$$+\frac{nibc}{f^3}d\omega\left\{\begin{array}{ll}-\frac{1}{2}g'\sin.v & -\frac{1}{4}h'k\sin.2v+\frac{1}{4}k(4h-2h''-2g+g'')\sin.\eta\\ +(\frac{1}{2}g-\frac{1}{4}g'')\sin.(\eta-v) & +\frac{1}{8}k(6h-h''+2g-g'')\sin.(\eta-2v)\\ +(\frac{1}{2}g-\frac{1}{4}g'')\sin.(\eta+v) & +\frac{1}{8}k(2h-3h''+2g-g'')\sin.(\eta+2v)\\ +\frac{1}{4}(3g'-g''')\sin.(2\eta-v) & +\frac{1}{4}k(2h'-2h'''-g'+g''')\sin.2\eta\\ +\frac{1}{4}(g'-3g''')\sin.(2\eta+v) & +\frac{1}{8}k(3h'-h'''+g'-g''')\sin.(2\eta-2v)\\ & +\frac{1}{8}k(h-3h'''+g-g''')\sin.(2\eta+2v)\end{array}\right\}$$

§. LXXXV. Pro motu aphelii autem habebimus negligendis ſimili modo minimis terminis, & pro $a\sqrt{a}$ ſcribendo $ib\sqrt{b}$;

$$d\varphi-dv=\frac{nibbd\omega}{f^3k}\left(M(2\sin.v+\tfrac{1}{2}k\sin.2v)-N\cos.v\right);$$

quæ expreſſio, ſi loco M & N valores eruti ſubſtituantur, abibit in formam ſequentem

$$\frac{d\varphi-dv}{d\omega}=-\frac{nibb}{cck}\Big(\tfrac{1}{2}\cos.(\eta-v)-\tfrac{1}{2}\cos.(\eta+v)+\tfrac{1}{4}k\cos.(\eta-2v)-\tfrac{1}{4}k\cos.(\eta+2v)\Big)$$

$$-\frac{nib^3}{f^3k}\left\{\begin{array}{ll}+g\cos.v & +\frac{1}{2}k(g'+h')\cos.\eta\\ +\frac{1}{2}g'\cos.(\eta-v)+\frac{1}{2}k(g+h) & +\frac{1}{4}k(g'+h')\cos.(\eta-2v)\\ +\frac{1}{2}g'\cos.(\eta+v) & +\frac{1}{4}k(g'+h')\cos.(\eta+2v)\\ +\frac{1}{2}g''\cos.(2\eta-v)+\frac{1}{2}k(g+h)\cos.2v & +\frac{1}{2}k(g''+h'')\cos.2\eta\\ +\frac{1}{2}g''\cos.(2\eta+v) & +\frac{1}{4}k(g''+h'')\cos.(2\eta-2v)\\ & +\frac{1}{4}k(g''+h'')\cos.(2\eta+2v)\end{array}\right\}$$

$$+\frac{nibbc}{f^3k}\left\{\begin{array}{ll}+\frac{1}{2}g'\cos.v & +\frac{1}{4}k(2h+h'')\cos.\eta\\ +\frac{1}{4}(6g-g'')\cos.(\eta-v)+\frac{1}{4}h'k & +\frac{1}{8}k(6h-h''-2g-g'')\cos.(\eta-2v)\\ +\frac{1}{4}(2g-3g'')\cos.(\eta+v) & -\frac{1}{8}k(2h-3h''+2g-g'')\cos.(\eta+2v)\\ +\frac{1}{4}(3g'-g''')\cos.(2\eta-v)+\frac{1}{4}h'k\cos.2v & +\frac{1}{4}k(h'+h''')\cos.2\eta\\ -\frac{1}{4}(g'-3g''')\cos.(2\eta+v) & +\frac{1}{8}k(3h'-h'''+g'-g''')\cos.(2\eta-2v)\\ & -\frac{1}{8}k(h(h'-3h'''+g'-g''')\cos.(2+2v)\end{array}\right\}$$

§. LXXXVI. Reſtat ut ſimili modo variationes, quibus cum longitudo lineæ nodorum π, tum inclinatio G ſunt obnoxiæ, exprimamus: Ac neglecta quidem excentricitate planetæ perturbantis e, ut ſit $y=c$ & $x=b(1+k\cos.v)$, erit $d\pi=-nibbcd\omega(1+k\cos.v)(\frac{1}{z^3}-\frac{1}{c^3})\sin.(\varphi-\pi)\sin.(\theta-\pi)$;

d. l tang.

$d.\,l\,tang.\,\rho=-nibbcd\omega(1+k\cos.v)(\frac{1}{z^3}-\frac{1}{c^3})\cos.(\varphi-\pi)\sin.(\theta-\pi)$; ubi valor ipſius $\frac{1}{z^3}$ debet ſubſtitui qui eſt poſito $e=o$

$$\frac{1}{z^3}=\frac{1}{f^3}\left\{\begin{array}{l}+g+hk\cos.v\\ +g'\cos.\eta+\frac{1}{2}h'k\cos.(\eta-v)+\frac{1}{2}h''k\cos.(2\eta-v)\\ +g''\cos.2\eta+\frac{1}{2}h'k\cos.(\eta+v)+\frac{1}{2}h''k\cos.(2\eta+v)\end{array}\right\}$$

Tum vero ob $\varphi-\theta=\eta$ eſt

$$\sin.(\varphi-\pi)\sin.(\theta-\pi)=\tfrac{1}{2}\cos.\eta-\tfrac{1}{2}\cos.(\varphi+\theta-2\pi)$$
$$\cos.(\varphi-\pi)\sin.(\theta-\pi)=-\tfrac{1}{2}\sin.\eta+\tfrac{1}{2}\sin.(\varphi+\theta-2\pi).$$

§. LXXXVII. Introducamus ad has formulas aliquanto ſimpliciores reddendas, argumentum latitudinis $\varphi-\pi$, ponamuſque $\varphi-\pi=\sigma$, eritque

$$d\pi=-nibbcd\omega(\tfrac{1}{z^3}-\tfrac{1}{c^3})\left\{\begin{array}{l}\frac{1}{2}\cos.\eta-\frac{1}{2}\cos.(\eta-2\sigma)\\ +\frac{1}{4}k\cos.(\eta-v)-\frac{1}{4}k\cos.(\eta-2\sigma-v)\\ +\frac{1}{4}k\cos.(\eta+v)-\frac{1}{4}k\cos.(\eta-2\sigma+v)\end{array}\right\}$$

& ſubſtituto pro $\frac{1}{z^3}$ valore:

$$d\pi=+\frac{nibb}{cc}d\omega\left\{\begin{array}{l}\frac{1}{2}\cos.\eta-\frac{1}{2}\cos.(\eta-2\sigma)\\ +\frac{1}{4}k\cos.(\eta-v)-\frac{1}{4}k\cos.(\eta-2\sigma-v)\\ +\frac{1}{4}k\cos.(\eta+v)-\frac{1}{4}k\cos.(\eta-2\sigma+v)\end{array}\right\}$$

$$-\frac{nibbc}{f^3}d\omega\left\{\begin{array}{ll}+\frac{1}{4}g' & +\frac{1}{8}k(2h+h''+2g+g'')\cos.(\eta-v)\\ +\frac{1}{4}(2g+g''\cos.\eta & +\frac{1}{8}k(2h+h''+2g+g'')\cos.(\eta+v)\\ +\frac{1}{4}g'+g''')\cos.2\eta & +\frac{1}{4}k(h'+g')\cos.v\\ -\frac{1}{2}g\cos.(\eta-2\sigma) & +\frac{1}{8}k(h'+h'''+g'+g''')\cos.(2\eta-v)\\ -\frac{1}{4}g'\cos.2\sigma & +\frac{1}{8}k(h'+h'''+g'+g''')\cos.(2\eta+v)\\ -\frac{1}{4}g'\cos.(2\eta-2\sigma) & -\frac{1}{8}k(2h+2g)\cos.(\eta-2\sigma-v)\\ -\frac{1}{4}g''\cos.(\eta+2\sigma) & -\frac{1}{8}k(2h+2g)\cos.(\eta-2\sigma+v)\\ & -\frac{1}{8}k(h''+g'')\cos.(\eta+2\sigma-v)\\ & -\frac{1}{8}k(h''+g'')\cos.(\eta+2\sigma+v)\\ & -\frac{1}{8}k(h'+g')\cos.(2\sigma-v)\\ & -\frac{1}{8}k(h'+g')\cos.(2\sigma+v)\\ & -\frac{1}{8}k(h'+g')\cos.(2\eta-2\sigma-v)\\ & -\frac{1}{8}k(h'+g')\cos.(2\eta-2\sigma+v)\end{array}\right\}$$

§. LXXXVIII. Simili vero modo æquatio differentialis pro inclinationis variatione erit :

$$d.l\,tang.\rho = nibbc\,d\omega\left(\tfrac{1}{f^3}-\tfrac{1}{c^3}\right)\left\{\begin{array}{l}\tfrac{1}{2}\,\mathit{fin}.\,\eta + \tfrac{1}{2}\,\mathit{fin}.\,(\eta - 2\sigma)\\ +\tfrac{1}{4}k\,\mathit{fin}.(\eta-\nu) + \tfrac{1}{4}k\,\mathit{fin}.(\eta-2\sigma-\nu)\\ +\tfrac{1}{4}k\,\mathit{fin}.(\eta+\nu) + \tfrac{1}{4}k\,\mathit{fin}.(\eta-2\sigma+\nu)\end{array}\right\}$$

hincque ob $d.\,l\,tang.\,\rho = \frac{d.\,tang.\,\rho}{tang.\,\rho}$ obtinebitur

$$\frac{d.\,tang\,\rho}{tang.\rho} = -\frac{nibb\,d\omega}{cc}\left\{\begin{array}{l}\tfrac{1}{2}\,\mathit{fin}.\,\eta + \tfrac{1}{2}\,\mathit{fin}.\,(\eta - 2\sigma)\\ +\tfrac{1}{4}k\,\mathit{fin}.(\eta-\nu) + \tfrac{1}{4}k\,\mathit{fin}.(\eta-2\sigma-\nu)\\ +\tfrac{1}{4}k\,\mathit{fin}.(\eta+\nu) + \tfrac{1}{4}k\,\mathit{fin}.(\eta-2\sigma+\nu)\end{array}\right\}$$

$$+\frac{nibbc}{f^3}d\omega\left\{\begin{array}{ll}+\tfrac{1}{4}(2g-g'')\,\mathit{fin}.\,\eta & +\tfrac{1}{8}k(2g+2h-g''-h'')\,\mathit{fin}.(\eta-\nu)\\ +\tfrac{1}{4}(g'-g''')\,\mathit{fin}.\,2\eta & +\tfrac{1}{8}k(2g+2h-g''-h'')\,\mathit{fin}.(\eta+\nu)\\ +\tfrac{1}{2}g\,\mathit{fin}.\,(\eta-2\sigma) & +\tfrac{1}{8}k(g'+h'-g'''-h''')\,\mathit{fin}.(2\eta-\nu)\\ -\tfrac{1}{4}g'\,\mathit{fin}.\,2\sigma & +\tfrac{1}{8}k(g'+h'-g'''-h''')\,\mathit{fin}.(2\eta+\nu)\\ +\tfrac{1}{4}g'\,\mathit{fin}.\,(2\eta-2\sigma) & +\tfrac{1}{4}k(g+h)\,\mathit{fin}.\,(\eta-2\sigma-\nu)\\ -\tfrac{1}{4}g''\,\mathit{fin}.\,(\eta+2\sigma) & +\tfrac{1}{4}k(g+h)\,\mathit{fin}.\,(\eta-2\sigma+\nu)\\ & -\tfrac{1}{8}k(g'+h')\,\mathit{fin}.\,(2\sigma-\nu)\\ & -\tfrac{1}{8}k(g'+h')\,\mathit{fin}.\,(2\sigma+\nu)\\ & +\tfrac{1}{8}k(g'+h')\,\mathit{fin}.\,(2\eta-2\sigma-\nu)\\ & +\tfrac{1}{8}k(g'+h')\,\mathit{fin}.\,(2\eta-2\sigma+\nu)\\ & -\tfrac{1}{8}k(g''+h'')\,\mathit{fin}.\,(\eta+2\sigma-\nu)\\ & -\tfrac{1}{8}k(g''+h'')\,\mathit{fin}.\,(\eta+2\sigma+\nu)\end{array}\right\}$$

Si quis vellet has formulas ad plures terminos continuare, lex eſt perſpicua, ſecundum quam hoc opus, quouſque libuerit perfici poſſet, verum pro noſtro inſtituto, ne his quidem terminis exhibitis omnibus indigebimus.

SECTIO VI.

Inveſtigatio inæqualitatum quibus ipſa orbita cujuſque Planetæ ab actione reliquorum Planetarum perturbatur.

§. LXXXIX. QUAMVIS igitur motus cujuſque Planetæ ab actione reliquorum perturbetur, is nihilo minus ſecundum ellipſin, in cujus alterutro foco Sol verſetur, fieri concipi poteſt, dummodo hæc ellipſis tanquam variabilis tam ratione magnitudinis & ſpeciei quam ratione ſitus lineæ abſidum conſideretur. Atque iſta perturbationum repræſentatio Aſtronomorum inſtituto maxime conveniens videtur, qui dum calculo elliptico jam ſunt aſſueti, huic curvæ inhærere malunt, quam alias curvas magis perplexas in calculum Aſtronomicum admittere. Quod propoſitum cum adeo in luna ſequi ſoleant, etiamſi ejus aberrationes à motu elliptico ſint enormes, id multo magis in motu planetarum principalium retinebitur, quemadmodum etiam Aſtronomi eorum orbitas jam mobiles aſſumſerunt contra indolem motus proprii Kepleriani.

§. XC. Ac primo quidem vidimus parametrum orbitæ cujuſque planetæ ab actione reliquorum continuo immutari. Notari ſcilicet debet ejus valor quidam medius, à quo verus mox in exceſſu mox in defectu diſcrepet; ita valorem medium ſemiparametri orbitæ planetæ, de quo quæritur, hic littera b deſignamus, dum littera p pro quovis tempore ejus valorem verum denotat. Quantum igitur p ob actionem certi alicujus planetæ ab b discrepet, ex æquatione differentiali supra §. LXXXIII evoluta per integrationem definiri poterit, ac si isti effectus, quatenus ab unoquoque planeta in parametrum propositi redundant, seorsim computentur, atque in unam summam colligantur, cognoscetur inversa perturbatio, quæ parametro illi ab actione omnium reliquorum planetarum inducitur, cujus collectionis fundamentum in eo est situm, quod singulæ perturbationes sint quam minimæ.

§. XCI. Totum autem integrationis formulæ §. LXXXIII datæ negotium huc reducitur, ut sequentium formularum simplicium: $d\omega\, \text{sin.}\, \eta$; $d\omega\, \text{sin.}\, 2\eta$; $d\omega\, \text{sin.}\, 3\eta$; $d\omega\, \text{sin.}\, (\eta \mp \nu)$; $d\omega\, \text{sin.}\, (2\eta \mp \nu)$ &c. integralia definiantur, quæ hac methodo investigo: Primo quia hic excentricitatem planetæ perturbantis negligimus, & motus anomaliæ veræ ν quam minime à motu longitudinis φ differt, si quidem motus aphelii certe est tardissimus, habebimus ex §. LXXV.

$$d\eta = (i - m)\, d\omega - 2\, i\, k\, d\omega\, \text{cos.}\, \nu, \ \&$$
$$d\nu = i\, d\omega - 2\, i\, k\, d\omega\, \text{cos.}\, \nu.$$

Jam pro prima formula $d\omega\, \text{sin.}\, \eta$, differentiale $d\omega$ ita ad $d\eta$ revoco ut sit

$$d\omega = \frac{d\eta}{i-m} + \frac{2\, i\, k\, d\omega}{i-m}\, \text{cos.}\, \nu, \text{ unde conficitur:}$$

$$d\omega\, \text{sin.}\, \eta = \frac{d\eta\, \text{sin.}\, \eta}{i-m} + \frac{i\, k\, d\omega}{i-m}\, \text{sin.}\, (\eta - \nu) + \frac{i\, k\, d\omega}{i-m}\, \text{sin.}\, (\eta + \nu),$$

quo pacto primum membrum jam redditum est integrabile.

§. XCII. Si idem valor pro $d\omega$ etiam in formulis $d\omega\, \text{sin.}\, 2\eta$ & $d\omega\, \text{sin.}\, 3\eta$ substituatur, erit simili modo

$$d\omega\, \text{sin.}\, 2\eta = \frac{d\eta\, \text{sin.}\, 2\eta}{i-m} + \frac{i\, k\, d\omega}{i-m}\, \text{sin.}\, (2\eta - \nu)$$
$$+ \frac{i\, k\, d\omega}{i-m}\, \text{sin.}\, (2\eta + \nu);$$

$$d\omega\, \text{sin.}\, 3\eta = \frac{d\eta\, \text{sin.}\, 3\eta}{i-m} + \frac{i\, k\, d\omega}{i-m}\, \text{sin.}\, (3\eta - \nu)$$
$$+ \frac{i\, k\, d\omega}{i-m}\, \text{sin.}\, (3\eta + \nu).$$

Integratis ergo partibus prioribus, habebimus:

$$\int d\omega\, \text{sin.}\, \eta = \frac{-\text{cos.}\, \eta}{i-m} + \frac{i\, k}{i-m} \int d\omega\, \text{sin.}\, (\eta - \nu)$$
$$+ \frac{i\, k}{i-m} \int d\omega\, \text{sin.}\, (\eta + \nu);$$

$$\int d\omega\, \text{sin.}\, 2\eta = \frac{-\text{cos.}\, 2\eta}{2(i-m)} + \frac{i\, k}{i-m} \int d\omega\, \text{sin.}\, (2\eta - \nu)$$
$$+ \frac{i\, k}{i-m} \int d\omega\, \text{sin.}\, (2\eta + \nu);$$

$$\int d\omega\, \text{sin.}\, 3\eta = \frac{-\text{cos.}\, 3\eta}{3(i-m)} + \frac{i\, k}{i-m} \int d\omega\, \text{sin.}\, (3\eta - \nu)$$
$$+ \frac{i\, k}{i-m} \int d\omega\, \text{sin.}\, (3\eta + \nu).$$

Sicque integrandæ restant reliquæ formulæ, quas nostra expressio pro dp inventa combinet, hæ autem formulæ quia per excentricitatem k sunt multiplicatæ, multo minores sunt prioribus partibus jam integratis, ideoque nisi precisio ultra necessitatem urgeri debeat, satis tuto omitti possent; si quidem jam ob similem caussam excentricitatem e neglleximus.

§. XCIII. Interim tamen quo darius perspiciatur, integrationem ex hac parte non impediri, atque pari facilitate perfici posse etiamsi nullos terminos rejecissemus, etiam horum integralia definiam: Pro $\int d\omega\, \text{sin.}$ $(\eta - \nu)$ igitur quæro primum

$$d\eta - d\nu = -m\,d\omega, \text{ ut fit } d\omega = \frac{-(d\eta - d\nu)}{m};$$

ficque erit $\int d\omega \,\mathit{fin.}\,(\eta - \nu) = \frac{+\mathit{cof.}\,(\eta-\nu)}{m}$.

Deinde pro $\int d\omega\,\mathit{fin.}\,(\eta+\nu)$ colligo

$$d\eta + d\nu = (2i - m)\,d\omega - 4ik\,d\omega\,\mathit{cof.}\,\nu;$$

unde erit $d\omega = \frac{d\eta + d\nu}{2i-m} + \frac{4ik\,d\omega\,\mathit{cof.}\,\nu}{2i-m}$, ideoque

$$\int d\omega\,\mathit{fin.}\,(\eta+\nu) = \frac{-\mathit{cof.}\,(\eta+\nu)}{2i-m} + \frac{4ik}{2i-m}\int d\omega\,\mathit{cof.}\,\nu\,\mathit{fin.}\,(\eta+\nu).$$

Sed quia in noftra formula $\int d\omega\,\mathit{fin.}\,(\eta+\nu)$ jam per k eft multiplicatum, pofterius membrum, quod adhuc integrari deberet, omittimus, qui produceret quantitatem per kk affectam. Hac omiffione pariter facta pro reliquis formulis, habebimus etiamnunc in differentialibus:

$$2d\eta - d\nu = (i - 2m)\,d\omega, \ \& \ 2d\eta + d\nu = (3i - 2m)\,d\omega,$$

ideoque $d\omega = \frac{2d\eta - d\nu}{i - 2m}$, & $d\omega = \frac{2d\eta + d\nu}{3i-2m}$.

§. XCIV. His igitur valoribus adhibitis adipifcemur facile formulas integrales fequentes:

$$\int d\omega\,\mathit{fin.}\,(\eta-\nu) = \frac{+\mathit{cof.}\,(\eta-\nu)}{m};$$

$$\int d\omega\,\mathit{fin.}\,(\eta+\nu) = \frac{-\mathit{cof.}\,(\eta+\nu)}{2i-m};$$

$$\int d\omega\,\mathit{fin.}\,(2\eta-\nu) = \frac{-\mathit{cof.}\,(2\eta-\nu)}{i-2m};$$

$$\int d\omega\,\mathit{fin.}\,(2\eta-\nu) = \frac{-\mathit{cof.}\,(2\eta+\nu)}{3i-2m};$$

atque ex his jam priora integralia completa reddentur:

$$\int d\omega\,\mathit{fin.}\,\eta = -\frac{\mathit{cof.}\,\eta}{i-m} + \frac{ik\,\mathit{cof.}\,(\eta-\nu)}{(i-m)\,m} - \frac{ik\,\mathit{cof.}\,(\eta+\nu)}{(i-m)(2i-m)},$$

$$\int d\omega\,\mathit{fin.}\,2\eta = -\frac{\mathit{cof.}\,2\eta}{2(i-m)} - \frac{ik\,\mathit{cof.}\,(2\eta-\nu)}{(i-m)(i-2m)} - \frac{ik\,\mathit{cof.}\,(2\eta+\nu)}{(i-m)(3i-2m)},$$

$$\int d\omega\,\mathit{fin.}\,3\eta = -\frac{\mathit{cof.}\,3\eta}{3(i-m)} - \frac{ik\,\mathit{cof.}\,(3\eta-\nu)}{(i-m)(2i-3m)} - \frac{ik\,\mathit{cof.}\,(3\eta+\nu)}{(i-m)(4i-3m)}.$$

&c.

Quæ integralia non folum ad valorem integralem ipfius p, fed etiam ipfius q inveniendum inferviunt.

§. XCV. Cum nimirum valor medius ipfius p debeat effe $= b$, in integratione circa adjectionem conftantis nullum erit dubium; fingulis igitur partibus integratis reperietur

$$\frac{p}{b} = 1 - \frac{2nibb}{cc}\left\{\frac{\mathit{cof.}\,\eta}{i-m} - \frac{(3i-m)\,k\,\mathit{cof.}\,(\eta-\nu)}{2(i-m)\,m} + \frac{(3i-m)\,k\,\mathit{cof.}\,(\eta+\nu)}{2(i-m)(2i-m)}\right\}$$

$$+ \frac{2nibbc}{f^1}\left\{\begin{aligned}
&+\frac{(2g-g'')\,\mathit{cof.}\,\eta}{2(i-m)} + \frac{ik(2g-g'')\,\mathit{cof.}\,(\eta-\nu)}{2(i-m)\,m} \\
&+\frac{(g'-g''')\,\mathit{cof.}\,2\eta}{4(i-m)} + \frac{k(2g-g''+2h-h'')\,\mathit{cof.}\,(\eta-\nu)}{4m} \\
&+\frac{(g''-g'''')\,\mathit{cof.}\,3\eta}{6(i-m)} - \frac{ik(g'-g''')\,\mathit{cof.}\,(2\eta-\nu)}{2(i-m)(i-2m)} \\
&\qquad - \frac{k(g'-g'''+h'-h''')\,\mathit{cof.}\,(2\eta-\nu)}{4(i-2m)} \\
&\qquad - \frac{ik(2g-g'')\,\mathit{cof.}\,(\eta+\nu)}{2(i-m)(2i-m)} \\
&\qquad - \frac{k(2g-g''+2h-h'')\,\mathit{cof.}\,(\eta+\nu)}{4(2i-m)} \\
&\qquad - \frac{ik(g'-g''')\,\mathit{cof.}\,(2\eta+\nu)}{2(i-m)(3i-2m)} \\
&\qquad - \frac{k(g'-g''+h'-h''')\,\mathit{cof.}\,(2\eta+\nu)}{4(3i-2m)}
\end{aligned}\right\}$$

Ac ſi terminos per excentricitatem k affectos, ut pote præ reliquis valde parvos negligamus, erit ſuccinctius:

$$\frac{p}{b} = 1 - \frac{2nibb\,cof.\,\eta}{(i-m)cc} + \frac{nibbc}{(i-m)f^3}\Big((2g - g'')\,cof.\,\eta + \tfrac{1}{2}(g' - g''')\,cof.\,2\eta + \tfrac{1}{3}(g'' - g'''')\,cof.\,3\eta + \&c.\Big)$$

ubi notandum eſſe $\frac{b}{c} = \sqrt[3]{\frac{mm}{ii}}$, & $ff = bb + cc$.

§. XCVI. Ope earundem formularum ſimplicium integralium etiam vera excentricitas orbitæ q per integrationem differentialis (§. LXXXIV) evoluti aſſignari poterit; modo adjiciatur $\int d\omega\, ſin.\, v = \int \frac{dv}{i} ſin.\, v = \frac{-cof.\, v}{i}$, ſi quidem porro ex his expreſſionibus minimis terminos excentricitatem k involventes negligere pergamus. Hinc igitur poſita excentricitate media $= k$, erit excentricitas vera:

$$q = k - \frac{nibb}{cc}\left(\frac{3\,cof.(\eta - v)}{2m} - \frac{cof.(\eta + v)}{2(2i - m)}\right);$$

$$- \frac{nib^3}{f^3}\left\{\frac{g\,cof.\,v}{i} + \frac{g'\,cof.(\eta - v)}{2m} - \frac{g'\,cof.(\eta + v)}{2(2i - m)} - \frac{g''\,cof.(2\eta - v)}{2(i - 2m)} + \frac{g''\,cof.(2\eta + v)}{2(3i - 2m)}\ \&c.\right\}$$

$$+ \frac{nibbc}{f^3}\left\{+\frac{g'\,cof.\,v}{2i} + \frac{(6g - g'')\,cof.(\eta - v)}{4m} - \frac{(2g - 3g'')\,cof.(\eta + v)}{4(2i - m)} - \frac{(3g' - g''')\,cof.(2\eta - v)}{4(i - 2m)} - \frac{(g' - 2g''')\,cof.(2\eta + v)}{4(3i - 2m)}\right\}$$

&c.

Ubi quidem aſſumimus excentricitatem mediam k tantam eſſe, ut ejus reſpectu iſtæ inæqualitates longe ſint minimæ; patet autem has inæqualitates non ab ipſa magnitudine media excentricitatis k pendere, ſed eaſdem prodire ſive k ſit major ſive minor. Quod ſecus accidit in variationibus lateris recti, quæ ſunt proportionales ipſi magnitudini mediæ parametri.

§. XCVII. Cognito jam ſemiparametro p & excentricitate q, ſemiaxis tranſverſus orbitæ facile definietur, cum ſit $= \frac{p}{1 - qq}$. Erit igitur variabilis tam ob variabilitatem ipſius p, quam ipſius q, ſed hæc poſterior tantum terminos producit per k affectos, unde his neglectis variatio axis tranſverſi potiſſimum pendebit a variatione parametri, hincque ergo erit

$$\text{Semiaxis tranſverſus} = \frac{b}{1 - kk} - \frac{2nibb}{(i-m)c^2}\cdot b\,cof.\,\eta; + \frac{nibbc}{(i-m)f^3}\cdot b\Big((2g - g'')\,cof.\,\eta + \tfrac{1}{2}(g' - g''')\,cof.\,2\eta + \tfrac{1}{3}(g'' - g'''')\,cof.\,3\eta + \&c.\Big)$$

Quare tam parameter & axis tranſverſus quam excentricitas, variationes tantum ſubeunt periodicas, quæ poſt certa temporis intervalla ad ſtatum priſtinum revertantur, neque perpetua ſive incrementa ſive decrementa capiunt; ſed quantum certis temporibus fuerint aucta, tandundem aliis temporibus diminuentur. Ceterum ex hac applicatione ad axem tranſverſum patet, valorem inventum pro q, etſi terminos tantum primi ordinis continet, tamen æque longe productum eſſe æſtimandum atque valorem ipſius p in quo terminos primi & ſecundi ordinis evolvimus.

§. XCVIII. Denique definiendus occurrit motus aphelii, in quo præcipuus effectus actionis mutuæ planetarum, quem quidem obſervationes evidenter manifeſtarint, cernitur; is autem per integrationem formulæ (§. LXXXV) datæ determinabitur. Aliæ autem hic adſunt formulæ ſimplices integrandæ, quarum integrationem quoque ad ſecundum ordinem continuari oporter, uti circa parametrum fecimus, non quo termini ſecundi ordinis præ primo minus negligi queant, ſed quia secundus ordo continet partes omnino constantes, unde per integrationem hujusmodi termini $\alpha\omega$ nascuntur, qui quantumvis coëfficiens α fuerit parvus, tamen cum tempore continuo crescunt. Quia enim angulus ω est tempori proportionalis, hi termini motum medium aphelii declarabunt; in quorum idcirco investigatione vel minima particula perperam negligitur. At terminis hujus formæ exceptis, reliqui ad secundum ordinem pertinentes, quia periodicas inæqualitates continent, & præ primo ordine valde sunt parvi sine errore omitti poterunt; cum etiam levis error in loco aphelii commissus nullius sit momenti.

§. XCIX. Simili igitur modo integrationem instituendo, ante omnia sequentes formulas expendere oportet

$$d\nu = i\,d\omega - 2ik\,d\omega\, cos.\,\nu;$$

$$d\eta - d\nu = -m\,d\omega - o;$$

$$d\eta + d\nu = (2i - m)\,d\omega - 4ik\,d\omega\, cos.\,\nu;$$

$$2d\eta - d\nu = (i - 2m)\,d\omega - 2ik\,d\omega\, cos.\,\nu;$$

$$2d\eta + d\nu = (3i - 2m)\,d\omega - 6ik\,d\omega\, cos.\,\nu;$$

$$d\omega = \frac{d\nu}{i} + 2k\,d\omega\, cos.\,\nu;$$

$$d\omega = \frac{-(d\eta - d\nu)}{m};$$

$$d\omega = \frac{d\eta + d\nu}{2i - m} + \frac{4ik\,d\omega\, cos.\,\nu}{2i - m};$$

$$d\omega = \frac{2d\eta - d\nu}{i - 2m} + \frac{2ik\,d\omega\, cos.\,\nu}{i - 2m};$$

$$d\omega = \frac{2d\eta + d\nu}{3i - 2m} + \frac{6ik\,d\omega\, cos.\,\nu}{3i - 2m};$$

tum pro terminis secundi ordinis:

$$d\omega = \frac{d\nu}{i} = \frac{d\eta}{i - m} = \frac{-(d\eta - 2d\nu)}{i + m} = \frac{d\eta + 2d\nu}{3i - m}$$

$$= \frac{-(2d\eta - 2d\nu)}{2m} = \frac{2d\eta + 2d\nu}{2(2i - m)}$$

Hinc omittendis terminis secundi ordinis, qui non sunt formæ $\alpha\omega$ fiet $\int d\omega\, cos.\,\nu = \frac{sin.\,\nu}{i} + k\int d\omega\,(1 + cos.\,2\nu) = \frac{sin.\,\nu}{i} + k\omega$;

$$\int d\omega\, cos.\,(\eta - \nu) = -\frac{sin.\,(\eta - \nu)}{m},$$

$$\int d\omega\, cos.\,(2\eta - \nu) = +\frac{sin.\,(2\eta - \nu)}{i - 2m},$$

$$\int d\omega\, cos.\,(\eta + \nu) = \frac{+sin.\,(\eta + \nu)}{2i - m},$$

$$\int d\omega\, cos.\,(2\eta + \nu) = \frac{+sin.\,(2\eta + \nu)}{3i - 2m}.$$

quæ formulæ ad motum aphelii definiendum sufficiunt.

§. C. Ex his igitur differentiale (§. LXXXV) integratum præbebit motum aphelii sequenti modo expressum:

$$\varphi - \nu = Const. + \frac{nibb}{cck}\left(\frac{3\,sin.\,(\eta - \nu)}{2m} + \frac{sin.\,(\eta + \nu)}{2(2i - m)}\right)$$

$$- \frac{nib^3}{f^3}\left\{\begin{array}{l} + \frac{g\,sin.\,\nu}{i} - \frac{g'\,sin.\,(\eta - \nu)}{2m} + \frac{g'\,sin.\,(\eta + \nu)}{2(2i - m)} \\ + \frac{1}{2}k(3g + h)\,\omega + \frac{g''\,sin.\,(2\eta - \nu)}{2(i - 2m)} + \frac{g''\,sin.\,(2\eta + \nu)}{2(3i - 2m)} \end{array}\right\}$$

$$+ \frac{nibbc}{f^3k}\left\{\begin{array}{l} + \frac{g'\,sin.\,\nu}{2i} - \frac{(6g - g'')\,sin.\,(\eta - \nu)}{4m} + \frac{(3g' - g''')\,sin.\,(2\eta - \nu)}{4(i - 2m)} \\ + \frac{1}{4}k(2g' + h')\,\omega - \frac{(2g - 3g'')\,sin.\,(\eta + \nu)}{4(2i - m)} - \frac{(g' - 3g''')\,sin.\,(2\eta + \nu)}{4(3i - 2m)} \end{array}\right\}$$

Hujus expressionis pars præcipua formæ $\alpha\omega$ motum medium aphelii præbet, qui ergo uti perspicuum est non à quantitate excentricitatis pendet. Tempore scilicet quo ſol ſecundum motum medium percurrit angulum $= \omega$ aphelium planetæ proferetur per angulum $\frac{nibbc}{4f^3}(2g' + h')\omega - \frac{nib^3}{2f^3}(3g + h)\omega$ reliqui vero termini inæqualitates periodicas aphelii complectuntur, quæ eo evadunt majores quo minor fuerit excentricitas orbitæ.

§. CI. Præter hunc autem motum uniformem, quo aphelium profertur, ejus locus ad quodvis tempus corrigi debet per inæqualitates periodicas, quæ ſinibus angulorum $\nu, n \mp \nu, 2n \mp \nu,$ &c. ſunt proportionales: atque in hunc finem longitudo aphelii ita exprimetur:

$$\varphi - \nu = Conſt. + \frac{nibbc}{4f^3}(2g' + h')\omega - \frac{nib^3}{2f^3}(3g + h)\omega;$$

$$+ \frac{nibb}{2cck}\left(\frac{3\,\mathit{ſin.}\,(n - \nu)}{m} + \frac{\mathit{ſin.}\,(n + \nu)}{2i - m}\right);$$

$$- \frac{nib^3}{4f^3k}\left\{\frac{2g\,\mathit{ſin.}\,\nu}{i} - \frac{g'\,\mathit{ſin.}\,(n - \nu)}{m} + \frac{g'\,\mathit{ſin.}\,(n + \nu)}{2i - m} + \frac{g''\,\mathit{ſin.}\,(2n - \nu)}{i - 2m} + \frac{g''\,\mathit{ſin.}\,(2n + \nu)}{3i - 2m}\right\}$$

$$+ \frac{nibbc}{4f^3k}\left\{\frac{2g'\,\mathit{ſin.}\,\nu}{i} - \frac{(6g - g'')\,\mathit{ſin.}\,(n - \nu)}{m} - \frac{(2g - 3g'')\,\mathit{ſin.}\,(n + \nu)}{2i - m} + \frac{(3g' - g''')\,\mathit{ſin.}\,(2n - \nu)}{i - 2m} - \frac{(g' - 3g''')\,\mathit{ſin.}\,(2n + \nu)}{3i - 2m}\right\}$$

Cujus expreſſionis pars prima exhibet longitudinem mediam aphelii ad quodvis tempus, cui porro ſi applicentur inæqualitates reliqua parte contentæ, impetrabitur locus aphelii verus. Quodſi ponatur $\omega = 360$, ex prima parte innoteſcet motus aphelii annuus reſpectu ſtellarum fixarum.

§. CII. Quia in motu Lunæ inveſtigatio motus ejus apogei tantam diligentiam ac ſagacitatem, torque calculos intricatos exigebat, dubium hic oriri poteſt, an

hoc modo verus motus apheliorum eliciatur? Quodſi enim idem calculus ad Lunam transferretur, formula inventa ſemiſſem tantum veri motus apogei prope modum eſſet oſtenſura. Verum in hac applicatione ad Lunam numerus n ſeu potius termini hunc numerum continentes incomparabiliter prodeunt majores, quam noſtro caſu, atque termini quadratum numeri n involventes demum veram motus apogei quantitatem complent. Hic autem ob valores terminorum numero n affectorum minimos, nullum eſt dubium, quin terminos, qui ejus quadratum complecterentur, ſine ullius erroris ſenſibilis metu prætermittere queamus. Deinde etiam ex formulis generalioribus evidens eſt, excentricitatem planetæ perturbantis e nihil ad motum aphelii conferre.

SECTIO VII.

Investigatio Anomaliæ veræ quatenus ea ad quodvis tempus ab actione Planetarum mutua perturbatur.

§. CIII. In superiori sectione formulas eruimus, quibus ad quodvis tempus veri valores cum parametri & excentricitatis orbitæ, tum etiam vera longitudo aphelii definiuntur; in has autem formulas præter angulum n potissimum ingreditur angulus v qui planetæ anomaliam veram designat. Præcipuum opus igitur adhuc perficiendum in hoc consistit, ut methodum tradamus ad quodvis tempus anomaliam veram inveniendi; quæ cum, si nullæ adsint perturbationes ex anomalia media colligi soleat; hic quoque anomaliam planetæ mediam in computum introduci conveniet, quæ quoniam uniformiter cum tempore crescit; ad quodvis tempus tempore crescit, ad quodvis tempus expedite assignatur; sive quod eodem redit anomalia media reperitur, si à planetæ longitudine media, aphelii locus medius subtrahatur. Quæstio ergo hac sectione enodanda determinationem anomaliæ veræ v ex data anomalia media postulat.

§. CIV. Si nullæ adessent vires turbantes, foret $d\varphi = dv$, atque anomalia vera v ex hac æquatione

$$d\varphi = dv = \frac{a\,d\omega\sqrt{ap}}{xx}, \text{ seu } d\omega = \frac{p\sqrt{p}}{a\sqrt{a}} \cdot \frac{dv}{(1 - q\cos. v)^2}$$

definiri deberet; essent enim p & q quantitates constantes, & $d\omega$ incremento anomaliæ mediæ proportionale. In nostro autem casu neque quantitates p & q sunt con-

stantes, neque $d\varphi = dv$, unde manifestum est relationem inter anomalias mediam & veram quoque ab actione planetarum mutua perturbari. Interim tamen hæc relatio erit petenda ex æquatione $d\varphi = \frac{a\,d\omega\sqrt{ap}}{xx}$, seu hæc $d\omega = \frac{p\sqrt{p}}{a\sqrt{a}} \cdot \frac{d\varphi}{(1-q\cos. v)^2}$ substituendo pro $d\varphi$ valorem, qui ipsi ex æquatione differentiali motus aphelii convenit; hæcque æquatio in §. LXXXV habetur evoluta vi cujus cum non sit $d\varphi - dv = 0$; ponamus brevitatis gratia loco hujus æquationis differentialis $d\varphi - dv = nV\,d\omega$, eritque

$$d\omega = \frac{p\sqrt{p}}{a\sqrt{a}} \cdot \frac{dv}{(1 - q\cos. v)^2} + \frac{p\sqrt{p}}{a\sqrt{a}} \cdot \frac{nV\,d\omega}{(1-q\cos. v)^2}.$$

§. CV. Cum jam p & q non sint quantitates constantes, eorumque valores in superiori sectione sint definiti, ponamus quoque brevitatis gratia

$$p = b(1 + nP), \ \& \ q = k + nQ;$$

ita ut nP, nQ, & nV sint effectus perturbationis, eritque ob numerum n minimum $p\sqrt{p} = b^{\frac{3}{2}}(1 + \frac{3}{2}nP)$, & quia posuimus $\frac{a\sqrt{a}}{b\sqrt{b}} = i$ habebimus

$$\frac{p\sqrt{p}}{a\sqrt{a}} = \frac{1}{i}(1 + \tfrac{3}{2}nP).$$

Deinde fractio $\frac{1}{(1-q\cos. v)^2}$ in seriem conversa dat proxime

$$(1-qq)^{-\frac{3}{2}}(1 + 2q\cos. v + \tfrac{3}{2}qq\cos. 2v + q^3\cos. 3v \ \&c.)$$

quæ ponendo $k + nQ$ loco q, & negligendo terminos per nn & nkk affectos abit in hanc:

$$(1-kk)^{-\frac{3}{2}}(1 + 2k\cos. v + \tfrac{3}{2}kk\cos. 2v + k^3\cos. 3v)$$
$$+ 2nQ\cos. v + 3nkQ\cos. 2v + 3nkQ.$$

Hinque erit

$$\frac{p\sqrt{p}}{a\sqrt{a}} \cdot \frac{dv}{(1-q\cos. v)^2} = \frac{dv(1 + 2k\cos. v + \frac{3}{2}kk\cos. 2v + k^3\cos. 3v)}{i(1-kk)\sqrt{(1-kk)}}$$

$+\frac{n\,d\omega}{i}(2Q\,cos.\,\nu+\tfrac{3}{2}P+3kP\,cos.\,\nu+3kQ+3kQ\,cos.\,2\nu).$

§. CVI. In parte altera autem formulæ integrandæ tam p quam q pro constantibus haberi possunt, eritque ergo ea pari

$$\tfrac{n}{i}\,d\omega\,(V+2\,k\,V\,cos.\,\nu),$$

& quia in his particulis minimis est $d\nu=i\,d\omega(1-2k\,cos.\,\nu)$ obtinebimus æquationem sequentem:

$$d\omega=\frac{d\nu\,(1+2\,k\,cos.\,\nu+\tfrac{3}{2}\,k\,k\,cos.\,2\,\nu+k^3\,cos.\,3\,\nu)}{i\,(1-kk)\,\sqrt{(1-kk)}};$$

$$+\,n\,d\omega\,(2\,Q\,cos.\,\nu+k\,Q+k\,Q\,cos.\,2\,\nu+\tfrac{3}{2}\,P)$$
$$+\,\frac{n\,d\omega}{i}\,(V+2\,k\,V\,cos.\,\nu);$$

cujus pars principalis integrata deducet ad hanc æquationem integralem:

$$\omega=\frac{\nu+2\,k\,sin.\,\nu+\tfrac{3}{4}\,k\,k\,sin.\,2\,\nu+\tfrac{1}{3}\,k^3\,sin.\,3\,\nu}{i\,(1-kk)\,\sqrt{(1-kk)}};$$

$$+\,n\int d\omega\left(\tfrac{3}{2}\,P+2\,Q\,cos.\,\nu+k\,Q+k\,Q\,cos.\,2\,\nu\right.$$
$$\left.+\,\tfrac{1}{i}\,V+\tfrac{2}{i}\,k\,V\,cos.\,\nu\right);$$

cujus postremæ partis non amplius erit difficile integrale eruere.

§. CVII. Pro integratione hujus postremæ formulæ notandum est partem $n\int V\,d\omega$ exprimere motum aphelii, cujus ergo integrale jam supra §. CI est inventum. Reliquas partes tantisper indicemus signo summatorio, ac pro anomalia vera quæsita sequentem nanciscemur æquationem:

$$\nu=i(1-kk)^{\frac{3}{2}}\,\omega-2k\,sin.\,\nu-\tfrac{3}{4}kk\,sin.\,2\nu-\tfrac{1}{3}k^3\,sin.\,3\nu-n\int V\,d\omega;$$
$$-\,i\,n\int d\omega\,(\tfrac{3}{2}\,P+2\,Q\,cos.\,\nu+\tfrac{2}{i}\,k\,V\,cos.\,\nu);$$

in hac enim ultima parte perspicuum est terminos $k\,Q$ & $k\,Q\,cos.\,\nu$ præ P & Q posse rejici, at vero $k\,V\,cos.\,\nu$ iisdem

iisdem esse quasi homogeneum unde tantum opus est valores supra pro P, Q, & V inventos substituere. Hic autem primo observo terminum $i\,(1-kk)^{\frac{3}{2}}\,\omega$ cum partibus formæ $\alpha\,d\omega$, quas posteriora membra integralia forte continent, designare anomaliam mediam, quæ ad quodvis tempus facile colligitur. Si ergo anomaliam mediam ponamus $=\eta$, habemus hic æquationem inter η & ν, per cujus resolutionem non difficulter pro quavis anomalia media ejus respondens anomalia vera elicietur.

§. CVIII. Statuamus ad abbreviandum:

$$\tfrac{3}{2}P+2\,Q\,cos.\,\nu=A+B\,cos.\,2\,\nu\quad+C\,cos.\,\eta$$
$$+\,\tfrac{2}{i}\,k\,V\,cos.\,\nu\quad+D\,cos.\,2\,\eta\quad+E\,cos.\,(\eta-2\,\nu)$$
$$+\,F\,cos.\,(\eta+2\,\nu)+G\,cos.\,(2\,\eta-2\,\nu)$$
$$+\,H\,cos.\,(2\eta+2\nu)$$

atque horum coëfficientium valores ex superioribus formulis colliguntur:

$$A=\frac{bbc}{f^3}\cdot g'-\frac{2\,b^3}{f^3}\,g;\quad B=\frac{bbc}{f^3}\cdot g'-\frac{2\,b^3}{f^3}\,g;$$

$$C=\left\{\begin{aligned}&-\frac{ibb}{cc^3}\left(\frac{3}{i-m}+\frac{3}{2\,m}-\frac{1}{2\,(2\,i-m)}+\frac{1}{i}\right)-\frac{i\,b^3}{2\,f^3}\\&\qquad\left(\frac{2\,g'}{i}+\frac{g'}{m}+\frac{g'}{2\,i-m}\right)\\&+\frac{ibbc}{4f^3}\left(\frac{6\,(2\,g-g'')}{i-m}+\frac{(6\,g-g'')}{m}+\frac{(6\,g-g'')}{i}\right.\\&\qquad\left.-\frac{(2\,g-3\,g'')}{2\,i-m}-\frac{(2\,g-3\,g'')}{i}\right)\end{aligned}\right\}$$

$$D=\left\{\begin{aligned}&-\frac{i\,b^3}{2f^3}\left(\frac{-g''}{i-2\,m}+\frac{2\,g''}{i}+\frac{g''}{3\,i-2\,m}\right)\\&+\frac{ibbc}{4f^3}\left(\frac{3\,(g'-g''')}{i-m}-\frac{(3\,g'-g''')}{i-2\,m}+\frac{(3\,g'-g''')}{i}\right.\\&\qquad\left.-\frac{(g'-3\,g''')}{3\,i-2\,m}-\frac{(g'-3\,g''')}{i}\right)\end{aligned}\right\}$$

$$E = \frac{-ibb}{2cc}\left(\frac{3}{m}+\frac{3}{i}\right) - \frac{ib^3}{2f^3}\left(\frac{g'}{m}+\frac{g'}{i}\right) + \frac{ibbc}{4f^3}\left(\frac{6g-g''}{m}+\frac{6g-g''}{i}\right)$$

$$F = \frac{-ibb}{2cc}\left(-\frac{1}{2i-m}-\frac{1}{i}\right) - \frac{ib^3}{2f^3}\left(\frac{g'}{2i-m}+\frac{g'}{i}\right) + \frac{ibbc}{4f^3}\left(-\frac{(2g-3g'')}{2i-m}-\frac{(2g-3g'')}{i}\right)$$

$$G = \frac{-ib^3}{2f^3}\left(-\frac{g''}{i-2m}+\frac{g''}{i}\right) + \frac{ibbc}{4f^3}\left(-\frac{(3g'-g''')}{i-2m}+\frac{(3g'-g''')}{i}\right)$$

$$H = \frac{-ib^3}{2f^3}\left(\frac{+g''}{3i-2m}+\frac{g''}{i}\right) + \frac{ibbc}{4f^3}\left(-\frac{(g'-3g''')}{3i-2m}-\frac{(g'-3g''')}{i}\right)$$

§. CIX. Valoribus autem horum coëfficientum definitis facile erit ſingulorum terminorum integralia exhibere, quia ultra ordinem primum ea deducere non eſt opus: Erit itaque

$$\int d\omega\left(\tfrac{3}{2}P + 2Q\,cof.\,\nu + \tfrac{2}{i}kV\,cof.\,\nu\right) = A\omega + \frac{B}{2i}\int in.\,2\nu + \frac{C}{i-m}\int in.\,\eta + \frac{D}{2(i-m)}\int in.\,2\eta$$
$$- \frac{E}{i+m}\int in.\,(\eta-2\nu) + \frac{F}{3i-m}\int in.\,(\eta+2\nu) - \frac{G}{2m}\int in.\,(2\eta-2\nu) + \frac{H}{2(2i-m)}\int in.\,(2\eta+2\nu).$$

Deinde ſi ponamus ſimili modo ad abbreviandum

$$\int V d\omega = \Delta\omega + \frac{\alpha}{k}\int in.\,\nu + \frac{\beta}{k}\int in.\,(\eta-\nu) + \frac{\gamma}{k}\int in.\,(\eta+\nu) + \frac{\delta}{k}\int in.\,(2\eta-\nu) + \frac{\varepsilon}{k}\int in.\,(2\eta+\nu);$$

erunt hi coëfficientes ex §. CI.

$$\Delta = \frac{ibbc}{4f^3}(2g'+h') - \frac{ib^3}{2f^3}(3g+h);$$

$$\alpha = \frac{bbc}{2f^3}\cdot g' - \frac{b^3}{f^3}\cdot g;$$

$$\beta = \frac{bb}{cc}\cdot\frac{3i}{2m} + \frac{b^3}{2f^3}\cdot\frac{ig'}{m} - \frac{bbc}{4f^3}\cdot\frac{(6g-g'')i}{m};$$

$$\gamma = \frac{bb}{2cc}\cdot\frac{i}{2i-m} - \frac{b^3}{2f^3}\cdot\frac{ig'}{2i-m} - \frac{bbc}{4f^3}\cdot\frac{(2g-3g'')i}{2i-m};$$

$$\delta = -\frac{b^3}{2f^3}\cdot\frac{ig''}{i-2m} + \frac{bbc}{4f^3}\cdot\frac{(3g'-g''')i}{i-2m};$$

$$\varepsilon = -\frac{b^3}{2f^3}\cdot\frac{ig''}{3i-2m} - \frac{bbc}{4f^3}\cdot\frac{(g-3g''')i}{3i-2m}.$$

§. CX. Si jam hos valores determinaverimus, habebimus primo anomaliam mediam:

$$\aleph = i(1-kk)^{\frac{3}{2}}\omega - n\Delta\omega - inA\omega;$$

qua cognita anomalia vera ν ita debet definiri ut ſit;

$$\nu = \aleph - 2k\int in.\,\nu - \tfrac{3}{4}kk\int in.\,2\nu - \tfrac{1}{3}k^3\int in.\,3\nu;$$
$$-\frac{n}{k}\left(\alpha\int in.\,\nu + \beta\int in.\,(\eta-\nu) + \gamma\int in.\,(\eta+\nu) + \delta\int in.\,(2\eta-\nu) + \varepsilon\int in.\,(2\eta+\nu)\right);$$

$$-n\left\{\begin{array}{l}\frac{1}{2}B\int in.\,2\nu + \frac{Ci}{i-m}\int in.\,\eta + \frac{Di}{2(i-m)}\int in.\,2\eta \\ -\frac{Ei}{i+m}\int in\,(\eta-2\nu) + \frac{Fi}{3i-m}\int in.\,(\eta+2\nu) \\ -\frac{Gi}{2m}\int in.\,(2\eta-2\nu) + \frac{Hi}{2(2i-m)}\int in.\,(2\eta+2\nu)\end{array}\right\}$$

Si eſſet $n = o$, nota eſt operatio, qua ex data anomalia media $\aleph$ elicitur vera ν; cum igitur ſit n fractio valde parva, per eandem operationem, omittendis primum terminis per n affectis quæratur anomalia vera ν mediæ ȣ conveniens, eaque deinceps per terminos fractione n affectos corrigatur. Tum si eam accuratius definire velimus, valorem pro ν modo inventum in expressione illa pro ν reperta substituamus, ex eoque denuo ν determinemus.

§. CXI. Facilius autem per consuetas tabulas anomaliarum totum hoc negotium expediri potest. Cum enim perturbationes sunt minimæ, sufficiet pro iis anomaliam veram ν proxime saltem nosse, ejusque ergo loco anomalia media ipsa ȣ uti licebit, namque errores, qui hoc modo committentur, ad sequentem terminorum, quos negligimus, ordinem pertinerent. Tum valor horum terminorum minimorum ad anomaliam mediam ȣ referatur, seu ex data anomalia media ȣ quæratur anomalia media correcta ȣ$'$ ut sit

$$\text{ȣ}' = \text{ȣ} - \tfrac{n}{k}\Big(\alpha\, \mathit{fin.}\, \text{ȣ} + \beta\, \mathit{fin.}\, (\eta - \text{ȣ}) + \gamma\, \mathit{fin.}\, (\eta + \text{ȣ}) + \delta\, \mathit{fin.}\, (2\eta - \text{ȣ}) + \varepsilon\, \mathit{fin.}\, (2\eta + \text{ȣ})\Big);$$

ubi quidem partem posteriorem, utpote præ hac valde parvam omitto, atque jam pro data excentricitate k ex tabulis consuetis quæratur anomalia vera quæ huic anomaliæ mediæ correctæ respondeat; hocque modo obtinebitur ipsa illa anomalia vera ν, qua pro evolutione omnium formularum hactenus inventarum indigemus, erit scilicet

$$\nu = \text{ȣ}' - 2\,k\, \mathit{fin.}\, \nu - \tfrac{1}{4}\, kk\, \mathit{fin.}\, 2\nu - \tfrac{1}{3}\, k^3\, \mathit{fin.}\, 3\,\nu.$$

§. CXII. In Tabulis autem Astronomicis pro data quavis anomalia media non tam ei respondens anomalia vera; quam differentia, quæ prostaphæresis seu æquatio centri vocatur, exhiberi solet, neque etiam pro nostro scopo quicquam in hoc instituto immutari est opus. Ad manus igitur sit tabula more solito adornata, quæ pro excentricitate k cuique anomaliæ mediæ respondentem æquationem centri exhibeat. Antequam autem hac tabula utamur, anomalia media planetæ ȣ ad datum tempus collecta, per inæqualitates supra expositas & tam ab ea ipsa quam ab angulo η, cujus valorem quoque ex motu medio utriusque planetæ collegisse sufficit, pendentes corrigatur, ut obtineatur anomalia media correcta ȣ$'$. Tum in dicta Tabula quæratur æquatio isti anomaliæ mediæ ȣ$'$ conveniens, quæ sit $= \pm Æ$, qua inventa statim habebitur anomalia vera quæsita $\nu = \text{ȣ}' \pm Æ$, qua in determinatione & evolutione omnium formularum supra inventarum uti opportebit. Simul vero hæc æquatio $\pm Æ$ ex tabula desumta verum valorem formulæ $-\,2\,k\, \mathit{fin.}\, \nu - \tfrac{1}{4}\, kk\, \mathit{fin.}\, 2\,\nu - \tfrac{1}{3}\, k^3\, \mathit{fin.}\, 3\,\nu$ exhibebit, id quod pro sequenti calculo probe notasse conducet.

§. CXIII. Ad Anomaliam mediam autem pro dato tempore colligendam motum aphelii medium duntaxat nosse opportet, qui membro primo formulæ §. CI erutæ continetur, ex quo habemus:

$$\text{Long. aphelii mediam} = \mathit{Const.} + \frac{n\,i\,b\,b\,c}{4\,f^3}\,(2\,g' + h')\,\omega - \frac{n\,i\,b^3}{2\,f^3}\,(3\,g + h)\,\omega,$$

seu abbreviationem ante introductum adhibendo erit:

$$\text{Long. aphelii media} = \mathit{Const.} + n\,\Delta\,\omega.$$

Quoniam autem omnes planetæ ad motum medium aphelii aliquid conferunt, singulorum effectus exquiri debet, ut inde ad quodvis tempus propositum longitudo aphelii media rite obtineatur. Vel cum ex collatione recentiorum observationum cum antiquis motus medius aphelii cujusque planetæ satis accurate jam sit exploratus, eo potius uti conveniet. Quare cum hinc ad tempus propositum longitudo media aphelii sit definita, ea a longitudine media ipsius planetæ subtracta præbebit ejus anomaliam mediam ȣ pro eodem tempore proposito, quæ etiam in nonnullis tabulis immediate exprimi solet.

§. CXIV. Deinde si perturbationes, quæ ab actione certi cujusdam planetæ proficiscuntur, indagare velimus, primum ex collatione semi parametri ejus orbitæ c, cum semi parametro b planetæ examinandi ope formularum §. LXXII & LXXVIII datarum computentur valores litterarum g, g', g'', &c. ex hisque porro per rationem mediorum motuum i & m valores litterarum α, β, γ, δ, ε, itemque A, B, C, D, E, F, G, H, qui omnes in meris numeris expressi prodibunt. Tum etiam massa Planetæ per massam Solis divisa dabit fractionem n. Quibus inventis ad tempus propositum colligatur longitudo media planetæ perturbati & perturbantis, quia posteriori à priori ablata remanebit angulus, quo loco n uti licebit in indagatione correctionum anomaliæ mediæ (§. CXI). Vel quod expediet, utriusque planetæ longitudo per tabulas ordinarias definiatur, ac differentia pro angulo n assumatur, quandoquidem hic valor à vero nonnisi in minutiis discrepabit.

SECTIO VIII.

Expositio Universi Calculi quo verus Planetæ locus in orbita ob actionem reliquorum planetarum perturbatus assignatur.

§. CXV. PRIMA operatio in hoc consistet, ut pro quolibet planeta, à cujus actione motus planetæ propositi perturbatur, ope formularum §. LXXII & LXXVIII expositarum primo valores litterarum g, g', g'', &c. (litteris enim reliquis h, h' l, l', &c. ibidem adhibitis carere possumus), tum vero ex his porro valores litterarum A, B, C, D, &c. ex §. CVIII, & litterarum quoque Δ, α, β, γ, δ, &c. ex §. CIX per calculum evolvantur: pro quo calculo recordari debemus, fractionem n obtineri, si massa planetæ perturbantis per massam solis dividatur: deinde si motus diurnus medius solis unitate exponatur, exprimet littera i motum diurnum medium planetæ perturbati, & m planetæ perturbantis. Calculus quidem pro valoribus illarum litterarum instituendus admodum est molestus, verumtamen per subsidia indicata satis exacte absolvi poterit.

§. CXVI. Statim autem ex valore A cognoscetur, quantum aphelium ab actione cujusdam planetæ promoveatur; si enim pro angulo ω ponamus 360° terminus $n\Delta\omega$ dabit motum aphelii annuum, ac si hunc valorem ab actione cujuslibet planetæ deducamus, omnes conjunctim ostendent verum motum annuum planetæ respectu stellarum fixarum, qui vix quicquam ab eo, quem per obſervationes cognovimus, diſcrepare deprehendetur. Cognito autem tam aphelii quam ipſius, planetæ motu medio ad quodvis tempus propoſitum tam hujus planetæ longitudo media quam anomalia media facile aſſignabitur. Statuamus ergo ejus longitudinem mediam $= \zeta$ & anomaliam mediam $=$ ȣ; tum vero excentricitas media ſit $= k$.

§. CXVII. Deinde hæc anomalia media ȣ ex tabulis mediorum motuum deſumta corrigi debet per formulam §. CXI allatam, ut obtineatur anomalia media correcta ȣ$'$. Vel ſi tabulæ mediorum motuum loco anomaliæ mediæ exhibeant locum aphelii medium, eædem correctiones ſignis verſis ad aphelium applicari debebunt: hoc autem modo reperietur ipſa longitudo aphelii vera, unde hæc correctio magis eſt naturalis priori anomaliæ illata. Quare ex longitudine aphelii media quæratur longitudo ejus vera per hanc formulam

$$\text{Longitudo Aphelii vera} = \text{Longitudini Aphelii mediæ} + \tfrac{n}{k}\Big(\alpha\, \mathit{ſin.}\, \text{ȣ} + \beta\, \mathit{ſin.}\, (\eta - \text{ȣ}) + \gamma\, \mathit{ſin.}\, \eta + \text{ȣ}) + \delta\, \mathit{ſin.}\, (2\eta - \text{ȣ}) + \varepsilon\, \mathit{ſin.}\, (2\eta + \text{ȣ})\Big);$$

quæ correctio, quia per excentricitatem k eſt diviſa ſatis notabilis eſſe poteſt. Tum iſta Longitudo aphelii vera ſubtrahatur à longitudine planetæ media ζ, ut obtineatur anomalia media ejus correcta ȣ$'$.

§. CVXIII. Tertio in promtu eſto tabula æquationum centri more ſolito ad excentricitatem k computata, ex qua pro anomalia media ȣ$'$ excerpatur æquatio centri reſpondens quæ ſit $\pm$ Æ, atque hinc reperietur Anomalia vera $=$ ȣ$' \pm$ Æ quæ ob duplicem cauſam ab anomalia vera, quæ more ſolito ex tabulis æquationum colligitur nullo reſpectu ad perturbationes habito; diſcrepat, primo enim etſi ex eadem tabula deſumta eſt, tamen alii anomaliæ mediæ ac vulgo reſpondet, ideoque

que tantumdem diſcrepat; deinde quia alii anomaliæ mediæ reſpondet, etiam æquatio $\pm$ Æ erit diverſa. Manifeſtum autem eſt hoc poſterius diſcrimen multo fore minus priore; cum hoc adeo eo majus evadat, quo minor fuerit excentricitas k, tum vero æquatio $\pm$ Æ diminuatur. Etſi ergo in calculo perturbationum non adeo accurate noſſe opus eſt anomaliam veram ν, tamen correctio anomaliæ mediæ ſeu loci aphelii neutiquam negligi poteſt.

§. CXIX. Definita hoc modo anomalia vera ν ſtatim locum planetæ in orbita aſſignare poterimus, ita ut non opus habeamus ante variationem parametri & excentricitatis exquirere: quantum enim hæ variationes ad locum planetæ in orbita perturbandum conferunt, id jam ſumus complexi in expreſſione pro loco aphelii vero ſupra §. CI inventa. Nam quia jam valorem ipſius ν exacte expreſſum habemus, erit longitudo vera

$$\varphi = \nu + n \int V\, d\omega + \mathit{Conſt.}$$

Si ergo pro $\int V\, d\omega$ valorem §. CIX poſitum & pro ν valorem §. CX aſſignatum ſubſtituamus, conſequemur

$$\varphi = \mathit{Conſt.} + i\,(1 - kk)^{\frac{1}{2}}\,\omega - i\,n\,A\,\omega - 2\,k\, \mathit{ſin.}\, \nu - \tfrac{1}{4}\, k\, k\, \mathit{ſin.}\, 2\,\nu - \tfrac{1}{3}\, k^3\, \mathit{ſin.}\, 3\,\nu - \frac{nB}{2}\, \mathit{ſin.}\, 2\,\nu - \frac{n\,C\,i}{i-m}\, \mathit{ſin.}\, \eta - \frac{n\,D\,i}{2\,(i-m)}\, \mathit{ſin.}\, 2\,\eta + \frac{n\,E\,i}{i+m}\, \mathit{ſin.}\, (\eta - 2\,\nu) - \frac{n\,F\,i}{3\,i-m}\, \mathit{ſin.}\, (\eta + 2\nu) + \frac{nGi}{2\,m}\, \mathit{ſin.}\, (2\eta - 2\nu) - \frac{n\,H\,i}{2(2i-m)}\, \mathit{ſin.}\, (2\eta + 2\nu)$$

neque igitur hic amplius inæqualitates illæ majores in forma $n \int V\, d\omega$ contenta aliter ingrediuntur, niſi quatenus illis ipſa anomalia vera ν jam eſt immutata.

§. CXX. Prima portio hujus expreſſionis $i\,\omega\,((1 - kk)^{\frac{1}{2}} - n\,A)$ motum medium hujus planetæ exponit, quem ergo etiam ab actione planetarum aliquantillum perturbari manifestum est; hinc si longitudo planetæ media ponatur $= \zeta$, erit $\zeta = Const. + i(1 - kk)^{\frac{3}{2}}\,\omega - i\,n\,A\,\omega$. Deinde vidimus portionem $-2\,k\,\mathit{sin.}\,\nu - \frac{5}{4}\,kk\,\mathit{sin.}\,2\,\nu - \frac{1}{3}\,k^3\,\mathit{sin.}\,3\,\nu$ designare æquationem centri $\pm Æ$ quæ in tabulis ordinariis anomaliæ mediæ correctæ υ' respondet, dummodo hæ tabulæ excentricitati k sint justo calculo superstructæ. Cum igitur tam longitudo media ζ quam ista æquatio $\pm Æ$ constet, habebitur longitudo vera planetæ in sua orbita:

$$\phi = \zeta \pm Æ - \frac{n\,B}{2}\,\mathit{sin.}\,2\,\nu - \frac{n\,C\,i}{i-m}\,\mathit{sin.}\,\eta - \frac{n\,D\,i}{2(i-m)}\,\mathit{sin.}\,2\,\eta$$

$$+ \frac{n\,E\,i}{i+m}\,\mathit{sin.}\,(\eta - 2\,\nu) - \frac{n\,F\,i}{3\,i-m}\,\mathit{sin.}\,(\eta + 2\,\nu)$$

$$+ \frac{n\,G\,i}{2\,m}\,\mathit{sin.}\,(2\,\eta - 2\,\nu) - \frac{n\,H\,i}{2(2\,i-m)}\,\mathit{sin.}\,(2\,\eta + 2\,\nu)$$

ubi portio $\zeta \pm Æ$ exhibet longitudinem modo ordinario inventam, nisi quatenus anomalia media hic est correcta, tum vero reliqui termini continent ceteras inæqualitates ab actione planetæ perturbantis profectas, quarum quidem portio quædam jam in ipsa æquatione centri $\pm Æ$ ob anomaliam mediam correctam comprehenditur.

§. CXXI. Actionum ergo planetæ perturbantis ad duplicem effectum perduximus, dum altero longitudo aphelii seu anomalia media, altero vero ipsa longitudo perturbatur. Quia vero & priori effectus valde est parvus, uterque commode ad unum revocari poterit. Cum enim anomalia vera tantumdem immutetur quantum anomalia media, si ν denotet eam ipsam anomaliam veram, quæ anomaliæ mediæ non correctæ seu naturali υ respondet, in expressione pro vero planetæ loco inventa, loco ν scribi oportet $\nu - \frac{n}{k}\big(\alpha\,\mathit{sin.}\,\nu + \beta\,\mathit{sin.}$

$(\eta - \nu) + \gamma\,\mathit{sin.}\,(\eta + \nu) + \delta\,\mathit{sin.}\,(2\,\eta - \nu) + \epsilon\,\mathit{sin.}\,(2\,\eta + \nu)\big)$ quæ mutatio quidem in terminis minimis nullam variationem sensibilem gignit. At si jam $\pm Æ$ denotet æquationem centri ipsi anomaliæ mediæ υ convenientem, quia est $\pm Æ = -2\,k\,\mathit{sin.}\,\nu - \frac{5}{4}\,kk\,\mathit{sin.}\,2\,\nu - \frac{1}{3}\,k^3\,\mathit{sin.}\,3\,\nu$, in primo termino mutatio sensibilis orietur, ideoque loco $\pm Æ$ scribi debebit

$$\pm Æ + n\big(\alpha\,\mathit{sin}\,2\,\nu + (\beta + \gamma)\,\mathit{sin.}\,\eta + (\delta + \epsilon)\,\mathit{sin.}\,2\,\eta$$
$$+ \beta\,\mathit{sin.}\,(\eta - 2\,\nu) + \gamma\,\mathit{sin.}\,(\eta + 2\,\nu) + \delta\,\mathit{sin.}\,(2\,\eta - 2\,\nu)$$
$$+ \epsilon\,\mathit{sin.}\,(2\,\eta + 2\,\nu)\big);$$

sicque jam $\pm Æ$ denotabit æquationem centri anomaliæ mediæ naturali υ respondentem, & anomalia vera ν erit etiam ea quæ more solito sumitur scilicet $\nu = \upsilon \pm Æ$.

§. CXXII. Hinc igitur faciliorem modum adipiscimur effectum perturbationis in loco planetæ determinandi. More scilicet solito ad datum tempus colligatur anomalia media υ, eique ex tabula ordinaria capiatur respondens æquatio centri $\pm Æ$, indeque formetur anomalia vera $\nu = \upsilon \pm Æ$. Qua stabilita, si longitudo planetæ media fuerit $= \zeta$ & η designet angulum, qui relinquitur si a longitudine planetæ perturbati longitudo planetæ perturbantis subtrahatur, habebitur longitudo planetæ perturbati vera.

$$\phi = \zeta \pm Æ + n\Big\{\Big(\alpha - \frac{B}{2}\Big)\,\mathit{sin.}\,2\,\nu + \Big(\beta + \gamma - \frac{C\,i}{i-m}\Big)\,\mathit{sin.}\,\eta$$
$$+ \Big(\delta + \epsilon - \frac{D\,i}{2(i-m)}\Big)\,\mathit{sin.}\,2\,\eta$$
$$+ \Big(\beta + \frac{E\,i}{i+m}\Big)\,\mathit{sin.}\,(\eta - 2\,\nu) + \Big(\gamma - \frac{F\,i}{3\,i-m}\Big)\,\mathit{sin.}\,(\eta + 2\,\nu)$$
$$+ \Big(\delta + \frac{G\,i}{2\,m}\Big)\,\mathit{sin.}\,(2\,\eta - 2\,\nu) + \Big(\epsilon - \frac{H\,i}{2(2\,i-m)}\Big)\,\mathit{sin.}\,(2\,\eta + 2\,\nu)\Big\}.$$

ubi $\zeta + Æ$ exprimit longitudinem planetæ, quam tabulæ ordinariæ præbent, totaque perturbatio jam in terminis annexis continetur.

§. CXXIII. Hic statim observo fieri $\alpha - \frac{B}{2} = 0$, unde si ad reliquos terminos contrahendos ponatur:

$$\varphi = \zeta \pm Æ + nB' \sin. \eta + nC' \sin. 2\eta + nD' \sin. (\eta - 2\nu) + nE' \sin. (\eta + 2\nu) + nF' \sin. (2\eta - 2\nu) + nG' \sin. (2\eta + 2\nu) + \&c.$$

per valores supra exhibitos reperiemus

$$B' = \left\{ \begin{array}{l} \frac{bb}{cc}\left(\frac{3i^3}{m(i-m)^2} - \frac{ii}{(i-m)(2i-m)}\right) + \frac{b^3}{f^3} \cdot \frac{2i^3g'}{m(i-m)(2i-m)} \\ - \frac{bbc}{2f^3}\left(\frac{3(2g-g'')ii}{(i-m)^2} + \frac{(6g-g'')ii}{m(i-m)} - \frac{(2g-3g'')ii}{(i-m)(2i-m)}\right) \end{array} \right\}$$

$$C' = \left\{ \begin{array}{l} - \frac{b^3}{f^3} \cdot \frac{i^3 g''}{(i-m)(i-2m)(3i-2m)} \\ - \frac{bbc}{4f^3}\left(\frac{3(g'-g''')ii}{2(i-m)^2} - \frac{(3g'-g''')ii}{(i-m)(i-2m)} - \frac{(g'-3g''')ii}{(i-m)(3i-2m)}\right) \end{array} \right\}$$

$D' = 0$; $E' = 0$; $F' = 0$; & $G' = 0$.

Hanc ob rem tota correctio ita contrahitur, ut tantum duobus terminis constet; sitque $\varphi = \zeta \pm Æ + n B' \sin. \eta + n C' \sin. 2\eta$; siquidem in perturbationibus excentricitatem k rejicimus.

§. CXXIV. Distantia vera planetæ à sole x nunc quoque facile definiri poterit; cum enim sit $x = \frac{p}{1 - q \cos. \nu}$ si ponamus ut supra $p = b(1 + nP)$ & $q = k + nQ$, erit ob nP & nQ minima:

$$x = \frac{b}{1 - k \cos. \nu} + n.b(P + Q \cos. \nu).$$

Supra autem jam valores quantitatum P & Q assignavimus, hic vero pro ν capi debet ea anomalia vera, quæ

anomaliæ mediæ correctæ ϑ' respondet: sin autem anomalia vera tabulari uti velimus, eamque littera ν indicemus, pro ν in ista formula scribere debemus

$$\nu - \frac{n}{k}\left(\alpha \sin. \nu + \beta \sin. (\eta - \nu) + \gamma \sin. (\eta + \nu) + \delta \sin. (2\eta - \nu) + \varepsilon \sin. (2\eta + \nu)\right);$$

ideoque pro $k \cos. \nu$ scribi opportebit:

$$k \cos. \nu + \frac{n}{2}\left(\alpha - \alpha \cos. 2\nu - (\beta - \gamma) \cos. \eta + \beta \cos. (\eta - 2\nu) + \gamma \cos. (\eta + 2\nu) - (\delta - \varepsilon) \cos. 2\eta + \delta \cos. (2\eta - 2\nu) - \varepsilon \cos. (2\eta + 2\nu)\right).$$

Hinc si ponamus $k \cos. \nu = k \cos. \nu + nR$, erit

$$x = \frac{b}{1 - k \cos. \nu} + nb(P + Q \cos. \nu + R),$$ ubi $\frac{b}{1 - k \cos. \nu}$ distantiam ex tabulis more solito erutam exprimit; neque vero plerumque operæ est pretium pro distantia hanc correctionem adhibere.

SECTIO IX.

Evolutio Inæqualitatum quibus cum linea nodorum tum inclinatio ab actione Planetarum afficitur.

§. CXXV. SUPRA in §. LXXXVII & LXXXVIII formulas exhibuimus differentiales, quibus mutatio momentanea tam in situ lineæ nodorum quam in inclinatione orbitæ planetæ perturbati ad orbitam perturbantis, quam tanquam fixam considero exprimitur. Productæ autem sunt istæ formulæ usque ad terminos excentricitate simplici k affectos, omissis iis, qui vel per quadratum altioremve potestatem ejusdem excentricitatis k, vel per excentricitatem orbitæ planetæ perturbantis e sunt multiplicati, quos autem si quis laborem suscipere velit eidem methodo insistendo non esset difficile insuper adjicere: neque etiam tum istarum formularum integratio majori premeretur difficultate. Verum quia actio planetarum est minima, hic adeo terminos excentricitatem k involventes rejicere licebit, sicque expressiones integrales & facilius invenientur, & multo fient simpliciores.

§. CXXVI. Quod igitur primum ad longitudinem nodi attinet, quam respectu stellarum fixarum littera π indicavimus, in ejus differentiale ingreditur angulus σ, qui denotat argumentum latitudinis $\varphi - \pi$. Cum ergo hoc calculo negligamus, & differentiale ipsius $d\pi$ præ $d\varphi$ sit minimum, tuto assumere licet $d\sigma = d\varphi = i\,d\omega$, & quia porro est $d\eta = (i-m)\,d\omega$,

integrando obtinebimus pro longitudine lineæ nodorum:

$$\pi = Const. - \frac{nbbc}{4f^3}\cdot g'i\omega + \frac{nibb}{cc}\left(\frac{\sin.\eta}{2(i-m)} + \frac{\sin.(\eta-2\sigma)}{2(i+m)}\right)$$
$$-\frac{nibbc}{4f^3}\left\{\begin{matrix}\frac{(2g+g'')\sin.\eta}{i-m} + \frac{(g'+g''')\sin.2\eta}{2(i-m)} + \frac{2g\sin.(\eta-2\sigma)}{i+m}\\ -\frac{g'\sin.2\sigma}{2i} + \frac{g'\sin.(2\eta-2\sigma)}{2m} - \frac{g''\sin.(\eta+2\sigma)}{3i-m}\end{matrix}\right\}$$

Hinc ergo erit longitudo media nodi $= Const. - \frac{nbbc}{4f^3}\cdot g'i\omega$, & quia g' semper est quantitas positiva, patet lineam nodorum semper regredi, & quidem singulis annis per angulum $= \frac{90^\circ\,nbbc}{f^3}\cdot ig'$ graduum, ponendo $= 360^\circ$.

§. CXXVII. Formulam pro differentiali $\frac{d.\,tang.\,G}{tang.\,G}$ inventam, quia etiam est valde parva, loco $tang.\,G$ poterimus per tangentem inclinationis mediæ multiplicare, sit igitur inclinatio media $= \lambda$, denotante G inclinationem veram, atque integrando obtinebimus

$$\frac{tang.\,G}{tang.\,\lambda} = 1 + \frac{nibb}{2cc}\left(\frac{\cos.\eta}{i-m} - \frac{\cos.(\eta-2\sigma)}{i+m}\right)$$
$$+\frac{nibbc}{4f^3}\left\{\begin{matrix}\frac{(2g-g'')\cos.\eta}{i-m} + \frac{(g'-g''')\cos.2\eta}{2(i-m)} - \frac{2g\cos.(\eta-2\sigma)}{i+m}\\ +\frac{g'\cos.2\sigma}{2i} - \frac{g'\cos.(2\eta-2\sigma)}{2m} - \frac{g''\cos.(\eta+2\sigma)}{3i-m}\end{matrix}\right\}$$

Cum igitur inclinatio vera G minime discrepet à media λ, ponamus $G = \lambda + d\lambda$, eritque $tang.\,G = tang.\,\lambda + \frac{d\lambda}{\cos.\lambda^2}$, & $\frac{tang.\,G}{tang.\,\lambda} = 1 + \frac{d\lambda}{\sin.\lambda\cos.\lambda} = 1 + \frac{2\,d\lambda}{\sin.2\lambda}$; qua formula cum illa expressione collata eliciemus valorem ipsius $d\lambda$, quo substituto reperietur

$$G = \lambda + \frac{nibb\,fin.\,2\lambda}{4cc}\left(\frac{cof.\,u}{i-m} - \frac{cof.\,(u-2\sigma)}{i+m}\right)$$

$$-\frac{nibbc\,fin.\,2\lambda}{8f^3}\left\{\begin{array}{l} +\frac{(2g-g'')\,cof.\,u}{i-m} - \frac{2g\,cof.\,(u-2\sigma)}{i+m} - \frac{g'\,cof\,(2u-2\sigma)}{2m} \\ +\frac{(g'-g''')\,cof.\,2u}{2(i-m)} - \frac{g'\,cof.\,2\sigma}{2i} - \frac{g''\,cof.\,(u+2\sigma)}{3i-m} \end{array}\right\}$$

§. CXXVIII. Inæqualitates igitur istæ non solum ob fractionem minimam n sed etiam ob *fin.* 2λ erunt tam exiguæ, ut nullo modo observari queant: atque etiam inæqualitates periodicæ in lineâ nodorum vix unquam in sensus occurrant, unde in usu astronomico tuto negligi poterunt. Tantum ergo notasse sufficiet motum lineæ nodorum medium, qui continetur hac formula:

$$\pi = Conft. - \frac{nbbc}{4f^3} g' i \omega;$$

unde constat lineam nodorum motu uniformi contra signorum seriem recedere. Etsi enim hic motus singulis annis sit tardissimus ut percipi nequeat, tamen successione plurium annorum, quia continuo accumulatur, maxime sensibilis evadere potest. Effectus autem qui inde in phænomena Astronomica redundat, in hoc potissimum cernetur, quod si pro planeta perturbato terra accipiatur, latitudo stellarum fixarum aliquantillum immutetur, qui effectus propterea imprimis meretur, ut accuratius evolvatur.

§. CXXIX. Ista autem latitudinis mutatio pendebit à longitudine cujusque stellæ fixæ ratione nodorum: Posita enim longitudine nodi ascendentis terræ super orbita planetæ perturbantis $= \pi$, quæ convenit cum longitudine nodi descendentis ejusdem planetæ super ecliptica, si longitudo cujuspiam stellæ fixæ fuerit $= \pi$, ejus latitudo si fuerit borealis post tempus, quo sol arcum ω absolvit, diminuetur particula $= \frac{nbbc}{4f^3} g' i \omega\, fin.\, \lambda$, sin autem latitudo fuerit australis tantumdem augebitur. Contrarium eveniet si longitudo stellæ fuerit $180^\circ + \pi$, tum enim eodem tempore, cui solis motus ω respondet, ejus latitudo si fuerit borealis augebitur particula $\frac{nbbc}{4f^3} g'.\, i\, \omega\, fin.\, \lambda$, sin autem sit australis tantumdem diminuetur. At si longitudo stellæ 90° distet à nodis tum ejus latitudo nullam patietur mutationem. In genere autem si longitudo stellæ fixæ fuerit $= \xi$, eodem tempore ejus latitudo si fuerit borealis diminuetur particula $= \frac{nbbc}{4f^3} g' i \omega\, fin.\, \lambda\, cof.\, (\xi - \pi)$ sin autrem latitudo sit australis tantumdem augebitur.

§. CXXX. Maxime igitur notabiles effectus, qui ab actione planetarum in terram exercentur, sunt primo ista exigua mutatio in latitudine stellarum fixarum, quæ autem cum observationibus vetustis circa latitudinem stellarum fixarum institutis minus fidere liceat, utrum veritati sit conformis? non tam facile explorari potest. Interim tamen studiosa collatio veterum observationum cum recentioribus vix dubitare sinit, quin in quibus stellis fixis latitudo parumper sit immutata, quod phænomenum sine dubio actioni planetarum est tribuendum. Deinde maxime conspicuus effectus cernitur in motu aphelii, cujus consensus cum veritate facillime explorari potest, quandoquidem ex observationibus certum est, aphelium terræ quotannis per spatiolum 11″ circiter promoveri; similique modo motus apheliorum in reliquis planetis ab eorum actione mutua oriundus cum observationibus comparari poterit. Reliqui effectus in plerisque planetis minus perceptibiles consistunt in mutatione excentricitatis, in inæqualitatibus periodicis loci apheliorum, unde anomalia media afficitur ac denique in variatione parametri orbitarum; quibus cognitis, loca planetarum per præcepta vulgaria Astronomica facile assignari poterunt.

PARS ALTERA

Continens Applicationem Theoriæ ad motum Terræ ejusque perturbationes ab actione reliquorum Planetarum oriundas.

1. IN parte superiori Theoriam actionis planetarum mutuæ ita in genere constitui, ut ex ea inæqualitates cujusque planetæ, quæ ejus motui ab actione reliquorum planetarum inducuntur, definiri atque assignari queant. Quas inæqualitates ita ad commodum calculi astronomici traduxi, ut pateat, quantum primo latus rectum seu parameter orbitæ, tum vero excentricitas, tertio locus aphelii, & quarto positio plani, quod orbita in cœlo occupat, quovis tempore immutetur. Cognitis enim his variationibus, manifesto apparebit, quantum motus planetæ quovis tempore à regulis Keplerianis recedere, & quales correctiones Tabulis consuetis adhiberi debeant, ut ad quodvis tempus verus planetæ locus in cœlo assignari queat.

2. Labor autem foret nimis operosus, limitesque huic dissertationi præfixos longe excederet, si hanc Theoriam ad singulos planetas accommodare vellem. Ipsa quoque Illustrissima Academia Regia tam prolixum opus non requirit, dum postquam Theoria perturbationum solide fuerit stabilita ejus applicationem tantum ad motum Terræ exigit: cujus præcepto morem gesturus

cunctas perturbationes, quibus terra in motu ſuo ob actionem reliquorum planetarum eſt obnoxia, data opera determinabo. Ex hac autem applicatione facile perſpicietur, quomodo per eandem Theoriam & reliquorum planetarum omnium perturbationes, quas ſibi mutuo induunt, definiri oporteat.

3. Ad motum autem terræ perturbandum reliqui planetæ omnes concurrunt, ſingulorumque effectus ſecundum præcepta ſuperiora ſeorſim inveſtigari conveniet, quod opus pro ſingulis ſimili calculo abſolvetur. Quoniam igitur terram in locum planetæ perturbati conſtituimus, littera i perpetuo unitatem denotabit: atque ex tabulis ſolaribus pro ejus excentricitate media aſſumemus $k = 0,0168$. Quanquam enim cunctis inæqualitatibus rite determinatis demum verum valorem excentricitatis mediæ k definire licet, tamen in ipſa harum inæqualitatum inveſtigatione valore ipſius k proxime vero tuto uti poterimus, quandoquidem hic minimas aberrationes merito negligimus. Interim valor $k=0,0168$ tam prope ad veritatem accedere videtur, ut error nullius certe ſit momenti. Habebimus igitur conſtanter $i = 1$ & $k = 0,0168$, neque quicquam præterea ex terræ theoria repeti eſt neceſſe, propterea quod non tam quantitas abſoluta ejus parametri quam ejus ratio ad parametrum cujuſque alterius planetæ in computum ingreditur.

4. Quicunque planetarum pro perturbante aſſumitur, ejus primum vim abſolutam, ſeu rationem ejus maſſæ ad maſſam Solis noſſe oportet, quam rationem littera n indicavimus. Ex phænomenis quidem Satellitum Newtonus concluſit, ſi Saturnus ſit planeta perturbans fore $n = \frac{1}{3021}$, ſin autem ſit Jupiter eſſe $n = \frac{1}{1067}$; pro reliquis autem planetis, quoniam Satellitibus deſtituuntur, valor fractionis n ex phænomenis determinari nequit. Etſi autem Mars & Venus ratione voluminis terræ ſunt minores, fortaſſe ob majorem denſitatem ratione maſſæ non multum diſcrepant, foretque ergo pro illis $n = \frac{1}{170000}$; pro Marte tamen hanc fractionem ob celeb. Monnierii obſervationes notabiliter imminuere vellem, ut eſſet quaſi $n = \frac{1}{2000000}$, nullumque eſt dubium, quin pro Mercurio hæc fractio multo minor ſit accipienda forſitan $n = \frac{1}{100000000}$. Verum ex ipſa quantitate effectuum forte hæc accuratius definire licebit.

5. Porro pro quovis planeta noſſe oportet motum medium ſeu rationem anguli, dato tempore circa ſolem deſcripti ad motum medium ſolis pro eodem tempore. Hanc rationem littera m indicavimus, unde tabulas aſtronomicas conſulentes reperiemus

pro Saturno	$m = \frac{2}{59} = 0,0339$
pro Jove	$m = \frac{7}{83} = 0,0843$
pro Marte	$m = \frac{42}{79} = 0,5316$
pro Venere	$m = \frac{13}{8} = 1,6250$
pro Mercurio	$m = \frac{137}{33} = 4,1515$

Excentricitate horum planetarum littera e indicata non erit opus, ſiquidem vidimus perturbationes inde pendentes tam prodire exiguas, ut præ reliquis facile rejici queant. Saltem in hac applicatione ejus rationem non habebimus, etiamſi in Theoria non ſit neglecta; propterea quod ad motum apogei medium nihil plane confert.

6. Cum igitur ſit $m\frac{a\sqrt{a}}{c\sqrt{c}} = \frac{b\sqrt{b}}{c\sqrt{c}}$ ob $a = b$, hinc reliquas expreſſiones, quæ in calculum ingrediuntur, determinare poterimus.

Sic erit $\frac{bb}{cc} = \sqrt[3]{m^4}$; $\frac{bb}{ff} = \frac{\sqrt[3]{m^4}}{1+\sqrt[3]{m^4}}$; $\frac{cc}{ff} = \frac{1}{1+\sqrt[3]{m^4}}$;

hincque $\frac{b^3}{f^3} = \frac{mm}{(1+\sqrt[3]{m^4})^{\frac{1}{2}}}$; $\frac{bbc}{f^3} = \frac{\sqrt[3]{m^4}}{(1+\sqrt[3]{m^4})^{\frac{1}{2}}}$.

Tum etiam hinc elicientur valores numerorum μ & ν ſupra (§. LXXVII) introductorum, eritque:

$$\mu = \frac{2bc}{ff} = \frac{2\sqrt[3]{mm}}{1+\sqrt[3]{m^4}}; \ \&$$
$$\nu = \frac{cc-bb}{ff} = \frac{1-\sqrt[3]{m^4}}{1+\sqrt[3]{m^4}}.$$

Ex his autem neglecta excentricitate e habemus:

$$\frac{1}{r^3} = \frac{1}{f^3}(1-\tfrac{3}{2}(1-\nu)k\,cos.\,\nu) = \frac{1}{f^3}\left(1-\frac{3\sqrt[3]{m^4}}{1+\sqrt[3]{m^4}}k\,cos.\,\nu\right);$$

$s = \mu\,(1+\nu\,k\,cos.\,\nu)$; $ss = \mu^2\,(1+2\nu\,k\,cos.\,\nu)$;
$s^3 = \mu^3\,(1+3\,\nu\,k\,cos.\,\nu)$ &c.
atque $1-ss = \nu\nu - 2\mu\mu\nu\,k\,cos.\,\nu$, &
$$\frac{1}{1-ss} = \frac{1}{\nu^2} + \frac{2\mu\mu\,k\,cos.\,\nu}{\nu^3}.$$

7. Jam præcipuus labor in computo litterarum g, h, g', h', &c. conſiſtet, pro quibus primum ex §. LXXII valores expreſſionum

$P(1-ss)$; $\frac{1}{2}Q(1-ss)$; $\frac{1}{2}R(1-ss)$; $\frac{1}{2}S(1-ss)$ &c.

hincque ipſæ hæ litteræ P, Q, R, S, &c. colligi debent. Quæ ſingulæ cum habituræ ſint formam $A + B\,k\,cos.\,\nu$, erit porro

$$g + h\,k\,cos.\,\nu = P\,(1-\tfrac{3}{2}(1-\nu)\,k\,cos.\,\nu)$$
$$g' + h'\,k\,cos.\,\nu = Qs\,(1-\tfrac{3}{2}(1-\nu)\,k\,cos.\,\nu)$$
$$g'' + h''\,k\,cos.\,\nu = Rs^2\,(1-\tfrac{3}{2}(1-\nu)\,k\,cos.\,\nu)$$
$$g''' + h'''\,k\,cos.\,\nu = Ss^3\,(1-\tfrac{3}{2}(1-\nu)\,k\,cos.\,\nu)$$
&c.

quoniam excentricitatem e, ac proinde numeros inde pendentes l, l', l'', &c. negligimus. Negligimus vero etiam terminos quadratum k^2 ejuſque altiores poteſtates involventes, unde calculus in numeris ſatis expedite abſolvi poterit. Atque hoc modo omnia elementa, quæ ad perturbationes motus in orbita inveniendas ſpectant, erunt cognita.

8. Denique vero quod ad variationem plani orbitæ attinet id pro quovis planeta perturbante ad planum ejus orbitæ, quæ ſaltem ad tempus ut fixa ſpectatur, eſt relatum. Ex tabulis autem Aſtronomicis colligimus pro An. 1750.

Si orbita terræ referatur ad	Eſſe longitudinem nodi aſcendentis	Inclinationem orbitæ
Orbitam Saturni	9ˢ, 21°, 20′, 6″	2°, 30′, 10″
Orbitam Jovis	9, 8, 15, 50,	1, 19, 10
Orbitam Martis	7, 17, 56, 22,	1, 51, 0
Orbitam Veneris	8, 14, 23, 43,	3, 23, 20
Orbitam Mercurii	7, 18, 29, 0,	6, 59, 20

His igitur notatis perturbationes, quas quilibet planeta in motu terra producit, per calculum numericum inveſtigemus, unde quantum Theoria cum veritate conſentiat, facile erit judicare.

I.

Investigatio inæqualitatum motus Terræ ab actione Saturni oriundarum.

9. PRIMUM igitur Saturnus locum teneat planetæ perturbantis, atque ut vidimus pro eo habemus

$$n = \frac{1}{3021} \quad \& \quad m = 0,0339,$$

& quantitates hinc derivatas cum suis logarithmis:

$\frac{bb}{cc} = 0,01097 \qquad l\frac{bb}{cc} = 8,040266$
$\frac{b^3}{f^3} = 0,00114 \qquad l\frac{b^3}{f^3} = 7,055293$
$\frac{bbc}{f^3} = 0,01079 \qquad l\frac{bbc}{f^3} = 8,033159$

$\mu = 0,2072; \quad l\mu = 9,316425; \quad \mu\nu = 0,2027$
$\mu^2 = 0,0429; \quad l\mu^2 = 8,632850; \quad \mu^2\nu = 0,0420$
$\mu^3 = 0,0089; \quad l\mu^3 = 7,949275; \quad \mu^3\nu = 0,0087$
$\mu^4 = 0,0018; \quad l\mu^4 = 7,265700; \quad \mu^4\nu = 0,0018$
$\nu = 0,9783; \quad l\nu = 9,990465; \quad 1-\nu = 0,0217$
$\frac{1}{\nu\nu} = 1,0449; \quad l\frac{1}{\nu\nu} = 0,019070; \quad \frac{\mu\mu}{\nu^3} = 0,0459$

10. Ex his valoribus formabimus sequentes

$\frac{f^3}{r^3} = 1 - \frac{3}{2}(1-\nu)\,k\,cos.\,v = 1 - 0,0326\,k\,cos.\,v$
$s = 0,2072 + 0,2027\,k\,cos.\,v$
$s^2 = 0,0429 + 0,0840\,k\,cos.\,v$
$s^3 = 0,0089 + 0,0261\,k\,cos.\,v$
$s^4 = 0,0018 + 0,0071\,k\,cos.\,v$
$s^5 = 0,0003 + 0,0017\,k\,cos.\,v$
$s^6 = 0,0001 + 0,0003\,k\,cos.\,v$

Tum

Tum vero $\frac{1}{1-ss} = 1,0449 + 0,0918\,k\,cos.\,v$, unde valores $P(1-ss)$; $\frac{1}{2}Q(1-ss)$; $\frac{1}{2}R(1-ss)$, &c. colliguntur

$P(1-ss) = 0,99729 - 0,00535\,k\,cos.\,v$
$\frac{1}{2}Q(1-ss) = 0,75307 + 0,00609\,k\,cos.\,v$
$\frac{1}{2}R(1-ss) = 0,47518 + 0,01283\,k\,cos.\,v$
$\frac{1}{2}S(1-ss) = 0,28004 + 0,01326\,k\,cos.\,v$

11. Ex his deducitur:

$P = 1,0421 + 0,0859\,k\,cos.\,v$; hinc
$Q = 1,5746 + 0,1512\,k\,cos.\,v$; $Qs = 0,3257 + 0,3500\,k\,cos.\,v$
$R = 0,9930 + 0,1140\,k\,cos.\,v$; $Rs^2 = 0,0425 + 0,0877\,k\,cos.\,v$
$S = 0,5852 + 0,0790\,k\,cos.\,v$; $Ss^3 = 0,0052 + 0,0161\,k\,cos.\,v$

multiplicentur jam hæ formulæ per $1 - 0,0326\,k\,cos.\,v$, indeque pro litteris g, h, g', h', g'', h'', &c. sequentes obtinebuntur valores:

$g = 1,0421; \qquad h = 0,0519$
$g' = 0,3257; \qquad h' = 0,3394$
$g'' = 0,0425; \qquad h'' = 0,0863$
$g''' = 0,0052; \qquad h''' = 0,0159$
&c.

qui per se tantopere decrescunt, ut circa convergentiam seriei in quam supra terminum $\frac{1}{z^3}$ transformavimus nullum dubium superesse possit.

12. His valoribus inventis inquiramus primo in motum aphelii terræ, quatenus ab actione Saturni afficitur, & quoniam per (101) tempore quo Sol motu medio conficit angulum ω, aphelium terræ respectu stellarum fixarum promovetur per spatiolum

$$\frac{nbbc}{4f^3}(2g' + h')\,\omega - \frac{nb^3}{2f^3}(3g + h)\,\omega, \text{ ob}$$

$\frac{bbc}{4f^3} = 0,002698; \quad 2g' + h' = 0,9908$, erit

$\frac{bbc}{4f^3}(2g' + h') = 0,002673;$

$\frac{b^3}{2f^3} = 0,000570; \quad 3g + h = 3,1782$, erit

$\frac{b^3}{2f^3}(3g + h) = 0,001811.$

Hinc iſto tempore aphelium proferetur per ſpatium

$$0,000862\, n\,\omega = \frac{0,000862\,\omega}{3021}, \text{ ob } n = \frac{1}{3021}.$$

Tempore ergo unius anni, quo $\omega = 360° = 1296000''$, aphelium terræ à Saturno propellitur per ſpatium $= 0,370'' = 22'''$, ideoque tempore 100 annorum per ſpatium $= 37''$, ſi ergo terra tantum à Saturno perturbaretur, aphelium reſpectu ſtellarum fixarum promoveretur:

Tempore unius anni per ſpatium $22'''$,

Tempore 100 annorum per ſpatium $37''$.

13. Hinc ad quodvis tempus longitudo media aphelii terræ innoteſcit, quæ autem porro per inæqualitates periodicas corrigi debet. Pendent autem eæ à duobus angulis η & ν, quorum ille η habetur ſi longitudo Saturni θ à longitudine terræ φ ſubtrahatur, hic vero ν denotat anomaliam terræ veram. Cum igitur ſit $\frac{bb}{2cc} = 0,00548$; & $m = 0,0339$, hinc $2i - m = 1,9661$; $i - 2m = 0,9322$, & $3i - 2m = 2,9322$, ob $n = \frac{1}{3021}$ & $k\ 0,0168$, formula pro motu aphelii (§. CI) inventa ad angulos reducta dabit:

Longitudo Aphelii vera = Longitudini aphelii mediæ

$$+ 1973''\sin.(\eta - \nu) + 11''\sin.(\eta + \nu)$$
$$- 5''\sin.\nu + 22\sin.(\eta - \nu) - \tfrac{1}{2}\sin.(\eta + \nu)$$

$$+ 7\sin.\nu \mp 2010\sin.(\eta - \nu) - 11\sin.(\eta + \nu)$$
$$+ 12''\sin.(2\eta - \nu) - 1''\sin.(2\eta + \nu);$$

unde patet has inæqualitates tantum non ſe mutuo deſtruere dum eæ reducuntur ad

$$+ 2''\sin.\nu - 15''\sin.(\eta - \nu) + 12''\sin.(2\eta - \nu)$$

quare dum nunquam ad dimidium minutum aſſurgunt, tuto negligi poſſunt; ita ut ſufficiat effectum in motu aphelii medio notaſſe.

14. Variationes, quæ ab actione Saturni excentricitati & parametro inducuntur, tam ſunt exiguæ ut omnino ſentiri nequeant. Neque vero etiam has inæqualitates evolviſſe eſt opus, cum quoniam ſunt minimæ, ſupra (§. CXXIII) effectum inde in locum terræ redundantem expreſſerimus, ſumta ſcilicet æquatione, quæ ſecundum tabulas ordinarias anomaliæ mediæ convenit, quæ ſit $= \pm Æ$, & poſita longitudine terræ media $= \zeta$, vidimus fore longitudinem ejus veram

$$\varphi = \zeta \pm Æ + nB'\sin.\eta + nC'\sin.2\eta.$$

Ibidem autem valores litterarum B' & C' dedimus, ex quibus hos coëfficientes in minutis ſecundis colligimus;

$$nB' = -\tfrac{1}{2}'', \ \& \ nC' = 0;$$

unde patet longitudinem terræ regulis ordinariis computatam nullam ſenſibilem alterationem ab actione Saturni pati, cum ea vix dimidio minuto ſecundo mutari poſſit. Pro orbita igitur terræ nil aliud relinquitur, niſi exigua illa aphelii terræ promotio, cujus effectus poſt integrum ſeculum demum ad $37''$ exſurgit.

15. Tantum ergo ſupereſt, ut in mutationem plani, in quo orbita terræ verſatur, inquiramus; à Saturno autem linea nodorum, ſeu interſectio orbitarum terræ & Saturni contra ſignorum ordinem removebitur tem-

pore quo sol motu medio angulum ω absolvit per spatiolum $= \frac{nbbc}{4f^3} \cdot g' \omega = \frac{\omega}{3438000}$. Hinc ergo singulis annis linea nodorum super orbita Saturni regredietur per $0,377''$, seu $22'''$, sæculo autem elapso hic motus erit quasi $38''$. Inæqualitates periodicas, quibus locus nodi afficitur, quia nullius plane sunt momenti, hic non evolvo, multoque minus eas, quibus in genere inclinatio turbari est inventa: illæ enim nunquam ad minutum secundum, hæ vero ne ad tertium quidem assurgere reperientur. Si ergo terra à solo Saturno perturbantur, linea nodorum terræ super orbita Saturni retromoveretur

Tempore unius anni per spatium $22'''$,
Tempore unius seculi per spatium $38''$,

qui ergo motus motui aphelii proxime est æqualis.

16. Phænomena, quæ hinc in latitudinem stellarum fixarum fluunt, ita se habebunt. Cum sit inclinatio orbitæ terræ ad orbitam Saturni $\lambda = 2°, 30', 10''$, erit pro tempore unius anni $\frac{nbbc}{4f^3} g' . \omega \, fin. \lambda = 0,0164''$: Hinc stellarum fixarum, quarum longitudo est $9^s, 21°$, vel $3^s, 21°$, latitudo tempore unius anni mutabitur fere uno minuto tertio. Seculo autem elapso, mutatio latitudinis ita se habebit:

Si longitudo stellæ sit $9^s, 21°$ circiter

ejus latitudo $\left\{\begin{matrix}\text{decrescit}\\ \text{crescit}\end{matrix}\right\}$ $1'', 38'''$ si latitudo fuerit $\left\{\begin{matrix}\text{borealis}\\ \text{australis}\end{matrix}\right\}$

At si longitudo stellæ sit $3^s, 21°$ circiter

ejus latitudo $\left\{\begin{matrix}\text{crescit}\\ \text{decrescit}\end{matrix}\right\}$ $1'', 38'''$ si latitudo fuerit $\left\{\begin{matrix}\text{borealis}\\ \text{australis}\end{matrix}\right\}$

Hujusmodi ergo stellarum fixarum latitudo intervallo decem seculorum mutari potuit $16'', 24'''$, idque ob solam actionem Saturni.

II.

Investigatio inæqualitatum Terræ ab actione Jovis oriundarum.

17. COLLOCATO jam Jove in locum planetæ perturbantis habebimus

$$n = \frac{1}{1067}, \; \& \; m = 0,0843;$$

indeque quantitates derivatas cum logarithmis subscriptis

$\frac{bb}{cc} = 0,0369643$; $\frac{b^3}{f^3} = 0,0067303$; $\frac{bbc}{f^3} = 0,0350004$
8,5677705; 7,8280103; 8,544124

Porro erit $\mu = 0,37081$ & $\nu = 0,97871$, hincque

$s = 0,37081 + 0,34437 \, k \, cos. \, v$;
9,569151 9,537029

$s^2 = 0,13750 + 0,25539 \, k \, cos. \, v$;
6,138302 9,407210

$s^3 = 0,05099 + 0,14206 \, k \, cos. \, v$;
8,707453 9,152452

$s^4 = 0,01891 + 0,07023 \, k \, cos. \, v$;
8,276604 8,846542

$s^5 = 0,00701 + 0,03255 \, k \, cos. \, v$;
7,845755 8,512603

$s^6 = 0,00260 + 0,01449 \, k \, cos. \, v$;
7,414906 8,160935

$s^7 = 0,00096 + 0,00627 \, k \, cos. \, v$;
6,985057 7,797033

$s^8 = 0,00036 + 0,00266\ k\ cos.\ v;$
$\quad 6,553208 \quad 7,424176$

$s^9 = 0,00013 + 0,00111\ k\ cos.\ v;$
$\quad 6,122359 \quad 7,044479$

$s^{10} = 0,00005 + 0,00046\ k\ cos.\ v;$
$\quad 5,691510 \quad 6,659388$

$\frac{1}{1-ss} = 1,15943 + 0,34332\ k\ cos.\ v;$
$\quad 0,064244 \quad 9,535698$

$\frac{f^3}{r^3} = 1 - 0,10694\ k\ cos.\ v;$
$\quad 9,029140$

18. Ex his jam calculo ſecundum (§. LXXII) ſubducto invenitur

$P\ (1-ss) = 0,991111 - 0,017094\ k\ cos.\ v;$
$\quad 9,996122 \quad 8,232844$

$Qs(1-ss) = 0,563774 + 0,538394\ k\ cos.\ v;$
$\quad 9,751105 \quad 9,731100$

$Rs^2(1-ss) = 0,134856 + 0,262368\ k\ cos.\ v;$
$\quad 9,129870 \quad 9,418910$

$Ss^3(1-ss) = 0,030176 + 0,088736\ k\ cos.\ v;$
$\quad 8,479518 \quad 8,948100$

haſque formulas primum per $\frac{1}{1-ss}$ deinde per $\frac{f^3}{r^3}$ multiplicari oportet, hoc eſt conjunctim per

$1,15943 + 0,21933\ k\ cos.\ v;$
$0,064244 \quad 9,341098$

unde prodeant formæ $g + h\ k\ cos.\ v$. Facta igitur multiplicatione reperietur:

$g = 1,14912; \quad h = 0,19756;$
$g' = 0,65366; \quad h' = 0,74788;$
$g'' = 0,15636; \quad h'' = 0,33378;$
$g''' = 0,03498; \quad h''' = 0,10950;$

19. Hinc pro motu aphelii terræ medio erit

$2g' + h' = 2,05520; \quad 3g + h = 3,64492$
$\frac{bhc}{4f^3}(2g' + h') = 0,017985; \quad \frac{b^3}{2f^3}(3g + h) = 0,012265;$

unde tempore, quo ſol motu medio angulum ω conficit, aphelium terræ promovetur per ſpatium

$$0,005720\ n\omega = \frac{\omega}{186538}, \text{ ob } n = \frac{1}{1067}.$$

Ponamus jam $\omega = 360^\circ = 1296000''$, ut obtineamus aphelii motum annuum, qui prodibit $= 6'', 95 = 6'', 57'''$; & motus ſecularis $= 695'' = 11', 35''$. Quare ſi terra tantum à Jove perturbaretur, aphelium ejus reſpectu ſtellarum fixarum promoveretur

Tempore unius anni per ſpatium $6'', 57'''$,
Tempore centum annorum per ſpatium $11', 35''$.

Saturnus igitur & Jupiter conjunctim imprimunt aphelio terræ motum annuum $7'', 19'''$. Revera autem quotannis promoveri obſervatur per ſpatium $11''$ circiter.

20. Omiſſis mutationibus, quæ excentricitatem & parametrum afficiunt, quæramus ſtatim correctionem, quam locus terræ in orbita exigit, ac pro coëfficientibus B' & C' (§. CXXIII) obtinemus valores ſequentes:

$$B' = -0,03650; \quad C' = +0,01381.$$

Quare ſi η jam denotet angulum, qui oritur, ſi longitudo Jovis à longitudine terræ ſubtrahatur, longitudo terræ per tabulas ſolito more computatas ſequentem correctionem recipere debet:

$$-7'', 06\ ſin.\ \eta + 2'', 67\ ſin.\ 2\eta;$$

quæ ergo nunquam ad decem minuta ſecunda exſurgere poteſt. Verumtamen hæ correctiones maximi ſunt momenti, quandoquidem Theoria motus ſolis jam ad tantam perfectionem eſt perducta, ut in calculo vix unum minutum ſecundum negligere fas ſit. Deinde cum Luna fore tantumdem motum terræ perturbare sit inventa, neuter effectus per observationes rite comprobari potest, nisi utriusque vera quantitas per Theoriam sit explorata.

21. Motus lineæ nodorum motui aphelii tam exacte æqualis deprehenditur, ut differentia vix ad partes millionesimas ascendat, discrimen autem in hoc versatur, quod aphelium secundum signorum ordinem progreditur, dum nodi motum retrogradum tenent. Motus igitur hujus lineæ nodorum ita est comparatus ut retrogrediatur

Tempore unius anni per spatium $6'', 57'''$,
Tempore 100 annorum per spatium $11', 35''$.

Vicissim ergo linea nodorum orbitæ Jovis ad eclipticam relatæ tanto motu retrogredietur, quatenus ipsa terra actioni Jovis est subjecta: qui effectus probe distingui debet ab eo, quem reliqui planetæ actione sua immediate in Jovem exerunt, unde peculiaris lineæ nodorum Jovis motus oritur non pendens à mobilitate plani neclipticæ. Ex quo intelligitur motum observatum nodorum cujusque planetæ esse effectum mixtum partim ex mobilitate ejus propriæ orbitæ partim vero ex mobilitate ipsius eclipticæ oriundum, qui propterea modo magis rationalis definietur, si orbitæ planetarum non cum plano eclipticæ, utpote mobili, verum cum plano respectu stellarum fixo, veluti forsitan cum plano æquatoris solis comparentur.

22. Seorsim autem hæc mobilitas orbitæ terræ ab actione Jovis profecta sentiri debet in latitudine stellarum fixarum, quæ inde variabilis reddetur. Maximam verò mutationem subibunt eæ stellæ fixæ, quarum longitudo in nodos orbitæ Jovis incidit, & quæ est vel $9^s, 8^o$, vel $3^s, 8^o$; hæcque maxima mutatio ob inclinationem orbitæ Jovis $= 1^o, 19' \, 10''$ singulis annis valebit $0'', 16 = 9'''$, singulisque seculis $16''$; quæ propterea elapso quovis seculo ita se habebit:

Si longitudo stellæ sit $9^s, 8^o$

ejus latitudo $\left\{\begin{matrix}\text{decrescit}\\ \text{crescit}\end{matrix}\right\}$ $16''$ si latitudo fuerit $\left\{\begin{matrix}\text{borealis}\\ \text{australis}\end{matrix}\right\}$

At si longitudo stellæ sit $3^s, 8^o$

ejus latitudo $\left\{\begin{matrix}\text{crescit}\\ \text{decrescit}\end{matrix}\right\}$ $16''$ si latitudo fuerit $\left\{\begin{matrix}\text{borealis}\\ \text{australis}\end{matrix}\right\}$

pro reliquis stellis fixis hæc mutatio secularis in latitudine diminui debet in ratione sinus totius ad cosinum differentiæ longitudinis stellæ ab his duobus limitibus.