

20. Circa determinationes, quas haec ſecunda methodus ſuppeditat, ſequentia obſeruanda occurrunt:

Primo nonniſi in ordine primo, ſecundo, tertio et quinto omnes ſeries noſtrae formae definiuntur; in reliquis omnibus vna determinatio deeſt, quominus omnes ſeries eo pertinentes ſummari queant; ita vt ſi aliunde talis determinatio ſuppeteret, totum negotium confici poſſet.

Deinde etiam pro iis ordinibus, quibus $m+n$ eſt numerus par, imprimis notari meretur, quod haec methodus eiusdem relationes inter ſummas poteſtatum parium $\int\frac{1}{z^2}$, $\int\frac{1}{z^4}$, $\int\frac{1}{z^6}$ etc. patefaciat, quas olim ex principiis maxime diuerſis erueram; cum tamen hic quadraturae circuli, a qua hae ſummae [illegible]endent, pendent, nulla ratio habeatur. Ex quo etiam expectare licuisset, pro ordinibus, quibus $m + n$ est numerus impar, similem relationem inter summas potestatum imparium prodire debuisse, quod autem longe secus vsu venit, cum determinationum, quae pro his ordinibus reperiuntur, quaedam plane inter se conueniant, vt nihil prorsus inde concludi queat. Quod cum praeter omnem expectationem euenerit, iste defectus plenae determinationis omni attentione dignus est censendus.

21. Tertio obseruandum est, in omnibus ordinibus vnam seriem nostrae formae semper perfecte determinari, eam scilicet, quae formula $\int \frac{1}{z^{m+n-1}} \left(\frac{1}{y}\right)$ indicatur; cum autem eius determinationes, prouti $m + n$ fuerit numerus impar vel par, aliam legem sequantur, eas seorsim hic ob oculos ponamus:

Pro ordinibus quibus m + n *est numerus par.*

$$\int \frac{1}{z^3}\left(\frac{1}{y}\right) = \frac{3}{2}\int \frac{1}{z^2} \cdot \int \frac{1}{z^2} - \frac{5}{2}\int \frac{1}{z^4}$$

$$\int \frac{1}{z^5}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^2} \cdot \int \frac{1}{z^4} - \frac{1}{2}\int \frac{1}{z^3} \cdot \int \frac{1}{z^3} - \frac{7}{2}\int \frac{1}{z^6}$$

$$\int \frac{1}{z^7}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^2} \cdot \int \frac{1}{z^6} - \int \frac{1}{z^3} \cdot \int \frac{1}{z^5} + \frac{3}{2}\int \frac{1}{z^4} \cdot \int \frac{1}{z^4} - \frac{9}{2}\int \frac{1}{z^8}$$

$$\int \frac{1}{z^9}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^2} \cdot \int \frac{1}{z^8} - \int \frac{1}{z^3} \cdot \int \frac{1}{z^7} + 3\int \frac{1}{z^4} \cdot \int \frac{1}{z^6} - \frac{1}{2}\int \frac{1}{z^5}\int \frac{1}{z^5} - \frac{11}{2}\int \frac{1}{z^{10}}$$

$$\int \frac{1}{z^{11}}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^2} \cdot \int \frac{1}{z^{10}} - \int \frac{1}{z^3} \cdot \int \frac{1}{z^9} + 3\int \frac{1}{z^4} \cdot \int \frac{1}{z^8} - \int \frac{1}{z^5}\int \frac{1}{z^7} + \frac{3}{2}\int \frac{1}{z^6} \cdot \int \frac{1}{z^6} - \frac{13}{2}\int \frac{1}{z^{12}}$$

etc.

 quae

quae expressiones ita ad paritatem numeri m et n sunt adstrictae, vt ad impares per interpolationem transferri nequeant.

Pro ordinibus quibus m + n *est numerus impar.*

$$\int \frac{1}{z^2}\left(\frac{1}{y}\right) = 2\int \frac{1}{z^3}$$

$$\int \frac{1}{z^4}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^5} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^3}$$

$$\int \frac{1}{z^6}\left(\frac{1}{y}\right) = 4\int \frac{1}{z^7} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^5} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^4}$$

$$\int \frac{1}{z^8}\left(\frac{1}{y}\right) = 5\int \frac{1}{z^9} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^7} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^6} - \int \frac{1}{z^4}\cdot\int \frac{1}{z^5}$$

$$\int \frac{1}{z^{10}}\left(\frac{1}{y}\right) = 6\int \frac{1}{z^{11}} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^9} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^8} - \int \frac{1}{z^4}\cdot\int \frac{1}{z^7}$$
$$- \int \frac{1}{z^5}\cdot\int \frac{1}{z^6}$$

etc.

Hic autem nihil impedit, quominus hae expressiones etiam ad ordines pares transferantur.

22. Interpolatione autem rite instituta hae summationes pro omnibus ordinibus ita se habebunt:

$$2\int \frac{1}{z^2}\left(\frac{1}{y}\right) = 4\int \frac{1}{z^3}$$

$$2\int \frac{1}{z^3}\left(\frac{1}{y}\right) = 5\int \frac{1}{z^4} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^2}$$

$$2\int \frac{1}{z^4}\left(\frac{1}{y}\right) = 6\int \frac{1}{z^5} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^3}$$

$$2\int \frac{1}{z^5}\left(\frac{1}{y}\right) = 7\int \frac{1}{z^6} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^4} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^3}$$

$$2\int \frac{1}{z^6}\left(\frac{1}{y}\right) = 8\int \frac{1}{z^7} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^5} - 2\int \frac{1}{z^3}\cdot\int \frac{1}{z^4}$$

$$2\int \frac{1}{z^7}\left(\frac{1}{y}\right) = 9\int \frac{1}{z^8} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^6} - 2\int \frac{1}{z^3}\cdot\int \frac{1}{z^5} - \int \frac{1}{z^4}\cdot\int \frac{1}{z^4}$$

$$2\int \frac{1}{z^8}\left(\frac{1}{y}\right) = 10\int \frac{1}{z^9} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^7} - 2\int \frac{1}{z^3}\cdot\int \frac{1}{z^6} - 2\int \frac{1}{z^4}\cdot\int \frac{1}{z^5}$$

$$2\int \frac{1}{z^9}\left(\frac{1}{y}\right) = 11\int \frac{1}{z^{10}} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^8} - 2\int \frac{1}{z^3}\cdot\int \frac{1}{z^7} - 2\int \frac{1}{z^4}\cdot\int \frac{1}{z^6}$$
$$- \int \frac{1}{z^5}\cdot\int \frac{1}{z^5}$$

etc.

vnde

vnde in genere si ponatur $m+n=\lambda$ erit

$$2\int\frac{1}{z^{\lambda-1}}\left(\frac{1}{y}\right)=(\lambda+1)\int\frac{1}{z^{\lambda}}-\int\frac{1}{z^{2}}\cdot\int\frac{1}{z^{\lambda-2}}-\int\frac{1}{z^{3}}\cdot\int\frac{1}{z^{\lambda-3}}$$

$$-\int\frac{1}{z^{4}}\cdot\int\frac{1}{z^{\lambda-4}}\ldots\ldots-\int\frac{1}{z^{\lambda-2}}\cdot\int\frac{1}{z^{2}}.$$

23. Quo minus autem haec interpolatio in dubium vocari possit, comparentur hae expressiones pro ordinibus paribus cum ante exhibitis; indeque obtinebuntur sequentes relationes:

$$5\int\frac{1}{z^{4}}=2\int\frac{1}{z^{2}}\cdot\int\frac{1}{z^{2}}$$

$$7\int\frac{1}{z^{6}}=4\int\frac{1}{z^{2}}\cdot\int\frac{1}{z^{4}}$$

$$9\int\frac{1}{z^{8}}=4\int\frac{1}{z^{2}}\cdot\int\frac{1}{z^{6}}+2\int\frac{1}{z^{4}}\cdot\int\frac{1}{z^{4}}$$

$$11\int\frac{1}{z^{10}}=4\int\frac{1}{z^{2}}\cdot\int\frac{1}{z^{8}}+4\int\frac{1}{z^{4}}\cdot\int\frac{1}{z^{6}}$$

$$13\int\frac{1}{z^{12}}=4\int\frac{1}{z^{2}}\cdot\int\frac{1}{z^{10}}+4\int\frac{1}{z^{4}}\cdot\int\frac{1}{z^{8}}+2\int\frac{1}{z^{6}}\cdot\int\frac{1}{z^{6}}$$

etc.

quae cum iis, quas olim elicui, perfecte consentiunt. Si enim ponamus

$$\int\frac{1}{z^{2}}=\alpha\,\pi^{2},\ \int\frac{1}{z^{4}}=\beta\,\pi^{4},\ \int\frac{1}{z^{6}}=\gamma\,\pi^{6},\ \int\frac{1}{z^{8}}=\delta\,\pi^{8},$$

$$\int\frac{1}{z^{10}}=\varepsilon\,\pi^{10}\ \text{etc.}$$

erit vtique quemadmodum demonstraui

$$5\,\beta=2\,\alpha\,\alpha$$

$$7\,\gamma=4\,\alpha\,\beta$$

$$9\,\delta=4\,\alpha\,\gamma+2\,\beta\,\beta$$

$$11\,\varepsilon=4\,\alpha\,\delta+4\,\beta\,\gamma$$

$$13\,\zeta=4\,\alpha\,\varepsilon+4\,\beta\,\delta+2\,\gamma\,\gamma$$

$$15\,\eta=4\,\alpha\,\zeta+4\,\beta\,\varepsilon+4\,\gamma\,\delta$$

etc.

24. Prima methodus tantum pro ordinibus paribus vnius feriei in noftra forma generali contentae fummam praebuerat, quae pofito $m+n=2\mu$ ita fe habebat

$$\int \frac{1}{z^{\mu}}\left(\frac{1}{y^{\mu}}\right) = \tfrac{1}{2}\int\frac{1}{z^{\mu}} \cdot \int\frac{1}{z^{\mu}} + \tfrac{1}{2}\int\frac{1}{z^{2\mu}}.$$

Nunc autem ope fecundi methodi praeterea ex quouis ordine vnam feriem formae noftrae fummare valemus, atque adeo in ordine $m+n=5$ omnes has feries fummare licuit. Exquo fufpicari licet, hanc fummationem quoque in omnibus ordinibus fuccedere, etiamfi fecunda methodus negotium non penitus conficiat: plurimum autem iam praeftitum cenferi debet, quod fi cuiusque ordinis vnica feries praeter binas memoratas vndecunque fummari poffet, inde ftatim omnium reliquarum fummas confequi: Res quidem ita fe habet in ordinibus hic euolutis; at fi vlterius progrediamus, plures vna determinationes deficere deprehenduntur.

25. Quo autem clarius ratio aequationum, quas tam prima quam fecunda methodus pro quolibet ordine fuppeditat, perfpiciatur, formulas noftras adhuc fuccinctius ita repraefentemus, vt pro ordine quocunque $m+n=\lambda$ loco $\int\frac{1}{z^{\mu}}\cdot\frac{1}{z^{\nu}}$ fcribatur vel p^{μ} vel p^{ν}, quippe quae duae formulae ob $\mu+\nu=\lambda$ pro aequiualentibus funt habendae. Similique modo pro

pro $\int \frac{1}{z^{\lambda}}$ ſcribatur p^{λ}; tum vero loco formulae

$$\int \frac{1}{z^{\mu}}\left(\frac{1}{y^{\nu}}\right) \quad \text{ſeu} \quad \int \frac{1}{z^{\mu}}\left(\frac{1}{z^{\lambda-\mu}}\right)$$

ſcribatur q^{μ}; hincque aequationes ſingulorum ordinum magis euadent perſpicuae.

Pro ordine $m+n=3$

$$\begin{array}{l|l} q+q^{2}=p+p^{3} & q+q^{2}=2p^{2}+p-p^{3} \quad \text{ſeu} \quad q=p-p^{3} \\ & \;-1\;\;-2 \end{array}$$

Pro ordine $m+n=4$

$$\begin{array}{l|l} q+q^{3}=p+p^{4} & q+q^{2}+q^{3}=2p^{2}+p-p^{4} \\ & \qquad\;\; +1 \\ 2q^{2}=p^{2}+p^{4} & \quad q^{2}+2q^{3}=2p^{2}-p^{2}+p^{4} \\ & \;+1\;+2\;\;\;+2 \end{array}$$

Pro ordine $m+n=5$

$$\begin{array}{l|l} q+q^{4}=p+p^{5} & q+q^{2}+q^{3}+q^{4}=2p^{2}+2p^{4}+p-p^{5} \\ & \qquad\qquad -1 \qquad\qquad -2 \\ q^{2}+q^{3}=p^{2}+p^{5} & q^{2}+2q^{3}+3q^{4}=2p^{2}+6p^{4}-p^{2}+p^{5} \\ & \;\;-1\;\;-3 \qquad\qquad -6 \end{array}$$

Pro ordine $m+n=6$

$$\begin{array}{l|l} q+q^{5}=p+p^{6} & q+q^{2}+q^{3}+q^{4}+q^{5}=2p^{2}+2p^{4}+p-p^{6} \\ & \qquad\qquad\qquad +1 \\ q^{2}+q^{4}=p^{2}+p^{6} & q^{2}+2q^{3}+3q^{4}+4q^{5}=2p^{2}+6p^{4}-p^{2}+p^{6} \\ & \qquad\;\; +1\;\;+4 \qquad\qquad +2 \\ 2q^{3}=p^{3}+p^{6} & \quad q^{3}+3q^{4}+6q^{5}= \qquad 6p^{4}+p^{3}-p^{6} \\ & +1\;+3\;\;+6 \qquad\qquad +6 \end{array}$$

Pro

Pro ordine $m+n=7$

$q+q^6=p+p^7$	$q+q^2+q^3+q^4+q^5+q^6=2p^2+2p^4+2p^6+p-p^7$
	$-1 \quad -2$
$q^2+q^5=p^2+p^7$	$q^2+2q^3+3q^4+4q^5+5q^6=2p^2+6p^4+10p^6-p^2+p^7$
	$-1 \quad -5 \quad -10$
$q^3+q^4=p^3+p^7$	$q^3+3q^4+5q^5+10q^6=6p^4+20p^6+p^3-p^7$
	$-1 \quad +4 \quad -10 \quad -2 \quad -20.$

Pro ordine $m+n=8$

$q+q^7=p+p^8$	$q+q^2+q^3+q^4+q^5+q^6+q^7=2p^2+2p^4+2p^6+p-p^8$
	$+1$
$q^2+q^6=p^2+p^8$	$q^2+2q^3+3q^4+4q^5+5q^6+6q^7=2p^2+6p^4+10p^6-p^2+p^8$
	$+1 \quad +6 \quad +2$
$q^3+q^5=p^3+p^8$	$q^3+3q^4+6q^5+10q^6+15q^7=6p^4+20p^6+p^3-p^8$
	$+1 \quad +5 \quad +15 \quad +10$
$2q^4=p^4+p^8$	$q^4+4q^5+10q^6\ 20q^7=2p^4+20p^6-p^4+p^8$
	$+1 +4 \quad +10 +20 \quad +2 \quad +20$

hoc ergo modo iſtas aequationes quouſque lubuerit facile continuare licet.

Tertia Methodus ad huiusmodi ſeries perueniendi.

26. Haec methodus ſimilis fere eſt praecedenti, conſidero enim ſeriem

$$\int\frac{1}{z^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}+\dots\dots+\frac{1}{(z-1)^m}\right),$$

cuius valor modo ſuperiori expreſſus eſt

$$=\int\frac{1}{z^n}\left(\frac{1}{y^m}\right)-\int\frac{1}{z^{m+n}}$$

modo

modo autem in §. praec. vſitato $= q^n - p^{m+n}$; de quo notetur eſſe per methodum primam

$$q^m + q^n = p^m + p^{m+n} = p^n + p^{m+n} \text{ ob } p^m = p^n.$$

Iam huius formae

$$\frac{1}{z^n}\left(1 + \frac{1}{2^m} + \frac{1}{3^m} + \dots\dots + \frac{1}{(z-1)^m}\right)$$

quilibet terminus continetur in hac forma $\frac{1}{(x+a)^n x^m}$ quae vti vidimus §. 6. reſoluitur in has partes:

$$\frac{1}{a^n}\cdot\frac{1}{x^m} - \frac{n}{1\,a^{n+1}}\cdot\frac{1}{x^{m-1}} + \frac{n(n+1)}{1.2\,a^{n+2}}\cdot\frac{1}{x^{m-2}} - \frac{n(n+1)(n+2)}{1.\;2.\;3\;a^{n+3}}\cdot\frac{1}{x^{m-3}} \quad \text{et}$$

$$\pm \frac{1}{a^m}\cdot\frac{1}{(a+x)^n} \pm \frac{m}{1\,a^{m+1}}\cdot\frac{1}{(x+a)^{n-1}} \pm \frac{m(m+1)}{1.2\,a^{m+2}}\cdot\frac{1}{(x+a)^{n-2}} \pm \text{etc.}$$

vbi ſignorum ambiguorum valent ſigna ſuperiora, ſi m eſt numerus par, inferiora vero ſi m eſt numerus impar; tum vero vtramque progreſſionem eousque continuari conuenit, donec in ſuperiori factoris $\frac{1}{x}$, in inferiori vero factoris $\frac{1}{x+a}$ exponens euadat vnitas.

27. Vt igitur ſeriei propoſitae ſummam obtineamus, in formulae euolutae ſingulis terminis tam loco a quam loco x omnes numeros naturali ordine progredientes ab vnitate in infinitum ſucceſſiue ſcribi omnesque terminos inde oriundos in vnam ſummam colligi oportet. Tum autem pro terminis ſuperioris partis euolutae fore

$$\int\frac{1}{a^n}\cdot\frac{1}{x^m} = \int\frac{1}{z^n}\cdot\int\frac{1}{z^m} = p^n = p^m$$

$$\int\frac{1}{a^{n+1}}\cdot\frac{1}{x^{m-1}} = \int\frac{1}{z^{n+1}}\cdot\int\frac{1}{z^{m-1}} = p^{n+1} = p^{m-1}$$

$$\int\frac{1}{a^{n+2}}\cdot\frac{1}{x^{m-2}} = \int\frac{1}{z^{n+2}}\cdot\int\frac{1}{z^{m-2}} = p^{n+2} = p^{m-2}$$

etc.

pro terminis autem inferioris partis euolutae

$$\int\frac{1}{a^{m}}\cdot\frac{1}{(x+a)^{n}} = \int\frac{1}{z^{m}}\cdot\int\frac{1}{z^{n}} - \int\frac{1}{z^{m}}\left(\frac{1}{y^{n}}\right) = p^{m} - q^{m} = q^{n} - p^{m+n}$$

$$\int\frac{1}{a^{m+1}}\cdot\frac{1}{(x+a)^{n-1}} = \int\frac{1}{z^{m+1}}\int\frac{1}{z^{n-1}} - \int\frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right) = p^{m+1} - q^{m+1} = q^{n-1} - p^{m+n}$$

$$\int\frac{1}{a^{m+2}}\cdot\frac{1}{(x+a)^{n-2}} = \int\frac{1}{z^{m+2}}\cdot\int\frac{1}{z^{n-2}} - \int\frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right) = p^{m+2} - q^{m+2} = p^{n-2} - p^{m+n}$$

etc.

His igitur ſubſtitutis valor ſeriei noſtrae, qui eſt $= q^{n} - p^{m+n}$ euoluitur in ſequentem expreſſionem:

$$p^{m} - \frac{n}{1}p^{m-1} + \frac{n(n+1)}{1\cdot 2}p^{m-2} - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}p^{m-3} + \text{etc.}$$
$$\pm q^{n} \pm \frac{m}{1}q^{n-1} \pm \frac{m(m+1)}{1\cdot 2}q^{n-2} \pm \frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}q^{n-3}\ \text{etc.}$$
$$\mp p^{m+n} \mp \frac{m}{1}p^{m+n} \mp \frac{m(m+1)}{1\cdot 2}p^{m+n} \mp \frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}p^{m+n}\ \text{etc.}$$

Cum autem ſit $p^{\mu} = q^{\mu} + q^{m+n-\mu} - p^{m+n}$, habebimus

$$\begin{aligned} 0 = q^{m} &- \tfrac{n}{1}(q^{m-1} + q^{n+1}) + \tfrac{n(n+1)}{1\cdot 2}(q^{m-2} + q^{n+2}) - \tfrac{n(n+1)(n+2)}{1\cdot 2\cdot 3}(q^{m-3} + q^{n+3}) \\ &+ \tfrac{n}{1}\cdot p^{m+n} \qquad - \tfrac{n(n+1)}{1\cdot 2}p^{m+n} \qquad + \tfrac{n(n+1)(n+2)}{1\cdot 2\cdot 3}p^{m+n} \\ \pm q^{n} &\pm \tfrac{m}{1}q^{n-1} \qquad \pm \tfrac{m(m+1)}{1\cdot 2}q^{n-2} \qquad \pm \tfrac{m(m+1)(m+2)}{1\cdot 2\cdot 3}q^{n-3} \\ \mp p^{m+n} &\mp \tfrac{m}{1}p^{m+n} \qquad \mp \tfrac{m(m+1)}{1\cdot 2}p^{m+n} \qquad \mp \tfrac{m(m+1)(m+2)}{1\cdot 2\cdot 3}p^{m+n}. \end{aligned}$$

28.

28. Antequam ad ordines fupra confideratos defcendamus, euoluamus in genere aliquos cafus.

I. Sit igitur $m = 1$ et aequatio inuenta induet hanc formam:

$$0 = q - q^n - q^{n-1} - q^{n-2} \ldots\ldots - q + n p^{n+1}$$

feu $q^2 + q^3 + q^4 \ldots\ldots + q^n = n p^{n+1}$

exponente ordinis exiftente $n + 1$, ita vt fit

$$q^\mu + q^{n+1-\mu} = p^\mu + p^{n-1}.$$

II. Sit $m = 2$, ordinisque exponens $n + 2$, vt fit $q^\mu + q^{n+2-\mu} = p^\mu + p^{n+2}$ et aequatio noftra fiet

$$0 = q^2 - n(q + q^{n+1}) + q^n + 2q^{n-1} + 3q^{n-2} + 4q^{n-3} \ldots + nq$$
$$+ n p^{n+2} \qquad - \frac{n(n+1)}{1 \cdot 2} p^{n+2}$$

fiue

$$q^n + 2q^{n-1} + 3q^{n-2} \ldots (n-1)q^2 - nq^{n+1} = \frac{n(n-1)}{1 \cdot 2} p^{n+2}.$$

III. Sit $m = 3$, ordinisque exponens $n + 3$, vt fit $q^\mu + q^{n+3-\mu} = p^\mu + p^{n+3}$ et aequatio noftra erit

$$0 = q^3 - n(q^2 + q^{n+1}) + \frac{n(n+1)}{1 \cdot 2}(q + q^{n+2})$$
$$- \frac{n(n-1)}{1 \cdot 2} p^{n+3}$$
$$- q^n - 3q^{n-1} - 6q^{n-2} - 10q^{n-3} \ldots\ldots - \frac{n(n+1)}{1 \cdot 2} q$$
$$+ \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} p^{n+3}$$

fiue

$$q^n + 3q^{n-1} + 6q^{n-2} \ldots\ldots + \frac{(n-1)(n-2)}{1 \cdot 2} q^3 + \frac{n(n-1)}{1 \cdot 2} q^2 + n q^{n+1}$$
$$= \frac{n(nn+5)}{6} p^{n+3} \qquad - 1 \qquad + n \qquad - \frac{n(n+1)}{1 \cdot 2} q^{n+2}$$

vel hoc modo diftinctius:

$$q^n + 3q^{n-1} + 6q^{n-2} + 10q^{n-3} \ldots\ldots + \frac{n(n-1)}{1 \cdot 2} q^2$$
$$- q^3 + n(q^2 + q^{n+1}) - \frac{n(n+1)}{1 \cdot 2} q^{n+2} = \frac{n(nn+5)}{6} p^{n+3}.$$

IV. Sit $m = 4$, ordinisque exponens $n + 4$, et $q^\mu + q^{n+4-\mu} = p^\mu + p^{n+4}$, et aequatio noſtra fiet

$$\begin{array}{l} 0 = q^4 - n(q^3 + q^{n+1}) + \frac{n(n+1)}{1.\ 2}(q^2 + q^{n+2}) - \frac{n(n+1)(n+2)}{1.\ 2.\ 3}(q + q^{n+3}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{n(nn+5)}{1.\ 2.\ 3}p^{n+4} \\ + q^n + 4q^{n-1} + 10q^{n-2} + 20q^{n-3} \ldots\ldots + \frac{n(n+1)(n+2)}{1.\ 2.\ 3}q \\ \qquad\qquad\qquad\qquad\qquad\qquad - \frac{n(n+1)(n+2)(n+3)}{1.\ 2.\ 3.\ 4}p^{n+4} \end{array}$$

ſiue

$$\begin{array}{l} q^n + 4q^{n-1} + 10q^{n-2} \ldots + \frac{(n+1)n(n-1)}{1.\ 2.\ 3}q^2 = \frac{n(n-1)(nn+3n+14)}{24}p^{n+4} \\ + q^4 - n(q^3 + q^{n+1}) + \frac{n(n+1)}{1.\ 2}(q^2 + q^{n+2}) - \frac{n(n+1)(n+2)}{1.\ 2.\ 3}q^{n+3}. \end{array}$$

29. Euoluamus etiam ſimili modo quosdam caſus pro exponente n.

I. Sit primo $n = 1$, et ordinis exponens $m + 1$, eritque noſtra aequatio

$$\begin{array}{l} 0 = q^m - q^{m-1} + q^{m-2} - q^{m-3} \ldots \mp q \pm \tfrac{1}{2}]p^{m+1} \pm q \mp p^{m+1} \\ \qquad\quad - q^2 \quad + q^3 \quad - q^4 \qquad\quad \mp q^m \end{array}$$

vnde patet ſi m fuerit numerus par, quo caſu ſuperiora ſigna valent, totam aequationem fieri identicam; at ſi m ſit numerus impar habebitur

$$q^2 - q^3 + q^4 - q^5 \ldots - q^m = \tfrac{1}{2}p^{m+1}.$$

II. Sit $n = 2$, ordinisque exponens $m + 2$, et aequatio noſtra erit

$$\begin{array}{l} 0 = q^m - 2q^{m-1} + 3q^{m-2} - 4q^{m-3} \ldots \mp mq \quad + \tfrac{1}{2}(m+2) \Big\} p^{m+2} \\ \qquad\quad - 2q^3 \quad + 3q^4 \quad - 4q^5 \ldots\ldots \mp mq^{m+1} - \tfrac{1}{2}(m+1) \Big\} \\ \pm q^2 \pm mq \mp (m+1)p^{m+2} \end{array}$$

vbi ſuperior ambiguitas valet pro valoribus paribus ipſius m, inferior pro imparibus. Iam pro variis valoribus ipſius m habebimus.

Primo

Primo pro valoribus paribus

$m=2;\; q^2-q^3=\frac{1}{2}p^4$

$m=4;\; q^2-q^3+q^4-q^5=\frac{1}{2}p^6$

$m=6;\; q^2-q^3+q^4-q^5+q^6-q^7=\frac{1}{2}p^8$

$m=8;\; q^2-q^3+q^4-q^5+q^6-q^7+q^8-q^9=\frac{1}{2}p^{10}$

etc.

Deinde pro valoribus imparibus:

$m=1;\; q^2=2p^3$

$m=3;\; 3q^2+q^3-3q^4=3p^5$

$m=5;\; 5q^2-q^3-q^4+3q^5-5q^6=4p^7$

$m=7;\; 7q^2-3q^3+q^4+3q^5-3q^6+5q^7-7q^8=5p^9$

$m=9;\; 9q^2-5q^3+3q^4-q^5-q^6+3q^7-5q^8+7q^9-9q^{10}=6p^{11}$

etc.

III. Si ponamus $n=3$ singuli casus commodius perpenduntur, suntque primo pro m numero impari

$m=1;\; q^2+q^3=3p^4$

$m=3;\; 6q^2+6q^3+3q^4-6q^5=7p^6$

$m=5;\; 15q^2-5q^3+6q^4-7q^5+10q^6-15q^7=13p^8$

$m=7;\; 28q^2-14q^3+13q^4-12q^5+13q^6-16q^7+21q^8-28q^9=21p^{10}$

etc.

pro ordinibus vero imparibus:

$m=2;\; 3q^2+q^3-3q^4=3p^5$

$m=4;\; 10q^2-2q^3-2q^4+6q^5-10q^6=8p^7$

$m=6;\; 21q^2-9q^3+3q^4+3q^5-9q^6+15q^7-21q^8=15p^9$

$m=8;\; 36q^2-20q^3+12q^4-4q^5-4q^6+12q^7-20q^8+28q^9-36q^{10}=24p^{11}$

etc.

hic pro vtroque ordine forma aequationis in genere expressa ita se habebit:

ſi m ſit numerus par

$$(m+1)q^2-(m-3)q^3+(m-5)q^4-(m-7)q^5 \ldots\ldots -(m+1)q^{m+2}=\frac{m+4}{2}p^{m+3}$$

ſi m ſit numerus impar

$$m(m+1)q^2-(mm-3m)q^3+(mm-5m+12)q^4-(mm-7m+24)q^5$$
$$+(mm-9m+40)q^5-(mm-11m+60)q^7\ldots\ldots -m(m+1)q^{m+2}$$
$$=\frac{mm+4m+7}{2}p^{m+3}.$$

30. Nunc iam ſingulos ordines percurramus, atque aequationes per methodum ſecundam inuentas ope formulae $p^{\mu}=q^{\mu}+q^{m+n-\nu}-p^{m+n}$ ad ſimiles formas, cuiusmodi hic ſumus nacti reducamus.

Ordo $m+n=3$

meth. I.	meth. II.	meth. III.
$q+q^2=p+p^4$	$q^2=2p^3$	$q^2=2p^3$

Ordo $m+n=4$

$q+q^3=p+p^4$	$3q^2-q^3=4p^4$	$q^2+q^3=3p^4$
$2q^2=p^2+p^4$	$4q^2-4q^3=2p^4$	$2q^2-2q^3=p^4$

ergo $q^2=\frac{7}{4}p^4$ et $q^3=\frac{5}{4}p^4$ ob $p^2=\frac{5}{2}p^4$ ergo $q^3=\frac{1}{2}p^2$

Ordo $m+n=5$

$q+q^4=p+p^5$	$q^2+q^3+q^4=4p^5$	$q^2+q^3+q^4=4p^5$
$q^2+q^3=p^2+p^5$	$0=0$	$3q^2+q^3-3q^4=3p^5$

$q^4=3p^5-p^2$; $q^3=-\frac{9}{2}p^5+3p^2$; $q^2=\frac{11}{2}p^5-2p^2$.

Ordo $m+n=6$

$q+q^5=p+p^6$	$3q^2-q^3+3q^4-q^5=6p^6$	$q^2+q^3+q^4+q^5=5p^6$
$q^2+q^4=p^2+p^6$	$8q^2-2q^3+5q^4-8q^5=8p^6$	$4q^2+2q^3+q^4-4q^5=6p^6$
$2q^3=p^3+p^6$	$12q^2+6q^4-12q^5=14p^6$	$6q^2+3q^4-6q^5=7p^6$
		$q^2-q^3+q^4-q^5=\frac{1}{2}p^6$

Ergo

Ergo

$$q^5 = p^2 - \tfrac{1}{2}p^3;\ q^4 = p^3 - \tfrac{1}{6}p^6;\ q^3 = \tfrac{1}{2}p^3 + \tfrac{1}{2}p^6;\ q^2 = p^2 - p^3 + \tfrac{4}{3}p^6$$

$$q^5 = \tfrac{7}{2}p^6 - p^2 - \tfrac{1}{2}p^3$$

$$q^5 = -\tfrac{7}{2}p^6 + 3p^2 - \tfrac{1}{2}p^3 \quad \text{Ob } p^2 = \tfrac{7}{4}p^6 \text{ erit}$$

$$q^5 = \tfrac{7}{4}p^6 - \tfrac{1}{2}p^3;\ q^4 = -\tfrac{1}{7}p^6 + p^3;\ q^3 = \tfrac{1}{2}p^6 + \tfrac{1}{2}p^3;\ q^2 = \tfrac{37}{12}p^6 - p^3.$$

Ordo $m + n = 7$

$$\begin{array}{l|l|r}
q + q^6 = p + p^7 & q^2 + q^3 + q^4 + q^5 + q^6 = 6p^7 & q^2 + q^3 + q^4 + q^5 + q^6 = 6p^7 \\
q^2 + q^5 = p^2 + p^7 & 4q^3 + 3q^4 - 2q^5 = 6p^7 & 5q^2 + 3q^3 + 2q^4 + q^5 - 5q^6 = 10p^7 \\
q^3 + q^4 = p^3 + p^7 & 4q^3 + 8q^4 - 2q^5 = 6p^7 & 10q^2 + 2q^3 + q^4 + 4q^5 - 10q^6 = 14p^7 \\
 & & 10q^2 - 2q^3 - 2q^4 + 6q^5 - 10q^6 = 8p^7 \\
 & & 5q^2 - q^3 - q^4 + 3q^5 - 5q^6 = 4p^7
\end{array}$$

vnde concluditur

$$\begin{aligned}
q^6 &= +4p^7 - p^2 - p^3 \\
q^5 &= -10p^7 + 5p^2 + 2p^3 \\
q^4 &= +18p^7 - 10p^2 \\
q^3 &= -17p^7 + 10p^2 + p^3 \\
q^2 &= +11p^7 - 4p^2 - 2p^3.
\end{aligned}$$

Ordo $m + n = 8$

$$\begin{array}{l|r}
q + q^7 = p + p^8 & 3q^2 - q^3 + 3q^4 - q^5 + 3q^6 - q^7 = 8p^8 \\
q^2 + q^6 = p^2 + p^8 & 12q^2 - 2q^3 - 9q^4 - 4q^5 + 7q^6 - 12q^7 = 18p^8 \\
q^3 + q^5 = p^3 + p^8 & 30q^2 \quad + 9q^4 - 6q^5 + 15q^6 - 30q^7 = 38p^8 \\
2q^4 = p^4 + p^8 & 40q^2 \quad + 4q^4 - 8q^5 + 20q^6 - 40q^7 = 42p^8
\end{array}$$

$$\begin{aligned}
q^2 + q^3 + q^4 + q^5 + q^6 + q^7 &= 7p^8 \\
6q^2 + 4q^3 + 3q^4 + 2q^5 + q^6 - 6q^7 &= 15p^8 \\
15q^2 + 5q^3 + 3q^4 + q^5 + 5q^6 - 15q^7 &= 25p^8 \\
20q^2 \ * \ + 2q^4 - 4q^5 + 10q^6 - 20q^7 &= 21p^8 \\
5q^2 - 5q^3 + 6q^4 - 7q^5 + 10q^6 - 15q^7 &= 13p^8 \\
q^2 - q^3 + q^4 - q^5 + q^6 - q^7 &= \tfrac{1}{2}p^8
\end{aligned}$$

hinc

hinc reperitur

$$q^7 = \tfrac{9}{2} p^8 - p^2 - p^3 - \tfrac{1}{2} p^5$$
$$q^7 = -\tfrac{9}{2} p^8 + 3 p^2 - p^3 + \tfrac{1}{2} p^4$$
$$q^7 = p^2 - p^3 + \tfrac{1}{2} p^4$$

reliquae aequationes omnes coalescunt in hanc vnicam:

$$4 q^5 + 10 q^6 = 20 p^8 - 9 p^4 \text{ ob } 7 p^8 = 6 p^4$$

neque ergo haec tertia methodus plenam determinationem suppeditat, cum tamen pro casibus $m+n=5$ et $m+n=7$ esset largita.

Ordo $m+n=9$.

Hic ope tertiae methodi omnia determinentur idque vnico modo; vti sequitur

$$q^8 = +5 p^9 - p^2 - p^3 - p^4$$
$$q^7 = -\tfrac{35}{2} p^9 + 7 p^2 + 2 p^3 + 4 p^4$$
$$q^6 = +\tfrac{85}{2} p^9 - 21 p^2 \; * \; - 6 p^4$$
$$q^5 = -\tfrac{125}{2} p^9 + 35 p^2 \; * \; + 5 p^4$$
$$q^4 = +\tfrac{127}{2} p^9 - 35 p^2 \; * \; - 4 p^4$$
$$q^3 = -\tfrac{83}{2} p^9 + 21 p^2 + p^3 + 6 p^4$$
$$q^2 = +\tfrac{37}{2} p^9 - 6 p^2 - 2 p^3 - 4 p^4$$

praetermisso ordine decimo obseruo etiam vndecimum perfecte determinari posse; calculo enim subducto reperitur

$$q^{10} = 6 p^{11} - p^2 - p^3 - p^4 - p^5$$
$$q^9 = -27 p^{11} + 9 p^2 + 2 p^3 + 6 p^4 + 4 p^5$$
$$q^8 = +83 p^{11} - 36 p^2 \; * \; - 15 p^4 - 6 p^5$$
$$q^7 = -\tfrac{329}{2} p^{11} + 84 p^2 \; * \; + 21 p^4 + 4 p^5$$

$q^6 =$

$$q^6 = +\tfrac{463}{2}p^{11} - 126p^2 \;*\; - 21p^4$$
$$q^5 = -\tfrac{461}{2}p^{11} + 126p^2 \;*\; + 21p^4 + p^5$$
$$q^4 = +\tfrac{331}{2}p^{11} - 84p^2 \;*\; - 20p^4 - 4p^5$$
$$q^3 = -82p^{11} + 36p^2 + p^3 + 15p^4 + 6p^5$$
$$q^2 = +28p^{11} - 8p^2 - 2p^3 - 6p^4 - 4p^5.$$

31. Si has aequationes attentius contemplemur, in coefficientibus primi termini p^{11} haud difficulter sequentem ordinem deprehendimus;

$m+n=11$, $6 = \frac{11+1}{2}$	$m+n=9$, $5 = \frac{9+1}{2}$	$m+n=7$, $4 = \frac{7+1}{2}$	$m+n=5$, $3 = \frac{5+1}{2}$
$2.\,27 = 10.\,6 - 6$	$2.\,\frac{35}{2} = 8.\,5 - 5$	$2.\,10 = 6.\,4 - 4$	$2.\,\frac{9}{2} = 4.\,3 - 3$
$3.\,83 = 9.\,27 + 6$	$3.\,\frac{85}{2} = 7.\,\frac{35}{2} + 5$	$3.\,18 = 5.\,10 + 4$	$3.\,\frac{11}{2} = 3.\,\frac{9}{2} + 3$
$4.\,\frac{329}{2} = 8.\,83 - 6$	$4.\,\frac{125}{2} = 6.\,\frac{85}{2} - 5$	$4.\,17 = 4.\,18 - 4$	$4.\,2 = 2.\,\frac{11}{2} - 3$
$5.\,\frac{463}{2} = 7.\,\frac{329}{2} + 6$	$5.\,\frac{127}{2} = 5.\,\frac{125}{2} + 5$	$5.\,11 = 3.\,17 + 4$	est enim
$6.\,\frac{461}{2} = 6.\,\frac{463}{2} - 6$	$6.\,\frac{83}{2} = 4.\,\frac{127}{2} - 5$	$6.\,3 = 2.\,11 - 4$	$q = 2p^5 + p + p^2$
$7.\,\frac{331}{2} = 5.\,\frac{461}{2} + 6$	$7.\,\frac{37}{2} = 3.\,\frac{83}{2} + 5$	est enim $q = -3p^7 + p + p^2 + p^3$	
$8.\,82 = 4.\,\frac{331}{2} - 6$	$8.\,4 = 2.\,\frac{37}{2} - 5$		
$9.\,28 = 3.\,82 + 6$	est enim $q = -4p^9 + p + p^2 + p^3 + p^4$		
$10.\,5 = 2.\,28 - 6$			

est enim $q = -5p + p + p^2 + p^3 + p^4 + p^5$

32. Tentemus hinc aequationes pro ordine $m+n=13$ deriuare, quandoquidem simul lex progressionis pro altioribus ordinibus imparibus perspicua redditur:

$$
\begin{array}{llllll}
q^{12} = +Ap^{13} - & p^2 - p^3 - & p^4 + p^5 - p^6 \\
q^{11} = -Bp^{13} + & 11p^2 + 2p^3 + & 8p^4 + 4p^5 + \beta p^6 \\
q^{10} = +Cp^{13} - & 55p^2 \; * & - 28p^4 - 6p^5 - \gamma p^6 \\
q^{9} = -Dp^{13} + & 165p^2 \; * & +(56+1)p^4 + 4p^5 + \delta p^6 \\
q^{8} = +Ep^{13} - & 330p^2 \; * & -(70+8)p^4 \; * - \varepsilon p^6 \\
q^{7} = -Fp^{13} + & 462p^2 \; * & +(56+28)p^4 \; * + \zeta p^6 \\
q^{6} = +Gp^{13} - & 462p^2 \; * & -(28+56)p^4 \; * - \eta p^6 \\
q^{5} = -Hp^{13} + & 330p^2 \; * & +(8+70\, p^4 + p^5 + \theta p^6 \\
q^{4} = +Ip^{13} - & 165p^2 \; * & - 56p^4 - 4p^5 - \iota p^6 \\
q^{3} = -Kp^{13} + & 55p^2 + p^3 + & 28p^4 + 6p^5 + \kappa p^6 \\
q^{2} = +Lp^{13} - & (11-1)p^2 - 2p^3 - & 8p^4 - 4p^5 - \lambda p^6
\end{array}
$$

pro incognitis est

$$
\begin{array}{lll}
A = \frac{13+1}{2}; & A = 7 & \lambda = \beta \\
2B = 12A - 7; & B = \frac{77}{2} & \kappa = \gamma \\
3C = 11B + 7; & C = \frac{287}{2} & \iota = \delta \\
4D = 10C - 7; & D = 357 & \theta = \varepsilon \\
5E = 9D + 7; & E = 644 & \eta = \zeta - 1 \\
6F = 8E - 7; & F = \frac{1715}{2} & \text{videturque esse} \\
7G = 7F + 7; & G = \frac{1717}{2} & \beta = 6; \; \lambda = 6 \\
8H = 6G - 7; & H = 643 & \gamma = 15; \; \kappa = 15 \\
9I = 5H + 7; & I = 358 & \delta = 20; \; \iota = 20 \\
10K = 4I - 7; & K = \frac{285}{2} & \varepsilon = 15; \; \theta = 15 \\
11L = 3K + 7; & L = \frac{79}{2} & \zeta = 6 + 1; \; \eta = 6
\end{array}
$$

33. Quo ordo harum aequationum clarius perſpiciatur, atque anomaliae hic occurentes euaneſcant, ſecundum ſingulas ordines impares iſtas aequationes ita repraeſentemus:

Ordo

Ordo $m + n = 3$

$q^3 = A p^3$	$A = \frac{3+1}{2} = 2$
$q = -B p^3 + p$	$2 B = 2 A - 2$

Ordo $m + n = 5$

$q^4 = + A p^5 - p^2$	$A = \frac{5+1}{2} = 3$
$q^3 = - B p^5 + 3 p^2$	$2 B = 4 A - 3$
$q^2 = + C p^5 - 3 p^2 + p^2$	$3 C = 3 B + 3$
$q = - D p^5 + p^2 + p$	$4 D = 2 B - 3$

Ordo $m + n = 7$

$q^6 = + A p^7 - p^2 - p^3$	$A = \frac{7+1}{2} = 4$
$q^5 = - B p^7 + 5 p^2 + 2 p^3$	$2 B = 6 A - 4$
$q = + C p^7 - 10 p^2$ *	$3 C = 5 B + 4$
$q^3 = - D p^7 + 10 p^2$ * $+ p^3$	$4 D = 4 C - 4$
$q^2 = + E p^7 - 5 p^2 - 2 p^3 + p^2$	$5 E = 3 D + 4$
$q = - F p^7 + p^2 + p^3 + p$	$6 F = 2 E - 4$

Ordo $m + n = 9$

$q^8 = + A p^9 - p^2 - p^3 - p^4$	$A = \frac{9+1}{2} = 5$
$q^7 = - B p^9 + 7 p^2 + 2 p^3 + 4 p^4$	$2 B = 8 A - 5$
$q^6 = + C p^9 - 21 p^2$ * $- 6 p^4$	$3 C = 7 B + 5$
$q^5 = - D p^9 + 35 p^2$ * $+ (4+1) p^4$	$4 D = 6 C - 5$
$q^4 = + E p^9 - 35 p^2$ * $- (1+4) p^4 + p^4$	$5 E = 5 D + 5$
$q^3 = - F p^9 + 21 p^2$ * $+ 6 p^4 + p^3$	$6 F = 4 E - 5$
$q^2 = + G p^9 - 7 p^2 - 2 p^3 - 4 p^4 + p^2$	$7 G = 3 F + 5$
$q^2 = - H p^9 + p^2 + p^3 + p^4 + p$	$8 H = 2 G - 5$

Ordo $m + n = 11$

$q^{10} = +Ap^{11} - p^2 - p^3 - p^4 - p^5$	$A = \frac{11+1}{2} = 6$
$q^9 = -Bp^{11} + 9p^2 + 2p^3 + 6p^4 + 4p^5$	$2B = 10A - 6$
$q^8 = +Cp^{11} - 36p^2 * -15p^4 - 6p^5$	$3C = 9B + 6$
$q^7 = -Dp^{11} + 84p^2 * +(20+1)p^4 + 4p^5$	$4D = 8C - 6$
$q^6 = +Ep^{11} - 126p^2 * -(15+6)p^4 *$	$5E = 7D + 6$
$q^5 = -Fp^{11} + 126p^2 * +(6+15)p^4 * + p^5$	$6F = 6E - 6$
$q^4 = +Gp^{11} - 84p^2 * -(1+20)p^4 - 4p^5 + p^4$	$7G = 5F + 6$
$q^3 = -Hp^{11} + 36p^2 * + 15p^4 + 6p^5 + p^3$	$8H = 4G - 6$
$q^2 = +Ip^{11} - 9p^2 - 2p^3 - 6p^4 - 4p^5 + p^2$	$9I = 3H + 6$
$q = -Kp^{11} + p^2 + p^3 + p^4 + p^5 + p$	$10K = 2T - 6$

Ordo $m + n = 13$

$q^{12} = +Ap^{13} - p^2 - p^3 - p^4 - p^5 - p^6$	$A = \frac{13+1}{2} = 7$
$q^{11} = -Bp^{13} + 11p^2 + 2p^3 + 8p^4 + 4p^5 + 6p^6$	$2B = 12A - 7$
$q^{10} = +Cp^{13} - 55p^2 * - 28p^4 - 6p^5 - 15p^6$	$3C = 11B + 7$
$q^9 = -Dp^{13} + 165p^2 * +(56+1)p^4 + 4p^5 + 20p^6$	$4D = 10C - 7$
$q^8 = +Ep^{13} - 330p^4 * -(70+8)p^4 * - 15p^6$	$5E = 9D + 7$
$q^7 = -Fp^{13} + 462p^2 * +(56+28)p^4 * +(6+1)p^6$	$6F = 8E - 7$
$q^6 = +Gp^{13} - 462p^2 * -(28+56)p^4 * -(1+6)p^6 + p^6$	$7G = 7F + 7$
$q^5 = -Hp^{13} + 330p^2 * +(8+76)p^4 * + 15p^6 + p^5$	$8H = 6G - 7$
$q^4 = +Ip^{13} - 165p^2 * -(1+56)p^4 - 4p^5 - 20p^6 + p^4$	$9I = 5H + 7$
$q^3 = -Kp^{13} + 55p^2 * + 28p^4 + 6p^5 + 15p^6 + p^3$	$10K = 4I - 7$
$q^2 = +Lp^{13} + 11p^2 - 2p^3 - 8p^4 - 4p^5 - 6p^6 + p^2$	$11L = 3K + 7$
$q = -Mp^{13} + p^2 + p^3 + p^4 + p^5 + p^6 + p$	$12M = 2L - 7$

34. Hic coefficientes ipſius p^7 a lege ſequentium ordine parium recedere videntur, cum ii in ordine $m + n = 13$ ex binomii ad dignitatem 11 eleuati coefficientibus formentur, dum ſequentes ex digni-

dignitatibus 8, 6 formantur. At iidem illo coefficiente hoc modo repraesentari possunt, vt cum sequentium lege cohaereant:

$$-1+(10+1)-(45+10)+(120+45)-(210+120) \text{ etc.}$$

hac ergo ratione aequationes ordinis $m+n=15$ exhibebo.

Ordo $m+n=15$

$$\begin{array}{llllll}
q^{14} = +Ap^{15} - p^{2} & -p^{3} - & p^{4} & -p^{5} - & p^{6} - & p^{7} \\
q^{13} = -Bp^{15} + (12+1)p^{2} & +2p^{3} + & 10\,p^{4} & +4p^{5} + & 8p^{6} + & 6p^{7} \\
q^{12} = +Cp^{15} - (66+12)p^{2} & * \; - & 45\,p^{4} & -6p^{5} - & 28p^{6} - & 15p^{7} \\
q^{11} = -Dp^{15} + (220+66)p^{2} & * & +(120+1)p^{4} & +4p^{5} + & 56p^{6} + & 20p^{7} \\
q^{10} = +Ep^{15} - (495+220)p^{2} & * & -(210+10)p^{4} & * \; - & 70p^{6} - & 15p^{7} \\
q^{9} = -Fp^{15} + (792+495)p^{2} & * & +(252+45)p^{4} & * & +(56+1)p^{6} + & 6p^{7} \\
q^{8} = +Gp^{15} - (924+792)p^{2} & * & -(210+120)p^{4} & * & -(28+8)p^{6} & * \\
q^{7} = -Hp^{15} + (792+924)p^{2} & * & +(120+210)p^{4} & * & +(8+28)p^{6} & * \; +p^{7} \\
q^{6} = +Ip^{15} - (495+792)p^{2} & * & -(45+252)p^{4} & * & -(1+56)p^{6} - & 6p^{7} + p^{6} \\
q^{5} = -Kp^{15} + (220+495)p^{2} & * & +(10+210)p^{4} & * \; + & 70p^{6} + & 15p^{7} + p^{5} \\
q^{4} = +Lp^{15} - (66+220)p^{2} & * & -(1+120)p^{4} - 4p^{5} & - & 56p^{6} - & 20p^{7} + p^{5} \\
q^{3} = -Mp^{15} + (12+66)p^{2} & * \; + & 45p^{4} + 6p^{5} & + & 28p^{6} + & 15p^{7} + p^{3} \\
q^{2} = +Np^{15} - (1+12)\;p^{2} & -2p^{3} - & 10p^{4} - 4p^{5} & - & 8p^{6} - & 6p^{7} - p^{2} \\
q = -Op^{15} + \;\; p^{2} & + p^{3} + & p^{4} + p^{5} & + & p^{6} + & p^{7} + p
\end{array}$$

Atque nunc lex progressionis non nimis est complexa, eamque facile ad altiores ordines accommodare licet.

35. Cum autem haec lex, quatenus inductioni innititur, minus certa videri posset, omnia plane dubia tollentur, si loco potestatum parium ipsius p impares introducantur. Cum scilicet pro ordine vndecimo

decimo potestatibus p^2, p^4, aequiualeant impares p^9 et p^7. Deinde etiam coefficientes A, B, C, D etc. multo simpliciori lege exhiberi possunt, quae immediate ex coefficientibus binomii ad eandem dignitatem, cuius ordinis istae aequationes quaeruntur, eleuati, fluit, hoc modo:

Ordo $m + n = 11$

p^{11}	p^1	p^3	p^5	p^7	p^9	
$q^{10} = \frac{1}{2}(1+11)$	-1	-1	-1	-1	-1	$+p = p^{10}$
$q^9 = \frac{1}{2}(1-55)$	*	$+2$	$+4$	$+6$	$+8+1$	
$q^8 = \frac{1}{2})1+165)$	*	-1	-6	-15	$-28-8$	$+p^3 = p^8$
$q^7 = \frac{1}{2}(1-330)$	*	*	$+4$	$+20+1$	$+56+28$	
$q^6 = \frac{1}{2}(1+462)$	*	*	-1	$-15-6$	$-70-56$	$+p^5 = p^6$
$q^5 = \frac{1}{2}(1-462)$	*	*	$+1$	$+6+15$	$+56+70$	
$q^4 = \frac{1}{2}(1+330)$	*	*	-4	$-1-20$	$-28-56$	$+p^7 = p^4$
$q^3 = \frac{1}{2}(1-165)$	*	$+1$	$+6$	$+15$	$+8+28$	
$q^2 = \frac{1}{2}(1+55)$	*	-2	-4	-6	$-1-8$	$+p^9 = p^2$
$q^1 = \frac{1}{2}(1-11)$	$+1$	$+1$	$+1$	$+1$	$+1$	

In qualibet scilicet columna verticali coefficientes binomii ad dignitatem vnitate inferiorem eleuati tum a summo deorsum, quum ad imo sursum scribuntur, et vbi bini concurrunt in vnam summam colliguntur.

36. Hinc iam licebit pro omnibus ordinibus imparibus rem in genere definire; at quoniam hic coefficientes binomii potestatum occurrunt, vt breuitati consulamus, scribamus

$$\frac{n(n-1)(n-2)\ldots(n-\nu+1)}{1.\quad 2.\quad 3\ldots\ldots\nu} = n(\nu);$$

vt

vt fit

$$n(0)=1;\ n(1)=n;\ n(2)=\frac{n(n-1)}{1.\ 2};\ n(3)=\frac{n(n-1)(n-2)}{1.\ 2.\ 3}\ \text{etc.}$$

vbi obſeruaſſe iuuabit, ſi ν fuerit maior quam n, eſſe ſemper $n(\nu)=0$, at ſi $\nu=n$ fore $n(n)=1$ et generatim: $n(\nu)=n(n-\nu)$. Hac igitur notandi rationi recepta aequationes generales ita ſe habebunt

Ordo $m+n=\lambda$.

$$q^{\lambda-1}=p^{\lambda-1}+\tfrac{1}{2}(1+\lambda(1))p^{\lambda}\left.\begin{matrix}+0(0)\\+0(\lambda-2)\end{matrix}\right\}p\left.\begin{matrix}+2(0)\\+2(\lambda-2)\end{matrix}\right\}p^{3}\left.\begin{matrix}+4(0)\\+4(\lambda-2)\end{matrix}\right\}p^{5}\left.\begin{matrix}+6(0)\\+6(\lambda-2)\end{matrix}\right\}p^{7}\ \text{etc.}$$

$$q^{\lambda-2}=\ *\ +\tfrac{1}{2}(1-\lambda(2))p^{\lambda}\left.\begin{matrix}-0(1)\\-0(\lambda-3)\end{matrix}\right\}p\left.\begin{matrix}-2(1)\\-2(\lambda-3)\end{matrix}\right\}p^{3}\left.\begin{matrix}-4(1)\\-4(\lambda-3)\end{matrix}\right\}p^{5}\left.\begin{matrix}-6(1)\\-6(\lambda-3)\end{matrix}\right\}p^{7}\ \text{etc.}$$

$$q^{\lambda-3}=p^{\lambda-3}+\tfrac{1}{2}(1+\lambda(3))p^{\lambda}\left.\begin{matrix}+0(2)\\+0(\lambda-4)\end{matrix}\right\}p\left.\begin{matrix}+2(2)\\+2(\lambda-4)\end{matrix}\right\}p^{3}\left.\begin{matrix}+4(2)\\+4(\lambda-4)\end{matrix}\right\}p^{5}\left.\begin{matrix}+6(2)\\+6(\lambda-4)\end{matrix}\right\}p^{7}\ \text{etc.}$$

$$q^{\lambda-4}=\ *\ +\tfrac{1}{2}(1-\lambda(4))p^{\lambda}\left.\begin{matrix}-0(3)\\-0(\lambda-5)\end{matrix}\right\}p\left.\begin{matrix}-2(3)\\-2(\lambda-5)\end{matrix}\right\}p^{3}\left.\begin{matrix}-4(3)\\-4(\lambda-5)\end{matrix}\right\}p^{5}\left.\begin{matrix}-6(3)\\-6(\lambda-5)\end{matrix}\right\}p^{7}\ \text{etc.}$$

$$q^{\lambda-5}=p^{\lambda-5}+\tfrac{1}{2}(1+\lambda(5))p^{\lambda}\left.\begin{matrix}+0(4)\\+0(\lambda-6)\end{matrix}\right\}p\left.\begin{matrix}+2(4)\\+2(\lambda-6)\end{matrix}\right\}p^{3}\left.\begin{matrix}+4(4)\\+4(\lambda-6)\end{matrix}\right\}p^{5}\left.\begin{matrix}+6(4)\\+6(\lambda-6)\end{matrix}\right\}p^{7}\ \text{etc.}$$

vnde concludimus fore in genere

I. Si ν fit numerus impar

$$q^{\lambda-\nu}=p^{\lambda-\nu}+\tfrac{1}{2}(1+\lambda(\nu))p^{\lambda}\left.\begin{matrix}+0(\nu-1)\\+0(\lambda-\nu-1)\end{matrix}\right\}p\left.\begin{matrix}+2(\nu-1)\\+2(\lambda-\nu-1)\end{matrix}\right\}p^{3}\left.\begin{matrix}+4(\nu-1)\\+4(\lambda-\nu-1)\end{matrix}\right\}p^{5}\left.\begin{matrix}+6(\nu-1)\\+6(\lambda-\nu-1)\end{matrix}\right\}p^{7}\ \text{etc.}$$

II. ſi ν fit numerus par

$$q^{\lambda-\nu}=\ *\ +\tfrac{1}{2}(1-\lambda(\nu))p^{\lambda}\left.\begin{matrix}-0(\nu-1)\\-0(\lambda-\nu-1)\end{matrix}\right\}p\left.\begin{matrix}-2(\nu-1)\\-2(\lambda-\nu-1)\end{matrix}\right\}p^{3}\left.\begin{matrix}-4(\nu-1)\\-4(\lambda-\nu-1)\end{matrix}\right\}p^{5}\left.\begin{matrix}-6(\nu-1)\\-6(\lambda-\nu-1)\end{matrix}\right\}p^{7}\ \text{etc.}$$

Termini

Termini autem harum aequationum non vltra formulam $p^{\lambda-2}$, quae vltimum praebet terminum, continuari debent.

37. Hae autem ſummationes locum non habent, niſi exponens ordinis $m+n=\lambda$ fuerit numerus impar: ideoque harum aequationum ope ſummas omnium ſerierum in hac forma contentarum

$$q^m = 1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right) \text{ etc.}$$

exhibere licet, ſi modo $m+n=\lambda$ ſit numerus impar. Iſtae autem ſummae definiuntur per ſummas poteſtatum reciprocarum, quas littera p ſequenti modo repraeſento vt ſit

$$p^\lambda = 1+\frac{1}{2^\lambda}+\frac{1}{3^\lambda}+\frac{1}{4^\lambda}+\frac{1}{5^\lambda}+\text{etc. atque}$$

$$p^m \cdot p^n = \left(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\text{etc.}\right)\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.}\right)$$

Duos autem hic caſus diſtingui oportet, prout m fuerit numerus vel par vel impar; habebitur enim

I. Caſu quo m eſt numerus par

$$q^m = p^m + \tfrac{1}{2}(1+\lambda(n))p^\lambda \left.\begin{matrix}+0(n-1)\\+0(m-1)\end{matrix}\right\} p \left.\begin{matrix}+2(n-1)\\+2(m-1)\end{matrix}\right\} p^3 \left.\begin{matrix}+4(n-1)\\+4(m-1)\end{matrix}\right\} p^5$$

$$\left.\begin{matrix}+6(n-1)\\+6(m-1)\end{matrix}\right\} p^7 \cdots\cdots \left.\begin{matrix}+(\lambda-3)(n-1)\\+(\lambda-3)(m-1)\end{matrix}\right\} p^{\lambda-2}$$

II.

II. Casu quo m est numerus impar

$$q^m = * + \tfrac{1}{2}(1-\lambda)(n)p^{\lambda} \begin{matrix}-0(n-1)\\-0(m-1)\end{matrix}\Big\} p \begin{matrix}-2(n-1)\\-2(m-1)\end{matrix}\Big\} p^3 \begin{matrix}-4(n-1)\\-4(m-1)\end{matrix}\Big\} p^5$$

$$\begin{matrix}-6(n-1)\\-6(m-1)\end{matrix}\Big\} p^7 \ldots\ldots \begin{matrix}-(\lambda-3)(n-1)\\-(\lambda-3)(m-1)\end{matrix}\Big\} p^{\lambda-2}$$

Secundum hanc legem terminus vltimum sequens fieret $\begin{matrix}(\lambda-1)(n-1)\\(\lambda-1)(m-1)\end{matrix}\Big\} p^{\lambda}$, vbi notetur semper esse $(\lambda-1)(n-1)+(\lambda-1)(m-1)=\lambda(n)=\lambda(m)$.

38. At si ordinis exponens $m+n=\lambda$ fuerit numerus par, hae formulae neutiquam locum habere possunt, cum casu imparitatis formae p, p^3, p^5, p^7 etc. ob $p^m = p^n$ etiam has pares p^2, p^4, p^6 etc. in se complectantur, quod autem casu quo $m+n$ est numerus par, non euenit. Tres autem methodi hic vsitatae summis ordinum parium definiendis non sufficiunt, cum etiam tertia pro ordine octauo non omnes determinationes suppeditet. Etsi autem pro ordinibus quarto et sexto summae supra sunt assignatae, in iis tamen nulla lex perspicitur, vnde pro ordinibus sequentibus coniecturam deducere liceret. Ratio huius discriminis manifesto in eo est sita, quod pro ordinibus paribus, binae quaeuis harum formularum p^{λ}, p^2, p^4, p^6 etc. inter se comparari queant, haeque comparationes per methodos nostras indicentur; quocirca eae determinationes, quibus indigemus, deficere sunt censendae. Eo magis igitur est mirandum, quod in ordinibus imparibus

 nulla

nulla plane ratio aſſignalibis inter formulas p^{λ}, p^{2}, p^{3} p^{4} etc. iniercedat. Interim tamen nullum eſt dubium, quin aliae dentur methodi, quibus ſeries ordinum parium ſummari queant, etiamſi tres hic expoſitae minime ſufficiant.

PHYSICO-