INSIGNES
PROPRIETATES SERIERVM
SVB HOC TERMINO GENERALI CONTENTARVM

$$x = \tfrac{1}{2}\left(a + \tfrac{b}{\sqrt{k}}\right)(p + q\sqrt{k})^n + \tfrac{1}{2}\left(a - \tfrac{b}{\sqrt{k}}\right)(p - q\sqrt{k})^n.$$

Auctore

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Statim adparet, has feries effe recurrentes fecundi ordinis et quemlibet terminum per binos praecedentes determinari; cuiusmodi feries etfi iam fatis fuperque funt pertractatae, tamen nonnullas earum infignes proprietates hic proponam; inprimis autem operam dabo, vt omnia calculo fuccincto facillime expediantur, dum alias ad calculos non parum complicatos perueniri folet.

§. 1. Breuitatis autem gratia ftatim ponamus

$$\tfrac{1}{2}\left(a + \tfrac{b}{\sqrt{k}}\right) = f;\ \tfrac{1}{2}\left(a - \tfrac{b}{\sqrt{k}}\right) = g$$

$$p + q\sqrt{k} = v;\ p - q\sqrt{k} = u$$

tum vero fit etiam $p^2 - kq^2 = r$; ita, vt fit $p = \sqrt{(kq^2 + r)}$. Hoc modo formula noftra ita contrahetur

$$k = fv^n + gu^n$$

inde autem mox fequentes fluunt relationes:

$f + g$

$$f+g=a;\; f-g=\frac{b}{\sqrt{k}} \text{ et } fg=\tfrac{1}{4}\left(a^2-\frac{b^2}{k}\right)$$

$$v+u=2p;\; v-u=2q\sqrt{k};\; vu=p^2-kq^2=r.$$

§. 2. Praeterea quo terminos huius feriei facilius menti repraefentare poffimus; loco x fcribamus hoc fignum $[n]$, quippe quo terminus ex exponente n oriundus defignatur, ficque ipfa feries fequentibus conftabit terminis:

$$[0];\; [1];\; [2];\; [3] \text{ etc.}$$

vnde hic faltem termini initiales notentur, fcilicet

$$[0]=a;\; [1]=ap+bq;\; [2]=a(p^2+kq^2)+2bpq.$$

His igitur praenotatis fequentia problemata tractemus.

Problema Primum.

§. 3. *Definire legem, qua terni termini huius feriei immediate fe infequentes* $[n]$; $[n+1]$; $[n+2]$ *a fe inuicem pendent.*

Solutio.

Cum fit $[n]=f.v^n+g.u^n$; eodemque modo $[n+1]=f.v^{n+1}+g.u^{n+1}$ et $[n+2]=f.v^{n+2}+g.u^{n+2}$; confideretur haec formula $v^{n+1}(v+u)$, quae ob $v+u=2p$ fit $=2p.v^{n+1}$; eadem vero ob $vu=r$ fiue $u=\frac{r}{v}$ abit in $v^{n+2}+r.v^n$, ita vt nunc habeamus $2pv^{n+1}=v^{n+2}+r.v^n$ fiue $v^{n+2}=2p.v^{n+1}-r.v^n$; eodemque modo reperitur $u^{n+2}=2p.u^{n+1}-r.u^n$. Nunc igitur fi hae duae formulae addantur, $fv^{n+2}=2fp.v^{n+1}-fr.v^n$ et $g\,u^{n+2}=2gp.u^{n+1}-gr.u^n$

fumma

summa erit

$$[n+2]=2p[n+1]-r[n];$$

vnde patet, nostram seriem esse recurrentem, scala relationis existente $2p, -r$.

Coroll. 1.

§. 4. Si ergo dentur bini quicunque termini successiui huius seriei, qui sint P et Q, sequens terminus semper erit $=2pQ-rP$.

Coroll. 2.

§. 5. Quodsi ergo pro hac scala relationis $2p, -r$ dentur duo termini initiales, A et B, erit, vti ante ostendimus,

$$A=a \text{ et } B=ap+bq$$

ideoque $a=A$ et $b=\frac{B-Ap}{q}$, existente $kq^2=p^2-r$, ita, vt sit $\frac{b}{\sqrt{k}}=\frac{B-Ap}{\sqrt{(p^2-r)}}$ hincque ipse terminus generalis huius seriei innotescit, quippe qui est ipsa nostra formula proposita.

Coroll. 3.

§. 6. Ex formulis inuentis patet etiam fore

$$v^{n+2}+u^{n+2}=2p(v^{n+1}+u^{n+1})-r(v^n+u^n)$$

vnde deducuntur sequentes relationes

si	erit
$n=0$	$v^2+u^2=2p(v+u)-2r$
$n=1$	$v^3+u^3=2p(v^2+u^2)-r(v+u)$
$n=2$	$v^4+u^4=2p(v^3+u^3)-r(v^2+u^2)$
$n=3$	$v^5+u^5=2p(v^4+u^4)-r(v^3+u^3)$
$n=4$	$v^6+u^6=2p(v^5+u^5)-r(v^4+u^4)$
	etc.

Coroll.

Coroll. 4.

§. 7. Hinc ſi valores inuenti ſucceſſiue ſubſtituantur, reperiemus ſequentem progreſſionem

$$v^0 + u^0 = 2$$
$$v^1 + u^1 = 2p$$
$$v^2 + u^2 = 4p^2 - 2r$$
$$v^3 + u^3 = 8p^3 - 6pr$$
$$v^4 + u^4 = 16p^4 - 16p^2r + 2r^2$$
$$v^5 + u^5 = 32p^5 - 40p^3r + 10pr^2$$
$$v^6 + u^6 = 64p^6 - 96p^4r + 36p^2r^2 - 2r^3$$

etc.

Coroll. 5.

§. 8. Hae expreſſiones ſimpliciores reddentur ſi loco $2p$ ſcribamus litteram ſimplicem s tum enim iſta progreſſio reſultat

$$v^0 + u^0 = 2$$
$$v^1 + u^1 = s$$
$$v^2 + u^2 = s^2 - 2r$$
$$v^3 + u^3 = s^3 - 3rs$$
$$v^4 + u^4 = s^4 - 4rs^2 + 2r^2$$
$$v^5 + u^5 = s^5 - 5rs^3 + 5r^2s$$
$$v^6 + u^6 = s^6 - 6rs^4 + 9r^2s^2 - 2r^3$$

etc.

quae ſeries cum iam ſatis ſit pertractata aliunde nouimus, fore in genere

$$v^n + u^n = s^n - n.r\,s^{n-2} + \frac{n(n-3)}{1.\;2} r^2.\,s^{n-4}$$
$$- \frac{n(n-4)(n-5)}{1.\;2.\;3} r^3.\,s^{n-6}$$
$$+ \frac{n(n-5)(n-6)(n-7)}{1.\;2.\;3.\;4} r^4.\,s^{n-8}$$

etc.

 atque

atque hinc etiam sequens problema resolui poterit.

Problema Secundum.

§. 9. *Definire legem, qua terni termini huius seriei per saltum se inuicem insequentes inter se cohaerent, scilicet hi termini* $[n]$; $[n+\nu]$; et $[n+2\nu]$, *denotante* ν *indicem saltus, qua termini sumuntur.*

Solutio.

Cum ergo sit

$$[n]=f v^n+g.u^n, \text{ et } [n+\nu]=f.v^{n+\nu}+g.u^{n+\nu},$$

atque

$$[n+2\nu]=f.v^{n+2\nu}+g.u^{n+2\nu};$$

consideretur haec formula $v^{n+\nu}(v^\nu+u^\nu)$, ac ponamus per legem ante assignatam esse $v^\nu+u^\nu=2\pi$, existente $v^\nu.u^\nu=r^\nu$; et formula ista primo fiet $2\pi.v^{n+\nu}$ et quatenus $u^\nu=\frac{r^\nu}{v^\nu}$ eadem dat $v^{n+2\nu}+r^\nu v^n$ ita, vt iam sit

$$v^{n+2\nu}=2\pi.v^{n+\nu}-r^\nu.v^n$$

eodemque modo

$$u^{n+2\nu}=2\pi.u^{n+\nu}-r^\nu.u^n$$

quare ratiocinium, vt ante, instituendo nanciscimur istam relationem quaesitam

$$[n+2\nu]=2\pi[n+\nu]-r^\nu.[n]$$

ope cuius legis termini secundum eundem saltum procedentes

$$[n+3\nu]; [n+4\nu]; [n+5\nu] \text{ etc.}$$

facile reperiuntur.

Coroll.

Coroll. 1.

§. 10. Cum etiam, vti vidimus, formulae $v^\nu + u^\nu$ secundum eandem legem progrediuntur ac si breuitatis gratia loco 2π scribamus σ, progressio postremi corollarii etiam ad hos saltus adcommodabitur, si modo loco r scribatur r^ν; sic enim obtinebimus

$$v^0 + u^0 = 2$$
$$v^\nu + u^\nu = \sigma$$
$$v^{2\nu} + u^{2\nu} = \sigma^2 - 2.r^\nu$$
$$v^{3\nu} + u^{3\nu} = \sigma^3 - 3.r^\nu.\sigma$$
$$v^{4\nu} + u^{4\nu} = \sigma^4 - 4.r^\nu.\sigma^2 + 2r^{2\nu}$$
$$v^{5\nu} + u^{5\nu} = \sigma^5 - 5.r^\nu\sigma^3 + 5.r^{2\nu}\sigma$$
$$v^{6\nu} + u^{6\nu} = \sigma^6 - 6.r^\nu\sigma^4 + 9.r^{2\nu}\sigma^2 - 2r^{3\nu}.$$

Coroll. 2.

§. 11. Hoc ergo modo facile est, saltum siue numerum ν tantum efficere, quam quis voluerit, atque adeo ope huius problematis in serie nostra termini quantumuis ab initio remoti satis expedite definiri poterunt; id quod si seriem per singulos terminos actu continuare vellemus, nimis operosum calculum postularet.

Problema Tertium.

§. 12. Dato quocunque termino seriei nostrae $[n]$, inuenire eius immediate sequentem $[n+1]$.

Solutio.

Cum sit

$$[n] = fv^n + g.u^n \text{ et } [n+1] = f v.v^n + g.u.u^n,$$

 si

ſi haec a priore in u ducta ſubtrahatur, remanet

$$u[n]-[n+1]=f(u-v)\,v^n;$$

at ſi haec ab illa in v ducta ſubtrahatur, remanebit

$$v[n]-[n+1]=g(v-u)\,u^n.$$

Iam hac duae aequalitates in ſe inuicem ducantur et ob $v^n.\,u^n=r^n$ proueniet iſta aequatio

$$r[n]^2-2p[n][n+1]+[n+1]^2=-fg.\,(v-u)^2\,r^n.$$

At vero eſt

$$(v-u)^2=v^2+u^2-2vu=4p^2-4r;$$

ſicque habebimus

$$r[n]^2-2p[n][n+1]+[n+1]^2-4fg(p^2-r)\,r^n=0.$$

Quare terminus ſequens $[n+1]$ per aequationem quadraticam ex praecedente $[n]$ ita determinatur, vt ſit

$$[n+1]=p.[n]+\sqrt{((p^2-r)[n]^2-4fg(p^2-r)\,r^n)}.$$

Supra autem vidimus eſſe

$$fg=\tfrac{1}{4}\left(a^2-\tfrac{b^2}{k}\right) \text{ et } p^2-r=kq^2,$$

quibus valoribus ſubſtitutis ſolutio noſtra ita ſe habebit

$$[n+1]=p[n]+q\sqrt{(k[n]^2-(ka^2-b^2)\,r^n)}.$$

Coroll. I.

§. 13. Quilibet ergo terminus noſtrae ſeriei ita eſt comparatus, vt valor formulae

$$k[n]^2-(k\,a^2-b^2)\,r^n$$

certe ſit numerus quadratus, quandoquidem omnes termini noſtrae ſeriei ſunt rationales.

Coroll.

Coroll. 2.

§. 14. Si ponamus feriei binos terminos initiales A et B, ita, vt iis refpondeant exponentes $n=0$, et $n=1$ fupra vidimus effe

$$a = A \text{ et } b = \frac{B - Ap}{q}, \text{ ideoque}$$

$$ka^2 - b^2 = -\frac{B^2 + 2AB.p - A^2 r}{q^2};$$

quare habebimus

$$[n+1] = p[n] + \sqrt{(kq^2[n]^2 + (B^2 - 2AB.p + A^2 r) r^n)}.$$

Problema Quartum.

§. 15. *Dato quocunque feriei termino* $[n]$, *inuenire terminum dato interuallo ipfum fequentem fcilicet* $[n+\nu]$.

Solutio.

Ambo hi termini ita repraefententur

$$[n] = fv^n + g.u^n \text{ et } [n+\nu] = fv^\nu.v^n + gu^\nu.u^n$$

quarum prior nunc in u^ν, nunc in v^ν ducatur indeque pofterior fubtrahatur, et fequentes binae aequationes refultabunt:

$$u^\nu[n] - [n+\nu] = f(u^\nu - v^\nu)v^n \text{ et}$$

$$v^\nu[n] - [n+\nu] = g(v^\nu - u^\nu)u^n$$

fi iam vt ante fuerit $v^\nu + u^\nu = 2\pi$

vnde fequitur $v^{2\nu} + u^{2\nu} = 4\pi^2 - 2r^\nu$

duas illas aequationes inuicem multiplicando adipifcimur

$$r^\nu[n]^2 - 2\pi.[n][n+\nu] + [n+\nu]^2 = -fg(v^\nu - u^\nu)^2 r^n.$$

 Quia

Quia autem eſt

$$(v^{\nu}-u^{\nu})^2=v^{2\nu}+u^{2\nu}-2r^{\nu}=4\pi^2-4r^{\nu} \text{ et } fg=\tfrac{1}{4}(a^2-\tfrac{b}{k})$$

relatio inuenta erit

$$[n+\nu]^2-2\pi[n][n+1]+r^{\nu}[n]^2+(\pi^2-r^{\nu})(a^2-\tfrac{b^2}{k})r^n=0,$$

vnde iterum per extractionem radicis quadratae elicimus

$$[n+\nu]=\pi[n]\surd\left\{\begin{matrix}(\pi^2-r^{\nu})[n]^2\\-(a^2-\tfrac{b^2}{k})(\pi^2-r^{\nu})r^n\end{matrix}\right\}$$

cuius indoles quo clarius perſpiciatur, ſtatuamus $v^{\nu}=\pi+\xi\surd k$, eritque $u^{\nu}=\pi-\xi\surd k$, vnde vtique erit $v^{\nu}+u^{\nu}=2\pi$, at vero

$$v^{\nu}u^{\nu}=r^{\nu}=\pi^2-k\xi^2, \text{ ita, vt fit } \pi^2-r^{\nu}=k\xi^2;$$

quo valore ſubſtituto fit

$$[n+\nu]=\pi[n]+\xi\surd(k[n]^2-(ka^2-b^2)r^n).$$

Coroll. 1.

§. 16. Hic ergo denuo proprietas ante obſeruata inuoluitur, quod nempe iſta formula

$$(k[n]^2-(ka^2-b^2)r^n)$$

ſemper debeat eſſe quadratum.

Coroll. 2.

§. 17. Ex terminis autem initialibus A et B, ob $ka^2-b^2=\frac{-B^2+2ABp-A^2r}{q^2}$ vti ante inuenimus, habebimus

$$[n+\nu]=\pi[n]+\xi\surd(k[n]^2+(\tfrac{B^2-2ABp+A^2r}{q^2})r^n).$$

Coroll.

Coroll. 3.

§. 18. Si capiamus $n = 0$, vt fit $[n] = a$ aequatio nostra ita se habebit

$$[\nu] = \pi a + \xi b;$$

quae est insignis proprietas nostrae formulae, quae autem sponte se prodit ex eius indole; cum enim sit

$$[\nu] = f v^{\nu} + g.\, u^{\nu} = f(\pi + \xi \sqrt{k}) + g.(\pi - \xi \sqrt{k}); \text{ erit}$$

$$[\nu] = \pi(f + g) + \xi \sqrt{k}(f - g) = ,$$

quae formula ob $f + g = a$ et $f - g = \frac{b}{\sqrt{k}}$ reducitur ad $\pi a + \xi b$.

Problema Quintum.

§. 19. *Dato termino seriei quocunque $[n]$ ex sequentibus inuestigare eum, qui ab illo tantum distat, quantum ipse ab initio, hoc est inuenire terminum $[2n]$.*

Solutio.

Cum fit $[n] = f v^n + g.\, u^n$; quadratis sumendis erit

$$[n]^2 - f g.\, r^n = f^2 v^{2n} + g^2.\, u^{2n}$$

tum vero est

$$[2n] = f v^{2n} + g.\, u^{2n};$$

haec primo in g ducta ab illa subtrahatur et remanet

$$[n]^2 - 2 f g r^n - g[2n] = f(f - g) v^{2n}$$

deinde posterior aequatio in f ducta et a priore subtracta relinquit

$$[n]^2 - 2 f g r^n - f[2n] = g(g - f) u^{2n}$$

hae

hae iam duae aequationes in ſe ducantur et prodibit

$$[n]^4 - (f+g)[2n][n]^2 + fg[2n]^2 - 4fgr^n[n]^2$$
$$+ 2fgr^n(f+g)[2n] + 4f^2g^2 . r^{2n} = -fg(f-g)^2 r^{2n};$$

cum igitur ſit

$$f+g = a;\ fg = \tfrac{1}{4}\left(a^2 - \tfrac{b^2}{k}\right);\ f-g = \tfrac{b}{\sqrt{k}}$$

his valoribus ſubſtitutis aequatio hanc induet formam:

$$[n]^4 - a[2n][n]^2 + \tfrac{1}{4}\left(a^2 - \tfrac{b^2}{k}\right)[2n]^2$$
$$- \left(a^2 - \tfrac{b^2}{k}\right) r^n [n]^2 + \tfrac{1}{2}\left(a^2 - \tfrac{b^2}{k}\right) a . r^n [2n]$$
$$+ \tfrac{1}{4} a^2 \left(a^2 - \tfrac{b^2}{k}\right) r^{2n} = 0$$

vnde eliciamus $[2n]$, ſubductoque calculo, tandem reperitur, breuitatis gratia loco $a^2 - \frac{b^2}{k}$ ſcribendo c

$$[2n] = \tfrac{2a}{c}[n]^2 - ar^n + \tfrac{2b[n]}{ck}\sqrt{([n]^2 k - ck . r^n)}$$

et ſubſtituto valore ipſius c

$$[2n] = \tfrac{2ak}{a^2k - b^2}[n]^2 - ar^n$$
$$- \tfrac{2b[n]}{a^2k - b^2}\sqrt{(k[n]^2 - (a^2k - b^2)r^n)}.$$

Scholion.

§. 20. Quia haec ſolutio ad calculos non parum taedioſos eſt perducta, hoc negotium non modo multo facilius, ſed etiam generalius expediri poſſe obſeruaui. Quae methodus quo clarius percipiatur, ſequentia praemitto.

Hypotheſis.

§. 21. Quemadmodum formulae $fv^n + g$ charactere deſignauimus; ita iſtam formulam $fv^n - g$

illi adfinem hoc caractere $\overset{\prime}{[n]}$ indicemus; quae ergo quantitas ex illa nafcitur, fi littera g negatiue capiatur.

Corollarium.

§. 22. Cum fit

$$f v^n - g . u^n = \sqrt{([n]^2 - 4 f g . r^n)}$$

erit nouus nofter caracter

$$\overset{\prime}{[n]} = \sqrt{([n]^2 - 4 f g . r^n)},$$

quae formula ita eft irrationalis, vt per $\sqrt{k}$ multiplicata fiat rationalis; tum autem refultat formula illa

$$\sqrt{(k [n]^2 - 4 f g . k r^n)},$$

quam ante iam obferuauimus femper effe rationalem, cui ergo aequatur caracter $\overset{\prime}{[n]} \sqrt{k}$.

Lemma.

§. 23. Cum ergo fit

$$f v^n + g . u^n = [n] \text{ et } f v^n - g . u^n = \overset{\prime}{[n]}, \text{ erit}$$

$$v^n = \frac{[n] + \overset{\prime}{[n]}}{2 f} \text{ et } u^n = \frac{[n] - \overset{\prime}{[n]}}{2 g},$$

atque hinc adipifcimur

$$v^n + u^n = \frac{(f+g)[n] - (f-g)\overset{\prime}{[n]}}{2 f g}$$

quae eft ea ipfa formula, quam fupra, vbi loco n adhibuimus ν, per 2π indicauimus; tum erit

$$v^n - u^n = \frac{-(f-g)[n] + (f+g)\overset{\prime}{[n]}}{2 f g}$$

 quae

quae formula conuenit cum ea, quam casu $n = \nu$ supra per $2\xi\sqrt{k}$ denotauimus.

Problema Sextum.

§. 24. *Datis in serie nostra duobus quibuscunque terminis* $[n]$ *et* $[\nu]$ *inuenire terminum* $[n+\nu]$ *simulque eius adfinem* $[\acute{n}+\acute{\nu}]$, *siquidem etiam datae erunt formulae* $[\acute{n}]$ *et* $[\acute{\nu}]$.

Solutio.

Cum sit

$$[\nu] = f v^{\nu} + g.\, u^{\nu} \text{ et } [\acute{\nu}] = f v^{\nu} - g\, u^{\nu}$$

multiplicetur illa aequatio per $v^n + u^n$, haec vero per $v^n - u^n$, et obtinebimus binas sequentes aequationes:

$$[\nu](v^n + u^n) = f v^{n+\nu} + g.\, u^{\nu} v^n + f v^{\nu} u^n + g.\, u^{n+\nu}$$

$$[\acute{\nu}](v^n - u^n) = f v^{n+\nu} - g.\, u^{\nu} v^n - f v^{\nu} u^n + g.\, u^{n+\nu}$$

quae duae additae praebent

$$[\nu](v^n + u^n) + [\acute{\nu}](v^n - u^n) = 2(f v^{n+\nu} + g.\, u^{n+\nu}) = 2[n+\nu]$$

sicque iam assecuti sumus terminum quaesitum $[n+\nu]$; cum autem per lemma praemissum sit

$$v^n + u^n = \frac{(f+g)[n] - (f-g)[\acute{n}]}{2fg}$$

et $v^n - u^n = \frac{-(f-g)[n] + (f+g)[\acute{n}]}{2fg}$;

per formulas mere cognitas consequimur

$$[n+\nu] = \frac{(f+g)([n][\nu] + [\acute{n}][\acute{n}]) - (f-g)([\acute{n}][\nu] + [n][\acute{\nu}])}{4fg}.$$

Quia

Quia igitur

$$f+g=a \text{ et } f-g=\frac{b}{\sqrt{k}}, \text{ et } 4fg=a^2-\frac{b^2}{k}=\frac{ka^2-b^2}{k},$$

erit his valoribus ſubſtituendis terminus quaeſitus

$$[n+\nu]=\frac{ak[n][\nu]-b([n][\nu]'\sqrt{k}-b([\nu][n]')\sqrt{k}+ak[n]'[\nu]'}{ka^2-b^2}$$

quod ad formulam adfinem $[n+\nu]'$ attinet, iam innuimus eam ex priore naſci, dummodo loco g ſcribatur $-g$, atque hinc orietur

$$[n+\nu]'=\frac{-(f-g)([n][\nu]+[n]'[\nu]')+(f+g)([n]'[\nu]+[n]'[\nu]}{4fg}$$

quae in $\sqrt{k}$ ducta et loco f et g valoribus ſubſtitutis dabit

$$[n+\nu]'\sqrt{k}=\frac{-kb[n][\nu]+ak[n][\nu]'\sqrt{k}+ak[\nu][n]'\sqrt{k}-kb[n]'[\nu]'}{ka^2-b^2}.$$

Coroll. 1.

§. 25. Quo facilius hanc ſolutionem ad formam ſolitam reducere queamus; recordemur eſſe

$$[n]'\sqrt{k}=\sqrt{(k[n]^2-(ka^2-b^2)r^n)}$$

$$[\nu]'\sqrt{k}=\sqrt{(k[\nu]^2-(ka^2-b^2)r^\nu)}$$

et

$$[n+\nu]'\sqrt{k}=\sqrt{(k[n+\nu]^2-(ka^2-b^2)r^{n+\nu})}.$$

Coroll. 2.

§. 26. Si hic ſumamus $\nu=n$, vt prodeat caſus praecedente problemate tractatus, ſtatim reperimus

$$[2n]=\frac{ak[n]^2-2b[n][n]'\sqrt{k}+ak[n]'^2}{ka^2-b^2}$$

 et

et formulis affinibus elifis

$$[2n] = \frac{2ak[n]^2 - a(ka^2 - b^2)r^n - 2b[n]\sqrt{(k[n]^2 - (ka^2 - b^2)r^n)}}{ka^2 - b^2}$$

quae cum ante inuenta congruit.

Coroll. 3.

§. 27. Eodem cafu $\nu = n$ formula affinis colligitur:

$$[2n']\sqrt{k} = \sqrt{(k[2n]^2 - (ka^2 - b^2)r^{2n}} =$$
$$-2kb[n]^2 + 2a[n]\sqrt{(k[n]^2 - (ka^2 - b^2)r^n)}$$
$$+ b.(ka^2 - b^2)r^n$$
$$= \frac{-2kb[n]^2 + b(ka^2 - b^2)r^n + 2a[n]\sqrt{(k[n]^2 - (ka^2 - b^2)r^n)}}{ka^2 - b^2}.$$

Coroll. 4.

§. 28. Ope harum formularum iam facile erit terminos feriei ab initio quantumuis remotos affignare, quandoquidem ex binis quibuscunque $[n]$ et $[\nu]$ reperitur ftatim terminus $[n + \nu]$; interim tamen ex lege, quam fupra iam dedimus, pro terminis $[n]$, $[n + \nu]$, $[n + 2\nu]$; $[n + 3\nu]$, $[n + 4\nu]$ etc. haec feries multo facilius, quousque libuerit, continuari poteft; fi enim ponatur $v^\nu + u^\nu = 2\pi$; ita, vt fit

$$2\pi = \frac{(f+g)[n] - (f-g)[n']}{2fg}$$

$$= \frac{2ak[n] - 2b[n']\sqrt{k}}{ka^2 - b^2};$$

lex progreffionis ita fe habet

$$[n + 2\nu] = 2\pi[n + \nu] - r^\nu[n]$$
$$[n + 3\nu] = 2\pi[n + 2\nu] - r^\nu[n + \nu]$$
etc.

Proble-

Problema Septimum.

§. 29. *Si fuerit*

$$x = \tfrac{1}{2}\left(a + \tfrac{b}{\sqrt{k}}\right)(p + q\sqrt{k})^n$$
$$+ \tfrac{1}{2}\left(a - \tfrac{b}{\sqrt{k}}\right)(p - q\sqrt{k})^n$$

similique modo

$$y = \tfrac{1}{2}\left(\alpha + \tfrac{\beta}{\sqrt{k}}\right)(p + q\sqrt{k})^n$$
$$+ \tfrac{1}{2}\left(\alpha - \tfrac{\beta}{\sqrt{k}}\right)(p - q\sqrt{k})^n$$

inuenire aequationem inter **x** *et* **y**, *cui isti valores rationales et integri satisfaciant, quicunque integer pro* **n** *accipiatur.*

Solutio.

Ponamus iterum breuitatis gratia

$$\tfrac{1}{2}\left(a + \tfrac{b}{\sqrt{k}}\right) = f;\quad \tfrac{1}{2}\left(a - \tfrac{b}{\sqrt{k}}\right) = g$$
$$\tfrac{1}{2}\left(\alpha + \tfrac{\beta}{\sqrt{k}}\right) = \zeta;\quad \tfrac{1}{2}\left(\alpha - \tfrac{\beta}{\sqrt{k}}\right) = \eta$$

porro $p + q\sqrt{k} = v$; $p - q\sqrt{k} = u$; $p^2 = kq^2 + r$ et nunc habebimus

$$x = f v^n + g.\, u^n$$
$$y = \zeta v^n + \eta\, u^n$$

vnde eliminemus litteras v et u, ac primo quidem habebimus

$$\eta x - g y = (\eta f - \zeta g)\, v^n$$
et $$\zeta x - f y = (\zeta g - \eta f)\, u^n$$

quae duae formulae in se inuicem ductae ob $vu = r$ dabunt

$$\eta\zeta x^2 - (\zeta g + \eta f)xy + fg y^2 = -(\eta f - \zeta g)^2 r^n.$$

 Nunc

Nunc reſtituantur valores aſſumti ſcilicet

$$\zeta\eta = \tfrac{1}{4}\left(\alpha^2 - \tfrac{\beta^2}{k}\right);\quad fg = \tfrac{1}{4}\left(a^2 - \tfrac{b^2}{k}\right)$$

$$\zeta g + \eta f = \tfrac{1}{2}\left(a\alpha - \tfrac{b\beta}{k}\right)$$

et

$$(\eta f - \zeta g)^2 = \tfrac{1}{4k}\left(a^2\beta^2 - 2ab\alpha\beta + \alpha^2 b^2\right)$$

atque noſtra aequatio quaeſita erit

$$\tfrac{1}{4}\left(\alpha^2 - \tfrac{\beta^2}{k}\right)x^2 - \tfrac{1}{2}\left(a\alpha - \tfrac{b\beta}{k}\right)xy + \tfrac{1}{4}\left(a^2 - \tfrac{b^2}{k}\right)y^2$$
$$+ \tfrac{1}{4k}\left(a^2\beta^2 - 2ab\alpha\beta + \alpha^2 b^2\right)r^n = 0$$

ſiue per $4k$ multiplicando

$$(\alpha^2 k - \beta^2)x^2 - 2(a\alpha k - b\beta)xy + (a^2 k - b^2)y^2$$
$$+ (a\beta - \alpha b)^2 r^n = 0$$

quae tota eſt rationalis, eique ſatisfaciunt ipſi valores pro x et y aſſumti.

Corollarium.

§. 30. Comparetur haec aequatio inuenta cum forma generali

$$Ax^2 - 2Bxy + Cy^2 + Dr^n = 0$$

atque ſatisfieri oportet ſequentibus conditionibus

1°. $(a\beta - \alpha b)^2 \frac{A}{D} = (k\alpha^2 - \beta^2)$

2°. $(a\beta - \alpha b)^2 \frac{B}{D} = (ka\alpha - b\beta)$

3°. $(a\beta - \alpha b)^2 \frac{C}{D} = (ka^2 - b^2)$

quarum prima per tertiam diuiſa ſtatim ſuppeditat

$$k = \frac{Ab^2 - C\beta^2}{Aa^2 - C\alpha^2}.$$

Coroll. 2.

§. 31. Subſtituatur hic valor tam in prima, quam in ſecunda aequatione ac peruenietur ad iſtas aequalitates

$$(a\beta - \alpha b)^2 \frac{A}{D} = \frac{A \alpha^2 b^2 - a^2 \beta^2}{A a^2 - C \alpha^2}$$

$$(a\beta - \alpha b)^2 \frac{B}{D} = \frac{A a \alpha b^2 - C a \alpha \beta^2 - A a^2 b \beta + C \alpha^2 b \beta}{A a^2 - C \alpha^2}$$

quarum haec per illam diuiſa praebet

$$\frac{a}{\alpha} = \frac{C\beta - Bb}{B\beta - Ab}.$$

Coroll. 3.

§. 32. Statuamus nunc $a = \gamma \alpha$, et $b = \delta \beta$ vt prodeat

$$\gamma = \frac{C - B\delta}{B - A\delta};$$

vnde ſequitur

$$\gamma - \delta = \frac{A\delta^2 - 2B\delta + C}{B - A\delta}$$

et $\gamma + \delta = \frac{C - A\delta^2}{B - A\delta}$;

prima aequatio transformatur in hanc:

$$\frac{\gamma - \delta}{D} = \frac{-\gamma - \gamma}{A\alpha^2\gamma^2 - C\alpha^2}$$

vnde fit

$$\alpha^2 = \frac{-D(\gamma + \delta)}{(A\gamma^2 - C)(\gamma - \delta)}.$$

Coroll. 4.

§. 33. Valorem autem ipſius γ ſubſtituendo reperitur

$$A\gamma^2 - C = \frac{-(B^2 - AC)(C - A\delta^2)}{(B - A\delta)^2}$$

hinc-

hincque colligitur

$$\alpha^2 = \frac{D.(B - A\delta)^2}{(B^2 - AC)(A\delta^2 - 2B\delta + C)}$$

sicque omnibus conditionibus est satisfactum et nu quaestio huc reducitur, vt quaeratur numerus δ li lis, vt ista forma reuera fiat quadratum, quod fit si quadratum reddatur haec forma

$$D.(B^2 - AC)(A\delta^2 - 2B\delta + C).$$

Coroll. 5.

§. 34. Tali autem numero pro δ inuento bebitur numerus α, nec non γ; numerus autem arbitrio nostro permititur; hincque porro deduci $a = \gamma\alpha$ et $b = \delta\beta$; tum vero prodit

$$k = \beta^2 \frac{A\delta^2 - C}{Aa^2 - C\alpha^2}; \text{ indeque}$$

$$\frac{\beta^2}{k} = \frac{Aa^2 - C\alpha^2}{A\delta^2 - C}$$

$$\text{et } \frac{b^2}{k} = \frac{\delta^2\beta^2}{k} = \frac{Aa^2\delta^2 - C\alpha^2\delta^2}{A\delta^2 - C},$$

Quia ergo formulae $\frac{\beta}{\gamma k}$ et $\frac{b}{\gamma k}$ in calculo nostro tum occurrunt, nihil amplius arbitrarii restat.

Scholion.

§. 35. Hoc autem modo pro quouis valore exponentis n vnica tantum reperitur solutio, scilicet valores ipsarum x et y complectens, qui huic exponenti n respondent; verum quia num p et q adhuc arbitrio nostro sunt permissi, eos finitis modis ita definire licet, vt fiat $p^2 - kq^2$ quod quo facilius perspiciatur, quaeruntur num s et t, vt fit $s^2 - kt^2 = 1$ quibus inuentis vbi

loco $(p + q\sqrt{k})^n$ scribatur $(p + q\sqrt{k})^n (s + t\sqrt{k})^\lambda$
et loco $(p - q\sqrt{k})^n$ scribatur

$$(p - q\sqrt{k})^n (s - t\sqrt{k})^\lambda,$$

vbi exponens λ pro quouis numero n infinitis modis variari potest, sicque pro quouis exponente n infinitae solutiones nostris formulis exhibebuntur.

Scholion. 2.

§. 36. Inprimis autem hic mirandum videtur, quod littera δ, qua vniuersa solutio adstruitur, neque a numero r neque ab exponente n pendeat, sed ex solis litteris, A, B, C et D definiatur; interim tamen numerum r non omnino pro lubitu assumere licet, ita enim comparatus esse debet, vt fieri possit $p^2 - kq^2 = r$, id quod ab indole numeri k pendet. Supra autem inuenimus valorem $\frac{\beta^2}{k}$, vnde oritur

$$\frac{\beta}{\sqrt{k}} = \sqrt{\frac{A\alpha^2 - C\alpha^2}{A\delta^2 - C}} = \alpha.\sqrt{\frac{A\gamma^2 - C}{A\delta^2 - C}}$$

$$= \frac{\alpha}{B - A\delta}\sqrt{(BB - AC)}$$

$$= \frac{\alpha}{B - A\delta}\sqrt{(B^2 - AC)} = \frac{\alpha(B^2 - AC)}{(B - A\delta)\sqrt{(B^2 - AC)}}$$

quae si capiatur $\beta = \frac{\alpha(B^2 - AC)}{B - A\delta}$

erit $\sqrt{k} = \sqrt{B^2 - AC}$,

sicque per hunc numerum k determinatur indoles numerorum, quos pro r assumere licet.
