SVMMATIO PROGRESSIONVM

$\sin.\Phi^\lambda + \sin.2\Phi^\lambda + \sin.3\Phi^\lambda + \ldots\ldots + \sin.n\Phi^\lambda$
$\cos.\Phi^\lambda + \cos.2\Phi^\lambda + \cos.3\Phi^\lambda + \ldots\ldots + \cos.n\Phi^\lambda$.

Auctore

L. EVLERO.

§. 1.

Posito $\cos.\Phi + \sqrt{-1}\sin.\Phi = p$ et $\cos.\Phi - \sqrt{-1}\sin.\Phi = q$ notum eſt fore

$$\cos.n\Phi = \frac{p^n + q^n}{2} \text{ et } \sin.n\Phi = \frac{p^n - q^n}{2\sqrt{-1}}$$

tum vero etiam eſſe $pq = 1$. His poſitis euidens eſt, ſummationem harum ſerierum ſemper reduci poſſe ad has duas ſeries vel progreſſiones geometricas

$$p^\alpha + p^{2\alpha} + p^{3\alpha} + \ldots\ldots + p^{n\alpha} = \frac{p^{(n+1)\alpha} - p^\alpha}{p^\alpha - 1} = \frac{p^\alpha(1 - p^{n\alpha})}{1 - p^\alpha}$$

$$q^\alpha + q^{2\alpha} + q^{3\alpha} + \ldots\ldots + q^{n\alpha} = \frac{q^{(n+1)\alpha} - q^\alpha}{q^\alpha - 1} = \frac{q^\alpha(1 - q^{n\alpha})}{1 - q^\alpha}$$

§ 2. Quod ſi iam hae duae progreſſiones invicem addantur, vt prodeat iſta

$$\begin{array}{l} p^\alpha + p^{2\alpha} + p^{3\alpha} + \ldots\ldots + p^{n\alpha} \\ + q^\alpha + q^{2\alpha} + q^{3\alpha} + \ldots\ldots + q^{n\alpha} \end{array} \text{ eius ſumma erit } \begin{array}{l} \dfrac{p^\alpha - p^{(n+1)\alpha}}{1 - p^\alpha} \\ + \dfrac{q^\alpha - q^{(n+1)\alpha}}{1 - q^\alpha} \end{array}$$

=

$$= \frac{p^\alpha - p^{(n+1)\alpha} - p^\alpha q^\alpha + p^{(n+1)\alpha} q^\alpha + q^\alpha - q^{(n+1)\alpha} - p^\alpha q^\alpha + p^\alpha q^{(n+1)\alpha}}{1 - p^\alpha - q^\alpha + p^\alpha q^\alpha}$$

quae expressio ob $p\,q = 1$ transformatur in hanc

$$\frac{p^\alpha - p^{(n+1)\alpha} - 1 + p^{n\alpha} + q^\alpha - q^{(n+1)\alpha} - 1 + q^{n\alpha}}{2 - p^\alpha - q^\alpha}$$

quae porro ob

$$p^\alpha + q^\alpha = 2\cos.\alpha\Phi \text{ et } p^{(n+1)\alpha} + q^{(n+1)\alpha} = 2\cos.(n+1)\alpha\Phi$$
$$\text{et } p^{n\alpha} + q^{n\alpha} = 2\cos.n\alpha\Phi$$

reducitur ad hanc formam:

$$\frac{\cos.\alpha\Phi - \cos.(n+1)\alpha\Phi - 1 + \cos.n\alpha\Phi}{1 - \cos.\alpha\Phi} = -1 + \frac{\cos.(n\alpha\Phi) - \cos.(n+1)\alpha\Phi}{1 - \cos.\alpha\Phi}$$

quae ergo est summa seriei propositae.

§. 3. Sin autem altera nostrarum progressionum ab altera subtrahatur, vt habeatur ista

$$\left.\begin{array}{l} +p^\alpha + p^{2\alpha} + p^{3\alpha} + \ldots\ldots + p^{n\alpha} \\ -q^\alpha - q^{2\alpha} - q^{3\alpha} + \ldots\ldots - q^{n\alpha} \end{array}\right\} \text{ eius summa erit } \begin{array}{l} \dfrac{p^\alpha - p^{(n+1)\alpha}}{1 - p^\alpha} \\ \dfrac{-q^\alpha + q^{(n+1)\alpha}}{1 - q^\alpha} \end{array}$$

quae partes ad eundem denominatorem reductae producent

$$\left\{\begin{array}{l} +p^\alpha - p^{(n+1)\alpha} - p^\alpha q^\alpha + p^{(n+1)\alpha} q^\alpha \\ -q^\alpha + q^{(n+1)\alpha} + p^\alpha q^\alpha - q^{(n+1)\alpha} p^\alpha \end{array}\right\} : 1 - p^\alpha - q^\alpha + p^\alpha q^\alpha$$

ob $p\,q = 1$ vero haec expressio ad hanc reducitur

$$\begin{array}{l} p^\alpha - p^{(n+1)\alpha} - 1 + p^{n\alpha} \\ -q^\alpha + q^{(n+1)\alpha} + 1 - q^{n\alpha} \end{array} : 2 - p^\alpha - q^\alpha$$

et

et porro ob

$$p^{\alpha}-q^{\alpha}=2\sqrt{-1}\,\text{fin.}\,\alpha\Phi \text{ et } p^{(n+1)\alpha}-q^{(n+1)\alpha}=2\sqrt{-1}\,\text{fin.}\,(n+1)\alpha\Phi$$
$$\text{et } p^{n\alpha}-q^{n\alpha}=2\sqrt{-1}\,\text{fin.}\,n\alpha\Phi$$

in hanc transformatur expreffionem

$$\frac{(\text{fin.}\,\alpha\Phi-\text{fin.}\,(n+1)\alpha\Phi+\text{fin.}\,(n\alpha\Phi)}{1-\text{cof.}\,\alpha\Phi}\sqrt{-1}.$$

§. 4. Defignemus breuitatis gratia fummas harum ferierum, vltimo termino feu generali praefigendo fignum fummationis $\int$, ita vt bini cafus euoluti praebeant fequentes fummationes

$$\int(p^{n\alpha}+q^{n\alpha})=-1+\frac{\text{cof.}\,(n\alpha\Phi)-\text{cof.}\,(n+1)\alpha\Phi}{1-\text{cof.}\,\alpha\Phi}, \text{ et}$$
$$\int(p^{n\alpha}-q^{n\alpha})=\frac{(\text{fin.}\,\alpha\Phi+\text{fin.}\,(n\alpha\Phi)-\text{fin.}\,(n+1)\alpha\Phi)}{1-\text{cof.}\,\alpha\Phi}(\sqrt{-1})$$

ex quibus formulis facile erit omnes cafus propofitos deducere.

§. 5. Sit igitur primo $\lambda=1$, vt habeantur hae duae feries fummandae:

$$s=\text{fin.}\,\Phi+\text{fin.}\,2\Phi+\text{fin.}\,3\Phi+\ldots\ldots+\text{fin.}\,n\Phi \text{ fiue } s=\int\text{fin.}\,n\Phi \text{ et}$$
$$t=\text{cof.}\,\Phi+\text{cof.}\,2\Phi+\text{cof.}\,3\Phi+\ldots\ldots+\text{cof.}\,n\Phi \text{ fiue } t=\int\text{cof.}\,n\Phi$$

cum igitur fit

$$\text{fin.}\,n\Phi=\frac{p^n-q^n}{2\sqrt{-1}} \text{ et } \text{cof.}\,n\Phi=\frac{p^n+q^n}{2}$$

habebimus

$$2s\sqrt{-1}=\int(p^n-q^n) \text{ et } 2t=\int(p^n+q^n)$$

vnde ex paragrapho praecedente ftatim nancifcimur ob $\alpha=1$

$$2s\sqrt{-1}=\frac{(\text{fin.}\,\Phi+\text{fin.}\,n\Phi-\text{fin.}\,(n+1)\Phi}{1-\text{cof.}\,\Phi}\sqrt{-1} \text{ et}$$
$$2t=-1+\frac{\text{cof.}\,n\Phi-\text{cof.}\,(n+1)\Phi}{1-\text{cof.}\,\Phi} \text{ ideoque}$$

$s=$

$$s = \frac{\text{fin.}\,\Phi + \text{fin.}\,n\Phi - \text{fin.}\,(n+1)\Phi}{2\,(1 - \text{cof.}\,\Phi)} \quad \text{et} \quad t = -\tfrac{1}{2} + \frac{\text{cof.}\,n\Phi - \text{cof.}\,(n+1)\Phi}{2\,(1 - \text{cof.}\,\Phi)}.$$

§. 6. Sit nunc $\lambda = 2$ et ftatuamus iterum

$s = \text{fin.}\,\Phi^2 + \text{fin.}\,2\Phi^2 + \ldots\ldots + \text{fin.}\,n\Phi^2$ fiue $s = \int \text{fin.}\,n\Phi^2$ et

$t = \text{cof.}\,\Phi^2 + \text{cof.}\,2\Phi^2 + \ldots\ldots + \text{cof.}\,n\Phi^2$ fiue $t = \int \text{cof.}\,n\Phi^2$

cum nunc fit

$$\text{fin.}\,n\Phi^2 = \frac{p^{2n} - 2p^n q^n + q^{2n}}{-4} = \tfrac{1}{2} - \frac{p^{2n} - q^{2n}}{4} \quad \text{et}$$

$$\text{cof.}\,n\Phi^2 = \frac{p^{2n} + 2p^n q^n + q^{2n}}{4} = +\tfrac{1}{2} + \frac{p^{2n} + q^{2n}}{4}$$

habebimus has formulas

$$4s = 2\int 1 - \int (p^{2n} + q^{2n}) \quad \text{et} \quad 4t = 2\int 1 + \int (p^{2n} + q^{2n})$$

vbi cum numerus terminorum fit n, manifeftum eft fore $\int 1 = n$; hinc quia $\alpha = 2$, ex fuperioribus erit

$$\int (p^{2n} + q^{2n}) = -1 + \frac{\text{cof.}\,2n\Phi - \text{cof.}\,2(n+1)\Phi}{1 - \text{cof.}\,2\Phi}$$

quibus valoribus fubftitutis, facta diuifione per 4 obtinebimus

$$s = \tfrac{n}{2} + \tfrac{1}{4} - \frac{\text{cof.}\,2n\Phi - \text{cof.}\,2(n+1)\Phi}{4\,(1 - \text{cof.}\,2\Phi)} \quad \text{et} \quad t = \tfrac{n}{2} - \tfrac{1}{4} + \frac{\text{cof.}\,2n\Phi - \text{cof.}\,2(n+1)\Phi}{4\,(1 - \text{cof.}\,2\Phi)}$$

atque hinc ftatim liquet fore

$s + t = n$, prorfus vti rei natura poftulat.

§. 7. Statuamus nunc $\lambda = 3$ et feries fummandas ita repraefentemus

$s = \text{fin.}\,\Phi^3 + \text{fin.}\,2\Phi^3 + \ldots\ldots + \text{fin.}\,n\Phi^3$ fiue $s = \int \text{fin.}\,n\Phi^3$ et

$t = \text{cof.}\,\Phi^3 + \text{cof.}\,2\Phi^3 + \ldots\ldots + \text{cof.}\,n\Phi^3$ fiue $t = \int \text{cof.}\,n\Phi^3$.

 Cum

Cum nunc fit

$$\text{fin.}\, n\Phi^3 = \frac{p^{3n} - 3p^{2n}q^n + 3p^n q^{2n} - q^{3n}}{-8\sqrt{-1}} \text{ et}$$

$$\text{cof.}\, n\Phi^3 = \frac{p^{3n} + 3p^{2n}q^n + 3p^n q^{2n} + q^{3n}}{8}$$

vnde ob $pq = 1$ confequimur

$$s = \frac{-1}{8\sqrt{-1}}\int(p^{3n} - q^{3n})\frac{-3p^n + 3q^n}{-8\sqrt{-1}} = \frac{-1}{-8\sqrt{-1}}\int(p^{3n} - q^{3n}) + \frac{3}{8\sqrt{-1}}\int(p^n - q^n)$$

tum vero

$$t = +\tfrac{1}{8}\int(p^{3n} + q^{3n}) + \tfrac{3}{8}\int(p^n + q^n)$$

quod fi nunc valores fupra inuentos hic fubftituamus, ambae fummae quaefitae ita prodibunt expreffae.

$$s = -\frac{\text{fin.}\,3\Phi - \text{fin.}\,3n\Phi + \text{fin.}\,(3(n+1)\Phi)}{8(1 - \text{cof.}\,3\Phi)} + \frac{3\,\text{fin.}\,\Phi + 3\,\text{fin.}\,n\Phi - 3\,\text{fin.}\,(n+1)\Phi}{8(1 - \text{cof.}\,\Phi)}$$

$$t = -\tfrac{1}{2} + \frac{\text{cof.}\,3n\Phi - \text{cof.}\,3(n+1)\Phi}{8(1 - \text{cof.}\,3\Phi)} + \frac{3\,\text{cof.}\,n\Phi - 3\,\text{cof.}\,(n+1)\Phi}{8(1 - \text{cof.}\,\Phi)}.$$

§. 8. Sit nunc $\lambda = 4$ ita vt quaerantur hae fummae

$s = \text{fin.}\,\Phi^4 + \text{fin.}\,2\Phi^4 + \ldots\ldots + \text{fin.}\,n\Phi^4$ fiue $s = \int \text{fin.}\,n\Phi^4$ et
$t = \text{cof.}\,\Phi^4 + \text{cof.}\,2\Phi^4 + \ldots\ldots + \text{cof.}\,n\Phi^4$ fiue $t = \int \text{cof.}\,n\Phi^4$.

Cum igitur fit

$$\text{fin.}\, n\Phi^4 = \frac{p^{4n} - 4p^{3n}q^n + 6p^{2n}q^{2n} - 4p^n q^{3n} + q^{4n}}{+16}$$

$$\text{et } \text{cof.}\, n\Phi^4 = \frac{p^{4n} + 4p^{3n}q^n + 6p^{2n}q^{2n} + 4p^n q^{3n} + q^{4n}}{16}$$

ob $pq = 1$ fequuntur hi valores

$$s = \tfrac{1}{16}\int(p^{4n} + q^{4n}) - \tfrac{1}{4}\int(p^{2n} + q^{2n}) + \tfrac{3}{8}\int 1 \text{ et}$$
$$t = \tfrac{1}{16}\int(p^{4n} + q^{4n}) + \tfrac{1}{4}\int(p^{2n} + q^{2n}) + \tfrac{3}{8}\int 1$$

valo-

valoribus igitur quos ſupra dedimus ſubſtitutis erit:

$$f = \frac{3n}{8} + \frac{5}{16} + \frac{\cos.4n\Phi - \cos.4(n+1)\Phi}{16(1-\cos.4\Phi)} - \frac{\cos.2n\Phi + \cos.2(n+1)\Phi}{4(1-\cos.2\Phi)} \text{ et}$$

$$g = \frac{3n}{8} - \frac{5}{16} + \frac{\cos.4n\Phi - \cos.4(n+1)\Phi}{16(1-\cos.2\Phi)} + \frac{\cos.2n\Phi - \cos.2(n+1)\Phi}{4(1-\cos.2\Phi)}$$

ſicque facile erit etiam maiores valores exponentis λ euoluere.

§. 9. Quod ſi iam quaeratur, cuiusmodi ſummae hinc ſint proditurae, ſi iſtae ſeries in infinitum continuentur, non exigua circumſpectione erit vtendum. Primo enim ſi exponens λ fuerit numerus par, euidens eſt, ſumto pro n numero infinito, ſummas harum ſerierum etiam fore infinite magnas; verum ſi λ fuerit numerus impar, tum nihil eſt, quod has ſummas in infinitum augere poſſit, tota autem quaeſtio huc redigitur, vt valores formularum ſinus $n\alpha\Phi$ et coſ. $n\alpha\Phi$ aſſignentur quando pro n numeri infinite magni accipiuntur, perſpicuum autem eſt hos valores hoc caſu aeque a termino vsque ad terminum -1 variari poſſe, ac ſi n eſſet numerus finitus, vnde ſi res in ſe ſpectetur, nihil certi de his ſummis affirmari licet, cum quaecunque ſumma proferretur, ſi inſuper vnus pluresue termini adderentur, prorſus alia ſumma eſſet proditura; Interim tamen ab illuſtri Auctore praecedentis diſſertationis ſummae hoc caſu per rationes metaphyſicas perquam ingenioſe aſſignantur, quibus in analyſi perfecte acquieſcere queamus.

§. 10. Cum autem in his ſeriebus aeque ac in omnibus aliis non conuergentibus, notio ſummae

 pro-

proprie locum inuenire nequeat, quandoquidem quotcunque etiam termini actu addantur tamen nunquam ad summam determinatam peruenitur; Iam pridem validissimis rationibus innixus admonui his casibus voci summae alium significatum ad analysin magis accommodatum tribui debere, quam nouam notionem ita constitui debere censeo, vt summa cuiusque seriei infinitae, siue fuerit conuergens siue divergens, vocetur ea formula analytica, ex cuius euolutione eae series nascantur, hacque admissa definitione omnia dubia circa huiusmodi summationes sponte euanescunt.

§. 11. Quod quo clarius appareat, consideremus primam seriem supra exhibitam

$$s = \sin.\phi + \sin.2\phi + \sin.3\phi + \ldots\ldots + \sin.n\phi$$

pro qua inuenimus

$$\frac{\sin.\phi + \sin.n\phi - \sin.(n+1)\phi}{2(1-\cos.\phi)}$$

in quam expressionem formulae $\sin.n\phi$ et $\sin.(n+1)\phi$ propter vltimum terminum ingrediuntur, quod si ergo series reuera in infinitum continuetur; ob nullum terminum vltimum etiam hae formulae sponte excedunt, ita vt hoc casu fiat $s = \frac{\sin.\phi}{2(1-\cos.\phi)}$, quae etiam ea ipsa est formula, ex cuius euolutione ista series elicitur, vnde vi meae definitionis haec formula in analysi recte pro summa istius seriei haberi potest, quod idem de altera serie

$$t = \cos.\phi + \cos.2\phi + 3\phi + \ldots\ldots + \cos.n\phi$$

est

eſt tenendum, pro qua inuenimus

$$t = -\tfrac{1}{2} + \frac{\operatorname{cof.} n\Phi - \operatorname{cof.}(n+1)\Phi}{2(1-\operatorname{cof.}\Phi)}$$

omiſſo enim poſtremo membro vtpote a termino vltimo pendente, ſumma per meam definitionem vtique erit $t = -\tfrac{1}{2}$ quod cum non tam facile pateat, notandum eſt, hunc valorem natum eſſe ex formula $t = \frac{\operatorname{cof.}\Phi - 1}{2(1-\operatorname{cof.}\Phi)}$, quem valorem ſeriei propoſitae eſſe aequalem ita oſtendi poteſt; Multiplicetur vtrinque per $2 - 2\operatorname{cof.}\Phi$ fierique debebit

$$\operatorname{cof.}\Phi - 1 = \left\{\begin{array}{l} 2\operatorname{cof.}\Phi + 2\operatorname{cof.}2\Phi + 2\operatorname{cof.}3\Phi + 2\operatorname{cof.}4\Phi \\ -2\operatorname{cof.}\Phi^2 - 2\operatorname{cof.}\Phi\operatorname{cof.}2\Phi - 2\operatorname{cof.}\Phi\operatorname{cof.}3\Phi \end{array}\right.$$

cum nunc ſit in genere

$$2\operatorname{cof.}a\operatorname{cof.}b = \operatorname{cof.}(a-b) + \operatorname{cof.}(a+b) \text{ erit}$$

$$\begin{array}{ll} 2\operatorname{cof.}\Phi^2 = 1 + \operatorname{cof.}2\Phi & 2\operatorname{cof.}\Phi\operatorname{cof.}4\Phi = \operatorname{cof.}3\Phi + \operatorname{cof.}5\Phi \\ 2\operatorname{cof.}\Phi\operatorname{cof.}2\Phi = \operatorname{cof.}\Phi + \operatorname{cof.}3\Phi & 2\operatorname{cof.}\Phi\operatorname{cof.}5\Phi = \operatorname{cof.}4\Phi + \operatorname{cof.}6\Phi \\ 2\operatorname{cof.}\Phi\operatorname{cof.}3\Phi = \operatorname{cof.}2\Phi + \operatorname{cof.}4\Phi & 2\operatorname{cof.}\Phi\operatorname{cof.}6\Phi = \operatorname{cof.}5\Phi + \operatorname{cof.}7\Phi \\ & \text{etc.} \end{array}$$

quibus valoribus ſubſtitutis aequalitas manifeſto in oculos incurrit, prodibit enim

$$\begin{array}{rl} \operatorname{cof.}\Phi - 1 = & 2\operatorname{cof.}\Phi + 2\operatorname{cof.}2\Phi + 2\operatorname{cof.}3\Phi + 2\operatorname{cof.}4\Phi \\ & -1 - \operatorname{cof.}\Phi - \operatorname{cof.}2\Phi - \operatorname{cof.}3\Phi - \operatorname{cof.}4\Phi \text{ etc.} \\ & \qquad\quad - \operatorname{cof.}2\Phi - \operatorname{cof.}3\Phi - \operatorname{cof.}4\Phi \end{array}$$

§. 12. Iisdem obſeruatis cautelis etiam pro caſu $\lambda = 3$ quo poſueramus

$$s = \operatorname{fin.}\Phi^3 + \operatorname{fin.}2\Phi^3 + \operatorname{fin.}3\Phi^3 + \text{etc. in infinitum}$$

et $t = \operatorname{cof.}\Phi^3 + \operatorname{cof.}2\Phi^3 + \operatorname{cof.}3\Phi^3 + \operatorname{cof.}4\Phi^3 + \text{etc. in infinitum}$

ſum-

ſumma harum ſerierum infinitarum ita erunt expreſſae

$$s = -\frac{\text{ſin. } 3\,\Phi}{8\,(1 - \text{coſ. } 3\,\Phi)} + \frac{3\,\text{ſin. }\Phi}{8\,(1 - \text{coſ. }\Phi)} \text{ et } t = -\frac{1}{2}.$$

hic quidem non tam facile apparet, harum formularum euolutionem ad ipſas has ſeries perducere, nihilo vero minus certum eſt, perfectam aequalitatem locum habere, id quod, qui in hoc calculi genere ſatis ſunt exercitati, perſpicient. Interim tamen veritatem poſterioris ſummationis hoc modo oſtendiſſe iuuabit, cum ſit

$$\text{coſ. } a^3 = \tfrac{3}{4}\,\text{coſ. } a + \tfrac{1}{4}\,\text{coſ. } 3\,a$$

ſeries haec in duas ſequentes reſoluitur

$$t = \tfrac{3}{4}(\text{coſ. }\Phi + \text{coſ. } 2\,\Phi + \text{coſ. } 4\,\Phi \text{ etc.})$$
$$+ \tfrac{1}{4}(\text{coſ. } 3\,\Phi + \text{coſ. } 6\,\Phi + \text{coſ. } 9\,\Phi \text{ etc.})$$

prioris autem ſeriei ſumma ex precedentibus eſt $\frac{3}{4}\cdot -\frac{1}{2} = -\frac{3}{8}$ poſterioris vero ob eandem rationem ſumma eſt $\frac{1}{4}\cdot -\frac{1}{2} = -\frac{1}{8}$ vnde ambae coniunctim faciunt ſummam $-\frac{1}{2}$.

SVMMATIO GENERALIS

INFINITARVM ALIARVM PROGRESSIONVM AD HOC GENVS REFERENDARVM.

Theorema.

Si cognita fuerit ſummatio huius progreſſionis

$$Az + Bz^2 + Cz^3 + Dz^4 \ldots\ldots + Nz^n$$

tum

tum ſemper etiam has progreſſiones ſummare licebit

$$S = A x \,\text{ſin.}\, \Phi + B x^2 \,\text{ſin.}\, 2\Phi + C x^3 \,\text{ſin.}\, 3\Phi + \ldots\ldots N x^n \,\text{ſin.}\, n\Phi$$

et $$T = A x \,\text{coſ.}\, \Phi + B x^2 \,\text{coſ.}\, 2\Phi + C x^3 \,\text{coſ.}\, 3\Phi + \ldots\ldots N x^n \,\text{coſ.}\, n\Phi.$$

Demonſtratio.

Cum ſumma progreſſionis

$$A z + B z^2 + C z^3 \ldots\ldots + N z^n$$

ſit certa quaedam functio quantitatis variabilis z, deſignetur ea hac formula $\Delta : z$; tum vero ponendo vt ante

$$p = \text{coſ.}\,\Phi + \sqrt{-1}\,\text{ſin.}\,\Phi \quad \text{et} \quad q = \text{coſ.}\,\Phi - \sqrt{-1}\,\text{ſin.}\,\Phi,$$

vt fiat

$$\text{ſin.}\, n\Phi = \frac{1}{2\sqrt{-1}}(p^n - q^n) \quad \text{et} \quad \text{coſ.}\, n\Phi = \tfrac{1}{2}(p^n + q^n);$$

ſi hae formulae in propoſitis ſeriebus ſubſtituantur, earum ſummae ita expreſſae obtinebuntur

$$2S\sqrt{-1} = \Delta : px - \Delta : qx \quad \text{et} \quad 2T = \Delta : px + \Delta : qx$$

vbi notandum, in vtraque formula quantitates imaginarias litteris p et q inuolutas ſponte ſe deſtruere, ita, vt pro ſummis S et T valores reales ſint prodituri; atque haec ſummatio perinde ſuccedet, ſiue propoſitae ſeries in infinitum progrediantur, ſiue alicubi terminentur.

Exempl. I.

Sint omnes coëfficientes A, B, C .. $= 1$ et ſeries in infinitum continuetur; eritque $\Delta : z = \frac{z}{1-z}$; hinc ergo pro ſerie priore

$S = x\,\text{ſin.}\,\Phi + x^2\,\text{ſin.}\,2\Phi + x^3\,\text{ſin.}\,3\Phi + x^4\,\text{ſin.}\,4\Phi + \ldots$ in infi-

 habe-

habebitur

$$2 S \sqrt{-1} = \frac{p x}{1 - p x} - \frac{q x}{1 - q x} = \frac{(p - q) x}{1 - (p + q) x + p q x^2}$$

quae expreſſio ob $p - q = 2\sqrt{-1}\,\text{fin.}\,\Phi$ et $p + q = 2\,\text{cof.}\,\Phi$ et $p q = 1$ praebebit

$$S = \frac{x\,\text{fin.}\,\Phi}{1 - 2 x\,\text{cof.}\,\Phi + x^2}.$$

Pro altera vero ſerie

$$T = x\,\text{cof.}\,\Phi + x^2\,\text{cof.}\,2\,\Phi + x^3\,\text{cof.}\,3\,\Phi + \ldots\ldots \text{ in infinit.}$$

fiet $2\,T = \frac{p x}{1 - p x} + \frac{q x}{1 - q x} = \frac{(p + q) x - 2 p q . x^2}{1 - (p + q) x + p q . x^2}$

ſiue $T = \frac{x\,\text{cof.}\,\Phi - x^2}{1 - 2 x\,\text{cof.}\,\Phi + x^2}.$

Coroll. 1.

Hinc igitur ſi $x = 1$, oriuntur ſummationes ſupra datae, ſcilicet

$$S = \frac{\text{fin.}\,\Phi}{2(1 - \text{cof.}\,\Phi)} = \tfrac{1}{2}\,\text{cotang.}\,\tfrac{1}{2}\,\Phi$$

et $T = -\frac{1}{2}$

qui caſus eo magis eſt notatu dignus, quod ſinguli termini ſunt quantitates variabiles, cum tamen ſumma ſit quantitas conſtans.

Coroll. 2.

Semper autem in genere quantitatem x ita aſſumere licebit, vt ſumma ſeriei datae quantitati a fiat aequalis; pro priore ſerie ſinuum autem erit

$$\frac{x\,\text{fin.}\,\Phi}{1 - 2 x\,\text{cof.}\,\Phi + x^2} = a;$$

ac ſi hinc littera x determinetur, certo erit

$$a = x\,\text{fin.}\,\Phi + x^2\,\text{fin.}\,2\,\Phi + x^3\,\text{fin.}\,3\,\Phi + \ldots\ldots;$$

ſimi-

similique modo si statuatur

$$\frac{x\,\text{cos.}\,\Phi - x^2}{1 - 2x\,\text{cos.}\,\Phi + x^2} = a$$

ex eaque valor litterae x eruatur; etiam erit

$$a = x\,\text{cos.}\,\Phi + x^2\,\text{cos.}\,2\Phi + x^3\,\text{cos.}\,3\Phi + \ldots\ldots$$

Exempl. 2.

Sit nunc

$$\Delta : z = z + \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 + \tfrac{1}{4}z^4 + \ldots\ldots \text{ in infinit.}$$

$= \text{log.}\,\frac{1}{1-z}$; ita, vt series propositae iam sint

$$S = x\,\text{sin.}\,\Phi + \tfrac{1}{2}x^2\,\text{sin.}\,2\Phi + \tfrac{1}{3}x^3\,\text{sin.}\,3\Phi + \ldots\ldots$$

et $T = x\,\text{cos.}\,\Phi + \tfrac{1}{2}x^2\,\text{cos.}\,2\Phi + \tfrac{1}{3}x^3\,\text{cos.}\,3\Phi + \ldots\ldots$

habebimus

$$2S\sqrt{-1} = \text{log.}\,\frac{1}{1-px} - \text{log.}\,\frac{1}{1-qx} = \text{log.}\,\frac{1-qx}{1-px}$$

siue

$$2S\sqrt{-1} = \frac{1 - x\,\text{cos.}\,\Phi + x\sqrt{-1}\,\text{sin.}\,\Phi}{1 - x\,\text{cos.}\,\Phi - x\sqrt{-1}\,\text{sin.}\,\Phi}$$

pro cuius formulae reductione consideretur haec forma

$$\text{log.}\,\frac{f + g\sqrt{-1}}{f - g\sqrt{-1}}$$

de qua constat, si ponatur $\frac{g}{f} = \text{tang.}\,\omega$, hunc logarithmum fore $= 2\,\omega.\sqrt{-1}$; hinc ergo quaeratur angulus ω, vt sit

$$\text{tang.}\,\omega = \frac{x\,\text{sin.}\,\Phi}{1 - x\,\text{cos.}\,\Phi};$$

vnde protenus sequitur $S = \omega$; pro altera progressione cum sit

$$2T = \text{log.}\,\frac{1}{1-px} + \text{log.}\,\frac{1}{1-qx} = -\text{log.}\,(1 - 2x\,\text{cos.}\,\Phi + x^2)$$

 habe-

habetur

$$T = -\tfrac{1}{2}\log.(1 - 2x\,\mathrm{cof.}\,\Phi + x^2).$$

Coroll.

Pro priore ergo progreffione cum fit

$$\frac{x\,\mathrm{fin.}\,\Phi}{1 - x\,\mathrm{cof.}\,\Phi} = \mathrm{tang.}\,\omega$$

hinc elicitur

$$x = \frac{\mathrm{tang.}\,\omega}{\mathrm{fin.}\,\Phi + \mathrm{cof.}\,\Phi\,\mathrm{tang.}\,\omega} = \frac{\mathrm{fin.}\,\omega}{\mathrm{fin.}\,(\Phi + \omega)}$$

qua valore fubftituto nancifcimur hanc fummationem attentione maxime dignam

$$\omega = \frac{\mathrm{fin.}\,\omega\,\mathrm{fin.}\,\Phi}{\mathrm{fin.}\,(\Phi + \omega)} + \frac{\mathrm{fin.}\,\omega^2\,\mathrm{fin.}\,2\,\Phi}{2\,\mathrm{fin.}\,(\Phi + \omega)^2} + \frac{\mathrm{fin.}\,\omega^3\,\mathrm{fin.}\,3\,\Phi}{3.\,\mathrm{fin.}\,(\Phi + \omega)^3}$$
$$+ \frac{\mathrm{fin.}\,\omega^4\,\mathrm{fin.}\,4\,\Phi}{4.\,\mathrm{fin.}\,(\Phi + \omega)^4} + \ldots\ldots\ldots$$

vbi fi $\omega = \frac{\pi}{2}$, vt fit $\mathrm{fin.}\,\omega = 1$; et $\mathrm{fin.}\,(\Phi + \omega) = \mathrm{cof.}\,\Phi$ oritur haec fummatio maxime concinna

$$\frac{\pi}{2} = \frac{\mathrm{fin.}\,\Phi}{1.\,\mathrm{cof.}\,\Phi} + \frac{\mathrm{fin.}\,2\,\Phi}{2.\,\mathrm{cof.}\,\Phi^2} + \frac{\mathrm{fin.}\,3\,\Phi}{3.\,\mathrm{cof.}\,\Phi^3} + \ldots\ldots\ldots$$

quam feriem iam olim in calculo differentiali ex diuerfiffimis principiis fum adeptus; quae eo magis memorabilis erat vifa, quod vtcunque angulus Φ accipiatur, fumma feriei femper maneat eadem $= \frac{\pi}{2}$.

OBSER-