DE VALORE FORMVLAE INTEGRALIS

$$\int \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}} \frac{dz}{z} (lz)^{\mu}$$

CASV QVO POST INTEGRATIONEM PONITVR $z = 1$.

Auctore

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§. 1.

Ex consideratione innumerabilium arcuum circularium, qui communem habent vel sinum vel tangentem, iam olim summationem duarum serierum infinitarum deduxi, quae ob summam generalitatem maxime memoratu dignae videbantur. Si enim literae m et n numeros quoscunque denotant; posita diametri ratione ad peripheriam vt 1 ad π, illae duae summationes hoc modo se habebant:

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} \text{ etc} = \frac{\pi}{n \sin. \frac{m\pi}{n}}$$

et

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc} = \frac{\pi}{n \operatorname{tang}. \frac{m\pi}{n}}$$

atque

atque ex his duabus feriebus iam tum temporis elicueram fummationes omnium ferierum illarum, quarum denominatores fecundum poteftates numerorum naturalium progrediuntur, quemadmodum in introductione in analyfin infinitorum et alibi fufius expofui. Nunc autem eaedem feries me perduxerunt ad integrationem formulae in titulo expreffae, quae eo magis attentione digna videtur, quod huiusmodi integrationes aliis methodis neutiquam exfequi liceat.

§. 2. Statim autem patet: has duas feries infinitas oriri ex euolutione quarundam formularum integralium, fi poft integrationem quantitati variabili certus valor, veluti vnitas tribuatur; ita prior feries deducitur ex euolutione huius formulae integralis

$$\int \frac{z^{m-1} + z^{n-m-1}}{1 + z^n} dz$$

pofteriore vero ex euolutione iftius

$$\int \frac{z^{m-1} - z^{n-m-1}}{1 - z^n} dz$$

fi quidem poft integrationem ftatuatur $z = 1$. Deinceps autem ex ipfis principiis calculi integralis demonftraui, valorem integralis prioris harum duarum formularum, fi quidem ponatur $z = 1$, reduci ad hanc formulam fimplicem

$$\frac{\pi}{n \sin. \frac{m\pi}{n}}$$

integrale autem pofterius, eodem cafu $z = 1$, ad iftam

$$\frac{\pi}{n \tang. \frac{m\pi}{n}}$$ ita,

ita, vt ex ipsis calculi integralis principiis certum sit esse

$$\int \frac{z^{m-1}+z^{n-m-1}}{1+z^n}\,dz = \frac{\pi}{n\,\text{sin.}\frac{m\pi}{n}}$$

$$\int \frac{z^{m-1}-z^{n-m-1}}{1-z^n}\,dz = \frac{\pi}{n\,\text{tang.}\frac{m\pi}{n}}$$

si quidem post integrationem ita institutam, vt integrale euanescat posito $z=0$, statuatur $z=1$.

§. 3. Quo iam hanc duplicem integrationem ad formam propositam reducamus, faciamus $n=2\lambda$ et $m=\lambda-\omega$, vnde binae illae series infinitae hanc induent formam

$$\frac{1}{\lambda-\omega}+\frac{1}{\lambda+\omega}-\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}+\frac{1}{5\lambda+\omega}-\text{etc.}$$

et

$$\frac{1}{\lambda-\omega}-\frac{1}{\lambda+\omega}+\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}-\frac{1}{5\lambda+\omega}+\text{etc.}$$

harum igitur serierum prioris summa erit

$$\frac{\pi}{2\lambda\,\text{sin.}\frac{\pi(\lambda-\omega)}{2\lambda}} = \frac{\pi}{2\lambda\,\text{cos.}\frac{\pi\omega}{2\lambda}}$$

posterioris vero summa erit

$$\frac{\pi}{2\lambda\,\text{tang.}\frac{\pi(\lambda-\omega)}{2\lambda}} = \frac{\pi}{2\lambda\,\text{cotang.}\frac{\pi\omega}{2\lambda}} = \frac{\pi\,\text{tang.}\frac{\pi\omega}{2\lambda}}{2\lambda}.$$

Quod si ergo breuitatis gratia ponamus

$$\frac{\pi}{2\lambda\,\text{cos.}\frac{\pi\omega}{2\lambda}} = S, \quad \text{et} \quad \frac{\pi}{2\lambda}\,\text{tang.}\frac{\pi\omega}{2\lambda} = T,$$

habe-

habebimus sequentes duas integrationes

$$\int \frac{z^{\lambda - \omega} + z^{\lambda + \omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} = S, \text{ et}$$

$$\int \frac{z^{\lambda - \omega} - z^{\lambda + \omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} = T.$$

§. 4. Circa has binas integrationes ante omnia obseruo, eas perinde locum habere, siue pro literis λ et ω accipiantur numeri integri, siue fracti. Sint enim λ et ω numeri fracti quicunque, qui euadant integri, si multiplicentur per α, quo posito fiat $z = x^{\alpha}$, eritque $\frac{dz}{z} = \frac{\alpha dx}{x}$, et potestas quaecunque $z^{\vartheta} = x^{\alpha \vartheta}$; prior igitur formula erit

$$\int \frac{x^{\alpha(\lambda - \omega)} + x^{\alpha(\lambda + \omega)}}{1 + x^{2\alpha\lambda}} \cdot \frac{\alpha dx}{x}$$

vbi, cum iam omnes exponentes fiat numeri integri, valor huius formulae posito post integrationem $x = 1$, quandoquidem tunc etiam fit $z = 1$, a praec dente eo tantum differt, quod hic habeamus $\alpha\lambda$ et $\alpha\omega$ loco λ et ω, ac praeterea hic adsit factor α, quocirca valor istius formulae erit

$$\alpha \cdot \frac{\pi}{2\alpha\lambda \operatorname{cos.} \frac{\pi\omega}{2\lambda}} = \frac{\pi}{2\lambda \operatorname{cos.} \frac{\pi\omega}{2\lambda}}$$

qui ergo valor est $= S$ prorsus vt ante; quae identitas etiam manifesto est in altera formula, vnde patet, etiamsi pro λ et ω fractiones quaecunque accipiantur, integrationem hic exhibitam nihilo minus locum esse habituram; quae circumstantia probe

 notari

notari meretur, quoniam in ſequentibus literam ω tanquam variabilem ſumus tractaturi.

§. 5. Poſtquam igitur binae iſtae formulae integrales literis S et T indicatae fuerint integratae, ita, vt euaneſcant poſito $z = 0$, integralia ſpectari poterunt non ſolum vt functiones quantitatis z, ſed etiam vt functiones binarum variabilium z et ω, quandoquidem numerum ω tanquam quantitatem variabilem tractare licet, quin etiam exponentem λ pro quantitate variabili habere liceret; ſed quia hinc formulae integrales alius generis eſſent prodituraе, atque hic contemplari conſtitui, ſolam quantitatem ω, praeter ipſam variabilem z, hic vt quantitatem variabilem ſum tractaturus.

§. 6. Cum igitur ſit

$$S = \int \frac{z^{\lambda - \omega} + z^{\lambda + \omega}}{1 + z^{2\lambda}} \frac{dz}{z}$$

in qua integratione ſola z vt variabilis ſpectatur, erit vtique ſecundum ſignandi morem iam ſatis vſu receptum

$$\left(\frac{dS}{dz}\right) = \frac{z^{\lambda - \omega} + z^{\lambda + \omega}}{1 + z^{2\lambda}} \cdot \frac{1}{z}$$

haec iam formula denuo differentietur, poſita ſola litera ω variabili, eritque

$$\left(\frac{ddS}{dz\,d\omega}\right) = \frac{-z^{\lambda - \omega} + z^{\lambda + \omega}}{1 + z^{2\lambda}} \cdot \frac{1}{z}\, l z$$

quae formula ducta in dz, ac denuo integrata ſola z habita pro variabili, dabit

$$\int dz \left(\frac{ddS}{dz\,d\omega}\right) = \int - \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \frac{dz}{z} l z$$

vbi notetur esse

$$S = \frac{\pi}{2\lambda \operatorname{cof.} \frac{\pi\omega}{2\lambda}}$$

ita, vt hinc deducamus

$$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi \operatorname{fin.} \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \operatorname{cof.} \frac{\pi\omega^2}{2\lambda}},$$

hoc igitur valore fubftituto nancifcimur hanc integrationem

$$\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \frac{dz}{z} l z = \frac{\pi\pi \operatorname{fin.} \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \operatorname{cof.} \frac{\pi\omega^2}{2\lambda}}.$$

§. 7. Quod fi iam altera formula fimili modo tractetur, cum fit

$T = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda}$ erit

$$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{4\lambda\lambda \operatorname{cof.} \frac{\pi\omega^2}{2\lambda}}$$

ex formula autem integrali erit

$$\left(\frac{dT}{d\omega}\right) = \int - \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} . \frac{dz}{z} l z$$

vnde colligimus fequentem integrationem

$$\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 - z^{2\lambda}} \frac{dz}{z} l z = \frac{-\pi\pi}{4\lambda\lambda \operatorname{cof} \frac{\pi\omega^2}{2\lambda}}.$$

§. 8. Quoniam literas S et T etiam per feries expreffas dedimus, erit etiam per fimiles feries

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \text{etc.}$$

$$= \frac{\pi\pi \,\text{fin.}\, \frac{\pi\omega}{\lambda}}{4\lambda\lambda \,\text{cof.}\, \frac{\pi\omega^2}{2\lambda}}.$$

Similique modo etiam pro altera ferie

$$\left(\frac{dT}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} + \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \text{etc.}$$

$$= \frac{\pi\pi}{4\lambda\lambda \,\text{cof.}\, \frac{\pi\omega^2}{2\lambda}}$$

ficque fummas harum ferierum quoque duplici modo repraefentauimus, fcilicet per formulam euolutam quantitatem π inuoluentem, tum vero etiam per formulam integralem; quae ita eft comparata, vt eius integrale nulla methodo adhuc confueta affignari poffit.

§. 9. Applicemus has integrationes ad aliquot cafus particulares: ac primo quidem fumamus $\omega = 0$, quo quidem cafu prior integratio fponte in oculos incurrit, at pofterior praebet

$$\int \frac{2 z^\lambda}{1 - z^{2\lambda}} \frac{dz}{z}\, lz = -\frac{\pi\pi}{4\lambda\lambda}, \text{ fiue}$$

$$\int \frac{z^{\lambda-1}\, dz\, lz}{1 - z^{2\lambda}} = -\frac{\pi\pi}{8\lambda\lambda}$$

hincque fimul iftam fummationem adipifcimur

$$\frac{1}{\lambda\lambda} + \frac{1}{\lambda\lambda} + \frac{1}{9\lambda\lambda} + \frac{1}{9\lambda\lambda} + \frac{1}{25\lambda\lambda} + \frac{1}{25\lambda\lambda} + \text{etc.} = \frac{\pi\pi}{4\lambda\lambda}$$

fiue

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.} = \frac{\pi\pi}{8}$$

id quod iam dudum a me eft demonftratum.

§. 10.

§. 10. Hic ſtatim patet, perinde eſſe, quinam numerus pro λ accipiatur; ſit igitur $\lambda = 1$, et habebitur iſta integratio

$$\int \frac{dz\,lz}{1 - z^2} = - \frac{\pi\pi}{8},$$

ex qua ſequentia integralia ſimpliciora

$$\int \frac{dz\,lz}{1 - z} \quad \text{et} \quad \int \frac{dz\,lz}{1 + z}$$

deriuare licet, ope huius ratiocinii; ſtatuatur

$$\int \frac{z\,dz\,lz}{1 - zz} = \mathrm{P},$$

et poſito $zz = v$, vt ſit $z\,dz = \frac{dv}{2}$ et $lz = \frac{1}{2} lv$ prodibit

$$\frac{1}{4} \int \frac{dv\,lv}{1 + v} = \mathrm{P},$$

ſi ſcilicet poſt integrationem ſiat $v = 1$ quippe quo caſu etiam ſit $z = 1$; ſic igitur erit

$$\int \frac{dv\,lv}{1 - v} = 4\,\mathrm{P},$$

nunc prior illa formula addatur ad inuentam eritque

$$\int \frac{dz\,lz + z\,dz\,lz}{1 - zz} = \mathrm{P} - \frac{\pi\pi}{v}$$

haec autem formula ſponte reducitur ad hanc

$$\int \frac{dz\,lz}{1 - z} = \mathrm{P} - \frac{\pi\pi}{8}$$

modo autem vidimus eſſe

$$\int \frac{dv\,lv}{1 - v} \text{ ſiue } \int \frac{dz\,lz}{1 - z} = 4\,\mathrm{P}, \text{ ita, vt ſit } 4\,\mathrm{P} = \mathrm{P} - \frac{\pi\pi}{8},$$

vnde manifeſto ſit $\mathrm{P} = - \frac{\pi\pi}{25}$, ex quo ſequitur fore

$$\int \frac{dz\,lz}{1 - z} = - \frac{\pi\pi}{6};$$

ſimili modo erit

$$\int \frac{dz\,lz - z\,dz\,lz}{1 - zz} = - \mathrm{P} - \frac{\pi\pi}{8} = - \frac{\pi\pi}{12}$$

 quae,

quae, supra et infra per $1 - z$ diuidendo, praebet

$$\int \frac{dz\,lz}{1+z} = -\frac{\pi\pi}{12}$$

quare iam adepti sumus tres integrationes memoratu maxime dignas

$$\text{I. } \int \frac{dz\,lz}{1+z} = -\frac{\pi\pi}{12}$$

$$\text{II. } \int \frac{dz\,lz}{1-z} = -\frac{\pi\pi}{6}$$

$$\text{III. } \int \frac{dz\,lz}{1-zz} = -\frac{\pi\pi}{8}$$ quibus adiungi potest

$$\text{IV. } \int \frac{z\,dz\,lz}{1-zz} = -\frac{\pi\pi}{24}.$$

§. 11. Quemadmodum igitur hae formulae ex ipsis calculi integralis principiis sunt deductae, ita etiam earum veritas per resolutionem in series facile comprobatur; cum enim sit

$$\frac{1}{1+z} = 1 - z + zz - z^3 + z^4 - z^5 + \text{etc.},$$

et ingenere

$$\int z^n\,dz\,lz = \frac{z^{n+1}}{n+1}\,lz - \frac{z^{n+1}}{(n+1)^2}$$

qui valor posito $z = 1$ reducitur ad $\frac{1}{(n+1)^2}$, patet fore

$$\int \frac{dz\,lz}{1+z} = -1 + \tfrac{1}{4} - \tfrac{1}{9} + \tfrac{1}{16} - \tfrac{1}{25} + \text{etc.} = -\frac{\pi\pi}{12} \text{ siue}$$

$$1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \text{etc.} = \frac{\pi\pi}{12}$$

simili modo ob

$$\frac{1}{1-z} = 1 + z + zz + z^3 + z^4 + \text{etc. erit}$$

$$\int \frac{dz\,lz}{1-z} = -1 - \tfrac{1}{4} - \tfrac{1}{9} - \tfrac{1}{16} - \tfrac{1}{25} - \text{etc.} = -\frac{\pi\pi}{6}, \text{ seu}$$

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \text{etc.} = \frac{\pi\pi}{6},$$

tum

tum vero ob

$$\frac{1}{1-zz} = 1 + zz + z^4 + z^6 + z^8 + \text{etc. erit}$$

$$\int \frac{dz\,lz}{1-zz} = -1 - \tfrac{1}{9} - \tfrac{1}{25} - \tfrac{1}{49} - \tfrac{1}{81} - \text{etc.} = -\frac{\pi\pi}{8}, \text{ fiue}$$

$$1 + \tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49} + \tfrac{1}{81} = \frac{\pi\pi}{8}.$$

Eodem modo etiam

$$\int \frac{z\,dz\,lz}{1-zz} = -\tfrac{1}{4} - \tfrac{1}{16} - \tfrac{1}{36} - \tfrac{1}{64} - \text{etc.} = -\frac{\pi\pi}{24}$$

fiue $\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \text{etc.} = \frac{\pi\pi}{24}$

quae quidem fummationes iam funt notiffimae. Neque tamen quisquam adhuc methodo directa oftendit effe

$$\int \frac{dz\,lz}{1+z} = -\frac{\pi\pi}{12},$$

§. 12. Ponamus nunc $\omega = 1$, et noftrae integrationes has induent formas

$$1^\circ.\ \int \frac{-z^{\lambda-2}(1-zz)\,dz\,lz}{1+z^{2\lambda}} = \frac{\pi\pi\,\text{fin.}\,\frac{\pi}{2\lambda}}{4\lambda\lambda\,\text{cof.}\,\frac{\pi^2}{2\lambda}} \quad \text{et}$$

$$2^\circ.\ \int \frac{-z^{\lambda-2}(1+zz)\,dz\,lz}{1-z^{2\lambda}} = +\frac{\pi\pi}{4\lambda\lambda\,\text{cof.}\,\frac{\pi^2}{2\lambda}}$$

vnde pro diuerfis valoribus ipfius λ, quos quidem binario non minores accipere licet, fequentes obtinentur integrationes

I°. fi $\lambda = 2$ erit

$$1^\circ.\ \int \frac{-(1-zz)\,dz\,lz}{1+z^4} = \frac{\pi\pi}{8\sqrt{2}}$$

$$2^\circ.\ \int \frac{-(1+zz)\,dz\,lz}{1-z^4} = +\frac{\pi\pi}{8} \text{ fiue } \int \frac{-dz\,lz}{1-zz} = +\frac{\pi\pi}{8}$$

II°. fi $\lambda = 3$ habebimus

1. $\int \frac{-z(1+zz)dz\,lz}{1+z^6} = \frac{\pi\pi}{54}$, et

2. $\int \frac{-z(1+zz)dz\,lz}{1-z^6} = \int \frac{-z\,dz\,lz}{1-zz+z^4} = \frac{\pi\pi}{27}$.

Hae autem duae formulae ponendo $zz = v$ abibunt in sequentes

1. $\int \frac{-dv(1-v)lv}{1+v^3} = \frac{2\pi\pi}{27}$ etc.

2. $\int \frac{dv\,lv}{1-v+vv} = \frac{4\pi\pi}{27}$.

III°. Sit $\lambda = 4$ et consequemur

$$1°.\ \int \frac{-zz(1-zz)dz\,lz}{1+z^8} = \frac{\pi\pi\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{16(2+\sqrt{2})} = \frac{\pi\pi\sqrt{2-\sqrt{2}}}{32(2+\sqrt{2})} \text{ et}$$

$$2°.\ \int \frac{-zz(1+zz)dz\,lz}{1-z^8} = \int \frac{-zz\,dz\,lz}{(1-zz)(1+z^4)} = \frac{\pi\pi}{16(2+\sqrt{2})}$$

quae postrema forma reducitur ad hanc

$$\int -\frac{dz\,lz}{1-zz} + \int \frac{(1-zz)dz\,lz}{1+z^4} = \frac{\pi\pi}{8(2+\sqrt{2})}$$

est vero $\int \frac{-dz\,lz}{1-zz} = \frac{\pi\pi}{8}$ vnde reperitur

$$\int \frac{dz\,lz(1-zz)}{1+z^4} = -\frac{\pi\pi(1+\sqrt{2})}{8(2+\sqrt{2})} = -\frac{\pi\pi}{8\sqrt{2}}$$

qui valor iam in superiori casu $\lambda = 2$ est inuentus.

§. 13. Nihil autem impedit, quo minus etiam faciamus $\lambda = 1$, dummodo integralia ita capiantur vt euanescant, posito $z = 0$, tum autem reperiemus

1°. $\int \frac{-(1-zz)dz\,lz}{z(1+zz)} = \infty$ et

2°. $\int \frac{-(1+zz)dz\,lz}{z(1-zz)} = \infty$

vnde hinc nihil concludere licet. Ceterum etiam nostrae series supra inuentae manifesto declarant, earum

rum summas esse infinitas, quandoquidem primus terminus vtriusque $\frac{1}{(\lambda-\omega)^2}$ fit infinitus, sumto vti fecimus $\lambda = 1$ et $\omega = 1$.

§. 14. His casibus euolutis, vlterius progrediamur ac ponamus formulas integrales inuentas

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\frac{dz}{z}lz = S' \text{ et}$$

$$\int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\frac{dz}{z}lz = T'$$

ita vt fit

$$S' = \frac{\pi\pi \,\text{fin.}\, \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \,\text{cof.}\, \frac{\pi\omega^2}{2\lambda}}, \text{ et } T' = \frac{\pi\pi}{4\lambda\lambda \,\text{cof.}\, \frac{\pi\omega^2}{2\lambda}}$$

atque vt ante iam differentiemus solo numero ω pro variabili habito; quo facto sequentes nanciscimur integrationes

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\frac{dz}{z}(lz)^2 = \left(\frac{dS'}{d\omega}\right), \text{ et}$$

$$\int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\frac{dz}{z}(lz)^2 = \left(\frac{dT'}{d\omega}\right).$$

Hunc in finem ponamus breuitatis ergo angulum $\frac{\pi\omega}{2\lambda} = \Phi$ vt fit

$$S' = \frac{\pi\pi \,\text{fin.}\, \Phi}{4\lambda\lambda \,\text{cof.}\, \Phi^2} = \frac{\pi\pi}{4\lambda\lambda}\cdot\frac{\text{fin.}\, \Phi}{\text{cof.}\, \Phi^2} \text{ et}$$

$$T' = \frac{\pi\pi}{4\lambda\lambda}\cdot\frac{1}{\text{cof.}\, \Phi^2}, \text{ ac reperiemus}$$

$$d.\frac{\text{fin.}\, \Phi}{\text{cof.}\, \Phi^2} = \frac{\text{cof.}\, \Phi^2 + 2\,\text{fin.}\, \Phi^2}{\text{cof.}\, \Phi^3}d\Phi = \frac{1+\text{fin.}\, \Phi^2}{\text{cof.}\, \Phi^3}d\Phi$$

vbi est $d\Phi = \frac{\pi d\omega}{2\lambda}$; vnde colligimus

$$\left(\frac{dS'}{d\omega}\right)=\frac{\pi^3}{8\lambda^3}\left(\frac{1+\sin.\frac{\pi\omega^2}{2\lambda}}{\cos.\frac{\pi\omega^3}{2\lambda}}\right)=\frac{\pi^3}{8\lambda^3}\left(\frac{2}{\cos.\frac{\pi\omega^3}{2\lambda}}-\frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right)$$

ſimili modo ob $T'=\frac{\pi\pi}{4\lambda\lambda}\cdot\frac{1}{\cos.\Phi^2}$, erit

$$d.\frac{1}{\cos.\Phi^2}=\frac{2\,d\Phi\sin.\Phi}{\cos.\Phi^3},$$ hincque

$$\left(\frac{dT'}{d\omega}\right)=\frac{\pi^3}{8\lambda^3}\cdot\frac{2\sin.\frac{\pi\omega}{2\lambda}}{\cos.\frac{\pi\omega^3}{2\lambda}}.$$

conſequenter integrationes hinc natae erunt

$$\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\frac{dz}{z}(lz)^2=\frac{\pi^3}{8\lambda^3}\left(\frac{2}{\cos.\frac{\pi\omega^3}{2\pi}}-\frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right)$$

$$\int\frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\frac{dz}{z}(lz)^2=\frac{\pi^3}{8\lambda^3}\frac{2\sin.\frac{\pi\omega}{2\lambda}}{\cos.\frac{\pi\omega^3}{2\lambda}}.$$

§. 15. Si iam eodem modo ſeries §. 8. inventas denuo differentiemus, ſumta ſola ω variabili, perueniamus ad ſequentes ſummationes

$$\frac{\pi^3}{8\lambda^3}\left\{\frac{2}{\cos.\frac{\pi\omega^3}{2\lambda}}-\frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right\}=+\frac{2}{(\lambda-\omega)^3}+\frac{2}{(\lambda+\omega)^3}-\frac{2}{(3\lambda-\omega)^3}-\frac{2}{(3\lambda+\omega)^3}$$
$$+\frac{2}{(5\lambda-\omega)^3}+\frac{2}{(5\lambda+\omega)^3}-\text{etc.}$$

$$\frac{\pi^3}{8\lambda^3}\cdot\frac{2\sin.\frac{\pi\omega}{2\lambda}}{\cos.\frac{\pi\omega^3}{2\lambda}}=\frac{2}{(\lambda-\omega)^3}-\frac{2}{(\lambda+\omega)^3}+\frac{2}{(3\lambda-\omega)^3}-\frac{2}{(3\lambda+\omega)^3}+\frac{2}{(5\lambda-\omega)^3}-\text{etc.}$$

§. 16. Si iam hic ſumamus $\omega=0$ et $\lambda=1$, prior integratio hanc induit formam

$$\int\frac{2\,dz(lz)^2}{1+zz}=\frac{\pi^3}{8}=\frac{2}{1^3}+\frac{2}{1^3}-\frac{2}{3^3}-\frac{2}{3^3}+\frac{2}{5^3}+\frac{2}{5^3}-\frac{2}{7^3}-\frac{2}{7^3}+\text{etc.}$$

ita vt ſit

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}=\frac{\pi^3}{32}$$

quem-

quemadmodum iam dudum demonſtraui. Altera autem integratio hoc caſu in nihilum abit. Ex priori vero integrali

$$\int \frac{dz\, lz^2}{1+zz} = \frac{\pi^3}{16},$$

alia deriuare non licet, vti ſupra fecimus ex formula

$$\int \frac{dz\, lz}{1-zz} = -\frac{\pi\pi}{8},$$

propterea quod hic denominator $1+zz$ non habet factores reales.

§. 17. Sumamus igitur $\lambda = 2$ et $\omega = 1$, ac prior integratio dabit

$$\int \frac{(1+zz)\,dz\,(lz)^2}{1+z^4} = \frac{3\pi^3}{32\sqrt{2}};$$

ſeries autem hinc nata erit

$$\frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \text{etc.},$$ ita vt ſit

$$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} = \frac{3\pi^3}{64\sqrt{2}}$$

quae ſuperiori addita praebet

$$\frac{1}{1^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} = \frac{\pi^3(1+2\sqrt{2})}{128\sqrt{2}};$$

Altera vero integratio hoc caſu dat

$$\int \frac{dz\,(lz)^2}{1+zz} = \frac{\pi^3}{16}$$

quae cum paragrapho praecedenti perfecte congruit, quemadmodum etiam ſeries hinc nata eſt

$$\frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} \text{ etc.}$$

§. 18. Quo autem facilius ſequentes integrationes per continuam differentiationem elicere valea-

 mus,

mus, eas in genere repraeſentemus: et cum pro priore ſit

$$S = \frac{\pi}{2\lambda \cos \frac{\pi\omega}{2\lambda}}$$

integrationes hinc ortae ita ordine procedent:

I. $\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \frac{dz}{z} = S$

II. $\int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \frac{dz}{z} lz = \left(\frac{dS}{d\omega}\right)$

III. $\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \frac{dz}{z} (lz)^2 = \left(\frac{ddS}{d\omega^2}\right)$

IV. $\int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \frac{dz}{z} (lz)^3 = \left(\frac{d^3S}{d\omega^3}\right)$

V. $\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \frac{dz}{z} (lz)^4 = \left(\frac{d^4S}{d\omega^4}\right)$

VI. $\int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \frac{dz}{z} (lz)^5 = \left(\frac{d^5S}{d\omega^5}\right)$

VII. $\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \frac{dz}{z} (lz)^6 = \left(\frac{d^6S}{d\omega^6}\right)$

etc. etc. etc.

§. 19. Pro his differentiationibus continuis ω facilius abſoluendis, ponamus breuitatis ergo $\frac{\pi}{2\lambda} = \alpha$ vt ſit

$$S = \frac{\alpha}{\cos \alpha\omega};$$

tum vero ſit

$$\sin \alpha\omega = p \text{ et } \cos \alpha\omega = q,$$

erit-

eritque

$$dp = \alpha q\, d\omega \quad \text{et} \quad dq = -\alpha p\, d\omega;$$

Praeterea vero notetur esse

$$d.\, \frac{p^n}{q^{n+1}} = \alpha\, d\omega \left\{ \frac{n\, p^{n-1}}{q^n} + \frac{(n+1)\, p^{n+1}}{q^{n+2}} \right\}.$$

His praemissis ob $S = \alpha.\, \frac{1}{q}$ erit

$\left(\frac{dS}{d\omega}\right) = \alpha\alpha.\, \frac{p}{qq}$, deinde

$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{1}{q} + \frac{2pp}{q^3}\right)$, porro

$$\left(\frac{d^3 S}{d\omega^3}\right) = \alpha^4 \left(\frac{5p}{qq} + \frac{6p^3}{q^4}\right)$$

$$\left(\frac{d^4 S}{d\omega^4}\right) = \alpha^5 \left(\frac{5}{q} + \frac{28pp}{q^3} + \frac{24p^4}{q^5}\right)$$

$$\left(\frac{d^5 S}{d\omega^5}\right) = \alpha^6 \left(\frac{61p}{qq} + \frac{180p^3}{q^4} + \frac{120p^5}{q^6}\right)$$

$$\left(\frac{d^6 S}{d\omega^6}\right) = \alpha^7 \left(\frac{61}{q} + \frac{662pp}{q^3} + \frac{1320p^4}{q^5} + \frac{720p^6}{q^7}\right)$$

$$\left(\frac{d^7 S}{d\omega^7}\right) = \alpha^8 \left(\frac{1385p}{qq} + \frac{7266p^3}{q^4} + \frac{10920p^5}{q^6} + \frac{5040p^7}{q^8}\right)$$

hi autem valores ob $pp = 1 - qq$ ad sequentes reducuntur:

$$S = \alpha.\, \frac{1}{q}$$

$$\left(\frac{dS}{d\omega}\right) = \alpha\alpha p.\, \frac{1}{qq}$$

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{1.2}{q^3} - \frac{1}{q}\right)$$

$$\left(\frac{d^3 S}{d\omega^3}\right) = \alpha^4 p \left(\frac{1.2.3}{q^4} - \frac{1}{qq}\right)$$

$$\left(\frac{d^4 S}{d\omega^4}\right) = \alpha^5 \left(\frac{1.2.3.4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right)$$

$$\left(\frac{d^5 S}{d\omega^5}\right) = \alpha^6 p \left(\frac{1.2.3.4.5}{q^6} - \frac{60}{q^4} + \frac{1}{qq}\right)$$

$$\left(\frac{d^6 S}{d\omega^6}\right) = \alpha^7 \left(\frac{1 \ldots 6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right).$$

§. 20.

§. 20. Has posteriores formas reperire licet ope horum duorum lemmatum

$$\text{I. } d.\frac{1}{q^{n+1}} = \alpha\, d\omega \frac{(n+1)p}{q^{n+2}}; \quad \text{II. } d.\frac{p}{q^{n+1}} = \alpha\, d\omega \left\{\frac{n+1}{q^{n+2}} - \frac{n}{q^{n}}\right\}$$

hinc enim reperiemus

$$S = \alpha \tfrac{1}{q}$$

$$\left(\frac{dS}{d\omega}\right) = \alpha\,\alpha.\,\frac{p}{qq}$$

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3\left(\frac{2}{q^3} - \frac{1}{q}\right)$$

$$\left(\frac{d^3S}{d\omega^3}\right) = \alpha^4\left(\frac{2\cdot3\,p}{q^4} - \frac{p}{qq}\right)$$

$$\left(\frac{d^4S}{d\omega^4}\right) = \alpha^5\left(\frac{2\cdot3\cdot4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right)$$

$$\left(\frac{d^5S}{d\omega^5}\right) = \alpha^6\left(\frac{2\cdot3\cdot4\cdot5\,p}{q^6} - \frac{3\cdot20\,p}{q^4} + \frac{p}{qq}\right)$$

$$\left(\frac{d^6S}{d\omega^6}\right) = \alpha^7\left(\frac{2\cdot3\cdot4\cdot5\cdot6\,p}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right)$$

$$\left(\frac{d^7S}{d\omega^7}\right) = \alpha^8\left(\frac{2\cdots7\,q}{q^8} - \frac{5\cdot840}{q^6} + \frac{3\cdot182}{q^4} - \frac{p}{qq}\right).$$

§. 21. Ipsae autem series his formulis respondentes

$$S = \frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.}$$

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \frac{1}{(5\lambda+\omega)^2} - \text{etc.}$$

$$\left(\frac{ddS}{d\omega^2}\right) = \frac{1\cdot2}{(\lambda-\omega)^3} + \frac{1\cdot2}{(\lambda+\omega)^3} - \frac{1\cdot2}{(3\lambda-\omega)^3} - \frac{1\cdot2}{(3\lambda+\omega)^3} + \frac{1\cdot2}{(5\lambda-\omega)^3} + \text{etc.}$$

$$\left(\frac{d^3S}{d\omega^3}\right) = \frac{1\cdot2\cdot3}{(\lambda-\omega)^4} - \frac{1\cdot2\cdot3}{(\lambda+\omega)^4} - \frac{1\cdot2\cdot3}{(3\lambda-\omega)^4} + \frac{1\cdot2\cdot3}{(3\lambda+\omega)^4} + \frac{1\cdot2\cdot3}{(5\lambda-\omega)^4} - \text{etc.}$$

$$\left(\frac{d^4S}{d\omega^5}\right) = \frac{1\cdot2\cdot3\cdot4}{(\lambda-\omega)^5} + \frac{1\cdot2\cdot3\cdot4}{(\lambda+\omega)^5} - \frac{1\cdot2\cdot3\cdot4}{(3\lambda-\omega)^5} - \frac{1\cdot2\cdot3\cdot4}{(3\lambda+\omega)^5} + \frac{1\cdot2\cdot3\cdot4}{(5\lambda-\omega)^5} + \text{etc.}$$

$$\left(\frac{d^5S}{d\omega^5}\right) = \frac{1\cdot2\cdot3\cdot4\cdot5}{(\lambda-\omega)^6} - \frac{1\cdot2\cdot3\cdot4\cdot5}{(\lambda+\omega)^6} - \frac{1\cdot2\cdot3\cdot4\cdot5}{(3\lambda-\omega)^6} + \frac{1\cdot2\cdot3\cdot4\cdot5}{(3\lambda+\omega)^6} + \frac{1\cdot2\cdot3\cdot4\cdot5}{(5\lambda-\omega)^6} - \text{etc.}$$

$$\left(\frac{d^6S}{d\omega^6}\right) = \frac{1\cdots6}{(\lambda-\omega)^7} + \frac{1\cdots6}{(\lambda+\omega)^7} - \frac{1\cdots6}{(3\lambda-\omega)^7} - \frac{1\cdots6}{(3\lambda+\omega)^7} + \frac{1\cdots6}{(5\lambda-\omega)^7} - \text{etc.}$$

$$\left(\frac{d^7S}{d\omega^7}\right) = \frac{1\cdots7}{(\lambda-\omega)^8} - \frac{1\cdots7}{(\lambda+\omega)^8} - \frac{1\cdots7}{(3\lambda-\omega)^8} + \frac{1\cdots7}{(3\lambda+\omega)^8} + \frac{1\cdots7}{(5\lambda-\omega)^8} - \text{etc.}$$

etc. etc. etc.

Circa

Circa hos autem valores probe meminiſſe opportet, eſſe

$$\alpha = \frac{\pi}{2\lambda},\ p = \text{ſin.}\,\alpha\omega = \text{ſin.}\frac{\pi\omega}{2\lambda} \text{ et } q = \text{coſ.}\,\alpha\omega = \text{coſ.}\frac{\pi\omega}{2\lambda}.$$

§. 22. Eodem modo expediamus valores ſeu formulas integrales alterius generis, pro quibus eſt

$$T = \frac{\pi}{2\lambda}\,\text{tang.}\,\frac{\pi\omega}{2\lambda}$$

vnde continuo differentiando oriuntur ſequentes integrationes

$$\text{I. } \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \frac{dz}{z} = T$$

$$\text{II. } \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \frac{dz}{z}\, lz = \left(\frac{dT}{d\omega}\right)$$

$$\text{III. } \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \frac{dz}{z} (lz)^2 = \left(\frac{ddT}{d\omega^2}\right)$$

$$\text{IV. } \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \frac{dz}{z} (lz)^3 = \left(\frac{d^3T}{d\omega^3}\right)$$

$$\text{V. } \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \frac{dz}{z} (lz)^4 = \left(\frac{d^4T}{d\omega^4}\right)$$

$$\text{VI. } \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \frac{dz}{z} (lz)^5 = \left(\frac{d^5T}{d\omega^5}\right)$$

$$\text{VII. } \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \frac{dz}{z} (lz)^6 = \left(\frac{d^6T}{d\omega^6}\right).$$

§. 23. Ponatur iterum $\frac{\pi}{2\lambda} = \alpha$; ſin. $\alpha\omega = p$, et coſ $\alpha\omega = q$, vt ſit

$$T = \frac{\alpha p}{q},$$

quae

quae formula secundum lemmata §. 20. continuo differentiata dabit

$$T = \alpha . \frac{p}{q}$$

$$\left(\frac{dT}{d\omega}\right) = \alpha \alpha . \frac{1}{qq}$$

$$\left(\frac{ddT}{d\omega^2}\right) = \alpha^3 \frac{2p}{q^3}$$

$$\left(\frac{d^3T}{d\omega^3}\right) = \alpha^4 \left(\frac{6}{q^4} - \frac{4}{qq}\right)$$

$$\left(\frac{d^4T}{d\omega^4}\right) = \alpha^5 \left(\frac{24p}{q^5} - \frac{8p}{q^3}\right)$$

$$\left(\frac{d^5T}{d\omega^5}\right) = \alpha^6 \left(\frac{120}{q^6} - \frac{120}{q^4} + \frac{16}{qq}\right)$$

$$\left(\frac{d^6T}{d\omega^6}\right) = \alpha^7 \left(\frac{720p}{q^7} - \frac{480p}{q^5} + \frac{32p}{q^3}\right)$$

$$\left(\frac{d^7T}{d\omega^7}\right) = \alpha^8 \left(\frac{5040}{q^8} - \frac{6720}{q^6} + \frac{2016}{q^4} - \frac{64}{qq}\right).$$

§. 24. Series autem infinitae, quae hinc nascuntur, erunt

$$T = \frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.}$$

$$\left(\frac{dT}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} + \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \text{etc.}$$

$$\left(\frac{ddT}{d\omega^2}\right) = \frac{1.2}{(\lambda-\omega)^3} - \frac{1.2}{(\lambda+\omega)^3} + \frac{1.2}{(3\lambda-\omega)^3} - \frac{1.2}{(3\lambda+\omega)^3} + \frac{1.2}{(5\lambda-\omega)^3} - \text{etc.}$$

$$\left(\frac{d^3T}{d\omega^3}\right) = \frac{1.2.3}{(\lambda-\omega)^4} + \frac{1.2.3}{(\lambda+\omega)^4} + \frac{1.2.3}{(3\lambda-\omega)^4} + \frac{1.2.3}{(3\lambda+\omega)^4} + \frac{1.2.3}{(5\lambda-\omega)^4} + \text{etc.}$$

$$\left(\frac{d^4T}{d\omega^4}\right) = \frac{1.2.3.4}{(\lambda-\omega)^5} - \frac{1.2.3.4}{(\lambda+\omega)^5} + \frac{1.2.3.4}{(3\lambda-\omega)^5} - \frac{1.2.3.4}{(3\lambda+\omega)^5} + \frac{1.2.3.4}{(5\lambda-\omega)^5} - \text{etc.}$$

$$\left(\frac{d^5T}{d\omega^5}\right) = \frac{1.2.3.4.5}{(\lambda-\omega)^6} + \frac{1.2.3.4.5}{(\lambda+\omega)^6} + \frac{1.2.3.4.5}{(3\lambda-\omega)^6} + \frac{1.2.3.4.5}{(3\lambda+\omega)^6} + \frac{1.2.3.4.5}{(5\lambda-\omega)^6} + \text{etc.}$$

$$\left(\frac{d^6T}{d\omega^6}\right) = \frac{1\ldots6}{(\lambda-\omega)^7} - \frac{1\ldots6}{(\lambda+\omega)^7} + \frac{1\ldots6}{(3\lambda-\omega)^7} - \frac{1\ldots6}{(3\lambda+\omega)^7} + \frac{1\ldots6}{(5\lambda-\omega)^7} - \frac{1\ldots6}{(5\lambda+\omega)^7} + \text{etc.}$$

§. 25. Operae pretium erit, hinc casus simplicissimos euoluere, qui oriuntur ponendo $\lambda = 1$ et $\omega = 0$, ita vt sit $\alpha = \frac{\pi}{2}$, $p = 0$ et $q = 1$, vnde habebimus

Pro

Pro ordine priore	Pro ordine posteriore
$S = \frac{\pi}{2}$	$T = 0$
$\left(\frac{dS}{d\omega}\right) = 0$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{4}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{\pi^3}{8}$	$\left(\frac{ddT}{d\omega^2}\right) = 0$
$\left(\frac{d^3S}{d\omega^3}\right) = 0$	$\left(\frac{d^3T}{d\omega^2}\right) = \frac{\pi^4}{8}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{5\pi^5}{32}$	$\left(\frac{d^4T}{d\omega^4}\right) = 0$
$\left(\frac{d^5S}{d\omega^5}\right) = 0$	$\left(\frac{d^5T}{d\omega^5}\right) = \frac{\pi^6}{4}$
$\left(\frac{d^6S}{d\omega^6}\right) = \frac{61\pi^7}{128}$	$\left(\frac{d^6T}{d\omega^6}\right) = 0$
$\left(\frac{d^7S}{d\omega^7}\right) = 0$	$\left(\frac{d^7T}{d\omega^7}\right) = \frac{17\pi^8}{16}$
etc.	etc.

§. 26. Hinc ergo, omissis valoribus euanescentibus, ex priore ordine habebimus sequentes formulas integrales cum seriebus inde natis

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \text{etc.}$$

$$\int \frac{dz(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} + \text{etc.}$$

$$\int \frac{dz(lz)^6}{1+zz} = \frac{61\pi^7}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \text{etc.}$$

etc. etc. etc.

§. 27. Ex altero autem ordine pro eodem casu oriuntur

$$\int \frac{-dz\,lz}{1-zz} = \frac{\pi\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

$$\int \frac{-dz(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.}$$

$$\int \frac{-dz(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \frac{120}{13^6} + \text{etc.}$$

etc. etc. etc.

§. 28. Quemadmodum ex primo integrali ordinis posterioris deduximus has formulas

$$\int \frac{dz\, lz}{1-z} = -\frac{\pi\pi}{6} \quad \text{et} \quad \int \frac{dz\, lz}{1+z} = -\frac{\pi\pi}{12};$$

similes quoque formulae integrales ex sequentibus deduci possunt; cum enim sit

$$\int \frac{dz\,(lz)^3}{1-zz} = -\frac{\pi^4}{16},$$

ponamus esse

$$\int \frac{z\,dz\,(lz)^3}{1-zz} = P, \quad \text{eritque} \quad \int \frac{dz\,(lz)^3}{1-z} = P - \frac{\pi^4}{16}$$

$$\text{et} \quad \int \frac{dz\,(lz)^3}{1+z} = -P - \frac{\pi^4}{16}$$

nunc vero statuatur $zz = v$, vt sit $z\,dz = \frac{1}{2}dv$, et $lz = \frac{1}{2}lv$, ideoque $lz^3 = \frac{1}{8}lv^3$, quibus substitutis erit

$$P = \frac{1}{16}\int \frac{dv\,(lv)^3}{1-v} = \frac{1}{16}\left(P - \frac{\pi^4}{16}\right),$$

vnde fit

$$16P = P - \frac{\pi^4}{16} \quad \text{ideoque} \quad P = -\frac{\pi^4}{240}$$

sicque has duas habebimus integrationes nouas

$$\int \frac{dz\,(lz)^4}{1-z} = -\frac{\pi^4}{15}, \quad \text{et}$$

$$\int \frac{dz\,(lz)^3}{1+z} = -\frac{7\pi^4}{120}$$

hinc autem per series erit,

$$\int \frac{-dz\,(lz)^3}{1-z} = +\frac{\pi^4}{15} = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \text{etc.}\right) \text{ et}$$

$$\int \frac{-dz\,(lz)^3}{1+z} = +\frac{7\pi^4}{120} = 6\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{8^4} + \text{etc.}\right.$$

§. 29. Porro $\int \frac{dz\,(lz)^5}{1-zz} = -\frac{\pi^6}{8}$ ponamus esse

$$\int \frac{z\,dz\,(lz)^5}{1-zz} = P,$$

vt

vt hinc obtineamus

$$\int \frac{dz\,(lz)^5}{1-z} = P - \frac{\pi^5}{8}, \text{ et } \int \frac{dz\,(lz)^5}{1+z} = -P - \frac{\pi^6}{4}$$

nunc igitur ſtatuamus $zz = v$, eritque

$$P = \frac{1}{64}\int \frac{dv\,(lv)^5}{1-v} = \frac{1}{64}\left(P - \frac{\pi^6}{4}\right), \text{ vnde fit}$$

$$P = -\frac{\pi^6}{504},$$

nonaeque integrationes hinc deductae ſunt

$$\int \frac{dz\,(lz)^5}{1-z} = -\frac{8\pi^6}{63}, \text{ et}$$

$$\int \frac{dz\,(lz)^5}{1+z} = -\frac{31\pi^6}{252}$$

at vero per ſeries reperitur

$$\int \frac{dz\,(lz)^5}{1-z} = -\frac{8\pi^6}{63} = -120\left(1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \text{etc.}\right.$$

et

$$\int \frac{dz\,(lz)^5}{1+z} = -\frac{31\pi^6}{252} = -120\left(1 - \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \text{etc.}\right.$$

ita vt ſit

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \text{etc.} = \frac{\pi^6}{945} \text{ et}$$

$$1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \frac{1}{7^6} - \frac{1}{8^6} + \text{etc.} = \frac{31\pi^6}{30240} = \frac{31\pi^6}{32\cdot 945}.$$

§. 30. Conſideremus etiam caſus, quibus $\lambda = 2$ et $\omega = 1$ ita vt ſit $\alpha = \frac{\pi}{4}$, et $\alpha\omega = \frac{\pi}{4}$ hinc $p = q = \frac{1}{\sqrt{2}}$, vnde pro vtroque ordine ſequentes habebimus valores

Pro ordine priore	Pro ordine posteriore
$S = \frac{\pi}{2\sqrt{2}}$	$T = \frac{\pi}{4}$
$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi}{8\sqrt{2}}$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{8}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{3\pi^3}{32\sqrt{2}}$	$\left(\frac{ddT}{d\omega^2}\right) = \frac{\pi^3}{16}$
$\left(\frac{d^3S}{d\omega^3}\right) = \frac{11\pi^4}{128\sqrt{2}}$	$\left(\frac{d^3T}{d\omega^3}\right) = \frac{\pi^4}{16}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{57\pi^5}{512\sqrt{2}}$	$\left(\frac{d^4T}{d\omega^4}\right) = \frac{5\pi^5}{64}$
$\left(\frac{d^5S}{d\omega^5}\right) = \frac{361\pi^6}{2048\sqrt{2}}$	$\left(\frac{d^5T}{d\omega^5}\right) = \frac{\pi^6}{8}$
$\left(\frac{d^6S}{d\omega^6}\right) = \frac{2763\pi^7}{8192\sqrt{2}}$	$\left(\frac{d^6T}{d\omega^6}\right) = \frac{61\pi^7}{256}$
$\left(\frac{d^7S}{d\omega^7}\right) = \frac{24611\pi^8}{32768\sqrt{2}}$	$\left(\frac{d^7T}{d\omega^7}\right) = \frac{17\pi^8}{32}$
etc.	etc.

§. 31. Hinc igitur ſequentes integrationes, cum ſeriebus reſpondentibus reſultant; ac primo quidem ex ordine primo

$$\int \frac{(1+zz)dz}{1+z^4} = \frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \text{etc.}$$

$$\int \frac{-(1-zz)dz\,lz}{1+z^4} = \frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \text{etc.}$$

$$\int \frac{(1+zz)dz(lz)^2}{1+z^4} = \frac{3\pi^3}{32\sqrt{2}} = \frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \frac{2}{13^3} - \text{etc.}$$

$$\int \frac{-(1-zz)dz(lz)^3}{1+z^4} = \frac{11\pi^4}{128\sqrt{2}} = \frac{6}{1^4} - \frac{6}{3^4} - \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} - \frac{6}{11^4} - \frac{6}{13^4} + \text{etc.}$$

$$\int \frac{(1+zz)dz(lz)^4}{1+z^4} = \frac{57\pi^5}{512\sqrt{2}} = \frac{24}{1^5} + \frac{24}{3^5} - \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} + \frac{24}{11^5} - \frac{24}{13^5} - \text{etc.}$$

$$\int \frac{-(1-zz)dz(lz)^5}{1+z^4} = \frac{361\pi^6}{2048\sqrt{2}} = \frac{120}{1^6} - \frac{120}{3^6} - \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} - \frac{120}{11^6} - \frac{120}{13^6} + \text{etc.}$$

$$\int \frac{(1+zz)dz(lz)^6}{1+z^4} = \frac{2763\pi^7}{8192\sqrt{2}} = \frac{720}{1^7} + \frac{720}{3^7} - \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} + \frac{720}{11^7} - \frac{720}{13^7} - \text{etc.}$$

$$\int \frac{-(1-zz)dz(lz)^7}{1+z^4} = \frac{24611\pi^8}{32768\sqrt{2}} = \frac{5040}{1^8} - \frac{5040}{3^8} - \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} - \frac{5040}{11^8} - \frac{5040}{13^8} + \text{etc.}$$

etc.

§. 32.

§. 32 Eodem modo integrationes alterius ordinis cum seriebus erunt

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.}$$

$$\int \frac{-dz\,lz}{1-zz} = \frac{\pi\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

$$\int \frac{dz\,(lz)^2}{1+zz} = \frac{\pi^3}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.}$$

$$\int \frac{-dz\,(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.}$$

$$\int \frac{dz\,(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} + \frac{24}{13^5} - \text{etc.}$$

$$\int \frac{-dz\,(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \text{etc.}$$

$$\int \frac{dz\,(lz)^6}{1+zz} = \frac{61\pi^7}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \frac{720}{11^7} + \text{etc.}$$

$$\int \frac{-dz\,(lz)^7}{1-zz} = \frac{17\pi^8}{32} = \frac{5040}{1^8} + \frac{5040}{3^8} + \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} + \frac{5040}{11^8} + \text{etc.}$$

etc.

Hae autem series sunt eae ipsae, quas iam supra §§. 26 et 27. sumus consecuti.

§. 33. Praeterea autem ii casus imprimis notari merentur, quibus formulae integrales in formas simpliciores resolui possunt. Haec autem resolutio tantum spectat ad fractionem

$$\pm \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^n},$$

omisso factore $\frac{dz}{z}(lz)^\mu$; ad quod ostendendum sumamus primo $\lambda = 3$ et $\omega = 1$, vnde fit $\alpha = \frac{\pi}{6}$, $p = \sin.\frac{\pi}{4}$, et $q = \cos.\frac{\pi}{6}$, tum autem, in priori ordine occurrunt alternatim sequentes fractiones

1. $\frac{zz(1+zz)}{1+z^6} = \frac{zz}{1-zz+z^4}$,

quae poſito $zz = v$ abit in $\frac{v}{1 - v + vv}$ ergo cum ſit

$$\frac{dz}{z} = \frac{1}{2}\frac{dv}{v} \text{ et } lz = \frac{1}{2} lv,$$

hinc talis forma

$$\frac{1}{2^{2i+1}} \int \frac{dv\,(lv)^{2i}}{1 - v + vv}$$

integrari poterit caſu ſcilicet $v = 1$.

II. $-\frac{zz(1 - zz)}{1 + z^6} = +\frac{2}{3(1 + zz)} - \frac{(2 - zz)}{3(1 - zz + z^4)}$,

quae poſito $zz = v$ abit in $\frac{-2}{3(1 + v)} + \frac{2 - v}{3(1 - v + vv)}$

quae ergo forma ducta in $\frac{dz}{z}(lz)^{2i+1}$ vel in

$$\frac{1}{2^{2i+2}} \frac{dv}{v} (lv)^{2i+1}$$

ſemper integrari poteſt poſito $v = 1$.

§. 34. Eodem caſu ordo poſterior ſequentes ſuppeditat reſolutiones

I. $\frac{zz(1 - zz)}{1 - z^6} = \frac{zz}{1 + zz + z^4} = \frac{v}{1 + v + vv}$,

quae in $\frac{dz}{z}(lz)^{2i}$ vel in $\frac{1}{2^{2i+1}} \frac{dv}{v} (lz)^{2i}$ ducta ſemper eſt integrabilis

II. $\frac{-zz(1 + zz)}{1 - z^5} = \frac{-2}{3(1 - zz)} + \frac{2 + zz}{3(1 + zz + z^4)}$,

quae facto $zz = v$ fit

$$\frac{-2}{3(1 - v)} + \frac{2 + v}{3(1 + v + vv)}$$

quae ergo formulae in $\frac{dv}{v}(lv)^{2i+1}$ ductae fiunt integrabiles; quia autem in hac reſolutione numeratores per z vel v diuidere non licet, alia reſolutione eſt opus, quae reperitur

$zz-$

II. $\frac{-zz(1+zz)}{1-z^6} = \frac{-2zz}{3(1-zz)} - \frac{zz(1+2zz)}{3(1+zz+z^4)}$ fiue $\frac{-2v}{3(1-v)} \; \frac{-v(1+2v)}{3(1+v+vv)}$

quae formulae ductae in $\frac{dz}{z}(lz)^{2i+1}$ vel in $\frac{1}{2^{2i+2}}\frac{dv}{v}(lv)^{2i+1}$ integrationem quoque admittunt.

§. 35. Porro manente $\lambda = 3$ fumatur $\omega = 2$, vt fit $\alpha = \frac{\pi}{6}$; $p = \text{fin.}\,\frac{2\pi}{3}$ et $q = \text{cof.}\,\frac{\pi}{3}$ et ex ordine priore orientur fequentes reductiones.

I°. $\frac{z(1+z^4)}{1+z^6} = \frac{2z}{3(1+zz)} + \frac{z(1+zz)}{3(1-zz+z^4)}$,

vnde multiplicando per $\frac{dz}{z}lz^{2i}$ oriuntur formulae integrationem admittentes cafu $z = 1$.

II. $\frac{-z(1-z^4)}{1+z^6} = -\frac{z(1-zz)}{1-zz+z^4}$

quae per $\frac{dz}{z}(lz)^{2i+1}$ multiplicata integrari poterit cafu $z = 1$; ex ordine vero pofteriori fequentes prodibunt reductiones.

I. $\frac{z(1-z^4)}{1-z^6} = \frac{z(1+zz)}{1+zz+z^4}$,

quae ducta in $\frac{dz}{z}(lz)^{2i}$ fit integrabilis.

II. $\frac{-z(1+z^4)}{1-z^6} = \frac{-2z}{3(1-zz)} - \frac{z(1-zz)}{3(1+zz+z^4)}$,

quae formulae in $\frac{dz}{z}(lz)^{2i+1}$ ductae fiunt integrabiles.

§. 36. Operae iam erit pretium haec integralia actu euoluere, quare ex §. 33. eiusque numero I nancifcimur fequentes integrationes

1°. $\frac{1}{2}\int\frac{dv}{1-v+vv} = \alpha\frac{1}{q} = \frac{\pi}{3\sqrt{3}}$

2°. $\frac{1}{8}\int\frac{dv(lv)^2}{1-v+vv} = \alpha^3\left(\frac{2}{q^3} - \frac{1}{q}\right) = \frac{5\pi^3}{324\sqrt{3}}$

deinde

deinde vero ex eiusdem § numero II vbi etiam haec reductio locum habet

$$-\frac{zz(1-zz)}{1+z^6} = -\frac{2zz}{3(1+zz)} - \frac{zz(1-2zz)}{3(1-zz+z^4)} = -\frac{2v}{3(1+v)} - \frac{v(1-2v)}{3(1-v+vv)}$$

quae ducta in $\frac{1}{4}\frac{dv}{v}lv$ dabit

$$-\frac{1}{6}\int\frac{dv\,lv}{1+v} - \frac{1}{12}\int\frac{dv(1-2v)\,lv}{1-v+vv} = \alpha\alpha\frac{p}{qq} = \frac{\pi\pi}{54}$$

quarum formularum prior integrationem admittit, est enim

$$\int\frac{dv\,lv}{1+v} = -\frac{\pi\pi}{12}$$

vnde inuenitur posterior

$$\int\frac{dv(1-2v)\,lv}{1-v+vv} = -\frac{\pi\pi}{18}.$$

§. 37. Ex §. 34 eiusque numero I sequitur

$$1^{\circ}.\ \frac{1}{3}\int\frac{dv}{1+v+vv} = \frac{\alpha p}{q} = \frac{\pi}{6\sqrt{3}}$$

$$2^{\circ}.\ \frac{1}{3}\int\frac{dv(l.v)^2}{1+v+vv} = \alpha^3\frac{2p}{q^3} = \frac{\pi^3}{81\sqrt{3}}$$

deinde vero ex numero II fit

$$-\frac{1}{6}\int\frac{dv\,lv}{1-v} - \frac{1}{12}\int\frac{dv(1+2v)\,lv}{1+v+vv} = \alpha\alpha.\frac{1}{qq} = \frac{\pi\pi}{27}$$

supra autem inuenimus esse

$$\int\frac{dv\,lv}{1-v} = -\frac{\pi\pi}{6}$$

quo valore substituto fit

$$\int\frac{dv(1+2v)\,lv}{1+v+vv} = -\frac{\pi\pi}{9}$$

maxime igitur operae pretium est visum has postremas integrationes euoluisse.

§. 38. Quod si ambae formulae integrales

$$\int\frac{dv(1-2v)\,lv}{1-v+vv} \quad \text{et} \quad \int\frac{dv(1+2v)\,lv}{1+v+vv}$$

in

in ſeries conuertantur reperitur

$$\int \frac{dv(1-2v)\,lv}{1-v+vv} = -1 + \tfrac{1}{4} + \tfrac{2}{9} + \tfrac{1}{16} - \tfrac{1}{25} - \tfrac{2}{36} - \tfrac{1}{49} + \text{etc. et}$$

$$\int \frac{dv(1+2v)\,lv}{1+v+vv} = -1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} - \tfrac{1}{25} + \tfrac{2}{36} - \tfrac{1}{49} + \text{etc.}$$

vnde has duas ſummationes attentione noſtra non indignas aſſequimur

$$\text{I.}\quad 1 - \tfrac{1}{4} - \tfrac{2}{9} - \tfrac{1}{16} + \tfrac{1}{25} + \tfrac{2}{36} + \tfrac{1}{49} - \tfrac{1}{64} - \tfrac{2}{81} - \tfrac{1}{100} \text{ etc.} = \frac{\pi\pi}{18}$$

$$\text{II.}\quad 1 + \tfrac{1}{4} - \tfrac{2}{9} + \tfrac{1}{16} + \tfrac{1}{25} - \tfrac{2}{36} + \tfrac{1}{49} + \tfrac{1}{64} - \tfrac{2}{81} + \tfrac{1}{100} \text{ etc.} = \frac{\pi\pi}{9}$$

quarum prior a poſteriore ablata praebet

$$\tfrac{2}{4} + \tfrac{2}{16} - \tfrac{4}{36} + \tfrac{2}{64} + \tfrac{2}{100} \text{ etc.} = \frac{\pi\pi}{18},$$

cuius duplum perducit ad hanc

$$1 + \tfrac{1}{4} - \tfrac{2}{9} + \tfrac{1}{16} + \tfrac{1}{25} - \tfrac{2}{36} + \text{etc.} = \frac{\pi\pi}{9}$$

quae quoniam cum ſecunda congruit, veritas vtriusque ſummationis ſatis confirmatur; quod ſi vero ſecunda a duplo primae ſubtrahatur, remanebit iſta ſeries memorabilis

$$1 - \tfrac{3}{4} - \tfrac{2}{9} - \tfrac{3}{16} + \tfrac{1}{25} + \tfrac{6}{36} + \tfrac{1}{49} - \tfrac{3}{64} - \tfrac{2}{81} - \tfrac{3}{100} + \text{etc.} = 0$$

quae in periodos 6 terminos complectentes diſtributa, manifeſtum ordinem in numeratoribus declarat, quippe qui ſunt $1 - 3 - 2 - 3 + 1 + 6$.

Additamentum.

§. 39. Quemadmodum ſuperiores integrationes per continuam differentiationem formularum S et T deduximus, ita etiam per integrationem alias et prorſus ſingulares integrationes impetrabimus; ſi

 enim

enim vt ſupra fuerit $S = \int \frac{T\,dz}{z}$, exiſtente T formula illa

$$+ \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}},$$

quae praeter z etiam exponentem variabilem ω inuoluere concipitur, erit per naturam integralium duas variabiles inuoluentium

$$\int S\,d\omega = \int \frac{dz}{z} \int T\,d\omega,$$

vbi inpriore formula integrali $\int S\,d\omega$, vbi z pro conſtanti habetur, ſtatim ſcribi poteſt $z = 1$; hoc igitur lemmate praemiſſo, quia eſt

$$\int T\,d\omega = \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}\,l\,z},$$

ambas formulas ſupra tractatas nempe S et T hoc modo euoluamus, et quia vtramque triplici modo expreſſam dedimus; primo ſcilicet per ſeriem infinitam; ſecundo, per formulam finitam: ac tertio per formulam integralem, etiam quantitates, quae pro integralibus $\int S\,d\omega$ et $\int T\,d\omega$ reſultabunt, erunt inter ſe aequales.

§. 40. Incipiamus a formula S, et cum per ſeriem fuerit

$$S = \frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.}$$

erit

$$\int S\,d\omega = -l(\lambda-\omega) + l(\lambda+\omega) + l(3\lambda-\omega) - l(3\lambda+\omega) - \text{etc.} + C$$

quam conſtantem ita definire decet, vt integrale euaneſcat poſito $\omega = 0$, quo facto erit

$\int S\,d\omega$

$$\int S\,d\omega = l\frac{\lambda+\omega}{\lambda-\omega} + l\frac{3\lambda-\omega}{3\lambda+\omega} + l\frac{5\lambda+\omega}{5\lambda-\omega} + l\frac{7\lambda-\omega}{7\lambda+\omega} + \text{etc.}$$

quae expressio reducitur ad sequentem

$$\int S\,d\omega = l\frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega)(9\lambda+\omega)\text{ etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega)(9\lambda-\omega)\text{ etc.}}$$

Deinde quia per formulam finitam erat

$$S = \frac{\pi}{2\lambda\,\text{cos.}\,\frac{\pi\omega}{2\lambda}}, \text{ erit } \int S\,d\omega = \int\frac{\pi\,d\omega}{2\lambda\,\text{cos.}\,\frac{\pi\omega}{2\lambda}},$$

vbi si breuitatis gratia ponatur $\frac{\pi\omega}{2\lambda} = \Phi$, vt sit

$$d\omega = \frac{2\Phi\lambda}{\pi} \text{ erit } \int S\,d\omega = \int\frac{d\Phi}{\text{cos.}\,\Phi};$$

quia igitur nouimus esse

$$\int\frac{d\vartheta}{\text{sin.}\,\vartheta} = l\,\text{tang.}\,\tfrac{1}{2}\vartheta$$

sumamus sin. $\vartheta = \text{cos.}\,\Phi$ siue $\vartheta = 90^\circ - \Phi = \frac{\pi}{2} - \Phi$
eritque $d\vartheta = -d\Phi$ vnde fit

$$\int\frac{-d\Phi}{\text{cos.}\,\Phi} = l\,\text{tang.}\,(\tfrac{\pi}{4} - \tfrac{1}{2}\Phi),$$

quoniam autem est

$$\Phi = \frac{\pi\omega}{2\lambda} \text{ erit } \frac{\pi}{4} - \frac{1}{2}\Phi = \frac{\pi(\lambda-\omega)}{4\lambda},$$

vnde nostrum integrale erit

$$\int S\,d\omega = -l\,\text{tang.}\,\frac{\pi(\lambda-\omega)}{4\lambda} = +l\,\text{tang.}\,\frac{\pi(\lambda+\omega)}{4\lambda}$$

ex tertia autem formula integrali

$$S = \int\frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}}\,\frac{dz}{z}, \text{ colligitur fore}$$

$$\int S\,d\omega = \int\frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}}\,\frac{dz}{z\,lz}$$

quod integrale a termino $z = 0$ vsque ad terminum $z = 1$ extendi assumitur; sicque tres isti valores inuenti inter ſe erunt aequales. Ac ne ob conſtantes forte addendas vllum dubium ſuperſit, ſingulae iſtae expreſſiones ſponte euaneſcunt caſu $\omega = 0$.

§. 41. Conſideremus hinc primo aequalitatem inter formulam primam et ſecundam: et quia vtraque eſt logarithmus, erit

$$\text{tang.}\, \frac{\pi(\lambda+\omega)}{4\lambda} = \frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega)\text{ etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega)\text{ etc.}}$$

cum igitur huius fractionis numerator euaneſcat caſibus, vel $\omega = -\lambda$, vel $\omega = +3\lambda$, vel $\omega = -5\lambda$, vel $\omega = +7\lambda$ etc. euidens eſt iisdem caſibus quoque tangentem fieri $= 0$; denominator vero euaneſcit caſibus vel $\omega = \lambda$, vel $\omega = -3\lambda$ vel $\omega = +5\lambda$ vel $\omega = -\lambda$ etc. quibus ergo caſibus tangens in infinitum excreſcere debet, id quod etiam pulcherrime euenit. Ceterum haec expreſſio congruit cum ea, quam iam dudum inueni et in introductione expoſui.

§. 42. Productum autem iſtud infinitum per principia alibi ſtabilita ad formulas integrales reduci poteſt ope huius lemmatis latiſſime patentis.

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k)\text{ etc.}}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k)\text{ etc.}} = \frac{\int z^{c-1}\,dz\,(1-z^k)^{\frac{b-k}{k}}}{\int z^{c-1}\,dz\,(1-z^k)^{\frac{a-k}{k}}}$$

ſi quidem poſt vtramque integrationem fiat $z = 1$. Noſtro igitur caſu erit $a = \lambda + \omega$, $b = \lambda - \omega$, $c = 2\lambda$ et $k = 4\lambda$ vnde valor noſtri producti erit

$\int z^{\lambda-1}$

$$\frac{\int z^{2\lambda-1}\,dz\,(1-z^{4\lambda})^{\frac{-3\lambda-\omega}{4\lambda}}}{\int z^{2\lambda-1}\,dz\,(1-z^{4\lambda})^{\frac{-3\lambda+\omega}{4\lambda}}} = \text{tang.}\,\frac{\pi(\lambda+\omega)}{4\lambda}$$

formulae autem istae integrales concinniores euadant statuendo $z^{2\lambda}=y$, tum enim erit

$$\text{tang.}\,\frac{\pi(\lambda+\omega)}{4\lambda} = \frac{\int dy\,(1-yy)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int dy\,(1-yy)^{\frac{-3\lambda+\omega}{4\lambda}}}$$

quae expressio vtique omni attentione digna videtur. Denique ex formula integrali inuenta erit quoque

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\,\frac{dz}{z\,lz} = l\,\text{tang.}\,\frac{\pi(\lambda+\omega)}{4\lambda}.$$

§. 43. Operae erit pretium, etiam aliquot casus particulares euoluere: sit igitur primo $\lambda=2$ et $\omega=1$ ac per expressionem infinitam erit

$$\int S\,d\omega = l\,\frac{3.5}{1.7}.\frac{11.13}{9.15}.\frac{19.21}{17.23}.\frac{27.29}{25.31}.\frac{35.37}{33.39}.\frac{\text{etc.}}{\text{etc.}}$$

deinde per expressionem finitam habebimus

$$\int S\,d\omega = l\,\text{tang.}\,\frac{3\pi}{8}$$

at per formulam integralem

$$\int S\,d\omega = \int \frac{-(1-zz)}{1+z^4}.\frac{dz}{lz}.$$

Tum vero ex aequalitate duarum priorum expressionum

$$\text{tang.}\,\frac{3\pi}{8} = \frac{3.5}{1.7}.\frac{11.13}{9.15}.\frac{19.21}{17.23}\ \text{etc.}$$

hincque per binas formulas integrales

$$\text{tang.}\,\frac{3\pi}{8} = \frac{\int dy\,(1-yy)^{-\frac{5}{8}}}{\int dy\,(1-yy)^{-\frac{5}{8}}}.$$

§. 44. Ponamus nunc esse $\lambda = 3$ et $\omega = 1$ ac per expressionem infinitam erit

$$\int S\, d\omega = l\, \tfrac{2}{1} \cdot \tfrac{4}{5} \cdot \tfrac{8}{7} \cdot \tfrac{10}{11} \cdot \tfrac{14}{13} \cdot \tfrac{16}{17} \cdot \tfrac{20}{19} \cdot \tfrac{22}{23} \text{ etc.}$$

secundo, per expressionem finitam

$$\int S\, d\omega = l \operatorname{tang.} \tfrac{\pi}{3} = l \sqrt{3} = \tfrac{1}{2}\, l\, 3,$$

ita, vt futurum sit

$$\sqrt{3} = \frac{2 \cdot 4}{1 \cdot 5} \cdot \frac{8 \cdot 10}{7 \cdot 11} \cdot \frac{14 \cdot 16}{13 \cdot 17} \text{ etc.}$$

huiusque producti valor per formulas integrales erit

$$\frac{\int dy\,(1-yy)^{-\frac{5}{6}}}{\int dy\,(1-yy)^{-\frac{3}{2}}}.$$

Denique formula integralis praebebit

$$\int S\, d\omega = \int \frac{-z(1-zz)}{1+z^6} \frac{dz}{lz}.$$

§. 45. Eodem modo etiam euoluamus alteram formulam T cuius valor per seriem erat

$$T = \frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.}$$

vnde fit

$$\int T\, d\omega = -l(\lambda-\omega) - l(\lambda+\omega) - l(3\lambda-\omega) - l(3\lambda+\omega) - \text{etc.}$$

quae expressio vt euanescat posito $\omega = 0$, erit

$$\int T\, d\omega = l\, \frac{\lambda\lambda}{\lambda\lambda-\omega\omega} \frac{9\lambda\lambda}{9\lambda\lambda-\omega\omega} \frac{25\lambda\lambda}{25\lambda\lambda-\omega\omega} \text{ etc.}$$

deinde vero cum per formulam finitam fuerit

$$T = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda} \text{ erit}$$

$\int T\, d\omega = \int \frac{\pi\, d\omega}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda}$ vbi posito $\frac{\pi\omega}{2\lambda} = \Phi$ erit

$\int T\, d\omega = \int d\Phi \operatorname{tang.} \Phi = -l \operatorname{cos.} \Phi$ ita vt sit

$$\int T\, d\omega = -l \operatorname{cos.} \frac{\pi\omega}{2\lambda};$$

cuius

cuius valor casu $\omega = 0$ fit sponte $= 0$ denique per formulam integralem habebimus

$$\int T\, d\omega = -\frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z\, lz}$$

integrale itidem a termino $z = 0$ vsque ad terminum $z = 1$ extendi debet.

§. 46. Iam comparatio duorum priorum valorem hanc praebet aequationem

$$\frac{1}{\cos\frac{\pi\omega}{2\lambda}} = \frac{\lambda\lambda}{\lambda\lambda - \omega\omega} \cdot \frac{9\lambda\lambda}{9\lambda\lambda - \omega\omega} \cdot \frac{25\lambda\lambda}{25\lambda\lambda - \omega\omega} \cdot \frac{49\lambda\lambda}{49\lambda\lambda - \omega\omega} \text{ etc.}$$

$$\cos\frac{\pi\omega}{2\lambda} = \left(1 - \frac{\omega\omega}{\lambda\lambda}\right)\left(1 - \frac{\omega\omega}{9\lambda\lambda}\right)\left(1 - \frac{\omega\omega}{25\lambda\lambda}\right)\left(1 - \frac{\omega\omega}{49\lambda\lambda}\right) \text{ etc.}$$

vel si factores singuli iterum in simplices euoluantur, erit

$$\cos\frac{\pi\omega}{2\lambda} = \frac{\lambda+\omega}{\lambda} \cdot \frac{\lambda-\omega}{\lambda} \cdot \frac{3\lambda+\omega}{3\lambda} \cdot \frac{3\lambda-\omega}{3\lambda} \cdot \frac{5\lambda+\omega}{5\lambda} \cdot \frac{5\lambda-\omega}{5\lambda} \text{ etc.}$$

quae formula cum reductione generali supra allata comparata dat, $a = \lambda + \omega$, $b = \lambda$, $c = -\omega$ et $k = 2\lambda$ vnde colligimus

$$\cos\frac{\pi\omega}{2\lambda} = \frac{\int z^{-\omega-1}\, dz\, (1 - z^{2\lambda})^{-\frac{1}{2}}}{\int z^{-\omega-1}\, dz\, (1 - 1^{2\lambda})^{\frac{\omega-\lambda}{2\lambda}}}.$$

Vt autem exponentes negatiuos $z^{-\omega-1}$ euitemus, superius productum ita repraesentemus

$$\cos\frac{\pi\omega}{2\omega} = \frac{\lambda-\omega}{\lambda} \cdot \frac{\lambda+\omega}{\lambda} \cdot \frac{3\lambda-\omega}{3\lambda} \cdot \frac{3\lambda+\omega}{3\lambda} \text{ etc.}$$

eritque facta comparatione $a = \lambda - \omega$, $b = \lambda$ $c = +\omega$ et $k = 2\lambda$ sicque per formulas integrales erit

cos.

$$\text{cof.}\,\frac{\pi\omega}{2\lambda} = \frac{\int z^{\omega-1}\,dz\,(1-z^{2\lambda})^{-\frac{1}{2}}}{\int z^{\omega-1}\,dz\,(1-z^{2\lambda})^{\frac{-\lambda-\omega}{2\lambda}}}$$

quae expreſſio ad ſimpliciorem formam reduci nequit.

§. 47. Sit nunc etiam $\lambda = 2$ et $\omega = 1$ eruntque ternae noſtrae expreſſiones

I. $\int T\,d\omega = l\,\frac{4}{3}\cdot\frac{36}{35}\cdot\frac{100}{99}\cdot\frac{196}{195}$ ſiue

$\int T\,d\omega = l\,\frac{2\cdot 2}{1\cdot 3}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{10\cdot 10}{9\cdot 11}\cdot\frac{14\cdot 14}{13\cdot 15}$ etc.

II. $\int T\,d\omega = -l\,\text{cof.}\,\frac{\pi}{4} = +\frac{1}{2}\,l\,2$ ita, vt ſit

$\sqrt{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{10\cdot 10}{9\cdot 11}\cdot\frac{14\cdot 14}{13\cdot 15}$ etc.

quod productum per formulas integrales ita exprimitur

$$\frac{\int dz\,(1-z^4)^{-\frac{1}{2}}}{\int dz\,(1-z^4)^{-\frac{3}{4}}} = \sqrt{2}$$

III. $\int T\,d\omega = \int \frac{-(1+zz)}{1-z^4}\frac{dz}{z} = \int \frac{-dz}{(1-zz)\,l\,z}$

quod ergo integrale a termino $z = 0$, vsque ad $z = 1$ extenſum praebet eundem valorem $+\frac{1}{2}\,l\,2$, cuius aequalitatis ratio vtique difficillime patet.

§. 48. Sit denique vt ſupra $\lambda = 3$ et $\omega = 1$ ac ternae formulae ita ſe habebunt

I. $\int T\,d\omega = l\,\frac{9}{8}\cdot\frac{81}{80}\cdot\frac{225}{224}$ etc. $= l\,\frac{3\cdot 3}{2\cdot 4}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\frac{15\cdot 15}{14\cdot 16}\cdot\frac{21\cdot 21}{20\cdot 22}$ etc.

II. $\int T\,d\omega = -l\,\text{cof.}\,\frac{\pi}{6} = -l\,\frac{\sqrt{3}}{2} = +l\,\frac{2}{\sqrt{3}}$ ita vt ſit

$\frac{2}{\sqrt{3}} = \frac{3\cdot 3}{2\cdot 4}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\frac{15\cdot 15}{14\cdot 16}\cdot\frac{21\cdot 21}{20\cdot 22}$

ideo-

ideoque per binas formulas integrales

$$\frac{z}{\sqrt{3}} = \frac{\int dz(1-z^6)^{-\frac{1}{2}}}{\int dz(1-z^6)^{-\frac{1}{2}}}$$

$$\text{III.} \int T\,d\omega = \int \frac{-(1+zz)}{1-z^6} \frac{dz}{lz}$$

quae posito $zz = v$ abit in hanc

$$\int T\,d\omega = \int \frac{-dv(1+v)}{(1-v^3)lv}.$$

Hinc igitur patet hac methodo plane noua perueniri ad formulas integrales, quas per methodos adhuc cognitas nullo modo euoluere vel saltem inter se comparare licuit.