# EVOLVTIO FORMVLAE INTEGRALIS

$\int x^{f-1}\,dx\,(lx)^{\frac{m}{n}}$

## INTEGRATIONE A VALORE $x=0$ AD $x=1$ EXTENSA.

Auctore

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## Theorema I.

I.

Si $n$ denotat numerum integrum positiuum quemcunque et formulae $\int x^{f-1}\,dx\,(1-x^g)^n$ integratio a valore $x=0$ vsque ad $x=1$ extendatur, erit eius valor:

$$=\frac{g^n}{f}\cdot\frac{1.\quad 2.\quad 3.\quad \ldots\ldots\ldots\ n}{(f+g)(f+2g)(f+3g)\ldots\ldots(f+ng)}.$$

### Demonstratio.

Notum est in genere integrationem formulae $\int x^{f-1}dx\,(1-x^g)^m$ reduci posse ad integrationem huius $\int x^{f-1}\,dx\,(1-x^g)^{m-1}$ quoniam quantitates constantes A et B ita definire licet, vt fiat

$$\int x^{f-1}dx\,(1-x^g)^m = A\int x^{f-1}dx\,(1-x^g)^{m-1} + Bx^f(1-x^g)^m$$

M 2 sumtis

ſumtis enim differentialibus prodit haec aequatio:

$$x^{f-1}dx(1-x^g)^m = Ax^{f-1}dx(1-x^g)^{m-1} + Bfx^{f-1}dx(1-x^g)^m - Bmgx^{f+g-1}dx(1-x^g)^{m-1}$$

quae per $x^{f-1}dx(1-x^g)^{m-1}$ diuiſa dat:

$$1 - x^g = A + Bf(1-x^g) - Bmgx^g \text{ ſeu}$$

$$1 - x^g = A - Bmg + B(f+mg)(1-x^g)$$

quae aequatio vt conſiſtere poſſit, neceſſe eſt ſit

$$1 = B(f+mg) \quad \text{et} \quad A = Bmg$$

vnde colligimus $B = \frac{1}{f+mg}$ et $A = \frac{mg}{f+mg}$.

Quocirca habebimus ſequentem reductionem generalem:

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1} + \frac{1}{f+mg}\cdot x^f(1-x^g)^m$$

quae cum euaneſcat poſito $x = 0$; ſiquidem ſit $f > 0$, conſtantis additione haud eſt opus. Quare extenſo vtroque integrali vsque ad $x = 1$, pars integralis poſtrema ſponte euaneſcit, eritque pro caſu $x = 1$

$$\int x^{f-1}dx(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}dx(1-x^g)^{m-1}.$$

Cum igitur ſumto $m = 1$ ſit $\int x^{f-1}dx(1-x^g)^0 = \frac{1}{f}x^f = \frac{1}{f}$ poſito $x = 1$, nanciſcimur pro eodem caſu $x = 1$ ſequentes valores:

$$\int x^{f-1}dx(1-x^g)^1 = \frac{g}{f}\cdot\frac{1}{f+g}$$

$$\int x^{f-1}dx(1-x^g)^2 = \frac{g^2}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}$$

$$\int x^{f-1}dx(1-x^g)^3 = \frac{g^3}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}\cdot\frac{3}{f+3g}$$

hinc-

hincque pro numero quocunque integro positiuo $n$ concludimus fore

$$\int x^{f-1}dx(1-x^g)^n = \frac{g^n}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}\cdot\frac{3}{f+3g}\cdots\cdots\frac{n}{f+ng}$$

si modo numeri $f$ et $g$ sint positiui.

## Coroll. 1.

2. Hinc ergo viciſſim valor huiusmodi producti ex quotcunque factoribus formati, per formulam integralem exprimi poteſt, ita vt ſit

$$\frac{1.\quad 2.\quad 3.\ \ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots(f+ng)} = \frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$$

integrali hoc a valore $x=0$ vsque ad $x=1$ extenſo.

## Coroll. 2.

3. Quodſi ergo huiusmodi habeatur progreſſio:

$$\frac{1}{f+g};\ \frac{1.\quad 2}{(f+g)(f+2g)};\ \frac{1.\quad 2.\quad 3}{(f+g)(f+2g)(f+3g)};\ \frac{1.\quad 2.\quad 3.\quad 4}{(f+g)(f+2g)(f+3g)(f+4g)};\ \text{etc.}$$

eius terminus generalis qui indici indefinito $n$ conuenit commode hac forma integrali $\frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$ repraeſentatur, cuius ope ea progreſſio interpolari, eiusque termini indicibus fractis reſpondentes exhiberi poterunt.

## Coroll. 3.

4. Si loco $n$ ſcribamus $n-1$, habebimus:

$$\frac{1.\quad 2.\quad 3\ \ldots\ldots(n-1)}{(f+g)(f+2g)(f+3g)\ldots\ldots(f+(n-1)g)}=\frac{f}{g^{n-1}}\int x^{f-1}dx(1-x^{g})^{n-1}$$

quae per $\frac{n}{f+ng}$ multiplicata praebet

$$\frac{1.\quad 2.\quad 3\ \ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots\ldots(f+ng)}=\frac{f.\,ng}{g^{n}(f+ng)}\int x^{f-1}dx(1-x^{g})^{n-1}.$$

## Scholion I.

5. Hanc posteriorem formam immediate ex praecedente deriuare licuisset, cum modo demonstraverimus esse:

$$\int x^{f-1}dx(1-x^{g})^{n}=\frac{ng}{f+ng}\int x^{f-1}dx(1-x^{g})^{n-1}$$

siquidem vtrumque integrale a valore $x=0$ vsque ad $x=1$ extendatur; quam integralium determinationem in sequentibus vbique subintelligi oportet. Deinde etiam perpetuo est tenendum, quantitates $f$ et $g$ esse positiuas, quippe quam conditionem demonstratio allata absolute postulat. Quod autem ad numerum $n$ attinet, quatenus eo index cuiusque termini progressionis (§. 3.) designatur, nihil impedit, quominus eo numeri quicunque siue positiui siue negatiui denotentur, quandoquidem eius progressionis omnes termini etiam indicibus negatiuis respondentes per formulam integralem datam exhiberi censentur. Interim tamen probe tenendum est hanc reductionem

$$\int x^{f-1}dx(1-x^{g})^{m}=\frac{mg}{f+mg}\int x^{f-1}dx(1-x^{g})^{m-1}$$

non esse veritati consentaneam, nisi sit $m>0$; quia alioquin

alioquin pars algebraica $\frac{1}{f+mg}x^f(1-x^g)^m$ non euanefceret pofito $x=1$.

## Scholion 2.

6. Huiusmodi feries, quas tranfcendentes appellare licet, quia termini indicibus fractis refpondentes funt quantitates tranfcendentes, iam olim in Comment. Petrop. Tomo V. fufius fum profecutus; vnde hoc loco non tam iftas progreffiones, quam eximias formularum integralium comparationes, quae inde deriuantur, diligentius fum fcrutaturus. Cum fcilicet oftendiffem huius producti indefiniti $1.2.3\ldots n$ valorem hac formula integrali $\int dx\,(l\frac{1}{x})^n$ ab $x=0$ ad $x=1$ extenfa exprimi, quae res quoties $n$ eft numerus integer pofitiuus per ipfam integrationem eft manifefta, eos cafus examini fubieci, quibus pro $n$ numeri fracti accipiuntur; vbi quidem ex ipfa formula integrali neutiquam patet, ad quodnam genus quantitatum tranfcendentium hi termini referri debeant. Singulari autem artificio eosdem terminos ad quadraturas magis cognitas reduxi, quod propterea maxime dignum videtur, vt maiori ftudio perpendatur.

## Problema 1.

7. Cum demonftratum fit effe:

$$\frac{1.\;2.\;3\;\ldots\ldots\;n}{(f+g)(f+2g)(f+3g)\ldots\ldots(f+ng)}=\frac{f}{g^n}\int x^{f-1}dx(1-x^g)^n$$

inte-

integrali ab $x = 0$ ad $x = 1$ extenso; eiusdem producti casu quo $g = 0$ valorem per formulam integralem assignare.

## Solutio.

Posito $g = 0$ in formula integrali membrum $(1 - x^g)^n$ euanescit, simul vero etiam denominator $g^n$, vnde quaestio huc redit vt fractionis $\frac{(1 - x^g)^n}{g^n}$ valor definiatur casu $g = 0$, quo tam numerator quam denominator euanescit. Hunc in finem spectetur $g$ vt quantitas infinite parua, et cum sit $x^g = e^{g\,lx}$ fiet $x^g = 1 + g\,lx$ ideoque $(1 - x^g)^n = g^n(-lx)^n = g^n(l\frac{1}{x})^n$; ex quo pro hoc casu formula nostra integralis abit in $f\int x^{f-1}dx(l\frac{1}{x})^n$ ita vt iam habeatur

$$\frac{1.\ 2.\ 3\ldots\ldots n}{f^n} = f\int x^{f-1}dx\,(l\tfrac{1}{x})^n$$

seu $1.\ 2.\ 3\ldots\ldots n = f^{n+1}\int x^{f-1}dx(l\frac{1}{x})^n$.

## Coroll. 1.

8. Quoties $n$ est numerus integer positiuus, integratio formulae $\int x^{f-1}dx(l\frac{1}{x})^n$ succedit, eaque ab $x = 0$ ad $x = 1$ extensa reuera prodit id productum, cui istam formulam aequalem inuenimus. Sin autem pro $n$ capiantur numeri fracti eadem formula integralis inseruiet huic progressioni hypergeometricae interpolandae:

1; 1. 2; 1. 2. 3; 1. 2. 3. 4; 1. 2. 3. 4. 5; etc.

seu 1; 2; 6; 24; 120; 720; 5040; etc.

Coroll.

## Coroll. 2.

9 Si expreſſio modo inuenta per principalem diuidatur, orietur productum, cuius factores in progreſſione arithmetica quacunque progrediuntur:

$$(f+g)(f+2g)(f+3g)\ldots(f+ng)=f^n g^n.\frac{\int x^{f-1}dx\,(l\frac{1}{x})^n}{\int x^{f-1}dx(1-x^g)^n}$$

cuius ergo etiam valores, ſi $n$ ſit numerus fractus hinc aſſignare licebit.

## Coroll. 3.

10. Cum ſit

$$\int x^{f-1}dx\,(1-x^g)^n=\frac{n\,g}{f+n\,g}\int x^{f-1}dx\,(1-x^g)^{n-1}$$

erit etiam ſimili modo pro caſu $g=0$.

$$\int x^{f-1}dx\,(l\frac{1}{x})^n=\frac{n}{f}\int x^{f-1}dx\,(l\frac{1}{x})^{n-1}$$

hincque per iſtas alteras formulas integrales:

$$1.\,2.\,3\ldots\ldots n=nf^n\int x^{f-1}dx\,(l\frac{1}{x})^{n-1} \text{ et}$$

$$(f+g)(f+2g)\ldots(f+ng)=f^{n-1}g^{n-1}(f+ng).\frac{\int x^{f-1}dx\,(l\frac{1}{x})^{n-1}}{\int x^{f-1}dx\,(1-x^g)^{n-1}}.$$

## Scholion.

11. Cum inuenerimus eſſe:

$$1.\,2.\,3\ldots\ldots n=f^{n+1}\int x^{f-1}dx\,(l\frac{1}{x})^n$$

patet hanc formulam integralem non a valore quantitatis $f$ pendere, quod etiam facile perſpicitur ponendo $x^f=y$, vnde fit $f x^{f-1}dx=dy$, et $l\frac{1}{x}=$

 $-lx$

$-lx = -\frac{1}{f}ly = \frac{1}{f}l\frac{1}{y}$, ideoque $f^n(l\frac{1}{x})^n = (l\frac{1}{y})^n$, ita vt fit

$$1.\ 2.\ 3 \ldots\ldots n = \int dy\,(l\tfrac{1}{y})^n$$

quae formula ex priori nafcitur ponendo $f = 1$. Pro interpolatione ergo huiusmodi formarum totum negotium huc reducitur, vt iftius formulae integralis $\int dx\,(l\frac{1}{x})^n$ valores definiantur, quando exponens $n$ eft numerus fractus. Veluti fi $n$ fit $= \frac{1}{2}$, affignari oportet valorem huius formulae $\int dx \sqrt{l\frac{1}{x}}$, quem olim iam oftendi effe $= \frac{1}{2}\sqrt{\pi}$ denotante $\pi$ circuli peripheriam cuius diameter $= 1$: pro aliis autem numeris fractis eius valorem ad quadraturas curuarum algebraicarum altioris ordinis reuocare docui. Quae reductio cum minime fit obuia, atque tum folum locum habeat, quando formulae $\int dx\,(l\frac{1}{x})^n$ integratio a valore $x = 0$ ad $x = 1$ extenditur, fingulari attentione digna videtur. Etfi autem iam olim hoc argumentum tractaui, tamen quia per plures ambages eo fum perductus, idem hic refumere et concinnius euoluere conftitui.

## Theorema 2.

12. Si formulae integrales a valore $x = 0$ vsque ad $x = 1$ extendantur et $n$ denotet numerum integrum pofitiuum erit:

$$\frac{1.\quad 2.\quad 3 \ldots\ldots n}{(n+1)(n+2)(n+3)\ldots 2n} = \tfrac{1}{2}ng\int x^{f+ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}$$

quicunque numeri pofitiui loco $f$ et $g$ accipiantur.

Demon-

## Demonſtratio.

Cum ſupra (§. 4.) oſtenderimus eſſe:

$$\frac{1.\;2.\;3\ldots\ldots n}{(f+g)(f+2g)\ldots(f+ng)}=\frac{f.\,ng}{g^n(f+ng)}\int x^{f-1}dx(1-x^g)^{n-1}$$

habebimus ſi loco $n$ ſcribamus $2n$

$$\frac{1.\;2.\;3\ldots 2n}{(f+g)(f+2g)\ldots(f+2ng)}=\frac{f.\,2ng}{g^{2n}(f+2ng)}\int x^{f-1}dx(1-x^g)^{2n-1}$$

Diuidatur nunc prima aequatio per ſecundam, ac prodibit iſta tertia:

$$\frac{(f+(n+1)g)(f+(n+2)g)\ldots(f+2ng)}{(n+1)(n+2)\ldots\ldots 2n}=\frac{g^n(f+2ng)}{2(f+ng)}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}$$

At ſi in prima aequatione loco $f$ ſcribatur $f+ng$, orietur haec aequatio quarta:

$$\frac{1.\;2.\;3\ldots\ldots\ldots\ldots n}{(f+(n+1)g)(f+(n+2)g)\ldots(f+2ng)}=\frac{(f+ng)ng}{g^n(f+2ng)}\int x^{f+ng-1}dx(1-x^g)^{n-1}$$

Multiplicetur haec quarta aequatio per illam tertiam ac reperietur ipſa aequatio demonſtranda:

$$\frac{1.\;2.\;3\ldots\ldots n}{(n+1)(n+2)(n+3)\ldots 2n}=\tfrac{1}{2}ng\int x^{f+ng-1}dx(1-x^g)^{n-1}\cdot\frac{\int x^{f-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}}.$$

## Coroll. 1.

13. Si in prima aequatione ſtatuatur $f=n$ et $g=1$ orietur idem productum:

$$\frac{1.\;2.\;3\ldots\ldots n}{(n+1)(n+2)\ldots\ldots 2n}=\tfrac{1}{2}n\int x^{n-1}dx(1-x)^{n-1}$$

qua aequatione cum illa collata adipiſcimur:

$$\frac{\int x^{n-1}\,dx\,(1-x)^{n-1}}{g\int x^{f+ng-1}\,dx\,(1-x^g)^{n-1}} = \frac{\int x^{f-1}\,dx\,(1-x^g)^{n-1}}{\int x^{f-1}\,dx\,(1-x^g)^{2n-1}}.$$

## Coroll. 2.

14. Si in illa aequatione loco $x$ ſcribamus $x^g$, fiet

$$\frac{1.\ 2.\ 3 \ldots\ldots\ldots n}{(n+1)(n+2)\ldots\ldots 2n} = \tfrac{1}{2}\,n\,g \int x^{ng-1}\,dx\,(1-x^g)^{n-1}$$

ita vt iam conſequamur iſtam comparationem inter ſequentes formulas integrales:

$$\int x^{ng-1}\,dx(1-x^g)^{n-1} = \int x^{f+ng-1}\,dx(1-x^g)^{n-1}.\frac{\int x^{f-1}\,dx(1-x^g)^{n-1}}{\int x^{f-1}\,dx(1-x^g)^{2n-1}}.$$

## Coroll. 3.

15. Si in aequatione theorematis ponamus $g=0$ ob $(1-x^g)^m = g^m(l\tfrac{1}{x})^m$, poteſtates ipſius $g$ ſe deſtruent orieturque haec aequatio:

$$\frac{1.\ 2.\ 3\ldots n}{(n+1)(n+2)\ldots 2n} = \tfrac{1}{2}\,n\int x^{f-1}\,dx\,(l\tfrac{1}{x})^{n-1}.\frac{\int x^{f-1}\,dx\,(l\tfrac{1}{x})^{n-1}}{\int x^{f-1}\,dx\,(l\tfrac{1}{x})^{2n-1}}$$

vnde colligimus

$$\frac{(\int x^{f-1}\,dx\,(l\tfrac{1}{x})^{n-1})^2}{\int x^{f-1}\,dx\,(l\tfrac{1}{x})^{2n-1}} = g\int x^{ng-1}\,dx\,(1-x^g)^{n-1}$$

ſeu ob

$\int x^{f-1}\,dx(l\tfrac{1}{x})^{n-1} = \tfrac{f}{n}\int x^{f-1}\,dx(l\tfrac{1}{x})^n$ hanc

$$\frac{2f}{n}.\frac{(\int x^{f-1}\,dx(l\tfrac{1}{x})^n)^2}{\int x^{f-1}\,dx(l\tfrac{1}{x})^{2n}} = g\int x^{ng-1}\,dx\,(1-x^g)^{n-1}.$$

Coroll.

## Coroll. 4.

16. Ponamus hic $f=1$, $g=2$ et $n=\frac{m}{2}$ vt $m$ fit numerus integer positiuus, et ob $\int dx(l\frac{1}{x})^m = 1.2.3\ldots\ldots m$ erit

$$\frac{4}{m}\cdot\frac{(\int dx(l\frac{1}{x})^{\frac{m}{2}})^2}{1.2.3\ldots m} = 2\int x^{m-1}dx(1-x^2)^{\frac{m}{2}-1}$$

hincque

$$\int dx(l\tfrac{1}{x})^{\frac{m}{2}} = \sqrt{1.2.3\ldots m.\tfrac{m}{2}\int x^{m-1}dx(1-x^2)^{\frac{m}{2}-1}}$$

et fumendo $m=1$ ob $\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2}$ habebitur

$$\int dx\sqrt{l\tfrac{1}{x}} = \sqrt{\tfrac{1}{2}\int\frac{dx}{\sqrt{(1-xx)}}} = \tfrac{1}{2}\sqrt{\pi}.$$

## Scholion.

17. En ergo fuccinctam demonftrationem theorematis olim a me prolati, quod fit $\int dx\sqrt{l\frac{1}{x}}=\frac{1}{2}\sqrt{\pi}$, eamque ab interpolationis ratione, qua tum vfus fueram, libera. Deducta fcilicet hic ea ex hoc theoremate quo inueni effe:

$$\frac{(\int x^{f-1}dx(l\frac{1}{x})^{n-1})^2}{\int x^{f-1}dx(l\frac{1}{x})^{2n-1}} = g\int x^{ng-1}dx(1-x^g)^{n-1}$$

Principale autem theorema, vnde hoc eft deductum ita fe habet

$$g.\frac{\int x^{f-1}dx(1-x^g)^{n-1}.\int x^{f+ng-1}dx(1-x^g)^{n-1}}{\int x^{f-1}dx(1-x^g)^{2n-1}} = \int x^{n-1}dx(1-x)^{n-1}$$

vtrumque enim membrum per integrationem ab $x = 0$ ad $x = 1$ extensam euoluitur in hoc productum numericum:

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots (n-1)}{(n+1)(n+2)\ldots\ldots(2n-1)}$$

Ac si alteri membro speciem latius patentem tribuere velimus, theorema ita proponi poterit vt sit:

$$g \frac{\int x^{f-1} dx (1-x^g)^{n-1} \cdot \int x^{f+ng-1} dx (1-x^g)^{n-1}}{\int x^{f-1} dx (1-x^g)^{2n-1}}$$
$$= k \int x^{nk-1} dx (1-x^k)^{n-1}$$

hicque si capiatur $g = 0$, fit

$$\frac{(\int x^{f-1} dx (l\frac{1}{x})^{n-1})^2}{\int x^{f-1} dx (l\frac{1}{x})^{2n-1}} = k \int x^{nk-1} dx (1-x^k)^{n-1}.$$

Imprimis igitur notandum est, quod illa aequalitas subsistat, quicunque numeri loco $f$ et $g$ accipiantur casu quidem $f = g$, ea est manifesta, cum sit

$$\int x^{g-1} dx (1-x^g)^{n-1} = \frac{1-(1-x^g)^n}{ng} = \frac{1}{ng}$$

fiet enim

$$2g \int x^{ng+g-1} dx (1-x^g)^{n-1} = k \int x^{nk-1} dx (1-x^k)^{n-1}$$

et quia

$$\int x^{ng+g-1} dx (1-x^g)^{n-1} = \tfrac{1}{2} \int x^{ng-1} dx (1-x^g)^{n-1},$$

aequalitas est perspicua, quia $k$ pro lubitu accipere licet. Eodem autem modo, quo ad hoc theorema perueni, ad alia similia pertingere licet.

Theo-

## Theorema. 3.

18. Si sequentes formulae integrales a valore $x=0$ ad $x=1$ extendantur et $n$ denotet numerum integrum positiuum quemcunque, erit

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots n}{(2n+1)(2n+2)\ldots 3n} = \tfrac{2}{3} ng \int x^{f+2ng-1} dx (1-x^g)^{n-1} \cdot \frac{\int x^{f-1} dx (1-x^g)^{2n-1}}{\int x^{f-1} dx (1-x^g)^{3n-1}}$$

quicunque numeri positiui pro $f$ et $g$ accipiantur.

## Demonstratio.

In praecedente Theoremate iam vidimus esse:

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots 2n}{(f+g)(f+2g)\ldots(f+2ng)} = \frac{f \cdot 2ng}{g^{2n}(f+2ng)} \int x^{f-1} dx (1-x^g)^{2n-1}$$

simili autem modo, si in forma principali loco $n$ scribamus $3n$ habebimus:

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots 3n}{(f+g)(f+2g)\ldots(f+3ng)} = \frac{f \cdot 3ng}{g^{3n}(f+3ng)} \int x^{f-1} dx (1-x^g)^{3n-1}$$

ex quo illa aequatio per hanc diuisa producit:

$$\frac{(f+(2n+1)g)(f+(2n+2)g)\ldots\ldots(f+3ng)}{(2n+1)(2n+2)\ldots\ldots\ldots 3n} = \frac{2g^n(f+3ng)}{3(f+2ng)} \cdot \frac{\int x^{f-1} dx (1-x^g)^{2n-1}}{\int x^{f-1} dx (1-x^g)^{3n-1}}.$$

Verum si in aequatione principali (§. 4.) loco $f$ scribamus $f+2gn$ adipiscimur hanc aequationem:

$$\frac{1.\ 2.\ 3.\ \ldots\ldots\ldots\ n}{(f+(2n+1)g)(f+(2n+2)g)\ldots(f+3ng)} = \frac{(f+2ng).ng}{g^n(f+3ng)} \int x^{f+2ng-1}dx(1-x^g)^{n-1}.$$

Multiplicetur nunc haec aequatio per praecedentem, et orietur ipsa aequatio, quam demonstrari oportet:

$$\frac{1.\ 2.\ 3.\ \ldots\ n}{(2n+1)(2n+2)\ldots\ 3n} = \frac{2}{3}ng.\int x^{f+2ng-1}dx(1-x^g)^{n-1}.\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}}.$$

## Coroll. 1.

18. Eundem valorem ex aequatione principali nanciscimur ponendo $f = 2n$ et $g = 1$, ita vt sit:

$$\frac{1.\ 2.\ 3.\ \ldots\ldots\ n}{(2n+1)(2n+2)\ldots 3n} = \frac{2}{3}n\int x^{2n-1}dx(1-x)^{n-1}$$

quae formula integralis loco $x$ scribendo $x^k$ transformatur in hanc $\frac{2}{3}nk\int x^{2nk-1}dx(1-x^k)^{n-1}$, ita vt sit

$$g\int x^{f+2ng-1}dx(1-x^g)^{n-1}.\frac{\int x^{f-1}dx(1-x^g)^{2n-1}}{\int x^{f-1}dx(1-x^g)^{3n-1}} = k\int x^{2nk-1}dx(1-x^k)^{n-1}.$$

## Coroll. 2.

20. Si hic statuamus $g = 0$, ob $1 - x^g = g\,l\frac{1}{x}$ habebimus hanc aequationem:

$$\int x^{f-1}dx(l\tfrac{1}{x})^{n-1}.\frac{\int x^{f-1}dx(l\tfrac{1}{x})^{2n-1}}{\int x^{f-1}dx(l\tfrac{1}{x})^{3n-1}} = k\int x^{2nk-1}dx(1-x^k)^{n-1}$$

 cum

cum igitur ante inueniſſemus

$$\frac{\int x^{f-1}dx(l\tfrac{1}{x})^{n-1})^2}{\int x^{f-1}dx(l\tfrac{1}{x})^{2n-1}} = k\int x^{nk-1}dx(1-x^k)^{n-1}$$

habebimus has aequationes in ſe multiplicando:

$$\frac{(\int x^{f-1}dx(l\tfrac{1}{x})^{n-1})^3}{\int x^{f-1}dx(l\tfrac{1}{x})^{3n-1}} = k^2\int x^{nk-1}dx(1-x^k)^{n-1}.\int x^{2nk-1}dx(1-x^k)^{n-2}.$$

## Coroll. 3.

21. Sine vlla reſtrictione hic ponere licet $f=1$; tum ergo ſumto $n=\frac{1}{3}$ et $k=3$ erit

$$\frac{(\int dx(l\tfrac{1}{x})^{-\frac{2}{3}})^3}{\int dx(l\tfrac{1}{x})^0} = 9\int dx(1-x^3)^{-\frac{2}{3}}.\int xdx(1-x^3)^{-2}$$

et ob $\int dx(l\tfrac{1}{x})^{-\frac{2}{3}} = 3\int dx(l\tfrac{1}{x})^{\frac{1}{3}}$ et $\int dx(l\tfrac{1}{x})^0 = 1$,

$$(\int dx(l\tfrac{1}{x})^{\frac{1}{3}})^3 = \int dx(1-x^3)^{-\frac{2}{3}}.\int xdx(1-x^3)^{-\frac{2}{3}}$$

tum vero ſumto $n=\frac{2}{3}$ et $k=3$ erit

$$\frac{(\int dx(l\tfrac{1}{x})^{-\frac{1}{3}})^3}{\int dx(l\tfrac{1}{x})} = 9\int xdx(1-x^3)^{-\frac{1}{3}}.\int x^3dx(1-x^3)^{-\frac{1}{3}}$$

ſeu $(\int dx(l\tfrac{1}{x})^{\frac{2}{3}})^3 = 4\int xdx(1-x^3)^{-\frac{1}{3}}\int x^3dx(1-x^3)^{-\frac{1}{3}}$.

## Theorema generale.

22. Si ſequentes formulae integrales a valore $x=0$ vsque ad $x=1$ extendantur et $n$ denotet numerum integrum poſitiuum quemcunque, erit

$$\frac{1.\ 2.\ 3\ \ldots\ldots n}{(\lambda n+1)(\lambda n+2)\ldots(\lambda+1)n}=\frac{\lambda}{\lambda+1}ng\int x^{f+\lambda ng-1}dx(1-x^g)^{n-1}.\ \frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}$$

quicunque numeri poſitiui pro litteris $f$ et $g$ accipiantur.

## Demonſtratio.

Cum ſit vti ſupra oſtendimus:

$$\frac{1.\ 2\ \ldots\ldots\ n}{(f+g)(f+2g)\ldots\ldots(f+ng)}=\frac{f.\ ng}{g^n(f+ng)}\int x^{f-1}dx(1-x^g)^{n-1}$$

ſi hic loco $n$ ſcribamus primo $\lambda n$ tum vero $(\lambda+1)n$ nanciſcemur has duas aequationes

$$\frac{1.\ 2\ \ldots\ldots\ \lambda n}{(f+g)(f+2g)\ldots\ldots(f+\lambda ng)}=\frac{f.\ \lambda ng}{g^{\lambda n}(f+\lambda ng)}\int x^{f-1}dx(1-x^g)^{\lambda n-1}$$

$$\frac{1.\ 2\ \ldots\ldots\ (\lambda+1)n}{(f+g)(f+2g)\ldots\ldots(f+(\lambda+1)ng)}=\frac{f.\ (\lambda+1)ng}{g^{(\lambda+1)n}(f+(\lambda+1)ng)}\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}$$

quarum illa per hanc diuiſa praebet:

$$\frac{(f+\lambda ng+g)(f+\lambda ng+2g)\ldots\ldots(f+\lambda ng+ng)}{(\lambda n+1)(\lambda n+2)\ \ldots\ldots\ldots\ (\lambda n+n)}=g^n\frac{\lambda(f+\lambda ng+ng)}{(\lambda+1)(f+\lambda ng)}\cdot\frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}.$$

At ſi in aequatione prima loco $f$ ſcribamus $f+\lambda ng$ obtinebimus:

$$\frac{1 \cdot 2 \cdot \ldots \ldots \ldots \ldots n}{(f+\lambda ng+g)(f+\lambda ng+2g)\ldots(f+\lambda ng+ng)} = \frac{(f+\lambda ng)ng}{g^n(f+\lambda ng+ng)} \int x^{f+\lambda ng-1}dx(1-x^g)^{n-1}$$

quae duae aequationes in ſe ductae producunt ipſam aequalitatem demonſtrandam:

$$\frac{1 \cdot 2 \cdot \ldots \ldots n}{(\lambda n+1)(\lambda n+2)\ldots\ldots(\lambda n+n)} = \frac{\lambda ng}{\lambda+1} \int x^{f+\lambda ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{(\lambda+1)n-1}}.$$

## Coroll. 1.

23. Si in aequatione principali ſtatuamus $f = \lambda n$ et $g = 1$ reperiemus etiam:

$$\frac{1 \cdot 2 \cdot \ldots \ldots \ldots n}{(\lambda n+1)(\lambda n+2)\ldots\ldots(\lambda n+n)} = \frac{\lambda n}{\lambda+1} \int x^{\lambda n-1}dx(1-x)^{n-1}$$

quae forma loco $x$ ſcribendo $x^k$ abit in hanc:

$$\frac{\lambda nk}{\lambda+1} \int x^{\lambda nk-1}dx(1-x^k)^{n-1}$$

ita vt habeamus hoc theorema latiſſime patens:

$$g\int x^{f+\lambda ng-1}dx(1-x^g)^{n-1} \cdot \frac{\int x^{f-1}dx(1-x^g)^{\lambda n-1}}{\int x^{f-1}dx(1-x^g)^{\lambda n+n-1}} = k\int x^{\lambda nk-1}dx(1-x^k)^{n-1}.$$

## Coroll. 2.

24. Hoc iam theorema locum habet, etiamſi $n$ non ſit numerus integer, quin etiam cum nume-

rum $\lambda$ pro lubitu accipere liceat, loco $\lambda n$ scribamus $m$, et perueniemus ad hoc theorema:

$$\frac{\int x^{f-1}dx(1-x^g)^{m-1}}{\int x^{f-1}dx(1-x^g)^{m+n-1}} = \frac{k\int x^{mk-1}dx(1-x^k)^{n-1}}{g\int x^{f+mg-1}dx(1-x^g)^{n-1}}.$$

## Coroll. 3.

25. Si ponamus $g=0$; ob $1-x^g = g\,l\frac{1}{x}$, hoc theorema istam induet formam:

$$\frac{\int x^{f-1}dx(l\frac{1}{x})^{m-1}}{\int x^{f-1}dx(l\frac{1}{x})^{m+n-1}} = \frac{k\int x^{mk-1}dx(1-x^k)^{n-1}}{\int x^{f-1}dx(l\frac{1}{x})^{n-1}}$$

quae commodius ita repraesentatur:

$$\frac{\int x^{f-1}dx(l\frac{1}{x})^{n-1}\cdot\int x^{f-1}dx(l\frac{1}{x})^{m-1}}{\int x^{f-1}dx(l\frac{1}{x})^{m+n-1}} = k\int x^{mk-1}dx(1-x^k)^{n-1}$$

vbi euidens est numeros $m$ et $n$ inter se permutari posse.

## Scholion.

26. Duplicem ergo deteximus fontem, vnde innumerabiles formularum integralium comparationes haurire licet; alter fons §. 24. patefactus complectitur huiusmodi formulas integrales

$$\int x^{p-1}dx(1-x^g)^{q-1},$$

quas iam ante aliquod tempus pertractaui in obseruationibus circa integralia formularum

$$\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$$

a vaa valore $x = 0$ vsque ad $x = 1$ extensa, vbi ostendi primo litteras $p$ et $q$ inter se permutari posse, vt sit

$$\int x^{p-1} dx (1 - x^n)^{\frac{q}{n} - 1} = \int x^{q-1} dx (1 - x^n)^{\frac{p}{n} - 1}$$

tum vero etiam esse

$$\int \frac{x^{p-1} dx}{(1 - x^n)^{\frac{p}{n}}} = \frac{\pi}{n \sin. \frac{p\pi}{n}}$$

imprimis autem demonstraui esse:

$$\int \frac{x^{p-1} dx}{\sqrt[n]{(1-x^n)^{n-q}}} \cdot \int \frac{x^{p+q-1} dx}{\sqrt[n]{(1-x^n)^{n-r}}} = \int \frac{x^{p-1} dx}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{p+r-1} dx}{\sqrt[n]{(1-x^n)^{n-q}}}$$

in qua aequatione comparatio in §. 24. inuenta iam continetur; ita vt hinc nihil noui, quod non iam euolu[illegible], deduci queat. Alterum igitur fontem §. 25 indicatum hic potissimum inuestigandum suscipio, vbi cum sine vlla restrictione sumi queat $f = 1$, aequatio nostra primaria erit:

$$\frac{\int dx (l\frac{1}{x})^{n-1} \cdot \int dx (l\frac{1}{x})^{m-1}}{\int dx (l\frac{1}{x})^{m+n-1}} = k \int x^{mk-1} dx (1 - x^k)^{n-1}$$

cuius beneficio valores formulae integralis $\int dx (l\frac{1}{x})^{\lambda}$ quando $\lambda$ non est numerus integer ad quadraturas curuarum algebraicarum reuocare licebit; quandoquidem quoties $\lambda$ est numerus integer, integratio habetur absoluta, quoniam est

$$\int dx (l\frac{1}{x})^{\lambda} = 1 . 2 . 3 \ldots\ldots \lambda.$$

Maximi autem momenti quaestio versatur circa eos

 casus,

casus, quibus $\lambda$ est numerus fractus, quos ergo pro ratione denominationis hic successiue sum definiturus.

## Problema 2.

27. Denotante $i$ numerum integrum positiuum definire valorem formulae integralis $\int dx\,(l\frac{1}{x})^{\frac{i}{2}}$ integratione ab $x=0$ vsque ad $x=1$ extensa.

## Solutio.

In aequatione nostra generali faciamus $m=n$ eritque

$$\frac{(\int dx\,(l\frac{1}{x})^{n-1})^2}{\int dx\,(l\frac{1}{x})^{2n-1}} = k\int x^{nk-1}\,dx\,(1-x^k)^{n-1}$$

Sit iam $n-1=\frac{i}{2}$, et ob $2n-1=i+1$ erit

$$\int dx\,(l\frac{1}{x})^{2n-1} = 1.\,2.\,3\,\ldots\ldots\,(i+1)$$

sumatur porro $k=2$ vt sit $nk-1=i+1$, fietque

$$\frac{(\int dx\,\sqrt{(l\frac{1}{x})^i})^2}{1.\,2.\,3\,\ldots\,(i+1)} = 2\int x^{i+1}\,dx\,(1-x^2)^{\frac{i}{2}}$$

ideoque

$$\frac{\int dx\,\sqrt{(l\frac{1}{x})^i}}{\sqrt{(1.\,2.\,3\,\ldots\,(i+1))}} = \sqrt{2\int x^{i+1}\,dx\,\sqrt{(1-x^2)^i}}$$

vbi euidens est pro $i$ numeros tantum impares sumi conuenire, quoniam pro paribus euolutio per se est manifesta.

## Coroll. 1.

28. Omnes autem casus facile reducuntur ad $i=1$, vel adeo ad $i=-1$, dummodo enim $i+1$, non

non ſit numerus negatiuus reductio inuenta locum habet. Pro hoc ergo caſu erit:

$$\int\frac{dx}{\sqrt{l\frac{1}{x}}}=\sqrt{2}\int\frac{dx}{\sqrt{(1-xx)}}=\sqrt{\pi}\ \text{ob}\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2}.$$

## Coroll. 2.

29. Hoc autem caſu principali expedito ob $\int dx(l\frac{1}{x})^n=n\int dx(l\frac{1}{x})^{n-1}$ habebimus,

$$\int dx\sqrt{l\frac{1}{x}}=\frac{1}{2}\sqrt{\pi};\ \int dx(l\frac{1}{x})^{\frac{3}{2}}=\frac{1\cdot 3}{2\cdot 2}\sqrt{\pi}$$

atque in genere

$$\int dx(l\frac{1}{x})^{\frac{2n+1}{2}}=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdots\cdots\frac{(2n+1)}{2}\sqrt{\pi}.$$

## Problema 3.

30. Denotante $i$ numerum integrum poſitiuum definire valorem formulae integralis $\int dx(l\frac{1}{x})^{\frac{i}{2}-1}$ integratione ab $x=0$ ad $x=1$ extenſa.

## Solutio.

Inchoemus ab aequatione praecedentis problematis:

$$\frac{(\int dx(l\frac{1}{x})^{n-1})^2}{\int dx(l\frac{1}{x})^{2n-1}}=k\int x^{nk-1}dx(1-x^k)^{n-1}$$

atque in forma generali ſtatuamus $m=2n$, vt habeatur:

$$\frac{\int dx(l\frac{1}{x})^{n-1}\cdot\int dx(l\frac{1}{x})^{2n-1}}{\int dx(l\frac{1}{x})^{3n-1}}=k\int x^{2nk-1}dx(1-x^k)^{n-1}$$

ac

ac multiplicando has duas aequalitates adipiſcimur:

$$\frac{(\int dx(l\frac{1}{x})^{n-1})^3}{\int dx(l\frac{1}{x})^{3n-1}} = kk\int x^{nk-1}dx(1-x^k)^{n-1} \cdot \int x^{2nk-1}dx(1-x^k)^{n-1}.$$

Hic iam ponatur $n = \frac{i}{3}$ vt ſit

$$\int dx(l\frac{1}{x})^{i-1} = 1.\ 2.\ 3 \ldots\ldots (i-1)$$

ſumaturque $k = 3$ ac prodibit

$$\frac{(\int dx\sqrt[3]{(l\frac{1}{x})^{i-3}})^3}{1.2.3\ldots(i-1)} = 9\int x^{i-1}dx\sqrt[3]{(1-x^3)^{i-3}} \cdot \int x^{2i-1}dx\sqrt[3]{(1-x^3)^{i-3}}$$

vnde concludimus

$$\frac{\int dx\sqrt[3]{(l\frac{1}{x})^{i-3}}}{\sqrt[3]{1.\ 2.\ 3\ldots(i-1)}} = \sqrt[3]{9}\int\frac{x^{i-1}dx}{\sqrt[3]{(1-x^3)^{3-i}}} \cdot \int\frac{x^{2i-1}dx}{\sqrt[3]{(1-x^3)^{3-i}}}.$$

## Coroll. 1.

31. Bini hic occurrunt caſus principales, a quibus reliqui omnes pendent, ponendo ſcilicet vel $i = 1$ vel $i = 2$, qui ſunt:

$$\text{I. } \int\frac{dx}{\sqrt[3]{(l\frac{1}{x})^2}} = \sqrt[3]{9}\int\frac{dx}{\sqrt[3]{(1-x^3)^2}} \cdot \int\frac{x\,dx}{\sqrt[3]{(1-x^3)^2}}$$

$$\text{II. } \int\frac{dx}{\sqrt[3]{l\frac{1}{x}}} = \sqrt[3]{9}\int\frac{x\,dx}{\sqrt[3]{(1-x^3)}} \cdot \int\frac{x^3\,dx}{\sqrt[3]{(1-x^3)}}$$

quae poſterior forma ob $\int\frac{x^3\,dx}{\sqrt[3]{(1-x^3)}} = \frac{1}{3}\int\frac{dx}{\sqrt[3]{(1-x^3)}}$

abit

abit in

$$\int \frac{dx}{\sqrt[3]{l\frac{1}{x}}} = \sqrt[3]{3}\int \frac{dx}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x\,dx}{\sqrt[3]{(1-x^3)}}.$$

## Coroll. 2.

32. Si vti in obferuationibus meis ante allegatis breuitatis gratia ponamus $\int \frac{x^{p-1}\,dx}{\sqrt[3]{(1-x^3)^{3-q}}} = (\frac{p}{q})$, atque vt ibi pro hac claffe $(\frac{2}{1}) = \frac{\pi}{3\ \text{fin.}\frac{\pi}{3}} = \alpha$, tum vero $(\frac{1}{3}) = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = A$, erit

I. $\int \frac{dx}{\sqrt[3]{(l\frac{1}{x})^2}} = \sqrt[3]{9}(\frac{1}{1})(\frac{2}{1}) = \sqrt[3]{9\alpha A}$

II. $\int \frac{dx}{\sqrt[3]{(l\frac{1}{x})^1}} = \sqrt[3]{3}(\frac{1}{3})(\frac{2}{3}) = \sqrt[3]{\frac{3\alpha\alpha}{A}}.$

## Coroll. 3.

33. Pro cafu ergo priori habebimus,

$\int dx\sqrt[3]{(l\frac{1}{x})^{-2}} = \sqrt[3]{9\alpha A}$; $\int dx\sqrt[3]{l\frac{1}{x}} = \frac{1}{3}\sqrt[3]{9\alpha A}$ et

$\int dx\sqrt[3]{(l\frac{1}{x})^{3n+1}} = \frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}\cdots\cdots\frac{3n+1}{3}\sqrt[3]{9\alpha A}$

pro altero vero cafu

$\int dx\sqrt[3]{(l\frac{1}{x})^{-1}} = \sqrt[3]{\frac{3\alpha\alpha}{A}}$; $\int dx\sqrt[3]{(l\frac{1}{x})^2} = \frac{2}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}$ et

$\int dx\sqrt[3]{(l\frac{1}{x})^{3n-1}} = \frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}\cdots\cdots\cdots\frac{3n-1}{3}\sqrt[3]{\frac{3\alpha\alpha}{A}}.$

 Pro-

## Problema 4.

34. Denotante $i$ numerum integrum positiuum definire valorem formulae integralis $\int dx\left(l\frac{1}{x}\right)^{\frac{i}{4}-1}$ integratione ab $x=0$ ad $x=1$ extensa.

## Solutio.

In solutione problematis praecedentis perducti sumus ad hanc aequationem

$$\frac{\left(\int dx\left(l\frac{1}{x}\right)^{n-1}\right)^2}{\int dx\left(l\frac{1}{x}\right)^{2n-1}}=kk\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}}$$

forma generalis autem sumendo $m=3n$ praebet

$$\frac{\int dx\left(l\frac{1}{x}\right)^{n-1}\int dx\left(l\frac{1}{x}\right)^{3n-1}}{\int dx\left(l\frac{1}{x}\right)^{4n-1}}=k\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}}$$

quibus coniungendis adipiscimur,

$$\frac{\left(\int dx\left(l\frac{1}{x}\right)^{n-1}\right)^4}{\int dx\left(l\frac{1}{x}\right)^{4n-1}}=k^3\int\frac{x^{nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{2nk-1}dx}{(1-x^k)^{1-n}}\cdot\int\frac{x^{3nk-1}dx}{(1-x^k)^{1-n}}$$

Sit nunc $n=\frac{i}{4}$ et sumatur $k=4$ fietque

$$\frac{\int dx\left(l\frac{1}{x}\right)^{\frac{i}{4}-1}}{\sqrt[4]{1.2.3\ldots(i-1)}}=\sqrt[4]{4^3}\int\frac{x^{i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{2i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}\cdot\int\frac{x^{3i-1}dx}{\sqrt[4]{(1-x^4)^{4-i}}}$$

Coroll.

## Coroll 1.

35. Si igitur sit $i = 1$, habebimus

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{-3}} = \sqrt[4]{4^3} \int \frac{dx}{\sqrt[4]{(1-x^4)^3}} \cdot \int \frac{x\,dx}{\sqrt[4]{(1-x^4)^2}} \cdot \int \frac{xx\,dx}{\sqrt[4]{(1-x^4)^1}}$$

quae expressio si littera P designetur erit in genere

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{4n-3}} = \tfrac{1}{4} \cdot \tfrac{5}{4} \cdot \tfrac{9}{4} \ldots\ldots\ldots \tfrac{4n-3}{4} \cdot P.$$

## Coroll 2.

36. Pro altero casu principali sumamus $i = 3$ eritque

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{-1}} = \sqrt[4]{2 \cdot 4^3} \int \frac{x^2\,dx}{\sqrt[4]{(1-x^4)}} \cdot \int \frac{x^5\,dx}{\sqrt[4]{(1-x^4)}} \cdot \int \frac{x^8\,dx}{\sqrt[4]{(1-x^4)}}$$

seu facta reductione ad simpliciores formas

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{-1}} = \sqrt[4]{8} \int \frac{xx\,dx}{\sqrt[4]{(1-x^4)}} \cdot \int \frac{x\,dx}{\sqrt[4]{(1-x^4)}} \cdot \int \frac{dx}{\sqrt[4]{(1-x^4)}}$$

quae expresio si littera Q designetur erit generatim

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{4n-1}} = \tfrac{3}{4} \cdot \tfrac{7}{4} \cdot \tfrac{11}{4} \ldots\ldots\ldots \tfrac{4n-1}{4} \cdot Q.$$

## Scholion.

37. Si formulam integralem $\int \frac{x^{p-1}\,dx}{\sqrt[4]{(1-x^4)^{4-q}}}$ hoc signo $(\frac{p}{q})$ indicemus, solutio problematis ita se habebit

$$\int dx \sqrt[4]{(l\tfrac{1}{x})^{i-4}} = \sqrt[4]{1 \cdot 2 \cdot 3 \ldots\ldots (i-1) \cdot 4^3 (\tfrac{i}{i})(\tfrac{2i}{i})(\tfrac{3i}{i})}$$

 et

et pro binis casibus euolutis fit

$$P=\sqrt[4]{4^3\left(\frac{1}{1}\right)\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)} \text{ et } Q=\sqrt[4]{8\left(\frac{3}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)}.$$

Statuamus nunc pro iis formulis quae a circulo pendent:

$$\left(\frac{3}{1}\right)=\frac{\pi}{4\,\text{sin.}\,\frac{\pi}{4}}=\alpha \text{ et } \left(\frac{2}{2}\right)=\frac{\pi}{4\,\text{sin.}\,\frac{2\pi}{4}}=\beta$$

pro transcendentibus autem altioris ordinis

$$\left(\frac{2}{1}\right)=\int\frac{x\,dx}{\sqrt[4]{(1-x^4)^3}}=\int\frac{dx}{\sqrt[2]{(1-x^4)}}=A$$

quippe a qua omnes reliquae pendent ac reperimus,

$$P=\sqrt[4]{4^3\frac{\alpha\alpha}{\beta}.AA} \text{ et } Q=\sqrt[4]{4.\alpha\alpha\beta.\frac{1}{AA}}$$

vnde patet esse $PQ=4\alpha=\frac{\pi}{\text{sin.}\,\frac{\pi}{4}}$. Cum autem sit $\alpha=\frac{\pi}{2\sqrt{2}}$ et $\beta=\frac{\pi}{4}$ erit $P=\sqrt[4]{32\pi AA}$ et $Q=\sqrt[4]{\frac{\pi^3}{8AA}}$ et $\frac{P}{Q}=\frac{4A}{\sqrt{\pi}}$.

## Problema 5.

38. Denotante $i$ numerum integrum positiuum definire valorem formulae integralis $\int dx\sqrt[5]{(l\frac{1}{x})^{i-5}}$ integratione ab $x=0$ ad $x=1$ extensa.

## Solutio.

Ex praecedentibus solutionibus iam satis est perspicuum pro hoc casu tandem peruentum iri ad hanc formam

$\int dx$

$$\frac{\int dx \sqrt[5]{(l\tfrac{1}{x})^{i-5}}}{\sqrt[5]{1.2.3\ldots(i-1)}} = \sqrt[5]{5^4} \int \frac{x^{i-1}\,dx}{\sqrt[5]{(1-x^5)^{5-i}}} . \int \frac{x^{2i-1}\,dx}{\sqrt[5]{(1-x^5)^{5-i}}} .$$

$$\int \frac{x^{3i-1}\,dx}{\sqrt[5]{(1-x^5)^{5-i}}} . \int \frac{x^{4i-1}\,dx}{\sqrt[5]{(1-x^5)^{5-i}}}$$

quae formulae integrales ad classem quintam dissertationis meae supra allegatae sunt referendae. Quare si modo ibi recepto signum $(\tfrac{p}{q})$ denotet hanc formulam $\int \frac{x^{p-1}\,dx}{\sqrt[5]{(1-x^5)^{5-q}}}$, valorem quaesitum ita commodius exprimere licebit, vt sit

$$\int dx \sqrt[5]{(l\tfrac{1}{x})^{i-5}} = \sqrt[5]{1.2.3\ldots(i-1)\,5^4 (\tfrac{i}{i})(\tfrac{2i}{i})(\tfrac{3i}{i})(\tfrac{4i}{i})}$$

vbi quidem sufficit ipsi $i$ valores quinario minores tribuisse: quando autem numeratores quinarium superant tenendum est esse:

$(\frac{5+m}{i}) = \frac{m}{m+i}(\frac{m}{i})$ tum vero porro

$$(\tfrac{10+m}{i}) = \frac{m}{m+i} . \frac{m+5}{m+i+5} (\tfrac{m}{i})$$

$$(\tfrac{15+m}{i}) = \frac{m}{m+i} . \frac{m+5}{m+i+5} . \frac{m+10}{m+i+10} (\tfrac{m}{i})$$

Deinde vero pro hac classe binae formulae quadraturam circuli inuoluunt quae sint.

$$(\tfrac{4}{1}) = \frac{\pi}{5 \sin. \frac{\pi}{5}} = \alpha \text{ et } (\tfrac{3}{2}) = \frac{\pi}{5 \sin. \frac{2\pi}{5}} = \beta$$

duae autem quadraturas altiores continent quae ponantur:

$$\left(\tfrac{3}{1}\right)=\int\frac{xx\,dx}{\sqrt[5]{(1-x^5)^4}}=\int\frac{dx}{\sqrt[5]{(1-x^5)^2}}=A \text{ et}$$

$$\left(\tfrac{2}{2}\right)=\int\frac{x\,dx}{\sqrt[5]{(1-x^5)^2}}=B$$

atque ex his valores omnium reliquarum formularum huius claſſis aſſignaui ſcilicet:

$$\left(\tfrac{5}{1}\right)=1;\ \left(\tfrac{5}{2}\right)=\tfrac{1}{2};\ \left(\tfrac{5}{3}\right)=\tfrac{1}{3};\ \left(\tfrac{5}{4}\right)=\tfrac{1}{4};\ \left(\tfrac{5}{5}\right)=\tfrac{1}{5}$$

$$\left(\tfrac{4}{1}\right)=\alpha;\ \left(\tfrac{4}{2}\right)=\tfrac{\beta}{A};\ \left(\tfrac{4}{3}\right)=\tfrac{\beta}{2B};\ \left(\tfrac{4}{4}\right)=\tfrac{\alpha}{3A}$$

$$\left(\tfrac{3}{1}\right)=A;\ \left(\tfrac{3}{2}\right)=\beta;\ \left(\tfrac{3}{3}\right)=\tfrac{\beta\beta}{\alpha B}$$

$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha B}{\beta};\ \left(\tfrac{2}{2}\right)=B;$$

$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha A}{\beta}.$$

## Coroll. 1.

39. Sumto exponente $i=1$ erit:

$$\int dx\,\sqrt[5]{\left(l\tfrac{1}{x}\right)^{-4}}=\sqrt[5]{5^4\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)}=\sqrt[5]{5^4\cdot\tfrac{\alpha^3}{\beta^2}A^2B}$$

vnde in genere concludimus fore denotante $n$ numerum integrum quemcunque

$$\int dx\,\sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-4}}=\tfrac{1}{5}\cdot\tfrac{6}{5}\cdot\tfrac{11}{5}\cdots\tfrac{5n-4}{5}\cdot\sqrt[5]{5^4\cdot\tfrac{\alpha^3}{\beta^2}A^2B}.$$

## Coroll. 2.

40. Sit nunc $i=2$ et cum prodeat:

$$\int dx\,\sqrt[5]{\left(l\tfrac{1}{x}\right)^{-3}}=\sqrt[5]{1\cdot 5^4\left(\tfrac{2}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{8}{5}\right)}$$

ob $\left(\tfrac{6}{5}\right)=\tfrac{1}{5}\left(\tfrac{1}{5}\right)=\tfrac{1}{5}\left(\tfrac{2}{1}\right)$ et $\left(\tfrac{8}{5}\right)=\tfrac{3}{5}\left(\tfrac{3}{5}\right)$

erit

erit haec expressio

$$\sqrt[5]{5^3\left(\tfrac{2}{5}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{2}\right)} = \sqrt[5]{5^3.\,\alpha\beta.\,\tfrac{BB}{A}} \text{ et in genere}$$

$$\int dx\, \sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-3}} = \tfrac{2}{5}.\,\tfrac{7}{5}.\,\tfrac{12}{5} \ldots\ldots \tfrac{5n-3}{5}\sqrt[5]{5^3.\,\alpha\beta.\,\tfrac{BB}{A}}.$$

## Coroll 3.

51. Sit $i = 3$ et forma inuenta:

$$\int dx\, \sqrt[5]{\left(l\tfrac{1}{x}\right)^{-2}} = \sqrt[5]{2.\,5^4\left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{9}{3}\right)\left(\tfrac{12}{3}\right)} \text{ ob}$$

$$\left(\tfrac{6}{3}\right) = \tfrac{1}{4}\left(\tfrac{3}{1}\right);\quad \left(\tfrac{9}{3}\right) = \tfrac{4}{7}\left(\tfrac{4}{3}\right);\quad \left(\tfrac{12}{3}\right) = \tfrac{2}{5}.\,\tfrac{7}{10}\left(\tfrac{3}{2}\right)$$

abit in $\sqrt[5]{2.\,5^2\left(\tfrac{3}{3}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{3}{2}\right)} = \sqrt[5]{5^2.\,\tfrac{\beta^4}{\alpha}.\,\tfrac{A}{BB}}$

vnde in genere colligitur:

$$\int dx\, \sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-2}} = \tfrac{3}{5}.\,\tfrac{8}{5}.\,\tfrac{13}{5} \ldots\ldots \tfrac{5n-2}{5}\sqrt[5]{5^2.\,\tfrac{\beta^4}{\alpha}.\,\tfrac{A}{BB}}.$$

## Coroll. 4.

42. Posito denique $i = 4$ forma nostra:

$$\int dx\, \sqrt[5]{\left(l\tfrac{1}{x}\right)^{-1}} = \sqrt[5]{6.\,5^4\left(\tfrac{4}{4}\right)\left(\tfrac{8}{4}\right)\left(\tfrac{12}{4}\right)\left(\tfrac{16}{4}\right)} \text{ ob}$$

$$\left(\tfrac{8}{4}\right) = \tfrac{3}{7}\left(\tfrac{4}{3}\right);\quad \left(\tfrac{12}{4}\right) = \tfrac{2}{6}.\,\tfrac{7}{11}\left(\tfrac{4}{2}\right);\quad \left(\tfrac{16}{4}\right) = \tfrac{1}{5}.\,\tfrac{6}{10}.\,\tfrac{11}{15}\left(\tfrac{4}{1}\right)$$

transformabitur in hanc:

$$\sqrt[5]{6.\,5\left(\tfrac{4}{4}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{4}{1}\right)} = \sqrt[5]{5.\,\tfrac{\alpha\alpha\beta\beta}{AAB}}$$

ita vt sit in genere

$$\int dx\, \sqrt[5]{\left(l\tfrac{1}{x}\right)^{5n-1}} = \tfrac{4}{5}.\,\tfrac{9}{5}.\,\tfrac{14}{5} \ldots\ldots \tfrac{5n-1}{5}\sqrt[5]{5.\,\alpha\alpha\beta\beta.\,\tfrac{1}{AAB}}.$$

Scho-

## Scholion.

43. Si valorem formulae integralis $\int dx\,(l\frac{1}{x})^{\lambda}$ hoc signo $[\lambda]$ repraesentemus, casus hactenus euoluti praebent:

$$[-\tfrac{4}{5}] = \sqrt[5]{5^4 \cdot \tfrac{\alpha^3}{\beta^2} \cdot A^2 B}\,;\quad [+\tfrac{1}{5}] = \tfrac{1}{5}\sqrt[5]{5^4 \cdot \tfrac{\alpha^3}{\beta^2} \cdot A^2 B}$$

$$[-\tfrac{3}{5}] = \sqrt[5]{5^3 \cdot \alpha\beta \cdot \tfrac{BB}{A}}\,;\quad [+\tfrac{2}{5}] = \tfrac{2}{5}\sqrt[5]{5^3 \cdot \alpha\beta \cdot \tfrac{BB}{A}}$$

$$[-\tfrac{2}{5}] = \sqrt[5]{5^2 \cdot \tfrac{\beta^4}{\alpha} \cdot \tfrac{A}{BB}}\,;\quad [+\tfrac{3}{5}] = \tfrac{3}{5}\sqrt[5]{5^2 \cdot \tfrac{\beta^4}{\alpha} \cdot \tfrac{A}{BB}}$$

$$[-\tfrac{1}{5}] = \sqrt[5]{5 \cdot \alpha^2\beta^2 \cdot \tfrac{1}{AAB}}\,;\quad [+\tfrac{4}{5}] = \tfrac{4}{5}\sqrt[5]{5 \cdot \alpha^2\beta^2 \cdot \tfrac{1}{AAB}}$$

vnde binis, quarum indices simul sumti fiunt $= 0$ coniungendis colligimus.

$$[+\tfrac{1}{5}] \cdot [-\tfrac{1}{5}] = \alpha = \frac{\pi}{5 \sin. \frac{\pi}{5}}$$

$$[+\tfrac{2}{5}] \cdot [-\tfrac{2}{5}] = 2\beta = \frac{2\pi}{5 \sin. \frac{2\pi}{5}}$$

$$[+\tfrac{3}{5}] \cdot [-\tfrac{3}{5}] = 3\beta = \frac{3\pi}{5 \sin. \frac{3\pi}{5}}$$

$$[+\tfrac{4}{5}] \cdot [-\tfrac{4}{5}] = 4\alpha = \frac{4\pi}{5 \sin. \frac{4\pi}{5}}.$$

Ex antecedente autem problemate simili modo deducimus:

$$[-\tfrac{3}{4}] = P = \sqrt[4]{4^3 \cdot \tfrac{\alpha\alpha}{\beta} \cdot AA}\,;\quad [+\tfrac{1}{4}] = \tfrac{1}{4}\sqrt[4]{4^3 \cdot \tfrac{\alpha\alpha}{\beta} \cdot AA}$$

$$[-\tfrac{1}{4}] = Q = \sqrt[4]{4 \cdot \alpha\alpha\beta \cdot \tfrac{1}{AA}}\,;\quad [+\tfrac{3}{4}] = \tfrac{3}{4}\sqrt[4]{4 \cdot \alpha\alpha\beta \cdot \tfrac{1}{AA}}$$

hinc-

hincque

$$[+\tfrac{1}{4}].[-\tfrac{1}{4}] = \alpha = \frac{\pi}{4\,\text{fin.}\,\frac{\pi}{4}}$$

$$[+\tfrac{3}{4}].[-\tfrac{3}{4}] = 3\alpha = \frac{3\,\pi}{4\,\text{fin.}\,\frac{3\,\pi}{4}}$$

vnde in genere hoc Theorema adipifcimur quod fit

$$[\lambda].[-\lambda] = \frac{\lambda\,\pi}{\text{fin.}\,\lambda\,\pi}$$

cuius ratio ex methodo interpolandi olim expofita ita reddi poteft:

$$\text{cum fit } [\lambda] = \frac{1^{1-\lambda}.2^{\lambda}}{1+\lambda}.\frac{2^{1-\lambda}.3^{\lambda}}{2+\lambda}.\frac{3^{1-\lambda}.4^{\lambda}}{3+\lambda}\text{ etc.}$$

$$\text{erit } [-\lambda] = \frac{1^{1+\lambda}.2^{-\lambda}}{1-\lambda}.\frac{2^{1+\lambda}.3^{-\lambda}}{2-\lambda}.\frac{3^{1+\lambda}.4^{-\lambda}}{3-\lambda}\text{ etc.}$$

hincque

$$[\lambda].[-\lambda] = \frac{1.1}{1-\lambda\lambda}.\frac{2.2}{4-\lambda\lambda}.\frac{3.3}{9-\lambda\lambda}\text{ etc.} = \frac{\lambda\,\pi}{\text{fin.}\,\lambda\pi}$$

vti alibi demonftraui.

## Problema 6 generale.

44. Si litterae $i$ et $n$ denotent numeros integros pofitiuos definire valorem formulae integralis $\int dx\,(l\frac{1}{x})^{\frac{i-n}{n}}$ feu $\int dx\,\sqrt[n]{(l\frac{1}{x})^{i-n}}$, integratione ab $x=0$ ad $x=1$ extenfa.

## Solutio.

Methodus hactenus vſitata quaeſitum valorem ſequenti modo per quadraturas curuarum algebraicarum expreſſum exhibebit:

$$\frac{\int dx \sqrt[n]{(l\frac{1}{x})^{i-n}}}{\sqrt[n]{1.2.3.(i-1)}} = \sqrt[n]{n^{n-1}} \int \frac{x^{i-1}\,dx}{\sqrt[n]{(1-x^n)^{n-i}}} \cdot \int \frac{x^{2i-1}\,dx}{\sqrt[n]{(1-x^n)^{n-i}}} \cdots \int \frac{x^{(n-1)i-1}\,dx}{\sqrt[n]{(1-x^n)^{n-i}}}.$$

Quod ſi iam breuitatis gratia formulam integralem $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ hoc charactere $(\frac{p}{q})$, formulam vero $\int dx \sqrt[n]{(l\frac{1}{x})^m}$ iſthoc $[\frac{m}{n}]$ deſignemus, ita vt $[\frac{m}{n}]$ valorem huius producti indefiniti 1. 2. 3 .... $z$ denotet exiſtente $z = \frac{m}{n}$, ſuccinctius valor quaeſitus hoc modo expreſſus prodibit:

$$[\tfrac{i-n}{n}] = \sqrt[n]{1.2.3\ldots(i-1)\,n^{n-1}\cdot(\tfrac{i}{i})(\tfrac{2i}{i})(\tfrac{3i}{i})\ldots\ldots(\tfrac{ni-i}{i})}$$

vnde etiam colligitur

$$[\tfrac{i}{n}] = \tfrac{i}{n}\sqrt[n]{1.2.3\ldots(i-1)\,n^{n-1}\cdot(\tfrac{i}{i})(\tfrac{2i}{i})(\tfrac{3i}{i})\ldots(\tfrac{ni-i}{i})}.$$

Hic ſemper numerum $i$ ipſo $n$ minorem accepiſſe ſufficiet quoniam pro maioribus notum eſt eſſe:

$$[\tfrac{i+n}{n}] = \tfrac{i+n}{n}[\tfrac{i}{n}];\ \text{item}\ [\tfrac{i+2n}{n}] = \tfrac{i+n}{n}\cdot\tfrac{i+2n}{n}[\tfrac{i}{n}]\ \text{etc.}$$

hocque modo tota inueſtigatio ad eos tantum caſus reducitur, quibus fractionis $\frac{i}{n}$ numerator $i$ denominatore $n$ eſt minor. Praeterea vero de formulis integrategralibus $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}} = (\frac{p}{q})$, ſequentia notaſſe iuvabit:

I. Litteras $p$ et $q$ inter ſe eſſe permutabiles vt ſit $(\frac{p}{q}) = (\frac{q}{p})$.

II. Si alteruter numerorum $p$ vel $q$ ipſi exponenti $n$ aequetur, valorem formulae integralis fore algebraicum ſcilicet:

$$(\frac{n}{p}) = (\frac{p}{n}) = \frac{1}{p}. \text{ ſeu } (\frac{n}{q}) = (\frac{q}{n}) = \frac{1}{q}.$$

III. Si ſumma numerorum $p+q$ ipſi exponenti $n$ aequatur; formulae integralis $(\frac{p}{q})$ valorem per circulum exhiberi poſſe, cum ſit:

$$(\frac{p}{n-p}) = (\frac{n-p}{p}) = \frac{\pi}{n \text{ ſin.} \frac{p\pi}{n}} \text{ et } (\frac{q}{n-q}) = (\frac{n-q}{q}) = \frac{\pi}{n \text{ ſin.} \frac{q\pi}{n}}.$$

IV. Si alteruter numerorum $p$ vel $q$ maior ſit exponente $n$, formulam integralem $(\frac{p}{q})$ ad aliam revocari poſſe, cuius termini ſint ipſo $n$ minores, quod ſit ope huius reductionis

$$(\frac{p+n}{q}) = \frac{p}{p+q} (\frac{p}{q}).$$

V. Inter plures huiusmodi formulas integrales talem relationem intercedere vt ſit:

$$(\frac{p}{q})(\frac{p+q}{r}) = (\frac{p}{r})(\frac{p+r}{q}) = (\frac{q}{r})(\frac{q+r}{p})$$

cuius ope omnes reductiones reperiuntur quas in obſeruationibus circa has formulas expoſui.

## Coroll. 1.

45. Si hoc modo ope reductionis n°. IV. indicatae formam inuentam ad singulos casus accommodemus, eos sequenti ratione simplicissime exhibere poterimus. Ac primo quidem pro casu $n=2$, quo nulla opus est reductione habebimus:

$$[\tfrac{1}{2}] = \tfrac{1}{2}\sqrt[2]{2(\tfrac{1}{1})} = \tfrac{1}{2}\sqrt[2]{\frac{\pi}{\text{sin.}\,\frac{\pi}{2}}} = \tfrac{1}{2}\sqrt{\pi}.$$

## Coroll. 2.

46. Pro casu $n=3$ habebimus has reductiones:

$$[\tfrac{1}{3}] = \tfrac{1}{3}\sqrt[3]{3^2.(\tfrac{1}{1})(\tfrac{2}{1})}$$

$$[\tfrac{2}{3}] = \tfrac{2}{3}\sqrt[3]{3.\,1.(\tfrac{2}{2})(\tfrac{1}{2})}.$$

## Coroll. 3.

47. Pro casu $n=4$ hae tres reductiones obtinentur:

$$[\tfrac{1}{4}] = \tfrac{1}{4}\sqrt[4]{4^3(\tfrac{1}{1})(\tfrac{2}{1})(\tfrac{3}{1})}$$

$$[\tfrac{2}{4}] = \tfrac{2}{4}\sqrt[4]{4^2.\,2.(\tfrac{2}{2})^2(\tfrac{4}{2})} = \tfrac{1}{2}\sqrt[2]{4(\tfrac{2}{2})}\ \text{ob}\ (\tfrac{4}{2}) = \tfrac{1}{2}$$

$$[\tfrac{3}{4}] = \tfrac{3}{4}\sqrt[4]{4.\,1.\,2(\tfrac{3}{3})(\tfrac{2}{3})(\tfrac{1}{3})}$$

cum in media sit $(\frac{2}{2}) = (\frac{2}{4-2}) = \frac{\pi}{4}$ erit vtique vt ante

$$[\tfrac{2}{4}] = [\tfrac{1}{2}] = \tfrac{1}{2}\sqrt{\pi}.$$

Coroll.

## Coroll. 4.

48. Sit nunc $n = 5$, et prodeunt hae quatuor reductiones:

$$\left[\tfrac{1}{5}\right] = \tfrac{1}{5}\sqrt[5]{5^4}\cdot\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)$$

$$\left[\tfrac{2}{5}\right] = \tfrac{2}{5}\sqrt[5]{5^3\cdot 1}\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)$$

$$\left[\tfrac{3}{5}\right] = \tfrac{3}{5}\sqrt[5]{5^2\cdot 1\cdot 2}\left(\tfrac{3}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{2}{3}\right)$$

$$\left[\tfrac{4}{5}\right] = \tfrac{4}{5}\sqrt[5]{5\cdot 1\cdot 2\cdot 3}\left(\tfrac{4}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{1}{4}\right).$$

## Coroll. 5.

49. Sit $n = 6$, et habebimus has reductiones:

$$\left[\tfrac{1}{6}\right] = \tfrac{1}{6}\sqrt[6]{6^5}\cdot\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)$$

$$\left[\tfrac{2}{6}\right] = \tfrac{2}{6}\sqrt[6]{6^4\cdot 2}\left(\tfrac{2}{2}\right)^2\left(\tfrac{4}{2}\right)^2\left(\tfrac{6}{2}\right) = \tfrac{1}{3}\sqrt[3]{6^2}\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)$$

$$\left[\tfrac{3}{6}\right] = \tfrac{3}{6}\sqrt[6]{6^3\cdot 3\cdot 3}\left(\tfrac{3}{3}\right)^3\left(\tfrac{6}{3}\right)^2 = \tfrac{1}{2}\sqrt[2]{6}\left(\tfrac{3}{3}\right)$$

$$\left[\tfrac{4}{6}\right] = \tfrac{4}{6}\sqrt[6]{6^2\cdot 2\cdot 4\cdot 2}\left(\tfrac{4}{4}\right)^2\left(\tfrac{2}{4}\right)^2\left(\tfrac{6}{4}\right) = \tfrac{2}{3}\sqrt[3]{6\cdot 2}\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right)$$

$$\left[\tfrac{5}{6}\right] = \tfrac{5}{6}\sqrt[6]{6\cdot 1\cdot 2\cdot 3\cdot 4}\left(\tfrac{5}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{2}{5}\right)\left(\tfrac{1}{5}\right).$$

## Coroll. 6.

50. Posito $n = 7$ sequentes sex prodeunt aequationes:

$$\left[\tfrac{1}{7}\right] = \tfrac{1}{7}\sqrt[7]{7^6}\left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)\left(\tfrac{6}{1}\right)$$

$$\left[\tfrac{2}{7}\right] = \tfrac{2}{7}\sqrt[7]{7^5\cdot 1}\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)$$

$$\left[\tfrac{3}{7}\right] = \tfrac{3}{7}\sqrt[7]{7^4 . 1 . 2 \left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)}$$

$$\left[\tfrac{4}{7}\right] = \tfrac{4}{7}\sqrt[7]{7^3 . 1 . 2 . 3 \left(\tfrac{4}{4}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{3}{4}\right)}$$

$$\left[\tfrac{5}{7}\right] = \tfrac{5}{7}\sqrt[7]{7^2 . 1 . 2 . 3 . 4 \left(\tfrac{5}{5}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{1}{5}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{2}{5}\right)}$$

$$\left[\tfrac{6}{7}\right] = \tfrac{6}{7}\sqrt[7]{7 . 1 . 2 . 3 . 4 . 5 \left(\tfrac{6}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right)}.$$

## Coroll. 7.

51. Sit $n = 8$, et feptem hae reductiones impetrabuntur.

$$\left[\tfrac{1}{8}\right] = \tfrac{1}{8}\sqrt[8]{8^7 \left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)\left(\tfrac{6}{1}\right)\left(\tfrac{7}{1}\right)}$$

$$\left[\tfrac{2}{8}\right] = \tfrac{2}{8}\sqrt[8]{8^6 . 2 \left(\tfrac{2}{2}\right)^2\left(\tfrac{4}{2}\right)^2\left(\tfrac{6}{2}\right)^2\left(\tfrac{8}{2}\right)} = \tfrac{1}{4}\sqrt[4]{8^3 \left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)}$$

$$\left[\tfrac{3}{8}\right] = \tfrac{3}{8}\sqrt[8]{8^5 . 1 . 2 \left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{7}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)}$$

$$\left[\tfrac{4}{8}\right] = \tfrac{4}{8}\sqrt[8]{8^4 . 4 . 4 . 4 \left(\tfrac{4}{4}\right)^4\left(\tfrac{8}{4}\right)^3} = \tfrac{1}{2}\sqrt[2]{8 \left(\tfrac{4}{4}\right)}$$

$$\left[\tfrac{5}{8}\right] = \tfrac{5}{8}\sqrt[8]{8^3 . 1 . 2 . 3 . 4 \left(\tfrac{5}{5}\right)\left(\tfrac{2}{5}\right)\left(\tfrac{7}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{1}{5}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{3}{5}\right)}$$

$$\left[\tfrac{6}{8}\right] = \tfrac{6}{8}\sqrt[8]{8^2 . 4 . 2 . 6 . 4 . 2 \left(\tfrac{6}{6}\right)^2\left(\tfrac{4}{6}\right)^2\left(\tfrac{2}{6}\right)^2\left(\tfrac{8}{6}\right)} = \tfrac{3}{4}\sqrt[4]{8 . 2 . 4 \left(\tfrac{6}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{2}{6}\right)}$$

$$\left[\tfrac{7}{8}\right] = \tfrac{7}{8}\sqrt[8]{8 . 1 . 2 . 3 . 4 . 5 . 6 \left(\tfrac{7}{7}\right)\left(\tfrac{6}{7}\right)\left(\tfrac{5}{7}\right)\left(\tfrac{4}{7}\right)\left(\tfrac{3}{7}\right)\left(\tfrac{2}{7}\right)\left(\tfrac{1}{7}\right)}.$$

## Scholion.

52. Superfluum foret hos cafus vlterius euoluere cum ex allatis ordo iftarum formularum fatis perfpiciatur. Si enim in formula propofita $\left[\frac{m}{n}\right]$ numeri $m$ et $n$ fint inter fe primi lex eft manifefta, cum fiat

$$\left[\tfrac{m}{n}\right] = \tfrac{m}{n}\sqrt[n]{n^{n-m} . 1 . 2 \ldots (m-1) . \left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right) \ldots \left(\tfrac{n-1}{m}\right)}$$

fin

ſin autem hi numeri $m$ et $n$ communem habeant diuiſorem expediet quidem fractionem $\frac{m}{n}$ ad minimam formam reduci et ex caſibus praecedentibus quaeſitum valorem peti, interim tamen etiam operatio hoc modo inſtitui poterit. Cum expreſſio quaeſita certe hanc habeat formam

$$[\tfrac{m}{n}] = \tfrac{m}{n} \sqrt[n]{n^{n-m}}\, P.\, Q$$

vbi Q eſt productum ex $n-1$ formulis integralibus P vero productum ex aliquot numeris abſolutis, primum pro illo producto Q inueniendo, continuetur haec formularum ſeries $(\frac{m}{m})(\frac{2\,m}{m})(\frac{3\,m}{m})$ donec numerator ſuperet exponentem $n$, eiusque loco exceſſus ſupra $n$ ſcribatur, qui ſi ponatur $=\alpha$, vt iam formula noſtra ſit $(\frac{\alpha}{m})$; hic ipſe numerator $\alpha$ dabit factorem producti P tum hinc formularum ſeries porro ſtatuatur $(\frac{\alpha}{m})(\frac{\alpha+m}{m})(\frac{\alpha+2m}{m})$ etc. donec iterum ad numeratorem exponente $n$ maiorem perueniatur, formulaque prodeat $(\frac{n+\beta}{m})$ cuius loco ſcribi oportet $(\frac{\beta}{m})$, ſimulque hinc factor $\beta$ in productum P inferatur, ſicque progredi conueniet, donec pro Q prodierint $n-1$ formulae. Quae operationes quo facilius intelligantur, caſum formulae $[\frac{9}{12}] = \frac{9}{12}\sqrt[12]{12^3}\, P.\, Q$ hoc modo euoluamus, vbi inueſtigatio litterarum Q et P ita inſtituetur.

Pro Q. . . . $(\frac{9}{9})(\frac{6}{9})(\frac{3}{9})(\frac{12}{9})(\frac{9}{9})(\frac{6}{9})(\frac{3}{9})(\frac{12}{9})(\frac{9}{9})(\frac{6}{9})(\frac{3}{9})$

Pro P . . . . . 6. 3 9. 6. 3 9. 6. 3

ſicque

ſicque reperitur:

$$Q = \left(\tfrac{9}{9}\right)^3 \left(\tfrac{6}{9}\right)^3 \left(\tfrac{3}{9}\right)^3 \left(\tfrac{12}{9}\right)^2 \quad \text{et}$$

$$P = 6^3 . \ 3^3 . \ 9^2 .$$

Cum igitur ſit $\left(\tfrac{12}{9}\right) = \tfrac{1}{9}$ ſit $PQ = 6^3 . 3^3 \left(\tfrac{9}{9}\right) \left(\tfrac{6}{9}\right)^3 \left(\tfrac{3}{9}\right)^3$ ideoque

$$\left[\tfrac{9}{12}\right] = \tfrac{3}{4} \sqrt[4]{12 . 6 . 3 . \left(\tfrac{9}{9}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{9}\right)}.$$

## Theorema.

53. Quicunque numeri integri poſitiui litteris $m$ et $n$ indicentur, erit ſemper ſignandi modo ante expoſito:

$$\left[\tfrac{m}{n}\right] = \tfrac{m}{n} \sqrt[n]{n^{n-m} . 1 . 2 . 3 \ldots (m-1) \left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right) \ldots \left(\tfrac{n-1}{m}\right)}.$$

## Demonſtratio.

Pro caſu, quo $m$ et $n$ ſunt numeri inter ſe primi, veritas theorematis in antecedentibus eſt euicta, quod autem etiam locum habeat, ſi illi numeri $m$ et $n$ commune diuiſore gaudeant, inde quidem non liquet: verum ex hoc ipſo, quod pro caſibus, quibus $m$ et $n$ ſunt numeri primi, veritas conſtat, tuto concludere licet, theorema in genere eſſe verum. Minime quidem diffiteor hoc concludendi genus prorſus eſſe ſingulare, ac plerisque ſuſpectum videri debere. Quare quo nullum dubium relinquatur quoniam pro caſibus, quibus numeri $m$ et $n$ inter ſe ſunt compoſiti, geminam expreſſionem ſumus nacti, vtriusque conſenſum pro caſibus ante euolutis oſtendiſſe iuuabit. Inſigne autem iam ſuppeditat

peditat firmamentum cafus $m = n$, quo forma noſtra manifeſto vnitatem producit.

## Coroll. 1.

54. Primus caſus conſenſus demonſtrationem poſtulans eſt quo $m = 2$ et $n = 4$, pro quo ſupra §. 47. inuenimus

$$\left[\tfrac{2}{4}\right] = \tfrac{2}{4} \sqrt[4]{4^2} \cdot \left(\tfrac{2}{2}\right)^2$$ nunc autem vi theorematis eſt

$$\left[\tfrac{2}{4}\right] = \tfrac{2}{4} \sqrt[4]{4^2} \cdot 1 \left(\tfrac{1}{2}\right)\left(\tfrac{2}{2}\right)\left(\tfrac{3}{2}\right)$$

vnde comparatione inſtituta fit $\left(\tfrac{2}{2}\right) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)$ cuius veritas in Obſeruationibus ſupra allegatis eſt confirmata.

## Coroll. 2.

55. Si $m = 2$ et $n = 6$, ex ſuperioribus (49) eſt

$$\left[\tfrac{2}{6}\right] = \tfrac{2}{6} \sqrt[6]{6^4} \cdot \left(\tfrac{2}{2}\right)^2 \left(\tfrac{4}{2}\right)^2$$ nunc vero per theorema

$$\left[\tfrac{2}{6}\right] = \tfrac{2}{6} \sqrt[6]{6^4} \cdot 1 \left(\tfrac{1}{2}\right)\left(\tfrac{2}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{5}{2}\right)$$

ideoque neceſſe eſt ſit

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)$$

cuius veritas indidem patet.

## Coroll. 3.

56. Si $m = 3$ et $n = 6$, peruenitur ad hanc aequationem:

$$\left(\tfrac{3}{3}\right)^2 = 1 \cdot 2 \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{5}{3}\right)$$

at ſi $m = 4$ et $n = 6$ fit ſimili modo:

$$2^2 \left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right) = 1.\,2.\,3 \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right)$$

$$\text{ſeu } \left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right) = \tfrac{3}{2}\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right)$$

quod etiam verum deprehenditur.

## Coroll. 4.

57. Caſus $m = 2$ et $n = 8$ praebet hanc aequalitatem:

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)\left(\tfrac{7}{2}\right)$$

at caſus $m = 4$ et $n = 8$ hanc:

$$\left(\tfrac{4}{4}\right)^3 = 1.\,2.\,3 \left(\tfrac{1}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{7}{4}\right)$$

caſus denique $m = 6$ et $n = 8$ iſtam

$$2.\,4 \left(\tfrac{6}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{2}{6}\right) = 1.\,3.\,5 \left(\tfrac{1}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{7}{6}\right)$$

quae etiam veritati ſunt conſentaneae.

## Scholion.

58. In genere autem ſi numeri $m$ et $n$ communem habeant factorem 2, et formula propoſita ſit $\left[\frac{2m}{2n}\right] = \left[\frac{m}{n}\right]$ quia eſt;

$$\left[\frac{m}{n}\right] = \frac{m}{n} \sqrt[n]{n^{n-m}}.\ 1.\,2.\,3 \ldots (m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right) \ldots \left(\frac{n-1}{m}\right)$$

erit eadem ad exponentem $2n$ reducta:

$$\frac{m}{n} \sqrt[2n]{2n^{2n-2m}}.\ 2^2.\,4^2.\,6^2 \ldots (2m-2)^2 \left(\frac{2}{2m}\right)^2\left(\frac{4}{2m}\right)^2\left(\frac{6}{2m}\right)^2 \ldots \left(\frac{2n-2}{m}\right)^2$$

Per theorema vero eadem expreſſio fit

$$\frac{m}{n} \sqrt[2n]{2n^{2n-2m}}.\ 1.\,2.\,3 \ldots (2m-1)\left(\frac{1}{2m}\right)\left(\frac{2}{2m}\right)\left(\frac{3}{2m}\right) \ldots \left(\frac{2n-1}{2m}\right)$$

vnde

vnde pro exponente $2n$ erit

$$2.4.6\ldots(2m-2)\left(\frac{2}{2m}\right)\left(\frac{4}{2m}\right)\left(\frac{6}{2m}\right)\ldots\left(\frac{2n-2}{2m}\right)=$$
$$1.3.5\ldots(2m-1)\left(\frac{1}{2m}\right)\left(\frac{3}{2m}\right)\left(\frac{5}{2m}\right)\ldots\left(\frac{2n-1}{2m}\right)$$

Simili modo si communis diuisor sit 3 pro exponente $3n$ reperietur

$$3^2.6^2.9^2\ldots(3m-3)^2\left(\frac{3}{3m}\right)^2\left(\frac{6}{3m}\right)^2\left(\frac{9}{3m}\right)^2\ldots\left(\frac{3n-3}{3m}\right)^2=$$
$$1.2.4.5\ldots(3m-2)(3m-1)\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\ldots\left(\frac{3n-1}{3m}\right)$$

quae aequatio concinnius ita exhiberi potest:

$$\frac{1.\ 2.\ 4.\ 5.\ 7.\ 8.\ 10\ldots\ldots(3m-2)(3m-1)}{3^2.\quad 6^2.\quad 9^2\ldots\ldots(3m-3)^2}=$$
$$\frac{\left(\frac{3}{3m}\right)^2\qquad\left(\frac{6}{3m}\right)^2\ldots\ldots\left(\frac{3n-3}{3m}\right)^2}{\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\left(\frac{7}{3m}\right)\ldots\ldots\left(\frac{3n-2}{3m}\right)\left(\frac{3n-1}{3m}\right)}.$$

In genere autem si communis diuisor sit $d$ et exponens $dn$ habebitur.

$$\left[d.2d.3d\ldots(dm-d)\left(\frac{d}{dm}\right)\left(\frac{2d}{dm}\right)\left(\frac{3d}{dm}\right)\ldots\left(\frac{dn-d}{dm}\right)\right]^d=$$
$$1.2.3.4\ldots(dm-1)\left(\frac{1}{dm}\right)\left(\frac{2}{dm}\right)\left(\frac{3}{dm}\right)\ldots\left(\frac{dn-1}{dm}\right)$$

quae aequatio facile ad quosuis casus accommodari potest vnde sequens Theorema notari meretur.

## Theorema.

59. Si $\alpha$ fuerit diuisor communis numerorum $m$ et $n$ haecque formula $\left(\frac{p}{q}\right)$ denotet valorem integralis

gralis $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ ab $x = 0$ vsque ad $x = 1$ extenſi, erit

$$[\alpha . 2\alpha . 3\alpha \ldots\ldots (m-\alpha)(\tfrac{\alpha}{m})(\tfrac{2\alpha}{m})(\tfrac{3\alpha}{m}) \ldots\ldots (\tfrac{n-\alpha}{m})]^{\alpha} =$$

$$1 . 2 . 3 \ldots\ldots (m-1)(\tfrac{1}{m})(\tfrac{2}{m})(\tfrac{3}{m}) \ldots\ldots (\tfrac{n-1}{m}).$$

## Demonſtratio.

Ex praecedente ſcholio veritas huius theorematis perſpicitur, cum enim ibi diuiſor communis eſſet $= d$, binique numeri propoſiti $dm$ et $dn$ horum loco hic ſcripſi $m$ et $n$ loco diuiſoris eorum autem $d$ litteram $\alpha$ quam diuiſoris rationem aequalitas enunciata ita complectitur, vt in progreſſione arithmetica $\alpha$, $2\alpha$, $3\alpha$, etc. continuata occurrere aſſumantur ipſi numeri $m$ et $n$ ideoque etiam $m-\alpha$ et $n-\alpha$. Ceterum fateri cogor hanc demonſtrationem vtpote inductioni potiſſimum innixam, neutiquam pro rigoroſa haberi poſſe: cum autem nihilominus de eius veritate ſimus conuicti, hoc theorema eo maiori attentione dignum videtur, interim tamen nullum eſt dubium, quin vberior huiusmodi formularum integralium euolutio tandem perfectam demonſtrationem ſit largitura. quod autem iam ante nobis hanc veritatem perſpicere licuerit, inſigne hinc ſpecimen analyticae inueſtigationis elucet.

Coroll.

## Coroll. 1.

60. Si loco fignorum adhibitorum ipfas formulas integrales fubftituamus, theorema noftrum ita fe habebit vt fit:

$$\alpha.2\alpha.3\alpha\ldots(m-\alpha)\int\frac{x^{\alpha-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdot\int\frac{x^{2\alpha-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdots\cdots\int\frac{x^{n-\alpha-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}}=$$

$$\sqrt[\alpha]{1.2.3\ldots(m-1)\int\frac{dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdot\int\frac{x\,dx}{\sqrt[n]{(1-x^n)^{n-m}}}\cdots\cdots\int\frac{x^{n-2}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}}}.$$

## Coroll. 2.

61. Vel fi ad abbreuiandum ftatuamus $\sqrt[n]{(1-x^n)^{n-m}}=X$ erit

$$\alpha.2\alpha.3\alpha\ldots(m-\alpha)\int\frac{x^{\alpha-1}\,dx}{X}\cdot\int\frac{x^{2\alpha-1}\,dx}{X}\cdots\cdot\int\frac{x^{n-\alpha-1}\,dx}{X}=$$

$$\sqrt[\alpha]{1.2.3\ldots(m-1)\int\frac{dx}{X}\cdot\int\frac{x\,dx}{X}\cdot\int\frac{x^2\,dx}{X}\cdots\cdots\int\frac{x^{n-2}\,dx}{X}}.$$

## Theorema generale.

62. Si binorum numerorum $m$ et $n$ diuifores communes fint $\alpha$, $\beta$, $\gamma$ etc. formulaque $(\frac{p}{q})$ denotet valorem integralis $\int\frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ ab $x=0$ ad $x=1$ extenfi fequentes expreffiones ex huiusmodi formulis integralibus formatae inter fe erunt aequales:

$$[\alpha.\ 2\alpha.\ 3\alpha \ldots\ldots (m-\alpha)(\tfrac{\alpha}{m})(\tfrac{2\alpha}{m})(\tfrac{3\alpha}{m})\ldots\ldots(\tfrac{n-\alpha}{m})]^{\alpha} =$$

$$[\beta.\ 2\beta.\ 3\beta \ldots\ldots (m-\beta)(\tfrac{\beta}{m})(\tfrac{2\beta}{m})(\tfrac{3\beta}{m})\ldots\ldots(\tfrac{n-\beta}{m})]^{\beta} =$$

$$[\gamma.\ 2\gamma.\ 3\gamma \ldots\ldots (m-\gamma)(\tfrac{\gamma}{m})(\tfrac{2\gamma}{m})(\tfrac{3\gamma}{m})\ldots\ldots(\tfrac{n-\gamma}{m})]^{\gamma} \text{ etc.}$$

## Demonſtratio.

Ex praecedente Theoremate huius veritas manifeſto ſequitur cum quaelibet harum expreſſionum ſeorſim aequetur huic:

$$1.\ 2.\ 3 \ldots\ldots (m-1)(\tfrac{1}{m})(\tfrac{2}{m})(\tfrac{3}{m})\ldots\ldots\ldots(\tfrac{n-1}{m})$$

quae vnitati vtpote minimo communi diuiſori numerorum $m$ et $n$ conuenit. Tot igitur huiusmodi expreſſiones inter ſe aequales exhiberi poſſunt, quot fuerint diuiſores communes binorum numerorum $m$ et $n$.

## Coroll. 1.

63. Cum ſit haec formula $(\frac{n}{m}) = \frac{1}{m}$, ideoque $m(\frac{n}{m}) = 1$; expreſſiones noſtrae aequales ſuccinctius hoc modo repraeſentari poſſunt:

$$[\alpha.\ 2\alpha.\ 3\alpha \ldots\ldots m(\tfrac{\alpha}{m})(\tfrac{2\alpha}{m})(\tfrac{3\alpha}{m})\ldots\ldots(\tfrac{n}{m})]^{\alpha} =$$

$$[\beta.\ 2\beta.\ 3\beta \ldots\ldots m(\tfrac{\beta}{m})(\tfrac{2\beta}{m})(\tfrac{3\beta}{m})\ldots\ldots(\tfrac{n}{m})]^{\beta} =$$

$$[\gamma.\ 2\gamma.\ 3\gamma \ldots\ldots m(\tfrac{\gamma}{m})(\tfrac{2\gamma}{m})(\tfrac{3\gamma}{m})\ldots\ldots(\tfrac{n}{m})]^{\gamma}.$$

Etſi enim hic factorum numerus eſt auctus, tamen ratio compoſitionis facilius in oculos incurrit.

Coroll.

## Coroll. 2.

64. Si ergo ſit $m = 6$ et $n = 12$ ob horum numerorum diuiſores communes 6, 3, 2, 1 quatuor ſequentes formae inter ſe aequales habebuntur:

$$\left[6\left(\tfrac{6}{6}\right)\left(\tfrac{12}{6}\right)\right]^6 = \left[3.\,6\left(\tfrac{3}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{9}{6}\right)\left(\tfrac{12}{6}\right)\right]^3 =$$

$$\left[2.\,4.\,6\left(\tfrac{2}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{8}{6}\right)\left(\tfrac{10}{6}\right)\left(\tfrac{12}{6}\right)\right]^2 =$$

$$1.\,2.\,3.\,4.\,5.\,6\left(\tfrac{1}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{3}{6}\right)\ldots\ldots\left(\tfrac{12}{6}\right).$$

## Coroll. 3.

65. Si vltima cum penultima combinetur, naſcetur haec aequatio:

$$\frac{1.\,3.\,5}{2.\,4.\,6} = \frac{\left(\tfrac{2}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{8}{6}\right)\left(\tfrac{10}{6}\right)\left(\tfrac{12}{6}\right)}{\left(\tfrac{1}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{7}{6}\right)\left(\tfrac{9}{6}\right)\left(\tfrac{11}{6}\right)}$$

vltima autem cum antepenultima comparata praebet:

$$\frac{1.\,2.\,4.\,5}{3.\,3.\,6.\,6} = \frac{\left(\tfrac{3}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{9}{6}\right)\left(\tfrac{9}{6}\right)\left(\tfrac{12}{6}\right)\left(\tfrac{12}{6}\right)}{\left(\tfrac{1}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{7}{6}\right)\left(\tfrac{8}{6}\right)\left(\tfrac{10}{6}\right)\left(\tfrac{11}{6}\right)}.$$

## Scholion.

66. Infinitae igitur hinc conſequuntur relationes inter formulas integrales formae:

$$\int \frac{x^{p-n}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}} = \left(\tfrac{p}{q}\right)$$

quae eo magis ſunt notatu dignae, quod ſingulari prorſus methodo ad eas hic ſumus perducti. Ac ſi quis de earum veritate adhuc dubitet, obſeruationes meas circa has formulas integrales conſulat, indeque

pro

pro quouis casu oblato de veritate facile conuincetur. Etsi autem illa tractatio huic confirmandae inseruit, tamen relationes hic erutae eo maioris sunt momenti, quod in iis certus ordo cernitur, eaeque per omnes classes, quantumuis exponentem $n$ accipere lubeat, facili negotio continuentur, in priori vero tractatione calculus pro classibus altioribus continuo fiat operosior et intricatior.

## SVPPLEMENTVM
### continens demonstrationem
### Theorematis §. 53. propositi.

Demonstrationem hanc altius peti conuenit; sumatur scilicet aequatio §. 25. data, quae posito $f = 1$ et mutatis litteris est:

$$\frac{\int dx\,(l\tfrac{1}{x})^{\nu-1}.\int dx\,(l\tfrac{1}{x})^{\mu-1}}{\int dx\,(l\tfrac{1}{x})^{\mu+\nu-1}} = \varkappa\int\frac{x^{\varkappa\mu-1}\,dx}{(1-x^{\varkappa})^{1-\nu}}$$

eaque per reductiones notas hac forma repraesentetur:

$$\frac{\int dx\,(l\tfrac{1}{x})^{\nu}.\int dx\,(l\tfrac{1}{x})^{\mu}}{\int dx\,(l\tfrac{1}{x})^{\mu+\nu}} = \frac{\varkappa\mu\nu}{\mu+\nu}\int\frac{x^{\varkappa\mu-1}\,dx}{(1-x^{\varkappa})^{1-\nu}}.$$

Statuatur nunc $\nu = \frac{m}{n}$ et $\mu = \frac{\lambda}{n}$ tum vero $\varkappa = n$ vt habeamus;

$$\frac{\int dx\,(l\tfrac{1}{x})^{\frac{m}{n}}.\int dx\,(l\tfrac{1}{x})^{\frac{\lambda}{n}}}{\int dx\,(l\tfrac{1}{x})^{\frac{\lambda+m}{n}}} = \frac{\lambda m}{\lambda+m}\int\frac{x^{\lambda-1}\,dx}{\sqrt[n]{(1-x^{n})^{n-m}}}$$

qua

quae breuitatis gratia, more ſupra vſitato, ita concinne referatur:

$$\frac{[\frac{m}{n}].[\frac{\lambda}{n}]}{[\frac{\lambda+m}{m}]} = \frac{\lambda m}{\lambda+m}.\left(\frac{\lambda}{m}\right)$$

Iam loco $\lambda$ ſucceſſiue ſcribantur numeri $1, 2, 3, 4 \ldots n$ omnesque hae aequationes, quarum numerus eſt $= n$ in ſe inuicem ducantur, et aequatio reſultans erit:

$$[\frac{m}{n}]^n.\frac{[\frac{1}{n}][\frac{2}{n}][\frac{3}{n}]\ldots\ldots[\frac{n}{n}]}{[\frac{m+1}{n}][\frac{m+2}{n}][\frac{m+3}{n}]\ldots[\frac{m+n}{n}]} =$$

$$m^n.\frac{1}{m+1}.\frac{2}{m+2}.\frac{3}{m+3}\ldots\ldots\frac{n}{m+n}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\left(\frac{n}{m}\right) =$$

$$m^n.\frac{1.\ 2.\ 3\ldots\ldots\ldots m}{(n+1)(n+2)(n+3)\ldots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\left(\frac{n}{m}\right).$$

Simili autem modo pars prior transformetur vt fit

$$[\frac{m}{n}]^n.\frac{[\frac{1}{n}][\frac{2}{n}][\frac{3}{n}]\ldots\ldots\ldots[\frac{m}{n}]}{[\frac{n+1}{n}][\frac{n+2}{n}][\frac{n+3}{n}]\ldots\ldots[\frac{n+m}{n}]}$$

cuius conuenientia cum forma praecedente multiplicando per crucem, vt aiunt, ſponte ſe prodit. Cum vero ex natura harum formularum fit

$$[\frac{n+1}{n}] = \frac{n+1}{n}[\frac{1}{n}];\ [\frac{n+2}{n}] = \frac{n+2}{n}[\frac{2}{n}];\ [\frac{n+3}{n}] = \frac{n+3}{n}[\frac{3}{n}] \text{ etc.}$$

ob harum formularum numerum $= m$, euadet haec prior pars:

$$[\frac{m}{n}]^n.\frac{n^m}{(n+1)(n+2)(n+3)\ldots(n+m)}$$

quae cum aequalis ſit parti alteri ante exhibitae:

$$m^n.\frac{1.\ 2.\ 3\ldots\ldots\ldots m}{(n+1)(n+2)(n+3)\ldots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\left(\frac{n}{m}\right)$$

 adipi-

adipiſcimur hanc aequationem:

$$\left[\frac{m}{n}\right]^n = \frac{m^n}{n^m} \cdot 1 \cdot 2 \cdot 3 \ldots\ldots m \left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)$$

ita vt ſit

$$\left[\frac{m}{n}\right] = m \sqrt[n]{\frac{1 \cdot 2 \cdot 3 \ldots\ldots m}{n^m} \left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)}$$

quae cum propoſita in §. 53. ob $\left(\frac{n}{m}\right) = \frac{1}{m}$ omnino congruit, ex quo eius veritas nunc quidem ex principiis certiſſimis eſt euicta.

## Demonſtratio Theorematis §. 59. propoſiti.

Etiam hoc Theorema firmiori demonſtratione indiget, quam ex aequalitate ante ſtabilita:

$$\frac{\left[\frac{m}{n}\right] \cdot \left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{n}\right]} = \frac{\lambda m}{\lambda + m}\left(\frac{\lambda}{m}\right)$$

ita adorno. Exiſtente $\alpha$ communi diuiſore numerorum $m$ et $n$, loco $\lambda$ ſucceſſiue ſcribantur numeri $\alpha$, $2\alpha$, $3\alpha$ etc. vsque ad $n$, quorum multitudo eſt $= \frac{n}{\alpha}$ atque omnes aequalitates hoc modo reſultantes in ſe inuicem ducantur, vt prodeat haec aequatio

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \cdot \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\ldots\ldots\ldots\ldots\left[\frac{n}{n}\right]}{\left[\frac{m+\alpha}{n}\right]\left[\frac{m+2\alpha}{n}\right]\left[\frac{m+3\alpha}{n}\right]\ldots\ldots\left[\frac{m+n}{n}\right]} =$$

$$m^{\frac{n}{\alpha}} \cdot \frac{\alpha}{m+\alpha} \cdot \frac{2\alpha}{m+2\alpha} \cdot \frac{3\alpha}{m+3\alpha} \ldots \frac{n}{m+n}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\left(\frac{n}{m}\right).$$

Iam

Iam prior pars in hanc formam ipſi aequalem transmutetur:

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \cdot \frac{\left[\frac{\alpha}{n}\right]\left[\frac{2\alpha}{n}\right]\left[\frac{3\alpha}{n}\right]\ldots\ldots\ldots\left[\frac{m}{n}\right]}{\left[\frac{n+\alpha}{n}\right]\left[\frac{n+2\alpha}{n}\right]\left[\frac{n+3\alpha}{n}\right]\ldots\left[\frac{n+m}{n}\right]}$$

quae ob $\left[\frac{n+\alpha}{n}\right] = \frac{n+\alpha}{n}\left[\frac{\alpha}{n}\right]$ ſicque de ceteris reducitur ad hanc:

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} \cdot \frac{n}{n+\alpha} \cdot \frac{n}{n+2\alpha} \cdot \frac{n}{n+3\alpha} \ldots\ldots \frac{n}{n+m}.$$

Poſterior vero aequationis pars ſimili modo transformatur in:

$$m^{\frac{n}{\alpha}} \cdot \frac{\alpha}{n+\alpha} \cdot \frac{2\alpha}{n+2\alpha} \cdot \frac{3\alpha}{n+3\alpha} \ldots \frac{m}{n+m}\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\left(\frac{n}{m}\right)$$

vnde enaſcitur haec aequatio:

$$\left[\frac{m}{n}\right]^{\frac{n}{\alpha}} n^{\frac{m}{\alpha}} = m^{\frac{n}{\alpha}} \cdot \alpha.\ 2\alpha.\ 3\alpha\ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\left(\frac{n}{m}\right)$$

hincque

$$\left[\frac{m}{n}\right] = m \sqrt[n]{\frac{1}{m^n}\left(\alpha.\ 2\alpha.\ 3\alpha\ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\left(\frac{n}{m}\right)\right)^{\alpha}}$$

quae expreſſio cum praecedente comparata praebet hanc aequationem:

$$\left(\alpha.\ 2\alpha,\ 3\alpha\ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\left(\frac{n}{m}\right)\right)^{\alpha} =$$

$$1.\ 2.\ 3\ldots\ldots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\left(\frac{n}{m}\right)$$

quod de omnibus diuiſoribus communibus binorum numerorum $m$ et $n$ eſt intelligendum.

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