E414

INVESTIGATIO PERTURBATIONUM

Quibus Planetarum motus ob actionem eorum mutuam afficiuntur.

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Sidera quod tantis cieant se viribus æquis
In motu terræ plurima signa docent.

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PRÆFATIO.

PLANETAS non ſolum ad Solem ſecundum inverſam diſtantiarum rationem duplicatam impelli, ſed etiam ſimili ratione ſe mutuo incitare ex perturbationibus motuum Saturni & Jovis manifeſto eſt perſpectum. Cum enim Tabulæ Aſtronomicæ ita conſtrui ſoleant, quaſi planetæ ad ſolum Solem ſollicitati ſecundum regulas Keplerianas in ellipſibus revolverentur, ſi ex iis loca Saturni vel etiam Jovis definiantur ea nonnunquam ad plura minuta prima à veritate aberrare deprehenduntur; neque jam ullum eſt dubium, quin iſti errores ab actione mutua, qua hi duo planetæ ſe invicem impellunt, proficiſcantur. Reliquorum quidem planetarum motus ac præcipue terræ regulis illis Keplerianis magis eſt conformis, ac ſi quando errores in eorum motu à Tabulis occurrunt, incertum plerumque eſt, utrum illi vel non recte conſtitutis Tabularum elementis, vel ipſarum obſervationum imperfectioni cuipiam potius ſint tribuendi, quam ipſi Theoriæ, cui Tabulæ innituntur. Interim tamen jam ipſa tabularum ratio, qua pro quolibet planeta tam lineæ abſidum quam lineæ nodorum motus peculiaris aſſignatur, manifeſtum indicium aberrationis cujuſdam à Theoria continet: ſi enim planetæ nullam aliam impulſionem præter eam qua ſecundum rationem quadrati diſtantiarum inverſam ad ſolem urgentur, ſuſtinerent, non ſolum circa ſolem tanquam focum perfectas deſcriberent ellipſes, ſed etiam perpetuo in eodem plano ferrentur, axeſque iſtarum ellipſium omnino fixi manerent; neque idcirco linea abſidum neque linea nodorum ulli obnoxia foret mutationi, ſaltem reſpectu ſtellarum fixarum. Ad hanc quoque normam computatæ ſunt à ſtretio Tabulæ Carolinæ, in quibus tam apheliis quam nodis ſingulorum planetarum in cœlo ſidereo loca fixa aſſignantur; at vero inſignis harum tabularum à veritate diſſenſus mox luculenter monſtravit, huic hypotheſi locum concedi non poſſe. Cum igitur certum ſit tam aphelium quam linea nodorum cujuſque planetæ motu peculiari per cœlum proferri, atque etiam reſpectu ſtellarum fixarum continuo mutari; minime amplius dubitare licet, quin præter eam vim conſtantem, qua ſinguli planetæ ad ſolem pelluntur, aliæ quoque vires in eos effectum quempiam exerant. Quemadmodum enim in Saturno & Jove præter alias perturbationes ab eorum ac-

tione mutua utriuſque lineæ abſidum & nodorum certus imprimi motus eſt inventus obſervationibus ſatis conſentaneus, ita multo minus dubitare poterimus, quin ſimilis variatio in apheliis & nodis reliquorum quoque planetarum ab eorum actione mutua proficiſcatur, etiamſi in cæteris motus horum planetarum elementis nulla alteratio perciperetur. Sunt autem effectus talium virium in loca apheliorum & nodorum ita comparati, ut etiamſi ſint minimi, tamen cum tempore continuo creſcant, & poſt ſatis longum intervallum ſenſibiles evadant dum reliquæ perturbationes inde oriundæ ſunt periodicæ, & poſt certas revolutiones iterum penitus in nihilum redigantur, unde fit ut ſi ſint minimæ, percipi omnino non poſſint. Interim tamen Theoria motus telluris, cujus elementa per obſervationes Solis multo accuratius definire licet quam reliquorum planetarum, haud obſcura talium minimarum perturbationum ſigna exhibet, dum excentricitas ejus orbitæ prouti alia atque alia tempeſtate per obſervationes inveſtigatur, modo aliquantum major modo minor deprehenditur, quæ inconſtantia ad integrum minutum aſſurgere videtur. Quibus perpenſis palam omnino eſt non ſolum motum Saturni ac Jovis, ſed etiam reliquorum planetarum ab aliis viribus præter eam qua lege conſtanti ad Solem pelluntur, perturbari, earumque adeo effectum ab Aſtronomis manifeſto eſſe obſervatum. Quamvis enim Aſtronomorum Princeps Halleyus Mercurii motum ab hujuſmodi perturbationibus prorſus immunem ſit arbitratus, propterea quod ob ſummam ſolis vicinitatem reliquorum planetarum vires præ vi Solis quaſi evaneſcere crediderit, qua ſententia fretus in tabulis ſuis etiam neque Aphelio Mercurii neque ejus lineæ nodorum motum ullum reſpectu ſtellarum fixarum adſcripſit: tamen ex poſtremo potiſſimum tranſitu hujus planetæ per ſolem Aſtronomi didicerunt Halleyi Tabulas inſigni emendatione ex hac parte indigere, dum aphelio Mercurii motum annuum quaſi 55″, ejuſque nodo 45″ ratione æquinoxii tribui debere eſt compertum; ex quo manifeſtum eſt etiam Mercurium actiones reliquorum planetarum ſentire ob eaſque certis perturbationibus eſſe obnoxium.

Cum igitur extra dubium ſit poſitum planetas in ſe invicem attrahendo agere, diſpiciendum eſt quamnam rationem eorum vires reſpectu ad diſtantias habito ſequantur. Ac præterquam quod conſtantia naturæ eandem legem quadratis diſtantiarum reciproce proportionalem, quam in Sole ſtabilitam cernimus, exigere videtur, Theoria Lunæ atque Satellitum Jovis & Saturni hanc ſuſpicionem pleniſſime

confirmat, ita ut amplius dubitare non liceat, quin Jupiter Saturnuſque ſuos Satellites, & Terra Lunam ad ſe alliciant viribus quadratis diſtantiarum reciproce proportionalibus; & quamquam motus Apogei Lunæ aberrationem tantillam ab hac lege innuere eſſet viſus, tamen à celeberrimo CLAIRAUT primum pulcherrimus conſenſus eſt evictus, ita ut jam audacter aſſeverare queamus non ſolum Solem ſed etiam cunctos planetas vi attractrice eſſe prædictos, qua omnia corpora ad maximas etiam diſtantias remota ad ſe præciſe ſecundum illam conſtantem legem attrahant. Quin etiam pari fere fiducia pronunciare licet, ſingulorum planetarum vires, quas ad diſtantias æquales exerunt, ipſorum maſſis eſſe proportionales, id quod communis centri gravitatis ſtatus poſtulare videtur: de cætero maſſam ſeu quantitatem materiæ, quam quiſque planeta continet, aliunde nobis cognoſcere non datur; nihilque impedit, quo minus id, cui vis abſoluta cujuſque planetæ revera eſt proportionalis, nomine maſſæ ejus deſignemus; hinc ſaltem nullus certe error eſt pertimeſcendus. Terra igitur perinde ac reliqui planetæ omnes non ſolum verſus Solem, ſed etiam verſus ſingulos reliquos planetas viribus legi iſti ſacræ conformibus ſollicitatur, à quibus ſine dubio præter promotionem illam apheliorum & nodorum aliæ vehementer exiguæ perturbationes efficiuntur, quarum inventio in Aſtronomia ſine dubio maximi eſt momenti.

Hinc naſcitur quæſtio latiſſime patens ad Mechanicam referenda, qua determinatio motus plurium corporum, quæ ſe mutuo attrahant in ratione reciproca duplicata diſtantiarum, requiritur; cujus ſolutio eo ardentius eſt expetenda, quod omnia incrementa Aſtronomiæ, quæ adhuc deſiderantur, ex ea derivanda videantur. Verum enodatio hujus quæſtionis tot tantiſque difficultatibus eſt involuta, ut ſi in genere ſpectetur, vires ingenii humani longe ſuperare videatur; etſi enim caſus duorum corporum facilem habeat ſolutionem, tamen ſtatim ac tria aſſumuntur, nulla adhuc inventa ſunt artificia, quorum ope ad motus determinationem pervenire licuerit, unde multo minus pro caſu plurium corporum quicquam ſperare poſſumus. Interim tamen cum perturbationes, quas planetæ ſibi mutuo inferunt, ſint perquam exiguæ, neque eæ, quæ ab actione unius oriuntur, à reliquis affici ſint cenſendæ; hinc non contemnendum ſubſidium impetramus aliquid ſaltem in hoc arduo negotio præſtandi, dum effectus ſingulorum planetarum ſeorſim inveſtigare licebit, & quoniam ſunt minimi, conſuetis calculi approximationibus, quarum in hujuſmodi quæſtionibus uberrimus

uberrimus

uberrimus ſolet eſſe uſus, totum negotium confici debebit. In hunc etiam modum Illustris Academia Regia Parisina iſtam quæſtionem tractandam judicavit, cujus præceptis ut pro viribus ſatisfaciam, operam dabo, ut primo hoc abſtruſſimum argumentum ex primis Mechanicæ fontibus dilucide evolvam; atque ad æquationes analyticas perducam: tum vero quibuſnam modis ex iis aliquid per approximationes concludi queat, accuratius inveſtigabo. Denique præcepta quæ elicuero, ad perturbationes motus terræ accommodabo, examinaturus, quantum ſingula hujus motus elementa ab actione reliquorum planetarum continuo immutentur, quod inſtitutum ſequentibus ſectionibus abſolvere conabor.

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SECTIO PRIMA.

Generalis investigatio motus corporis à viribus quibuscunque impulsi.

§. I. Sumatur pro lubitu cum planum, ad quod motus corporis referatur, quodque plano tabulæ repræsentari concipiatur, tum vero in hoc plano linea recta fixa CA, atque in hac ipsa punctum fixum C, ubi quasi motus spectator sit constitutus. Jam ad quodvis tempus, ubicunque corpus motum versetur veluti in R, de ejus loco R in illud planum demittatur perpendiculum RQ, ita ut punctum Q ejus locum ad hoc planum relatum exhibeat, deinde etiam ex puncto fixo C ad ambo loca *Fig. I.*

R & Q ducantur rectæ CR & CQ. Quo facto perspicuum est, si ad quodvis tempus assignare valeamus cum angulos ACQ & QCR, tum magnitudinem sive rectæ CQ sive CR, locum corporis R perfecte fore cognitum, indeque verum corporis motum innotescere; ita ut plena motus cognitio determinatione horum trium elementorum contineatur.

§. II. Hoc autem potissimum modo investigationem motus instituo, cum quod videtur maxime naturalis, tum vero præcipue quod ad consuetudinem Astronomorum, qua motus corporum cœlestium considerare solent, imprimis est accommodatus. Namque si planum fixum tanquam planum eclipticæ spectemus, rectamque CA tanquam rectam ad ejus quodpiam punctum fixum directam, corpore moto in R existente, angulus ACQ ejus longitudinem, angulus vero QCR ejus latitudinem referet; & quemadmodum recta CR ejus distantiam veram à puncto C denotat, ita recta CQ distantiam ejus curtatam designabit, quarum alteram tantum in calculum introduxisse sufficiet. Vocemus igitur pro quovis tempore proposito,

I. Longitudinem corporis seu angulum $ACQ = \varphi$.
II. Latitudinem ejus seu angulum $QCR \ldots = \psi$.
III. Distantiam curtatam seu rectam $CQ \ldots = x$.

§. III. Cognitis vero his tribus elementis, omnia, quæ ad motus notitiam pertinent, definiri poterunt. Primo enim ex distantia curtata $CQ = x$ & latitudine $QCR = \psi$ habetur distantia vera $CR = \psi$ seu $CR = \frac{x}{cos.\psi}$ posito sinu toto constanter $= 1$. Tum vero ipsa distantia corporis à plano erit $QR = x\ tang.\ \varphi$. Deinde si à puncto Q ad rectam fixam CA ducatur normalis QP, ex distantia curtata $CQ = x$ & longitudine $ACQ = \varphi$, elicitur $CP = x\ cos.\varphi$ & $PQ = x\ sin.\varphi$; atque hoc modo pro loco puncti R, uti in Geometria fieri solet, ternas obtinemus coordinatas inter se rectangulas CP, PQ & QR. A quibus cum etiam investigatio mechanica incipiat, has lineas tantisper peculiaribus signis indicemus, quoad calculum ad illa primaria elementa perducere licuerit. Sit igitur $CP = x\ cos.\varphi = p$; $PQ = x\ sin.\varphi = q$ & $QR = x\ tang.\psi = r$; sicque habebimus $x = \surd(pp+qq)$; $cos.\varphi = \frac{p}{\surd(pp+qq)}$; $sin.\varphi = \frac{q}{\surd(pp+qq)}$ & $tang.\psi = \frac{r}{\surd(pp+qq)}$.

§. IV. Hæc autem motus elementa ex sollicitatione virium quarum actioni corpus fuerit subjectum, secundum præcepta mechanica determinari oportet. A quibuscunque autem viribus corpus impellatur, eas semper per notam resolutionem ad ternas directiones determinatas revocare licet. Concipiamus igitur corpus à tribus viribus sollicitari, quarum prima urgeat secundum directionem RQ ad planum fixum normalem, binarum autem reliquarum directiones sint ipsi plano parallelæ; altera quidem habeat directionem distantiæ curtatæ QC parallelam, altera vero huic normalem, cui in plano fixo parallela sit recta QN ad CQ normalis. Istas vires statuamus acceleratrices, sive jam ad corporis massam applicatas, easque denotemus:

I. Vim acceleratricem secundum $QC = V$.
II. Vim acceleratricem secundum $QN = T$
III. Vim acceleratricem secundum $RQ = R$

Ita, ut, quomodo per has vires terna illa elementa φ, ψ & x determinentur, sit investigandum.

§. V. Cum autem regulæ mechanicæ ad ternas coordinatas normales, quarum directiones perpetuo maneant fixæ accommodatæ esse soleant, harum autem virium tertia tantum RQ cum una coordinatarum conveniat, dum diarum reliquarum directiones QC & QN maxime sunt variabiles, & has ad directiones fixas revocari conveniet. Dabit igitur vis V resoluta:

Secundum directionem $PC = V\, cos.\, \varphi$

& secundum QP vim $= V\, sin.\, \varphi$;

vis vero T simili modo resoluta:

Secundum directionem $PC = - T\, sin\, \varphi$

& secundum QP vim $= T\, cos.\, \varphi$.

Hinc itaque pro directionibus nostrarum ternarum coordinatarum PC, QP, & RQ obtinebimus tres vires acceleratrices sequentes quæ sunt:

I. Vis acceleratrix secundum $PC = V\, cos.\, \varphi - T\, sin.\, \varphi$

II. Vis acceleratrix secundum $QP = V\, sin.\, \varphi + T\, cos.\, \varphi$

III. Vis acceleratrix secundum $RQ = R$.

§. VI. Quoniam actio harum ternarum virium ad diminutionem coordinatarum respondentium tendit, accelerationes quæ corpori inde secundum easdem coordinatas inducuntur negativæ sunt concipiendæ. Cum igitur posito temporis elemento $= dt$, sint corporis celeritates secundum has coordinatas $\frac{dp}{dt}$; $\frac{dq}{dt}$ & $\frac{dr}{dt}$; si elementum temporis dt pro constanti assumamus, erunt ipsæ accelerationes secundum istas directiones $\frac{ddp}{dt^2}$; $\frac{ddq}{dt^2}$; $\frac{ddr}{dt^2}$, quæ viribus illis acceleratricibus negative sumtis debent esse proportionales. Proportionalitate ergo, uti fieri solet, stabilita obtinebimus ternas sequentes æquationes.

I. $ddp = - \frac{1}{2} dt^2 (V\, cos.\, \varphi - T\, sin.\, \varphi)$

II. $ddq = - \frac{1}{2} dt^2 (V\, sin.\, \varphi + T\, cos.\, \varphi)$

III. $ddr = - \frac{1}{2} R\, dt^2$

quarum æquationum resolutione tota motus determinatio continetur.

§. VII. Jam iterum ambas vires V & T commode à se invicem separare licet, ut pateat quid utraque seorsim præstet. Nam I $\times$ $cos.\, \varphi$ + II $\times$ $sin.\, \varphi$ dat

$$ddp\, cos.\, \varphi + ddq\, sin.\, \varphi = - \tfrac{1}{2} V\, dt^2$$

Deinde II $\times$ $cos.\, \varphi$ — I $\times$ $sin.\, \varphi$ præbet hanc æquationem

$$ddq\, cos.\, \varphi - ddp\, sin.\, \varphi = - \tfrac{1}{2} T\, dt^2.$$

at ex tertia vi R nascitur æquatio $ddr = - \frac{1}{2} R\, dt^2$. Nunc igitur recordandum est nos supra posuisse

$$p = cos.\, \varphi;\ q = x\, sin.\, \varphi\ \&\ r = x\, tang.\, \psi$$

unde loco quantitatum subsidiariarum p, q, r, elementa nostra principalia φ, ψ & x in calculum introduci poterunt. Tres autem emergent æquationes, quæ propterea his tribus elementis definiendis sufficient: atque ita tota investigatio à principiis mechanicis ad Analysin puram traducetur.

§. VIII. Cum sit $p = x\, cos.\, \varphi$ & $q = x\, sin.\, \varphi$ erit differentiando;

$$dp = dx\, cos.\, \varphi - x\, d\varphi\, sin.\, \varphi\ \&\ dq = dx\, sin.\, \varphi + x\, d\varphi\, cos.\, \varphi$$

denuoque differentiando

$$ddp = ddx\, cos.\, \varphi - 2dxd\varphi\, sin.\, \varphi - xd\varphi^2 cos.\, \varphi - xdd\varphi\, sin.\, \varphi$$

$$ddq = ddx\, sin.\, \varphi + 2dxd\varphi\, cos.\, \varphi - xd\varphi^2 sin.\, \varphi + xdd\varphi\, cos.\, \varphi$$

unde per combinationem elicitur

$$ddp\, cos.\, \varphi + ddq\, sin.\, \varphi = ddx - x\, d\varphi^2$$

$$\&\ ddq\, cos.\, \varphi - ddp\, sin.\, \varphi = 2\, dx\, d\varphi + x\, dd\varphi.$$

Valorem autem ipsius $r = x\, tang.\, \psi$ nulla adhibita evolutione tantisper retineamus, donec compererimus, quomodo aptissime eum tractari conveniat. Hoc itaque pacto totum negotium ad resolutionem trium sequentium æquationum erit perductum:

I. $ddx - x\, d\varphi^2 = - \frac{1}{2} V\, dt^2$

II. $2\, dx\, d\varphi + x\, dd\varphi = - \frac{1}{2} T\, dt^2$

III. $dd.\, x\, tang.\, \psi = - \frac{1}{2} R\, dt^2$.

§. IX. Cum hæ æquationes ſint differentiales ſecundi gradus, temporis differentiali dt ſumto conſtante, primum diſpiciendum eſt, quamnam proportionem differentialia prima $d\varphi$, $d\psi$ & dx cum inter ſe tum ad temporis differentiale dt teneant, quod etſi ſine introductione formularum integralium fieri nequit, quamdiu vires ſollicitantes V, T & R in genere conſideramus, tamen eæ proportionem quæſitam minus perturbare ſunt cenſendæ. Quin etiam in negotio quod ſuſcipimus, ipſæ vires V, T & R quantitates incognitas, φ, ψ & x cum tempore t implicare reperientur, quominus earum ſeparatio perfecta expectari poterit. Pro initio igitur contenti eſſe debebimus, formulas noſtras à contemplatione differentialium ſecundorum liberaſſe, & quocunque modo relationem differentialium primorum determinaſſe, ut deinceps, approximationum artificio in ſubſidium vocato, ipſarum quantitatum finitarum relationem inde colligere valeamus.

§. X. Tertiam quidem æquationem $dd.\ x\ tang.\ \psi = -\frac{1}{2} R\,dt^2$ tantiſper ſeponamus, poſtmodum inveſtigaturi, quo modo ejus ratio convenientiſſime haberi queat; ambas igitur priores, quæ ſunt:

$$\text{I. } ddx - x\,d\varphi^2 = -\tfrac{1}{2} V\,dt^2 \quad \&$$

$$\text{II. } 2\,dx\,d\varphi + x\,dd\varphi = -\tfrac{1}{2} T\,dt^2$$

accuratius perpendamus, ut inde relationem differentialium primorum dx, $d\varphi$ & dt eliciamus. Ac primo quidem prius membrum ſecundæ æquationis ſi per x multiplicetur, redditur integrale, proditque

$$d\,(xx\,d\varphi) = -\tfrac{1}{2} \int T x\,dt^2$$

$$\text{ſeu } xx\,d\varphi = \tfrac{1}{2}\,dt\,(C - \int T x\,dt)$$

unde ſi vis T ſecundum directionem QN trahens evaneſcat, oritur æquabilis arearum deſcriptio. Hinc autem patet eandem illam æquationem integrabilem reddi ſi multiplicetur non ſolum per x, ſed etiam inſuper per functionem

functionem quamcunque ipſius $xx\,d\varphi$; Multiplicetur ergo per $x\,d\varphi$; eritque integrale:

$$\tfrac{1}{2} x^4\,d\varphi^2 = -\tfrac{1}{2}\,dt^2 \int T x^3\,d\varphi;$$

$$\text{ſeu } x^4\,d\varphi^2 = dt^2\,(A - \int T x^3\,d\varphi);$$

unde invenitur

$$dt^2 = \frac{x^4\,d\varphi^2}{A - \int T x^3\,d\varphi} \quad \& \quad dt = \frac{xx\,d\varphi}{\sqrt{(A - \int T x^3\,d\varphi)}}.$$

§. XI. Cum igitur ſit $x\,d\varphi^2 = \frac{dt^2}{x^3}(A - \int T x^3\,d\varphi)$, hoc valore in prima æquatione ſubſtituto habebimus:

$$ddx = dt^2\,\left(\tfrac{A}{x^3} - \tfrac{1}{x^3}\int T x^3\,d\varphi - \tfrac{1}{2} V\right),$$

quæ per $2\,dx$ multiplicata & integrata præbet:

$dx^2 = dt^2\,(B - \frac{A}{xx} - 2\int \frac{dx}{x^3} \int T x^3\,d\varphi - \int V\,dx)$. At eſt

$$-2\int \tfrac{dx}{x^3} \int T x^3\,d\varphi = \tfrac{1}{xx}\int T x^3\,d\varphi - \int T x\,d\varphi$$

quo valore introducto erit:

$dx^2 = dt^2\,(B - \frac{A}{xx} + \frac{1}{xx}\int T x^3\,d\varphi - \int T x\,d\varphi - \int V\,dx)$, vel

$x^2\,dx^2 = dt^2\,(Bxx - A + \int T x^3\,d\varphi - xx\int(T x\,d\varphi + V\,dx))$,

unde nanciſcimur

$$dt = \frac{\pm\,x\,dx}{\sqrt{(Bxx - A + \int T x^3\,d\varphi - xx\int(T x\,d\varphi + V\,dx))}} \quad \&$$

$$d\varphi = \frac{\pm\,dx\sqrt{(A - \int T x^3\,d\varphi)}}{x\sqrt{(Bxx - A + \int T x^3\,d\varphi - xx\int(T x\,d\varphi + V\,dx))}}.$$

§. XII. Ambiguitas ſignorum, quam motus natura involvit, ita ab arbitrio noſtro pendet, ut poſitivum valeat, ſi motum ab eo loco, ubi corpus puncto fuit proximum, definire velimus, negativum vero ſi à diſtantia maxima diſceſſerit. Quoniam igitur in Aſtronomia uſus eſt, motum corporum à maxima diſtantia computare, valeat ſignum negativum, ut habeamus has duas æquationes:

$$dt = \frac{-\,x\,dx}{\sqrt{(Bxx - A + \int T x^3\,d\varphi - xx\int(T x\,d\varphi + V\,dx))}}$$

$$d\varphi = \frac{-\,dx\sqrt{(A - \int T x^3\,d\varphi)}}{x\sqrt{(Bxx - A + \int T x^3\,d\varphi - xx\int(T x\,d\varphi + V\,dx))}};$$

cujus posterioris loco & hæc primum inventa $d\varphi = \frac{dt}{xx}\sqrt{(A - \int T x^3 d\varphi)}$ usurpari potest. Sunt autem A & B quantitates constantes, per duplicem integrationem invectæ, quæ deinceps ad quemvis casum oblatæ accommodari debent.

§. XIII. Quando vis normalis T evanescit, alteraque vis V ad C tendens per solam distantiam x determinatur, utraque æquatio habebit variabiles separatas, ita ut non solum differentialium dt & $d\varphi$ ratio ad dx absolute possit assignari, sed etiam per integrationem ipsæ quantitates finitæ t & φ per distantiam x definiri: hocque ergo casu problematis solutio perfecta poterit exhiberi. Neque vero in genere hæ formulæ magis ad usum accommodari posse videntur. Sed contentos nos esse opportet, hoc pacto rationem differentialium dt, $d\varphi$ & dx elicuisse. Quamvis enim adsint formulæ integrales $\int T x\, d\varphi$, $\int T x^3 d\varphi$, & $\int V dx$ hanc ipsam rationem involventes, eæ tamen negotium approximationis non multum turbant, dummodo earum valores sint perquam exigui, propterea quod tunc sufficit rationes differentialium prope veras nosse. Verum ipsum approximationis negotium alias requirit considerationes, antequam cum successu suscipi queat, quas deinceps evolvemus.

§. XIV. Perductis igitur binis prioribus æquationibus differentio-differentialibus ad formulas simpliciter differentiales, quæ ad usum maxime videntur accommodatæ, tertiam quoque æquationem $ddr = -\frac{1}{2} R dt^2$ instituto convenientius transformare conemur. Quæ cum latitudinis ψ determinationem ob $r = x\, tang.\, \psi$ contineat commodissime ea instituetur, si more apud Astronomos recepto lineam nodorum cum inclinatione orbitæ ad planum assumtum in calculum introducamus. Hunc in finem consideretur quovis momento planum, quod puncto fixo C & spatiolo à corpore jam jam percurrendo

determinetur. Quodque pro isto saltem momento planum orbitæ appellare liceat. Sit igitur dum corpus versatur in R recta $C\Omega$ intersectio plani orbitæ & plani assumti, quæ linea nodorum vocari solet, atque ad latitudinem ψ commodius investigandam ponamus:

I. Longitudinem lineæ nodorum seu angulum $AC\Omega = \pi$,
II. Inclinationem plani orbitæ ad planum assumtum $= G$;

quæ duo nova elementa tanquam utcunque variabilia contemplor.

§. XV. Ad inclinationem autem definiendam ex punctis R & Q ad lineam nodorum $C\Omega$ ducantur normales RS & QS, quarum inclinatio seu angulus QSR inclinationem metietur, ita ut sit $QSR = G$. Deinde ob angulum $\Omega CQ = \varphi - \pi$ & $CQ = x$, erit $QS = x\, sin.\, (\varphi - \pi)$; unde fit $QR = x\, sin.\, (\varphi - \pi)\, tang.\, G$. Cum igitur habeamus $QR = r = x\, tang\, \psi$, erit:

$$tang.\, \psi = sin.\, (\varphi - \pi)\, tang.\, G;$$

sicque ex elementis φ, π & G, latitudo quæsita ψ reperietur. Quoniam autem loco latitudinis ψ duo nova elementa π & G æque ad locum corporis sequentem pertineant; unde differentiale ipsius ψ seu $tang.\, \psi$ idem prodire debet sive elementa ambo π & G sumantur constantia, sive ambo variabilia, ex qua proprietate relatio inter π & G innotescet, quæ locum quartæ æquationis sustinebit, si quidem jam quatuor elementa x, φ, π & G in calculo habemus.

§. XVI. Cum igitur positis π & G constantibus sit

$$d.\, tang.\, \psi = d\varphi\, cos.\, (\varphi - \pi)\, tang.\, G,$$

iisdem autem tanquam variabiles tractatis prodeat

$$d.\, tang.\, \psi = (d\varphi - d\pi)\, cos.\, (\varphi - \pi)\, tang.\, G + sin.\, (\varphi - \pi)\, d.\, tang.\, G;$$

his valoribus inter se æquatis obtinebimus

$$d.\,tang.\,G = \frac{d\pi\, cof.\,(\varphi - \pi)}{fin.\,(\varphi - \pi)}\, tang.\,G,$$

$$\text{feu}\ \frac{d.\,tang.\,G}{tang.\,G} = \frac{d\pi\, cof.\,(\varphi - \pi)}{fin.\,(\varphi - \pi)}.$$

Sufficit igitur longitudinem nodi π ejusque variationem determinavisse, indeque facile inclinatio orbitæ G definietur; est enim $\frac{d.\,tang.\,G.}{tang.\,G.}$ differentiale logarithmi ipsius G, quod si ita indicemus $d.\,l\,tang.\,G$, erit

$$d.\,l\,tang.\,G = \frac{d\pi\, cof.\,(\varphi - \pi)}{fin.\,(\varphi - \pi)} = d\pi\, cot.\,(\varphi - \pi).$$

Patet igitur nisi linea nodorum sit fixa ideoque $d\pi = 0$, inclinationem orbitæ continuis variationibus esse obnoxiam, quarum autem alteræ ex alteris facile definiri poterunt.

§. XVII. Propositum autem est invenire ddr, cujus valor ob $r = x\,tang.\,\psi$ est: $ddx\,tang.\,\psi + 2dx\,d.\,tang.\,\psi + x\,dd.\,tang.\,\psi = -\frac{1}{2}R\,dt^2$. Verum invenimus:

$$tang.\,\psi = fin.\,(\varphi - \pi)\ tang.\,G, \text{ atque}$$

$$d.\,tang.\,\psi = d\varphi\, cof.\,(\varphi - \pi)\ tang.\,G; \text{ unde ob}$$

$$d.\,tang.\,G = \frac{d\pi\, cof.\,(\varphi - \pi)}{fin.\,(\varphi - \pi)}\, tang.\,G, \text{ erit porro differentiando}$$

$$dd.\,tang.\,\psi = dd\varphi\, cof.\,(\varphi - \pi)\, tang.\,G - d\varphi\,(d\varphi - d\pi).$$

$$fin.\,(\varphi - \pi)\ tang.\,G + \frac{d\varphi\, d\pi\, cof.\,(\varphi - \pi)^2}{fin.\,(\varphi - \pi)}\, tang.\,G,$$

$$\text{five}\ dd.\,tang.\,\psi = dd\varphi\,\Big(cof.\,(\varphi - \pi) - d\varphi^2 fin.\,(\varphi - \pi) + \frac{d\varphi\, d\pi}{fin.\,(\varphi - \pi)}\Big)\, tang.\,G.$$

Quibus valoribus substitutis tertia æquatio induet hanc formam:

$$\left\{\begin{array}{l} +\,ddx\,fin.\,(\varphi - \pi) + 2\,dx\,d\varphi\, cof.\,(\varphi - \pi) \\ +\,x\,dd\varphi\, cof.\,(\varphi - \pi) - x\,d\varphi^2 fin.\,(\varphi - \pi) + \dfrac{x\,d\varphi\, d\pi}{fin.\,(\varphi - \pi)} \end{array}\right\} tang.\,G = -\tfrac{1}{2}R\,dt^2$$

At ex binis prioribus æquationibus erat: $ddx - x\,d\varphi^2 = -\frac{1}{2}V\,dt^2$ & $2\,dx\,d\varphi + x\,dd\varphi = -\frac{1}{2}T\,dt^2$; sicque fiet:

$$\Big(-\tfrac{1}{2}V\,dt^2 fin.\,(\varphi - \pi) - \tfrac{1}{2}T\,dt^2 cof.\,(\varphi - \pi) + \frac{x\,d\varphi\, d\pi}{fin.\,(\varphi - \pi)}\Big)\, tang.\,G = -\tfrac{1}{2}R\,dt^2, \text{ feu}$$

$$\frac{x\,d\varphi\, d\pi}{fin.\,(\varphi - \pi)} = \tfrac{1}{2}dt^2\Big(V fin.\,(\varphi - \pi) + T cof.\,(\varphi - \pi) - \frac{R}{tang.\,G}\Big).$$

§. XVIII. Ex viribus igitur sollicitantibus V, T & R quaterna nostra elementa x, φ, π & G ad quodvis tempus t per sequentes quaternas æquationes differentiales primi gradus determinantur.

$$\text{I.}\ dt = \frac{-x\,dx}{\surd(Bxx - A + \int Tx^3 d\varphi - xx\int(Tx\,d\varphi + V dx))}$$

$$\text{II.}\ d\varphi = \frac{dt}{xx}\surd\,(A - \int Tx^3 d\varphi).$$

$$\text{III.}\ d\pi = \frac{dt^2 fin.\,(\varphi - \pi)}{2\,x\,d\varphi}\Big(V fin.\,(\varphi - \pi) + T cof.\,(\varphi - \pi) - \frac{R}{tang.\,G}\Big).$$

$$\text{IV.}\ d.\,l.\,tang.\,G = \frac{d\pi\, cof.\,(\varphi - \pi)}{fin.\,(\varphi - \pi)}, \text{ feu } d.\,l.\,tang.\,G = \frac{dt^2 cof.\,(\varphi - \pi)}{2\,x\,d\varphi}\Big(V fin.\,(\varphi - \pi) + T cof.\,(\varphi - \pi) - \frac{R}{tang.\,G}\Big).$$

quæ æquationes in genere vix tractabiliores reddi posse videntur, nisi certa quædam virium sollicitantium relatio accedat.

SECTIO II.

Reductio harum formularum ad casum quo corpus imprimis ad punctum fixum urgetur vi in quadratis distantiarum reciproce proportionali, cujus respectu reliquæ vires sunt valde parvæ.

§. XIX. QUONIAM nobis est propositum in perturbationes motus planetarum quatenus ab eorum actione mutua oriuntur inquirere, tantum jam pro certo assumere licet, eorum motum, proxime saltem, regulis Keplerianis esse conformem ac perturbationes quas definiri oportet, vehementer esse exiguas. Moderatio igitur præcipua motus eorum efficitur à vi quadam ad punctum fixum secundum rationem reciprocam duplicatam distantiarum tendente, præ qua reliquæ vires quasi evanescant. Diserte enim fateri cogor, nisi hujusmodi vis inter reliquas vires sollicitantes longe emineat, nullo plane modo me perspicere, qua ratione ad aliqualem saltem motus cognitionem pertingere nobis liceat. Minime autem casui tribuendum videtur, quod hujusmodi motus, quorum investigatio vires nostras penitus superaret, etiamsi æque facile existere potuissent, in mundo non deprehendantur.

FIG. I. §. XX. Sit igitur C id punctum, ad quod vis illa principalis quadratis distantiarum reciproce proportionalis dirigatur, si quidem hactenus hoc punctum ab arbitrio nostro pendebat. Neque tamen hanc vim quadrato distantiæ veræ RC reciproce proportionalem statuamus quoniam ejus reductio ad nostras formulas denuo latitudinem involveret. Sed quoniam planum fixum semper ita assumere licet, ut corpus ab eo non nisi quam minime recedat, angulusque ψ perpetuus sit valde exiguus, casum ita stabiliamus, ut vis V secundum directionem QC sollicitans habeat partem eximiam quadrato distantiæ $QC = x$ reciproce proportionalem, cujus respectu tam reliqua pars quam binæ reliquæ vires T & R pro minimis haberi queant. Statuamus igitur $V = \frac{ff}{xx} + S$ ita ut vires S, T & R præ vi $\frac{ff}{xx}$ quasi evanescant.

§. XXI. Posito autem $V = \frac{ff}{xx} + S$ erit $\int V dx = \frac{-ff}{x} + \int S dx$ & $xx\int V dx = -ffx + xx\int S dx$, quo valore in nostris formulis surrogato habebimus,

$$\text{I. } dt = \frac{-x\,dx}{\sqrt{(Bxx + ffx - A + \int Tx^3 d\varphi - xx(\int Tx d\varphi + \int S dx))}}.$$

$$\text{II. } d\varphi = \frac{dt}{xx}\sqrt{(A - \int Tx^3 d\varphi)}, \text{ seu } d\varphi = \frac{-dx(A - \int Tx^3 d\varphi)}{x\sqrt{(Bxx + ffx - A + \int Tx^3 d\varphi - xx(\int Tx d\varphi + \int S dx))}}.$$

$$\text{III. } d\pi = \frac{dt^2 \sin.(\varphi - \pi)}{2x\,d\varphi} \times \left(\frac{xx}{ff}\sin.(\varphi-\pi) + S\sin.(\varphi-\pi) + T\cos.(\varphi-\pi) - \frac{R}{\text{tang.}\,G}\right).$$

$$\text{IV. } d.l.\,\text{tang.}\,G = \frac{d\pi\cos.(\varphi-\pi)}{\sin.(\varphi-\pi)}, \text{ seu } d.l.\,\text{tang.}\,G = \frac{dt^2\cos.(\varphi-\pi)}{2x\,d\varphi} \times \left(\frac{ff}{xx}\sin.(\varphi-\pi) + S\sin.(\varphi-\pi) + T\cos.(\varphi-\pi) - \frac{R}{\text{tang.}\,G}\right).$$

Unde latitudo ψ ita determinatur ut sit $\text{tang.}\,\psi = \sin.(\varphi - \pi)\,\text{tang.}\,G$. Hasque æquationes, quatenus termini litteras S, T & R involventes sunt minimi, ad institutum nostrum propius accommodari oportet.

§. XXII. Quo rationem parvitatis virium T & S facilius in calculum introducere queamus, contemplemur primum caſum, quo iſtæ vires penitus evaneſcunt; binæque priores æquationes ſequentem induent formam:

$$\text{I. } dt = \frac{-x\,dx}{\sqrt{(Bxx + ffx - A)}} \;\&$$

$$\text{II. } d\varphi = \frac{dt\sqrt{A}}{xx} = \frac{-dx\sqrt{A}}{x\sqrt{(Bxx + ffx - A)}};$$

quarum evolutio ita eſt in promtu, ut introducendo quodam angulo v, qui anomalia vera vocari ſolet, ſi ponatur $x = \frac{b}{1 - k\,coſ.\,v}$, conſtantes illæ A & B per has novas b & k ita definiri queant, ut formula irrationalis $\sqrt{(Bxx + ffx - A)}$ evaneſcat ſive angulus v ſit $= o$ ſive duobus rectis æqualis. Atque hinc oritur notiſſima motus elliptici ratio, pro quo littera b denotat ſemiparametrum ellipſis & k ejus excentricitatem, ſeu focorum diſtantiam per axem tranſverſum diviſam. Accedentibus autem viribus minimis T & S motus aliquantillum ab hac lege diſcrepabit.

§. XXIII. Diſcrimen ſcilicet in hoc conſiſtet, quod jam quantitates b & k non amplius futuræ ſunt conſtantes, ſed variabilitatem à viribus T & S oriundam implicent. Quamobrem ponamus $x = \frac{p}{1 - q\,coſ.\,v}$, ſitque ut ante v ejuſmodi angulus, quo ſive evaneſcente ſive ad 180° increſcente, diſtantia x fiat ſive maxima ſive minima: ſeu quod eodem redit ut fiat $dx = o$, ſi ſit $ſin.\,v = o$. Cum igitur ſit: $dx = -\frac{dt}{x} \times \sqrt{(Bxx + ffx - A + \int Tx^3 d\varphi - xx(\int Tx\,d\varphi + \int S\,dx))}$ formula irrationalis, poſito $x = \frac{p}{1 - q\,coſ.\,v}$, factorem $ſin.\,v$ involvere ſeu hujuſmodi formam $W\,ſin.\,v$ habere debebit: quod ut eveniat ipſæ quantitates variabiles p & q debito modo definiri conveniet. At poſito præterquam in

formulis integralibus $x = \frac{p}{1 - q\,coſ.\,v}$, habebimus: $dx = -\frac{dt}{p} \times$ $\sqrt{(Bpp + ffp(1 - q\,coſ.\,v) - A(1 - q\,coſ\,v)^2 + (1 - q\,coſ.\,v)^2 \int Tx^3 d\varphi - pp(\int Tx\,d\varphi + \int S\,dx))}$.

§. XXIV. Evolvamus hanc formulam ſecundum $coſ.\,v$ hoc modo:

$$dx = -\frac{dt}{p}\sqrt{\left\{\begin{array}{l} Bpp + ffp - A + \int Tx^3 d\varphi - pp(\int Tx\,d\varphi + \int S\,dx) \\ - ffpq\,coſ.\,v + 2Aq\,coſ.\,v - 2q\,coſ.\,v.\int Tx^3 d\varphi \\ - Aqq\,coſ.\,v^2 + qq\,coſ.\,v^2 \int Tx^3 d\varphi \end{array}\right\}}$$

jam ut in ſigno radicali ſin. v^2 ſeu $1 - coſ.\,v^2$ involvatur, reddamus terminos ipſum $coſ.\,v$ continentes nihilo æquales, unde diviſione per $2q\,coſ.\,v$ inſtituta fit:

$$-\tfrac{1}{2}ffp + A - \int Tx^3 d\varphi = o, \text{ ſeu } A - \int Tx^3 d\varphi = \tfrac{1}{2}ffp,$$

hocque valore ſubſtituto orietur:

$$dx = -\frac{dt}{p}\sqrt{\left\{\begin{array}{l} Bpp + \frac{1}{2}ffp - pp(\int Tx\,d\varphi + \int S\,dx) \\ - \frac{1}{2}ffpqq\,coſ.\,v^2. \end{array}\right\}}$$

Fiat porro:

$$Bpp + \tfrac{1}{2}ffp - pp(\int Tx\,d\varphi + \int S\,dx) = \tfrac{1}{2}ffpqq,$$

ſeu $\frac{1}{2}ffqq = Bp + \frac{1}{2}ff - p(\int Tx\,d\varphi + \int S\,dx)$ quo facto habebitur:

$$dx = -\frac{dt}{p}\sqrt{(\tfrac{1}{2}ffpqq - \tfrac{1}{2}ffpqq\,coſ.\,v^2)} = -\frac{fq\,dt\,ſin.\,v}{\sqrt{2p}}.$$

§. XXV. Poſito ergo $x = \frac{p}{1 - q\,coſ\,v}$, ut p exprimat ſemiparametrum, & q excentricitatem ellipſis, utramque ob perturbationes variabilem, atque v anomaliam veram, hæ quantitates ita per vires T & S determinari debent ut ſit:

$$p = \frac{2A - 2\int Tx^3 d\varphi}{ff}, \;\&$$

$$qq = \frac{ff + 2Bp - 2p(\int Tx\,d\varphi + \int S\,dx)}{ff}.$$

Cum autem hæ ipſæ quantitates, evaneſcentibus viribus T & S, evadant conſtantes, ſicque earum valores quaſi medii prodire debeant, ponantur hi $p = b$ & $qq = kk$, hincque conſtantes A & B inſtituto convenienter ita definientur ut ſit:

$$b = \frac{2A}{ff} \text{ ſeu } A = \tfrac{1}{2} b ff, \text{ \&}$$
$$kk = \frac{ff + 2Bb}{ff} \text{ ſeu } B = \frac{-ff(1-kk)}{2b}.$$

Hinc itaque habebimus:

$$p = b - \frac{2}{ff}\int T x^3 d\varphi,$$
$$qq = 1 - \frac{(1-kk)p}{b} - \frac{2p}{ff}\left(\int T x d\varphi + \int S dx\right).$$

§. XXVI. Valoribus igitur p & q ita ſtabilitis, ut earum variabilitas tantum à viribus T & S, quarum actio eſt valde parva, pendeat, obtinebimus inde:

Ipſam diſtantiam curtatam $x = \frac{p}{1-q\,coſ.\,v}$,
Ejuſque differentiale . . . $dx = \frac{-fq\,dt\,ſin.\,v}{\sqrt{2p}}$.

Tum vero erit ut ante invenimus:

$$d\varphi = \frac{dt}{xx}\sqrt{\left(A - \int Tx^3 d\varphi\right)} = \frac{dt}{xx}\sqrt{\left(\tfrac{1}{2}bff - \int Tx^3 d\varphi\right)},$$
vel etiam $d\varphi = \frac{fdt}{xx}\sqrt{\tfrac{1}{2}p} = \frac{fdt(1-coſ.\,v)^2}{p\sqrt{2p}}$.

Sicque commodius differentialia dx & $d\varphi$ per differentiale temporis dt habentur expreſſa. Hæ autem formulæ in locum binarum priorum æquationum (§. XXI) inventarum ſunt ſubſtituendæ; quod vero ad binas poſteriores attinet, quæ ad latitudinem ſpectant, eas deinceps ſeorſim conſiderabo, quia earum evolutio non tantis eſt ſubjecta difficultatibus. Hic igitur ita binis prioribus inhærebo, quaſi binæ poſteriores prorſus abeſſent.

§. XXVII. Verum conditio, quâ eſſe debet $dx = \frac{-fg\,dt\,ſin.\,v}{\sqrt{2p}}$ exiſtente $x = \frac{p}{1-q\,coſ.\,v}$, novam determinationem continet, quæ indolem anomaliæ veræ v & quemadmodum ejus differentiale ſit comparatum definiet. Cum enim ſit $1 - q\,coſ.\,v. = \frac{p}{x}$ erit, differentiando & pro dx valorem $-\frac{fq\,dt\,ſin.\,v}{\sqrt{2p}}$ ſubſtituendo:

$$-dq\,coſ.\,v + q\,dv\,ſin.\,v = \frac{dp}{x} + \frac{fq\,dt\,ſin.\,v}{xx}\sqrt{\tfrac{1}{2}p}, \text{ ſeu}$$
$$q\,dv\,ſin.\,v. = \frac{dp}{x} + dq\,coſ.\,v + \frac{fq\,dt\,ſin.\,v}{xx}\sqrt{\tfrac{1}{2}p}.$$

Per valores autem pro p & q ſupra inventos, etiam harum quantitatum differentialia innoteſcunt: erit enim

$$dp = -\frac{2Tx^3 d\varphi}{ff} = -\frac{Txdt}{f}\sqrt{2p} \text{ ob } d\varphi = \frac{fdt}{xx}\sqrt{\tfrac{1}{2}p}, \text{ \&}$$
$$d.\frac{qq}{p} = -\frac{dp}{pp} - \frac{2Txd\varphi}{ff} - \frac{2Sdx}{ff} = \frac{Txdt}{fpp}\sqrt{2p} - \frac{Tdt}{fx}\sqrt{2p} + \frac{2Sq\,dt\,ſin.\,v}{f\sqrt{2p}}$$

ſubſtitutis pro dp, $d\varphi$ & dx valoribus inventis. Qua formula porro evoluta oritur:

$$\frac{2qdq}{p} = -\frac{Tqqxdt}{fpp}\sqrt{2p} + \frac{Txdt}{fpp}\sqrt{2p} - \frac{Tdt}{fx}\sqrt{2p} + \frac{2Sq\,dt\,ſin.\,v.}{f\sqrt{2p}}$$

quæ ob $p = x(1-q\,coſ.\,v)$ reducitur ad hanc formam:

$$dq = \frac{Txdt}{2fp}(2\,coſ.\,v - q - q\,coſ.\,v^2)\sqrt{2p} + \frac{Sp\,dt\,ſin.\,v}{f\sqrt{2p}}.$$

§. XXVIII. Cum igitur ſit:

$$\frac{dp}{x} = -\frac{Tdt}{f}\sqrt{2p} = -\frac{2Txdt(1-q\,coſ.\,v)}{2fp}\sqrt{2p} \text{ erit}$$
$$\frac{dp}{x} + dq\,coſ.\,v = \frac{Txdt}{2fp}(2\,coſ.\,v^2 + q\,coſ.\,v - q\,coſ.\,v^3 - 2)\sqrt{2p} + \frac{Sp\,dt\,ſin.\,v\,coſ.\,v}{f\sqrt{2p}},$$

hæc forma ob $1 - coſ.\,v^2 = ſin.\,v^2$ perducitur ad iſtam

$$\frac{dp}{x} + dq\,coſ.\,v = -\frac{Tx\,dt\,ſin.\,v^2}{f\sqrt{2p}}(2 - q\,coſ.\,v) + \frac{Sp\,dt\,ſin.\,v\,coſ.\,v}{f\sqrt{2p}},$$

quo valore in ſuperiori expreſſione pro $q\,d\nu\,ſin.\nu$ inventa ſubſtituto, diviſione facta per $q\,ſin.\nu$, habebitur:

$$d\nu = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} - \frac{T x\,dt\,ſin.\nu}{fq\sqrt{2p}}(2 - q\,coſ.\nu) + \frac{S p\,dt\,coſ.\nu}{fq\sqrt{2p}}.$$

Hæc autem porro, ob $x(1 - q\,coſ.\nu) = p$, tranſit in formas ſequentes:

$$d\nu = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}}p - \frac{T\,dt\,ſin.\nu}{fq}\sqrt{2p} - \frac{Tx\,dt\,ſin.\nu.coſ.\nu.}{f\sqrt{2p}} + \frac{S p\,dt\,coſ.\nu}{fq\sqrt{2p}},\ \text{ſeu}$$

$$d\nu = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}}p - \frac{p\,dt(2T\,ſin.\nu - S\,coſ.\nu)}{fq\sqrt{2p}} - \frac{Tx\,dt\,ſin.\nu\,coſ.\nu}{f\sqrt{2p}},\ \text{vel etiam}$$

$$d\nu = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}}p + \frac{x\,dt\,(S\,coſ.\nu.(1 - q\,coſ.\nu) - T\,ſin.\nu.(2 - q\,coſ.\nu))}{fq\sqrt{2p}}.$$

§. XXIX. Quoniam porro eſt:

$$d\varphi = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p}\ \&\ dx = -\frac{fq\,dt\,ſin.\nu}{\sqrt{2p}}\ \text{erit:}$$

$$\int T x^3 d\varphi = f\int T x\,dt\sqrt{\tfrac{1}{2}p}$$

$$\int T x\,d\varphi = f\int \frac{T\,dt}{x}\sqrt{\tfrac{1}{2}p}\ \&$$

$$\int S\,dx = -f\int \frac{S q\,dt\,ſin.\nu}{\sqrt{2p}},\ \text{ideoque}$$

$$\int T x\,d\varphi + \int S\,dx = f\int \frac{dt}{\sqrt{2p}}(T(1 - q\,coſ.\nu) - Sq\,ſin.\nu).$$

His igitur valoribus ſubſtituendis non ſolum formulæ integrales, quæ etiamnunc in calculum ingrediuntur, ad differentiale temporis reducuntur, ſed etiam omnium quantitatum variabilium, quibus jam erit utendum, differentialia per idem temporis differentiale dt erunt expreſſa. Neque vero adhuc ulla approximatione ſumus uſi, unde hæ determinationes etiam locum habent, tametſi forte vires T & S non fuerint adeo exiguæ. Interim tamen parum ſubſidii inde conſequi licet, niſi iſtæ vires valde fuerint parvæ.

§. XXX. Hæ autem formulæ maxime videntur idoneæ, ad motus aberrationes à regulis Kepplerianis definiendas; referuntur enim ad motum in ellipſi continuo variabili tam ratione ejus parametri & excentricitatis quam ſitus lineæ abſidum. Quovis enim tempore minimo motus corporis ita conſiderari poteſt, quaſi fieret in ellipſi ſecundum regulas Keppleri, ac ſi pro quolibet tempore conſtet magnitudo iſtius ellipſis, ejuſque excentricitas una cum ſitu lineæ abſidum, ex formulis inventis verus corporis locus, quatenus ad planum aſſumtum refertur, aſſignari poterit. Aſſumo igitur ad tempus propoſitum t orbitæ ad planum noſtrum fixum relatæ eſſe

I. Semiparametrum orbitæ $= p$,

II. Excentricitatem ejus $= q$,

III. Atque anomaliam veram corporis $= \nu$.

Unde cum longitudo corporis poſita ſit $= \varphi$, longitudo abſidis ſummæ definietur angulo $= \varphi - \nu$.

§. XXXI. Ex his igitur elementis ellipticis primo deducitur diſtantia curtata $CQ = x$ ope formulæ $x = \frac{p}{1 - q\,coſ.\nu}$; unde poſito angulo $\nu = 0$ colligitur diſtantia abſidis ſummæ $= \frac{p}{1-q}$ & poſito $\nu = 180°$ diſtantia abſidis imæ à puncto $C = \frac{p}{1+q}$, quarum ſumma $\frac{2p}{1-qq}$ præbet axem tranſverſum orbitæ ellipticæ, cujus propterea ſemiſſis eſt $= \frac{p}{1-qq}$, & ſemiſſis diſtantiæ focorum $= \frac{pq}{1-qq}$: tum vero ſemi axis conjugatus erit $= \frac{p}{\sqrt{(1-qq)}}$. Deinde vero ipſa corporis longitudo ſeu angulus $ACQ = \varphi$ ita per hæc elementa determinatur, ut ſit:

$$d\varphi = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p},\ \text{ſeu}\ d\varphi = \frac{f\,dt\,(1 - q\,coſ.\nu)^2}{p\sqrt{2p}};$$

unde ea per integrationem elici poterit, dummodo lex conſtet, qua quantitates variabiles p, q & ν cum tempore mutentur, hanc autem variabilitatis legem jam eruimus.

§. XXXII. Si nullæ adeſſent vires perturbantes T & S, tam parameter orbitæ $2p$ quam excentricitas q eſſent quantitates conſtantes, atque anomalia vera v cum longitudine φ paria caperet incrementa; ſicque ob $d\varphi - dv = o$, linea abſidum immota maneret. Videamus igitur quales mutationes hæ quantitates ſubire debeant accedentibus iſtis viribus perturbatricibus T & S. Ac primo quidem ob $Tx^3 d\varphi = fTx\,dt\sqrt{\tfrac{1}{2}p}$, tempuſculo dt ſemiparametri p incrementum inventum eſt:

$$dp = -\frac{Tx\,dt}{f}\sqrt{2p} = -\frac{Tp\,dt\sqrt{2p}}{f(1-q\,cof.v)}$$

quod ergo tantum à vi T pendet, niſi quatenus variabilitas quantitatum q & v ſimul alteram vim S involvit. Hinc igitur erit:

$$\frac{dp}{p\sqrt{p}} = -\frac{T\,dt\sqrt{2}}{f(1-q\,cof.v)} \;\&\; \frac{2}{\sqrt{p}} = \int\frac{T\,dt\sqrt{2}}{f(1-q\,cof.v)}$$

ſeu cum medius valor ipſius p ſit $= b$ erit:

$$\frac{1}{\sqrt{p}} = \frac{1}{\sqrt{b}} + \frac{1}{f\sqrt{2}}\int\frac{T\,dt}{1-q\,cof.v}.$$

§. XXXIII. Secundo incrementum excentricitatis dq ita expreſſum invenimus pro tempuſculo dt, ut ſit:

$$dq = \frac{Tx\,dt(2\,cof.v - q - q\,cof.v^2) + Sp\,dt\,fin.v}{f\sqrt{2p}},$$

quo ſubſtituto pro x valore $\frac{p}{1-q\,cof.v}$ abit in

$$dq = \frac{p\,dt}{f\sqrt{2p}}\left(\frac{T}{1-q\,cof.v}(2\,cof.v - q - q\,cof.v^2) + S\,fin.v\right), \text{ ſeu}$$

$$dq = \frac{dt}{f}\left(T\,cof.v + S\,fin.v + T.\frac{cof.v - q}{1-q\,cof.v}\right)\sqrt{\tfrac{1}{2}p}, \text{ vel etiam}$$

$$dq = \frac{dt}{f}\left(2\,T\,cof.v + S\,fin.v - \frac{Tq\,fin.v^2}{1-q\,cof.v}\right)\sqrt{\tfrac{1}{2}p};$$

in qua formula ultimus terminus præ binis præcedentibus erit valde parvus, ſi quidem excentricitas q non adeo fuerit notabilis. Cognitis igitur viribus T & S cum anomalia vera, hinc facile colligitur, quantum excentricitas intervallo minimi tempuſculi dt immutetur, quemadmodum ex formula præcedente variatio ſemiparametri p inoteſcit.

§. XXXIV. Hinc etiam definiri poteſt variabilitas axis tranſverſi ellipſis, cum enim ejus ſemiſſis ſit $= \frac{p}{1-qq}$ erit ejus differentiale:

$$d.\frac{p}{1-qq} = \frac{(1-qq)dp + 2pq\,dq}{(1-qq)^2}.$$

Quod ſi jam valores pro dp & dq inventi ſubſtituantur, reperitur reductione rite facta:

$$d.\frac{p}{1-qq} = -\frac{p\,dt\sqrt{2p}}{f(1-qq)^2}\left(T - q(T\,cof.v + S\,fin.v)\right).$$

Quare ſi ſemi-axis tranſverſus ponatur $= r$, ut ſit $r = \frac{p}{1-qq}$, ob $\frac{p\sqrt{2p}}{(1-qq)^2} = rr\sqrt{\frac{2}{p}}$, fiet

$$dr = -\frac{rr\,dt}{f\sqrt{\frac{1}{2}p}}\left(T - q(T\,cof.v + S\,fin.v)\right), \text{ ſeu}$$

$$\frac{-dr}{rr} = d.\frac{1}{r} = \frac{dt\left(T - q(T\,cof.v + S\,fin.v)\right)}{f\sqrt{\frac{1}{2}p}}.$$

Unde, ſi ſemi-axis tranſverſus medius ponatur $= a$, fiet

$$\frac{1}{r} = \frac{1}{a} + \int\frac{dt}{f\sqrt{\frac{1}{2}p}}\left(T - q(T\,cof.v + S\,fin.v)\right).$$

§. XXXV. Denique mutatio inſtantanea lineæ abſidum eſt indaganda, cujus longitudo cum ſit $= \varphi - v$, erit ejus incrementum tempuſculo dt ortum $= d\varphi - dv$. Verum invenimus:

$$d\varphi = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} = \frac{f\,dt(1-q\,cof.v)^2}{p\sqrt{2p}}, \;\&$$

$$dv = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} + \frac{x\,dt\left(S\,cof.v(1-q\,cof.v) - T\,fin.v(2-q\,cof.v)\right)}{fq\sqrt{2p}}, \text{ ſeu}$$

$$dv = d\varphi + \tfrac{dt}{fq}\left(S\,cos.v - \frac{T(2 - q\,cos.v)\,sin.v}{1 - q\,cos.v}\right)\sqrt{\tfrac{1}{2}p}.$$

Unde colligimus

$$d\varphi - dv = \tfrac{dt}{fq}\left(\frac{T(2 - q\,cos.v)\,sin.v}{1 - q\,cos.v} - S\,cos.v\right)\sqrt{\tfrac{1}{2}p} =$$

$$\tfrac{dt}{fq}\left(\frac{T\,sin.v}{1 - q\,cos.v} + T\,sin.v - S\,cos.v\right)\sqrt{\tfrac{1}{2}p},$$ five

$$d\varphi - dv = \tfrac{dt}{fq}\left(2\,T\,sin.v - S\,cos.v + \frac{Tq\,sin.v\,cos.v}{1 - q\,cos.v}\right)\sqrt{\tfrac{1}{2}p}.$$

Hinc igitur patet motum lineæ abfidum eo fieri notabiliorem, quo minor fuerit excentricitas q; qua evanefcente etiam in infinitum abire videtur. Verum notandum eft, quo minor fuerit excentricitas, eo minus referre verum lineæ abfidum locum noffe.

§. XXXVI. Poftremo ad eadem elementa reducere poterimus æquationum principalium (§. XXI) binas pofteriores; cum enim fit $d\varphi = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p}$ erit $\frac{dt^2}{2x\,d\varphi} = \frac{x\,dt}{f\sqrt{2p}}$, mutationes momentaneæ, quas cum linea nodorum tum inclinatio orbitæ ad planum fixum fubibunt, ita per tempufculum minimum dt erunt expreffæ:

$$d\pi = \frac{x\,dt\,fin.(\varphi-\pi)}{f\sqrt{2p}}\left(\frac{ff}{xx}\,fin.(\varphi-\pi) + S\,fin.(\varphi-\pi) + T\,cof.(\varphi-\pi) - \frac{R}{tang.G}\right)$$

$$d\,l\,tang.G = \frac{x\,dt\,cof.(\varphi-\pi)}{f\sqrt{2p}}\left(\frac{ff}{xx}\,fin.(\varphi-\pi) + S\,fin.(\varphi-\pi) + T\,cof.(\varphi-\pi) - \frac{R}{tang.G}\right).$$

Unde latitudo ψ ita definitur ut fit $tang.\psi = fin.(\varphi-\pi)\,tang.G$. Hic quidem partes adfunt à viribus perturbatricibus non pendentes, verum hoc inde venit, quod vim quadratis diftantiarum reciproce proportionalem in plano fixo affumfimus. Si enim ea, uti rei natura poftulat, fecundum diftantiam veram RC affumatur, iftæ partes à vi $\frac{R}{tang.G}$ tollentur, id quod in applicatione fiet manifeftum.

SECTIO III.

SECTIO III.

Inveftigatio Virium quibus motus Planetæ principalis ab actione alius Planetæ perturbatur.

§. XXXVII. Cum perturbationes, quibus planetæ principales fe mutuo afficiunt, fint vehementer parvæ, dum uniufcujufque planetæ perturbationes inveftigamus, motum reliquorum tanquam regulis Kepleri perfecte conformem fpectare licebit: tantillus enim error, qui hac ratione in motu planetæ perturbantis admittitur, in effectu multo minorem, hoc eft evanefcentem producere eft cenfendus. Quoniam igitur quæftio ad planetas principales adftringitur, punctum fixum C in centro folis affumi conveniet; & quia motus planetæ perturbantis in plano fieri poteft judicari, hoc ipfum planum pro plano illo fixo, ad quod motum planetæ turbati referre conftituimus, commodiffime affumemus. Cum enim invenerimus, quomodo motus iftius planetæ refpectu hujus plani immutetur, facile erit perturbationes ad quodvis aliud planum in cœlo fixum traducere, ficque inconftantiam, cui planum orbitæ planetæ perturbantis eft obnoxium, exuere.

§. XXXVIII. Conveniat igitur planum tabulæ cum plano orbitæ planetæ perturbantis, in quo C fit centrum folis, & CA recta inde ad fixum cœli punctum ducta, unde longitudines numerentur. Ad datum ergo tempus t planeta perturbans fit in hujus plani puncto V, à quo ductâ rectâ CV, ponatur: FIG. II.

I. Distantia hujus planetæ à sole . . . $CV = y$

II. Longitudo ejus seu angulus ACV . . . $= \theta$

Quod si ergo vis, qua hic planeta in V ad solem urgetur sit $= \frac{ff}{yy}$, ejusque orbitæ semi latus rectum, seu semiparameter $= c$, ejus excentricitas $= e$, & anomalia vera $= u$, erit per formulas supra inventas:

$$y = \frac{c}{1 - e\,cos.u}; \; \& \; d\theta = du = \frac{f\,dt}{yy}\sqrt{\tfrac{1}{2}c} = \frac{f\,dt\,(1 - e\,cos.u)^2}{c\sqrt{2c}};$$

unde fit $dy = -\frac{c\,e\,du\,sin.u}{(1 - e\,cos.u)^2} = -\frac{e\,f\,dt\,sin.u}{\sqrt{2c}}$; sicque hæc differentialia ad elementum temporis dt habemus reducta.

§. XXXIX. Alter jam planeta, cujus perturbationes motus indagamus, extra hoc planum reperiatur in R unde ad id ducto perpendiculo RQ, junctisque rectis RC & QC, sit ut ante:

I. Ejus distantia à sole curtata seu recta $CQ = x$

II. Longitudo ejus seu angulus ACQ . . $= \varphi$

III. Latitudo ejus seu angulus QCR . . . $= \psi$

Unde fit ejus distantia à sole vera . . . $= \frac{x}{cos.\psi}$

Verum pro latitudine considerentur linea nodorum $C\Omega$ & inclinatio orbitæ ad planum orbitæ prioris planetæ, sitque

IV. Longitudo nodi ascendentis seu angulus $AC\Omega = \pi$

V. Inclinatio orbitæ seu angulus QSR . . . $= G$

ex quibus elementis latitudo, ita exprimitur, ut sit

$$tang.\,\psi = sin.\,(\varphi - \pi)\;tang.\,G.$$

Quod si porro jungantur rectæ QV & RV, erit

$$QV = \sqrt{(xx + yy - 2xy\,cos.\,(\varphi - \theta))}, \&$$

$$VR = \sqrt{\left(\frac{xx}{cos.\psi^2} + yy - 2xy\,cos.(\varphi - \theta)\right)} = z$$

ponamus enim brevitatis gratia hanc distantiam $VR = z$.

§. XL. Cum jam planeta R tam ad solem quam ad alterum planetam V attrahatur in ratione reciproca duplicata distantiarum, sit utraque vis ita comparata, ut ad distantiam d habeatur

vis acceleratrix ad solem tendens . . . $= \frac{E}{dd}$,

vis acceleratrix ad planetam tendens . . $= \frac{F}{dd}$.

Hinc itaque planeta in R urgebitur

primo ad solem secundum RC vi $= \frac{E\,cos.\psi^2}{xx}$;

deinde ad planetam in V secundum RV vi $= \frac{F}{zz}$.

Quoniam vero ipse sol quoque ad planetam V sollicitatur secundum CV vi acceleratrice $= \frac{F}{yy}$, ut solem in quiete retineamus hæc vis secundum directionem contrariam $Q\nu$ in planetam R transferri debet, hincque iste præterea sollicitabitur:

Secundum directionem $Q\nu$ vi $= \frac{F}{yy}$.

Quin etiam ipse planeta V omnino ad solem urgeri censendus est secundum VC vi $= \frac{E + F}{yy}$, quam modo posueramus $= \frac{ff}{yy}$.

§. XLI. Nunc vero ante omnia vires planetam R sollicitantes revocari debent ad directiones QC, QN & RQ, ut inde valores virium assumtarum V, T & R obtineantur. Ac primo quidem vis secundum $RC = \frac{E\,cos.\psi^2}{xx}$ præbet

Secundum directionem QC vim $= \frac{E\,cos.\psi^3}{xx}$ pro vi V;

Secundum directionem RQ vim $= \frac{E\,sin.\psi\,cos.\psi^2}{xx}$ pro vi R;

Secunda vis ſecundum $RV = \frac{F}{\zeta\zeta}$, ob $QR = x\, tang.\, \psi$, & $VR = \zeta$, præbet

Secundum directionem RQ vim $= \frac{F.QR}{\zeta^3} = \frac{Fx\, tang.\, \psi}{\zeta^3}$ pro R,

Tum vero ſecundum QV vim $= \frac{F.QV}{\zeta^3}$ quæ, ob

$$QV\, cof.\, CQV = -y\, cof.\, (\varphi - \theta) + x \;\&$$
$$QV\, fin.\, CQV = y\, fin.\, (\varphi - \theta),$$

reducitur ad binas ſequentes

Secundum QC vim $= \frac{-F}{\zeta^3}(y\, cof.\, (\varphi - \theta) - x)$ pro vi V,

Secundum QN vim $= \frac{Fy\, fin.\, (\varphi - \theta)}{\zeta^3}$ pro vi T;

Denique tertia vis ſecundum $Q\nu = \frac{F}{yy}$, ob $CQ\nu = \varphi - \theta$, dat

Secundum QC vim $= \dfrac{F\, cof.\, (\varphi - \theta)}{yy}$ pro vi V,

Secundum QN vim $= \dfrac{-F fin.\, (\varphi - \theta)}{yy}$ pro vi T.

§. XLII. Colligamus ſingulas has vires ad directiones QC, QN & RQ reductas, atque habebimus

$$\text{Vim } V = \frac{E\, cof.\, \psi^2}{xx} + \frac{Fx}{\zeta^3} - Fy\,(\tfrac{1}{\zeta^3} - \tfrac{1}{y^3})\, cof.\, (\varphi - \theta),$$

$$\text{Vim } T = Fy\,(\tfrac{1}{\zeta^3} - \tfrac{1}{y^3})\, fin.\, (\varphi - \theta),$$

$$\text{Vim } R = \frac{E\, fin.\, \psi\, cof.\, \psi^2}{xx} + \frac{Fx\, tang.\, \psi}{\zeta^3};$$

nihilque ſupereſt, niſi ut hæ expreſſiones in ſocum litterarum V, T & R ſubſtituantur. Quoniam vero angulus ψ ſemper eſt valde parvus, ejus coſinus proxime ad unitatem accedet; unde cum poſuerimus $V = \frac{ff}{xx} + S$, ita ut ſit

$$S = \frac{-ff}{xx} + \frac{E\, cof.\, \psi^2}{xx} + \frac{Fx}{\zeta^3} - Fy\,(\tfrac{1}{\zeta^3} - \tfrac{1}{y^3})\, cof.\, (\varphi - \theta);$$

pro ff aſſumi poterit E; & quamquam ante inveneramus $ff = E + F$, tamen quantitas F præ E tam eſt exigua, ut nullus plane error ſit metuendus, ſi ponamus $ff = E$; ita ut habeamus:

$$S = \frac{-E(1 - cof.\, \psi^2)}{xx} + \frac{Fx}{\zeta^3} - Fy\,(\tfrac{1}{\zeta^3} - \tfrac{1}{y^3})\, cof.\, (\varphi - \theta).$$

§. XLIII. Antequam autem hujus reductionis rationem habeamus, ſubſtitutio virium V, T & R in æquationibus variationem lineæ nodorum & inclinationis continentibus commode fieri poterit, quæ eo magis eſt notatu digna, quod uti jam innuimus termini à vi perturbatrice non pendentes deſtruantur. Cum enim ſit $tang.\, G = \frac{tang.\, \psi}{fin.\, (\varphi - \pi)}$ factoris $V\, fin.\, (\varphi - \pi) + T\, cof.\, (\varphi - \pi) - \frac{R}{tang.\, G.}$, qui illas expreſſiones ingreditur, valor ſatis concinne definietur, habebitur namque:

$$V\, fin.\, (\varphi - \pi) = \frac{E\, cof.\, \psi^2}{xx} fin.\, (\varphi - \pi) + \frac{Fx}{\zeta^3} fin.\, (\varphi - \pi) -$$
$$Fy\,(\tfrac{1}{\zeta^3} - \tfrac{1}{y^3})\, cof.\, (\varphi - \theta)\, fin.\, (\varphi - \pi),$$

$$T\, cof.\, (\varphi - \pi) = Fy\,(\tfrac{1}{\zeta^3} - \tfrac{1}{y^3})\, fin.\, (\varphi - \theta)\, cof.\, (\varphi - \pi),$$

$$-\frac{R}{tang.\, G.} = \frac{-E\, cof.\, \psi^2}{xx} fin.\, (\varphi - \pi) - \frac{Fx}{\zeta^3} fin.\, (\varphi - \pi);$$

Quare cum ſit $-cof.\, (\varphi - \theta)\, fin.\, (\varphi - \pi) + fin.\, (\varphi - \theta)\, cof.\, (\varphi - \pi) = -fin.\, (\theta - \pi)$, erit $V\, fin.\, (\varphi - \pi) + T\, cof.\, (\varphi - \pi) - \frac{R}{tang.\, G} = -Fy\,(\tfrac{1}{\zeta^3} - \tfrac{1}{y^3})\, fin.\, (\theta - \pi)$.

§ XLIV. Perſpicuum ergo eſt totam hanc expreſſionem, qua variatio in linea nodorum & inclinatione determinatur, unice à vi perturbante F pendere; cæterisque paribus sinui anguli $\theta - \pi$, qui oritur longitudinem nodi π a longitudine planetæ turbantis θ subtrahendo, esse proportionalem. Quodsi ergo ut ante semiparametrum orbitæ planetæ R ponamus $= p$, ex §. XXXVI, pro variatione tam lineæ nodorum, quam inclinationis, sequentes nanciscemur æquationes:

$$d\pi = -\frac{Fxy\,dt\,\text{fin.}(\varphi-\pi)\,\text{fin.}(\theta-\pi)}{f\sqrt{2p}}\left(\frac{1}{z^3}-\frac{1}{y^3}\right),$$

$$d.l.\,tang.\,G = -\frac{Fxy\,dt\,\text{cof.}(\varphi-\pi)\,\text{fin.}(\theta-\pi)}{f\sqrt{2p}}\left(\frac{1}{z^3}-\frac{1}{y^3}\right);$$

Ubi imprimis est memorabile has expressiones tanto simpliciores prodiisse. Quamvis autem eædem facilius ex ipsa virium sollicitantium indole erui potuissent, tamen earum derivationem ex formulis generalibus petere convenientius est visum.

§. XLV. Progrediamur ergo ad reliquas perturbationes, & cum posuerimus pro orbita planetæ R turbata:

Semiparametrum . . . . $= p$,
Excentricitatem . . . . $= q$,
Et anomiliam veram . . . . . $= v$; ita ut sit

$$x = \frac{p}{1-q\,\text{cof.}\,v} \;\&\; dx = \frac{-fq\,dt\,\text{fin.}\,v}{\sqrt{2p}},$$

primo variationem parametri ita invenimus expressam

$$dp = -\frac{Tx\,dt}{f}\sqrt{2p} = -\frac{Tp\,dt\sqrt{2p}}{f(1-q\,\text{cof.}\,v)},$$

quæ ideoque hanc induet formam:

$$dp = -\frac{Fxy\,dt\,\text{fin.}(\varphi-\theta)}{f}\left(\frac{1}{z^3}-\frac{1}{y^3}\right)\sqrt{2p},$$

neque enim adhuc pro x, y & z valores supra designatos substitui conveniet, quia illi nonsolum jam sunt

cogniti, sed etiam eorum differentialia per dt exhiberi possunt

$$\text{ob } y = \frac{c}{1-e\,\text{cof.}\,u},\; d\theta = du = \frac{f\,dt}{yy}\sqrt{\tfrac{1}{2}:c}, \;\&$$

$$dy = -\frac{fe\,dt\,\text{fin.}\,u}{\sqrt{2c}}.$$

§. XLVI. Secundo excentricitatis q variatio §. XXXIII est inventa

$$dq = \frac{dt}{f}\left(T\,\text{cof.}\,v + S\,\text{fin.}\,v + T.\frac{\text{cof.}\,v - q}{1-q\,\text{cof.}\,v}\right)\sqrt{\tfrac{1}{2}p}, \text{ feu}$$

$$dq = \frac{dt}{f}\left(2T\,\text{cof.}\,v + S\,\text{fin.}\,v - \frac{Tq\,\text{fin.}\,v^2}{1-q\,\text{cof.}\,v}\right)\sqrt{\tfrac{1}{2}p},$$

ubi recordari debemus esse:

$$S = \frac{-E(1-\text{cof.}\,\psi')}{xx} + \frac{Fx}{z^3} - Fy\left(\frac{1}{z^3}-\frac{1}{y^3}\right)\text{cof.}(\varphi-\theta), \;\&$$

$$T = Fy\left(\frac{1}{z^3}-\frac{1}{y^3}\right)\text{fin.}(\varphi-\theta);$$

quorum valorum substitutionem fieri non est opus: quod cum etiam in reliquis commode fieri nequeat, eas apponamus uti invenimus §. XXXV. Tertio scilicet pro motu lineæ absidum obtinuimus

$$d\varphi - dv = \frac{dt}{fq}\left(2T\,\text{fin.}\,v - S\,\text{cof.}\,v + \frac{Tq\,\text{fin.}\,v\,\text{cof.}\,v}{1-q\,\text{cof.}\,v}\right)\sqrt{\tfrac{1}{2}p};$$

& quia $d\varphi = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} = \frac{f\,dt\,(1-q\,\text{cof.}\,v)^2}{p.\sqrt{2p}}$,

erit incrementum momentaneum anomaliæ veræ

$$dv = \frac{f\,dt}{xx}\sqrt{\tfrac{1}{2}p} - \frac{dt}{fq}\left(2T\,\text{fin.}\,v - S\,\text{cof.}\,v + \frac{Tq\,\text{fin.}\,v\,\text{cof.}\,v}{1-q\,\text{cof.}\,v}\right)\sqrt{\tfrac{1}{2}p}.$$

§. XLVII. Nunc autem elementum temporis dt eliminari conveniet, cujus loco commodissime motus medius solis sive terræ introducitur. Ponamus ergo distantiam mediam terræ à sole esse $= a$, hocque motu medio absolvi tempusculo dt angulum $= d\omega$.

Quoniam hoc caſu excentricitas adeſt nulla, viſque ad ſolem tendens eſt $= \frac{ff}{aa} = \frac{E}{aa}$, erit ex principiis ante ſtabilitis: $d\omega = \frac{fdt}{aa} \sqrt{\frac{1}{2}a} = \frac{fdt}{a\sqrt{2a}}$ ideoque $fdt = ad\omega\sqrt{2a}$, &

$$\frac{dt}{F} = \frac{ad\omega\sqrt{2a}}{ff} = \frac{ad\omega\sqrt{2a}}{E}.$$

Tempore ergo abſoluto t eliminato, ejuſque loco angulo ω quem ſol motu medio interea abſolvit, noſtræ formulæ differentiales omnes ad elementum iſtius motus medii $d\omega$ reduci poterunt. Ac ſi inſuper ponatur $F = nE$, ubi n ſemper fractionem vehementer parvam denotabit, erit $\frac{T}{E} = ny\left(\frac{1}{z^3} - \frac{1}{y^3}\right) \text{ſin.}(\varphi - \theta)$,

$$\frac{S}{E} = -\frac{(1 - \text{coſ.}\,\varphi^2)}{xx} + \frac{nx}{z^3} - ny\left(\frac{1}{z^3} - \frac{1}{y^3}\right) \text{coſ.}(\varphi - \theta).$$

§. XLVIII. Subſtituto ergo pro dt iſto valore, habebimus primo pro planeta perturbante:

$d\theta = du = \frac{ad\omega}{yy}\sqrt{ac}$; & $dy = -aed\omega\,\text{ſin.}\,u\sqrt{\frac{a}{c}}$,

at pro planeta perturbato:

$d\varphi = \frac{ad\omega}{xx}\sqrt{ap}$; & $dx = -aqd\omega\,\text{ſin.}\,v\sqrt{\frac{a}{p}}$ porroque

$dp = -\frac{T}{E}.2axd\omega\sqrt{ap} = -2naxyd\omega\left(\frac{1}{z^3} - \frac{1}{y^3}\right)\text{ſin.}(\varphi - \theta).\sqrt{ap}$,

$$dq = ad\omega\left(\frac{2T}{E}\text{coſ.}\,v + \frac{S}{E}\text{ſin.}\,v - \frac{T}{E}.\frac{q\,\text{ſin.}\,v^2}{1 - q\,\text{coſ.}\,v}\right).\sqrt{ap},$$

$$d\varphi - dv = \frac{ad\omega}{q}\left(\frac{2T}{E}\text{ſin.}\,v - \frac{S}{E}\text{coſ.}\,v + \frac{T}{E}.\frac{q\,\text{ſin.}\,v\,\text{coſ.}\,v}{1 - q\,\text{coſ.}\,v}\right).\sqrt{ap}, \&$$

$$dv = \frac{ad\omega}{xx}.\sqrt{ap} - \frac{ad\omega}{q}\left(\frac{2T}{E}\text{ſin.}\,v - \frac{S}{E}\text{coſ.}\,v + \frac{T}{E}.\frac{q\,\text{ſin.}\,v.\,\text{coſ.}\,v}{1 - q\,\text{coſ.}\,v}\right).\sqrt{ap}.$$

Si ulterius ſemi-axis tranſverſus $\frac{p}{1 - qq}$ ponatur $= r$, erit

$$dr = \frac{-2arrd\omega\sqrt{a}}{\sqrt{p}}\left(\frac{T}{E} - \frac{T}{E}q\,\text{coſ.}\,v - \frac{S}{E}q\,\text{ſin.}\,v\right),$$

uti ex §. XXXIV colligere licet,

§. XLIX.

§. XLIX. Simili modo & mutationes, quas linea nodorum & inclinatio orbitæ tempuſculo, quo ſol ſecundum medium motum per angulum $d\omega$ progreditur ſubeunt, exprimi poterunt. Poſito enim $F = nE$, & ſolis diſtantia media à terra $= a$, quoniam invenimus:

$$\frac{dt}{f} = \frac{nad\omega\sqrt{2a}}{E} \text{ erit } \frac{Fdt}{f} = nad\omega\sqrt{2a}, \ \& \ \frac{Fdt}{f\sqrt{2p}} = nad\omega\sqrt{\frac{a}{p}};$$

formulæ ſupra exhibitæ abibunt in has:

$$d\pi = -naxyd\omega\,\text{coſ.}(\varphi - \pi)\,\text{ſin.}(\theta - \pi).\left(\frac{1}{z^3} - \frac{1}{y^3}\right).\sqrt{\frac{a}{p}}.$$

$$d.\,l.\,\text{tang.}\,G = -naxyd\omega\,\text{coſ.}(\varphi - \pi)\,\text{ſin.}(\theta - \pi).\left(\frac{1}{z^3} - \frac{1}{y^3}\right).\sqrt{\frac{a}{p}}.$$

Cum autem valor ipſius $d\pi$ jam fuerit inventus, erit ſuccinctius

$$d.\,l.\,\text{tang.}\,G = \frac{d.\,\text{tang.}\,G}{\text{tang.}\,G} = d\pi\frac{\text{coſ.}(\varphi - \pi)}{\text{ſin.}(\varphi - \pi)}.$$

Per has igitur formulas omnium quantitatum, quibus determinatio perturbationum continetur, incrementa, quæ tempuſculo per motum ſolis medium $d\omega$ expreſſo capiunt, definiri poterunt, neque hactenus ulla approximatione ſumus uſi, niſi quatenus pro motu planetæ perturbantis loco $E + F$ ſimpliciter E ſcripſimus, unde autem nulla aberratio à vero oriri poteſt.

§. L. At vero in calculi ſubſidium jam multo graviorem hypotheſin aſſumſimus, dum motum planetæ perturbantis, quem in V fingimus, tanquam regulis Kepleri perfecte conſentaneum ſpectamus; ſi enim iſte planeta eſſet Saturnus, cujus motum non mediocriter ab actione Jovis turbari novimus, nullum certè eſt dubium, quin ejus perturbationes effectus, qui ab ejus actione in motum reliquorum planetarum redundant, aliquantillum eſſent affecturæ. Interim tamen pro certo ſtatuere licet, iſtas variationes incomparabiliter futuras eſſe minores, neque effectum Saturni verum ſenſibiliter eſſe diſcrepaturum ab eo, quem regulas Keppleri exacte ſecutus, eſſet producturus; imprimis cum conſtet, univerſam perturbationem eſſe quam minimam, atque adeo nos contenti eſſe debeamus eam tantum vero proxime determinaſſe. Æque parvi autem momenti ſine ullo dubio æſtimanda erit ea aberratio, quæ dum pro $E+F$ tantum E ſeu 1 pro $1+n$ ſemper eſt fractio quam minima.

§. LI. Non parum paradoxon videri debet, quod etiamſi vis planetæ perturbantis, ſeu fractio n penitus evaneſceret, tamen pro altero planeta tam excentricitas q quam linea abſidum mutationibus eſſet obnoxia: propterea quod evaneſcente fractione n, quantitas $\frac{S}{E}$ non in nihilum abeat, ſed valorem $= \frac{-1+cos.\psi^3}{xx}$ retineat, quo utrumque differentiale dq & $d\varphi - dv$ afficitur. Verum perpendendum eſt, quod dum orbitam planetæ in aliud planum projicimus, projectio quidem quoque futura ſit ellipſis, ſed cujus focus non amplius futurus ſit in puncto C. Etiamſi ergo motus projectus fiat in ellipſi, areæque adeo circa punctum C deſcriptæ temporibus ſint proportionales, tamen quia in C non eſt focus ellipſis, motus regulis Keppleri non erit conformis. Cum autem nihilominus fingi poſſet quovis momento ellipſis focum habens in C, cujus elementum cum illius ellipſis elemento congruat, mirum non eſt hujus ellipſis fictæ tam excentricitatem quam poſitionem lineæ abſidum continuo variari: notatu autem eſt dignum parametrum ellipſis ſemper invariatum relinqui.

§. LII. Cum igitur caſu $n=o$ hoc incommodum penitus evitaremus, ſi motum planetæ non in plano alieno ſed proprio contemplaremur, idem quoque incommodum in genere evitabimus, ſi motum quovis

tempore ad planum orbitæ ipſum referamus, ita ut x diſtantiam veram CR & φ longitudinem planetæ in proprio plano denotaret. Hinc quidem ob orbitæ inconſtantiam alia incommoda naſcerentur, quæ autem, dummodo inclinatio G ſit valde parva, quemadmodum id quidem ſemper uſu venit, fere penitus removebuntur, propterea quod inde ipſis perturbationibus minima perturbatio inducetur. Quo circa totum negotium ita promtius expedietur ut primo dum variationes quantitatum p, q, & $\varphi - v$, exquiruntur, à quibus locus planetæ in propria orbita pendet, latitudo ψ prorſus negligatur ponendo $\psi = o$; deinde vero ſeorſim ad quodvis tempus tam poſitio lineæ nodorum quam inclinatio inveſtigetur: & his denique inventis more apud Aſtronomos recepto ex loco planetæ in propria orbita ejuſque argumento latitudinis ipſa latitudo eliciatur, cum reductione longitudinis.

§. LIII. Hanc ob cauſam quæſtionem noſtram commode bipartito pertractare licebit dum ſeorſim primo motus planetæ in propria orbita, quaſi eſſet plana, deinde vero hujus orbitæ poſitio reſpectu orbitæ planetæ perturbantis inveſtigatur. Primo igitur x denotabit diſtantiam veram planetæ à ſole, & φ ejus longitudinem; tum poſito:

$$z = \sqrt{(xx+yy-2xy\,cos.(\varphi-\theta))}, \text{ ob } \psi = o,$$

ſi brevitatis gratia ſtatuamus

$$M = y\left(\frac{1}{z^3}-\frac{1}{y^3}\right) sin.(\varphi-\theta)),$$

$$N = \frac{x}{z^3} - y\left(\frac{1}{z^3}-\frac{1}{y^3}\right) cos.(\varphi-\theta)); \text{ habebimus:}$$

$$y = \frac{c}{1-e\,cos.u};\ dy = -aed\omega\, sin.u.\sqrt{\frac{a}{c}};$$

$$d\theta = du = \frac{adu}{yy}\sqrt{ac};$$

$$x = \frac{p}{1-q\,cos.v};\ dx = -aqd\omega\, sin.v.\sqrt{\frac{a}{p}};$$

$d\varphi = \frac{a\,d\omega}{xx}\sqrt{ap}$; itemque:

$dp = -2nMax\,d\omega.\sqrt{ap}$:

$dq = na\,d\omega\left(2M\,coſ.\nu + N\,ſin.\nu - \frac{Mq\,ſin.\nu^2}{1-q\,coſ.\nu}\right)\sqrt{ap}$, &

$d\varphi - d\nu = \frac{na\,d\omega}{q}\left(2M\,ſin.\nu - N\,coſ.\nu + \frac{Mq\,ſin.\nu\,coſ.\nu}{1-q\,coſ.\nu}\right)\sqrt{ap}$;

ubi eſt $n = \frac{F}{E}$, cujus valor pro ſingulis planetis ex obſervationibus, quantum quidem id fieri licet, concludi debet.

§. LIV. Cum deinceps ex his formulis locus planetæ R in propria orbita cum ejus diſtantia à ſole fuerit inventus, porro inveſtigetur ad quodvis tempus propoſitum poſitio hujus orbitæ reſpectu plani fixi aſſumti, in quo orbita planetæ perturbantis verſatur, linea ſcilicet nodorum cum inclinatione mutua, ope formularum in §. XLIX exhibitarum; hincque facillime verus locus planetæ in cœlo aſſignabitur. In priori quidem inveſtigatione neglectio partis $\frac{-1+coſ.\psi^2}{xx}$ in valore N nullum errorem creat, quippe quæ per reductionem ad propriam orbitam compenſatur. Sed ob poſitionem $\psi = o$, valor ipſius ζ aliquantillum immutatur, ſed tam parum, niſi inclinatio ſit enormis, ut error præ ipſa quantitate ζ ſit vehementer exiguus. Quoniam igitur ipſa quantitas ζ aliter non intrat in calculum niſi per fractionem minimam n multiplicat errores illi denuo hinc diminuentur, ut tuto pro nihilo æſtimari queant.

SECTIO IV.

Conſiderationes neceſſariæ ad reſolutionem formularum inventarum expediendam.

§. LV. Ex his formulis primo ſine integrationis adjumento variationes horariæ, quas ſingula motus elementa capiunt intervallo unius horæ, ſatis exacte colligi poſſunt, cum enim tantillo tempore omnes variationes ſint quam minimæ, error plane erit imperceptibilis ſi ipſa differentialia tanquam variationes horarias ſpectemus. Cum igitur Sol ſecundum motum medium una hora angulum conficiat $= 2'\,28''$ ſeu accuratius $147\frac{5}{6}''$, ſi differentiali $d\omega$ hunc valorem tribuamus, ut ſit $d\omega = 147\frac{5}{6}''$, ſeu in partibus radii, pro quo unitatem aſſumimus, $d\omega = 0,00071672$, reliqua differentialia dp, dq, $d\varphi - d\nu$, $d\varphi$, dx, $d\pi$, $d.\,tang.\,G$, $d\theta = du$, & dy incrementa horaria iſtorum elementorum exhibebunt, quæ igitur ſine ulla integratione definire licebit, dummodo pro tempore propoſito ipſa hæc elementa fuerint cognita. Neque etiam error erit ſenſibilis, ſi hoc modo variationes diurnas tribuendo ipſi $d\omega$ valorem vicies quater majorem definire vellemus, dummodo neutrius planetæ motus tempore unius diei admodum ſit notabilis.

§. LVI. Si quis hunc laborem ſuſcipere vellet, totum negotium ſine integratione expedire poſſet. Cum enim pro dato quopiam tempore t explorati fuerint valores elementorum ſingulorum, quibus determinatio motus continetur, inventis ſingulorum incrementis horariis, colligemus hinc eorundem elementorum valores ad tempus $t+1$ hora; unde deinceps simili modo eadem elementa ad tempus $t+2$ horis obtinebimus, sicque ad tempora quotcunque horis remota progredi licebit. Interim tamen quia in singulis gradibus error quidam, etiamsi in se spectatus sit insensibilis committitur, is per continuam repetitionem ita accumulabitur, ut tandem satis notabilis evadat. Neque etiam is minoribus hora intervallis assumendis quo pacto quidem labor omnino insuperabilis redderetur, evitari posset, etsi enim singuli errores fierent multo minores tamen ob majorem operationum numerum tandem quoque ad magnitudinem notabilem excrescere possent.

§. LVII. Nisi igitur integratio in subsidium vocetur, sperari omnino nequit, ut singulas motus perturbationes unquam exacte definire valeamus, vel saltem ut tempore quantum vis magno interjecto error non fiat notabilis. Huc quoque accedet, quod priori methodo utentes ad nullum tempus, unde calculum inchoare vellemus, per observationes singulorum elementorum veros valores assignare valeremus, ideoque etiam hi errores sequentes operationes plurimum contaminarent. Quando autem integrationes in genere perficere licuerit, tum per observationes plurimas diversis temporibus institutas, dum eæ cum calculo generali conferentur, veri singulorum elementorum valores colligi poterunt, quibus semel definitis formulæ integratæ perpetuo usum desideratum præstabunt. Quocirca ad perfectam omnium perturbationum cognitionem absolute necessarium est, ut formulæ differentiales inventæ per integrationem ad determinationes finitas reducantur.

§. LVIII. Ac per reductiones quidem hactenus factas jam eximium commodum sumus consecuti, quod formulas differentio-differentiales, ad quas principia Mechanica nos immediate perduxerant, ad formulas simpliciter differentiales revocaverimus, quæ nullis amplius

formulis integralibus sint involutæ, quemadmodum usu venit in iis quas primum (§. XXI) elicueramus, quæ partim ob irrationalitatem partim ob integrales, vix tractari potuissent. Præcipuum autem commodum sine dubio in hoc consistere est censendum, quod omnes formulas ad similitudinem motus regularis, seu notissimis Keppleri regulis conformis explicuerimus, quæ reductio quoque ad usum Astronomicum maxime videtur accommodata. Neque etiam amplius premimur ejusmodi formulis irrationalibus, quarum valores ita sint vagi, ut modo in nihilum abire, modo etiam negativi fieri queant, uti in formulis §. XXI usu venerat.

§. LIX. Verum antequam integrationis negotium suscipiamus, nonnulla moneri est necesse, quibus operationes instituendæ dirigantur. Cum enim integrationem absolutam ac perfectam nullo modo sperare queamus, ad approximationes confugere cogimur, in quo negotio cum multa arbitrio nostro relinquantur, prouti alias atque alias particulas negligere velimus, præcipua cura hoc erit collocanda, ut nihil negligamus, unde error sensibilis resultare posset. Ex circumstantiis igitur judicari opportebit, quid ratione singulorum elementorum negligere liceat: ac primo quidem cum n sit fractio tantopere exigua, quippe qua ratio massæ planetæ perturbantis ad massam solis exprimitur, nullum est dubium, quin ejusmodi terminos, qui per quadratum hujus fractionis altioremve potestatem essent multiplicati, sine hæsitatione rejicere queamus. Ex ipsa autem hujus numeri n parvitate cognoscimus, perturbationes esse quam minimas, quæ adeo omnes evanescerent si esset $n = o$.

§. LX. Deinde etiam si orbitas singulorum planetarum consideremus, earum excentricitates tam parvas deprehendimus, ut in determinatione perturbationum, si non ipsæ, tamen earum quadrata altioresque potestates tuto negligi possint. De Mercurio quidem & Marte hic dubium suboriri posset, quorum planetarum excentricitas est maxima, è contrario eorum massæ tam sunt exiguæ præ massa Solis, ut totæ perturbationes inde oriundæ fere contemni queant. Quamquam autem in reliquorum planetarum motu proprio determinando quadratum excentricitatis perperam negligeretur, unde plerumque effectus satis sensibilis oriri solet, tamensi perturbationes quæ ab ejus actione in alios planetas redundant, investigentur, quoniam effectus quadrati insuper per fractionem n multiplicatur, is plane omnino imperceptibilis reddetur. Hinc quantitates e & q hujusque quantitatem mediam k tamquam tam parvas assumemus, ut in perturbationum investigatione earum quadrata certe ac plerumque etiam eæ ipsæ tuto rejici queant, quod dum evolutionem instituemus, clarius perspicietur.

§. LXI. Neque tamen pro planeta perturbato excentricitas q nimis parva concipi potest sed saltem tam magna, ut mutationes quas ipsi actio reliquorum planetarum inducere valet, præ tota ejus magnitudine tanquam minimæ spectari queant. Cum enim formula progressionem aphelii exprimens divisa sit per q, evidens est si valor ipsius q nimis esset exiguus, ac fortasse interdum plane evanesceret, motum aphelii maximis difficultatibus impeditum iri, ita ut si talis casus in mundo existeret, vix quicquam ex nostris formulis concludi liceret. Verum hic iterum ad insigne nostrum commodum usu venit, ut nullius planetæ excentricitas tam sit exigua, ut inde quicquam nobis sit extrimescendum. Cum enim Veneris excentricitas sit omnium minima, nullum tamen est dubium, quin mutationes, quas ea unquam ab actione reliquorum planetarum subit, præ ejus valore medio quasi evanescant. Quare si valor medius excentricitatis q ponatur $= k$, differentia inter k & q præ k seu fractio $\frac{k-q}{k}$ semper tanquam minima tuto spectari poterit.

§. LXII. Maximam autem in hac investigatione molestiam nobis facessit valor quantitatis $z = \sqrt{(xx + yy - 2xy.\ cos.(\phi - \theta))}$ qui eo magis est variabilis, quo propius amborum planetarum orbitæ ad se invicem accedunt. In calculo quidem hic valor, uti est irrationalis, relinqui non potest, quia integrationes instituendæ nullo modo succederent. Cum enim calculum aliter tractare non liceat, nisi ut omnes perturbationes ad sinus cosinusve certorum angulorum revocentur, omnino necesse est, ut quantitates M & N, quæ hanc quantitatem z involvunt, in ejusmodi series evolvant z, quæ hujusmodi sinus vel cosinus simpliciter contineant. Hoc enim solo modo integratio suscipi posse videtur, neque etiam ulla via patet, quemadmodum calculus ita expediri possit, ut valores integrales adhuc quantitatem z aliasve independentes in se contemplectentur.

§. LXIII. Insignem hic Analyseos, quatenus quidem etiamnunc est exculta, defectum agnoscere cogimur, quod aliter formularum inventarum integralia exhibere non valeamus, nisi per series, quarum singuli termini simplices sinus vel cosinus angulorum $\phi - \theta$, u, ν, $\phi - \pi$, $\theta - \pi$, ex iisque compositorum contineant. Fieri certe posset ut vera integralia vel quantitates ex his complexas neque in hujusmodi serie commode resolubiles continerent, vel etiam alios angulos veluti CVQ vel CQV qui utique si vis principalis ad planetam V tenderet, numerusque n foret prægrandis, primarias partes in calculo essent obtenturi. Atque cum ab his angulis, si n esset numerus valde magnus, totus calculus maximam partem penderet, eo minus dubitare possumus, quin iidem etiam præsenti casu, si calculum accurate expedire liceret, sint ingressuri. Interim tamen plane non patet, quomodo isti anguli per integrationem in calculum invehi queant. Quod si ergo ex hac parte fines analyseos extendere unquam contigerit, tum demum majores fructus pro Astronomia nobis polliceri poterimus.

§. LXIV. Eo igitur sumus redacti ut quantitatem surdam $z = \sqrt{(xx + yy - 2xy\,cos.(\varphi - \theta))}$ in seriem transformemus, ubi quidem eo potissimum est incumbendum, ut ista series quantum fieri potest reddatur convergens, atque secundum sinus vel cosinus certorum angulorum progrediatur. Et quoniam usus postulat convergentiam sive x sit majus, sive minus quam y, terminus tantum $2xy\,cos.(\varphi - \theta)$, qui modo affirmativus, modo in nihilum abire, modo negativus fieri potest, molestiam creat. Binomii ergo partem alteram constituo $xx + yy$, alteram vero $2xy\,cos.(\varphi - \theta)$; statuoque:

$$xx + yy = rr, \ \& \ \frac{2xy}{xx+yy} = s; \text{ ut sit}$$

$$z = \sqrt{(rr - rrs\,cos.(\varphi - \theta))} = r\sqrt{(1 - s\,cos.(\varphi - \theta))},$$

atque hic perspicuum est s semper esse unitate minus nisi sit $x = y$, & eo fieri minus, quo magis distantiæ x & y fuerint inter se inæquales: hinc ergo multo magis pars $s\,cos.(\varphi - \theta)$ minor erit parte 1, prout seriei convergentia postulat.

§. LXV. Quoniam angulus $\varphi - \theta$ tum frequenter occurret, ponamus ad abbreviandum $\varphi - \theta = \eta$, & cum sit $r = \sqrt{(xx + yy)}$ & $s = \frac{2xy}{xx+yy}$, erit $z = r\sqrt{(1 - s\,cos.\eta)}$, ideoque $\frac{1}{z^3} = \frac{1}{r^3}(1 - s\,cos.\eta)^{-\frac{3}{2}}$; unde formulam irrationalem $(1 - s\,cos.\eta)^{-\frac{3}{2}}$ in seriem evolvi oportet. Modo ergo communi adhibito reperiemus:

$$(1 + s\,cos.\eta)^{-\frac{3}{2}} = 1 + \frac{3}{2}s\,cos.\eta + \frac{3.5}{2.4}ss\,cos.\eta^2 + \frac{3.5.7}{2.4.6}s^3 cos.\eta^3 + \frac{3.5.7.9}{2.4.6.8} + s^4 cos.\eta^4 \ \&c.$$

quæ series, dummodo $s\,cos.\eta$ fuerit unitate minus, uti quidem semper usu venit, certe convergit. Interim ta-

men nisi $s\,cos.\eta$ sit valde parvum nimis lente convergit, quam ut aliquot terminis colligendis ejus summa satis exacte obtineri queat. Verum hic quoque commode accidit, ut dum hæc series per sequentes integrationes tractatur, multo promtiorem convergentiam acquirat, quod nisi eveniret, non perspicerem quomodo quæstioni propositæ satisfieri posset; necessitas tunc cogeret investigationi mutationum horariarum unice adhærescere, illasque pro quantumvis magnis temporis intervallis in unam summam colligere.

§. LXVI. Verum cum potestates ipsius $cos.\eta$ in differentiale $d\eta$ ductæ integrari nequeant, nisi prius in cosinus angulorum multiplorum 2η, 3η, &c. convertantur, quoniam differentiale $d\eta$ ad nostrum commune differentiale $d\omega$ reducere licet, eadem conditio postulat, ut hujus $cos.\eta$ potestates in cosinus angulorum multiplorum resolvantur id quod ope noti lemmatis: $cos.\alpha.\ cos.\beta = \frac{1}{2}cos.(\alpha + \beta) + \frac{1}{2}cos.(\alpha - \beta)$ facile præstatur, reperietur enim

$$cos.\eta = cos.\eta,$$
$$cos.\eta^2 = \frac{1}{2}cos.2\eta + \frac{1}{2},$$
$$cos.\eta^3 = \frac{1}{4}cos.3\eta + \frac{3}{4}cos.\eta,$$
$$cos.\eta^4 = \frac{1}{8}cos.4\eta + \frac{4}{8}cos.2\eta + \frac{3}{8},$$
$$cos.\eta^5 = \frac{1}{16}cos.5\eta + \frac{5}{16}cos.3\eta + \frac{10}{16}cos.\eta,$$
$$cos.\eta^6 = \frac{1}{32}cos.6\eta + \frac{6}{32}cos.4\eta + \frac{15}{32}cos.2\eta + \frac{10}{32},$$
$$cos.\eta^7 = \frac{1}{64}cos.7\eta + \frac{7}{64}cos.5\eta + \frac{21}{64}cos.3\eta + \frac{35}{64}cos.\eta,$$
$$cos.\eta^8 = \frac{1}{128}cos.8\eta + \frac{8}{128}cos.6\eta + \frac{28}{128}cos.4\eta + \frac{56}{128}cos.2\eta + \frac{35}{128}, \ \&c.$$

§. LXVII. Quodsi hi valores substituantur, obtinebitur $(1 - s\,cos.\eta)^{-\frac{3}{2}} =$

$$1 + \frac{3}{2}.s\,cos.\eta + \frac{3.5}{2.4}.\frac{1}{2}ss\,cos.2\eta + \frac{3.5.7}{2.4.6}.\frac{1}{4}.s^3 cos.3\eta + \frac{3.5.7.9}{2.4.6.8}.\frac{1}{8}s^4 cos.4\eta,$$
$$+ \frac{3.5}{2.4}.\frac{1}{2}ss + \frac{3.5.7}{2.4.6}.\frac{3}{4}s^3 cos.\eta + \frac{3.5.7.9}{2.4.6.8}.\frac{4}{8}s^4 cos.2\eta,$$
$$+ \frac{3.5.7.9}{2.4.6.8}.\frac{3}{8}s^4, \ \&c.$$

Unde si ponamus $(1 - s \cos. \eta)^{-\frac{3}{2}} = P + Q s \cos. \eta + R ss \cos. 2\eta + S s^3 \cos. 3\eta + T s^4 \cos. 4\eta +$, &c. hi coëfficientes assumti ita definientur ut sit:

$$P = 1 + \tfrac{3.5}{2.4}.\tfrac{1}{2} ss + \tfrac{3.5.7.9}{2.4.6.8}.\tfrac{3}{8} s^4 + \tfrac{3.5.7.9.11.13}{2.4.6.8.10.12}.\tfrac{10}{32} s^6 + \&c.$$
$$Q = \tfrac{3}{2}\left(1 + \tfrac{5.7}{4.6}.\tfrac{3}{4} ss + \tfrac{5.7.9.11}{4.6.8.10}.\tfrac{10}{16} s^4 + \tfrac{5.7.9.11.13.15}{4.6.8.10.12.14}.\tfrac{35}{64} s^6 + \&c.\right)$$
$$R = \tfrac{3.5}{2.4}\left(\tfrac{1}{2} + \tfrac{7.9}{6.8}.\tfrac{4}{8} ss + \tfrac{7.9.11.13}{6.8.10.12}.\tfrac{15}{32} s^4 + \tfrac{7.9.11.13.15.17}{6.8.10.12.14.16}.\tfrac{56}{128} s^6 + \&c.\right)$$
$$S = \tfrac{3.5.7}{2.4.6}\left(\tfrac{1}{4} + \tfrac{9.11}{8.10}.\tfrac{5}{16} ss + \tfrac{9.11.13.15}{8.10.12.14}.\tfrac{21}{64} s^4 + \&c.\right)$$
$$T = \tfrac{3.5.7.9}{2.4.6.8}\left(\tfrac{1}{8} + \tfrac{11.13}{10.12}.\tfrac{6}{32} ss + \&c.\right)$$
$$V = \tfrac{3.5.7.9.11}{2.4.6.8.10}\left(\tfrac{1}{16} + \&c.\right)$$

§. LXVIII. Hinc igitur habebimus:

$$\tfrac{1}{z^3} = \tfrac{1}{r^3}(P + Q s \cos. \eta + R ss \cos. 2\eta + S s^3 \cos. 3\eta + T s^4 \cos. 4\eta + \&c.)$$

atque series pro P, Q, R, S, &c., inventæ satis convergunt, ut hi valores per consuetas methodos approximando erui queant. Sequenti modo autem in alias formas transfundi possunt, quæ aptiores videntur

$$P = 1 + \tfrac{3.5}{4.4} ss + \tfrac{3.5.7.9}{4.4.8.8} s^4 + \tfrac{3.5.7.9.11.13}{4.4.8.8.12.12} s^6 + \&c.$$
$$\tfrac{1}{2} Q = \tfrac{3}{4}\left(1 + \tfrac{5.7}{4.8} ss + \tfrac{5.7.9.11}{4.8.8.12}. s^4 + \tfrac{5.7.9.11.13.15}{4.8.8.12.12.16} s^6 + \&c.\right)$$
$$\tfrac{1}{2} R = \tfrac{3.5}{4.4}\left(\tfrac{1}{2} + \tfrac{7.9}{8.8}.\tfrac{2}{3} ss + \tfrac{7.9.11.13}{8.8.12.12}.\tfrac{3}{4} s^4 + \tfrac{7.9.11.13.15.17}{8.8.12.12.16.16}.\tfrac{4}{5} s^6 + \&c.\right)$$
$$\tfrac{1}{2} S = \tfrac{3.5.7}{4.4.8}\left(\tfrac{1}{3} + \tfrac{9.11}{8.12}.\tfrac{2}{4} ss + \tfrac{9.11.13.15}{8.12.12.16}.\tfrac{3}{5} s^4 + \tfrac{9.11.13.15.17.19}{8.12.12.16.16.20}.\tfrac{4}{6} s^6 + \&c.\right)$$
$$\tfrac{1}{2} T = \tfrac{3.5.7.9}{4.4.8.8}\left(\tfrac{1.2}{3.4} + \tfrac{11.13}{12.12}.\tfrac{2.3}{4.5} ss + \tfrac{11.13.15.17}{12.12.16.16}.\tfrac{3.4}{5.6} s^4 + \tfrac{11.13.15.17.19.21}{12.12.16.16.20.20}.\tfrac{4.5}{6.7} s^6 + \&c.\right)$$
$$\tfrac{1}{2} V = \tfrac{3.5.7.9.11}{4.4.8.8.12}\left(\tfrac{1.2}{4.5} + \tfrac{13.15}{12.16}.\tfrac{2.3}{5.6}. ss + \tfrac{13.15.17.19}{12.16.16.20}.\tfrac{3.4}{6.7} s^4 + \&c.\right)$$
$$\tfrac{1}{2} X = \tfrac{3.5.7.9.11.13}{4.4.8.8.12.12}\left(\tfrac{1.2.3}{4.5.6} + \tfrac{15.17}{16.16}.\tfrac{2.3.4}{5.6.7} ss + \tfrac{15.17.19.21}{16.16.20.20}.\tfrac{3.4.5}{6.7.8} s^4 + \&c.\right)$$

§. LXIX. Nunc perspicuum est singulas has series in infinitum continuatas fieri geometricas, denominatore existente ss; quare eæ si per $1 - ss$ multiplicentur, multo magis convergentes reddentur. Hoc modo consequemur:

$$P(1-ss) = 1 - \tfrac{1.1}{4.4} ss - \tfrac{1.1.3.5}{4.4.8.8} s^4 - \tfrac{1.1.3.5.7.9}{4.4.8.8.12.12} s^6 - \&c.$$
$$\tfrac{1}{2} Q(1-ss) = \tfrac{3}{4}\left(1 + \tfrac{1.3}{4.8} ss + \tfrac{1.3.5.7}{4.8.8.12} s^4 + \tfrac{1.3.5.7.9.11}{4.8.8.12.12.16} s^6 + \&c.\right)$$
$$\tfrac{1}{2} R(1-ss) = \tfrac{3.5}{4.4}\left(\tfrac{1}{2} + \tfrac{3.5}{8.8}.\tfrac{2}{3} ss + \tfrac{3.5.7.9}{8.8.12.12}.\tfrac{3}{4} s^4 + \tfrac{3.5.7.9.11.13}{8.8.12.12.16.16}.\tfrac{4}{5} s^6 + \&c.\right)$$
$$\tfrac{1}{2} S(1-ss) = \tfrac{3.5.7}{4.4.8}\left(\tfrac{1}{3} + \tfrac{5.7}{8.12}.\tfrac{2}{4} ss + \tfrac{5.7.9.11}{8.12.12.16}.\tfrac{3}{5} s^4 + \tfrac{5.7.9.11.13.15}{8.12.12.16.16.20}.\tfrac{4}{6} s^6 + \&c.\right)$$
$$\tfrac{1}{2} T(1-ss) = \tfrac{3.5.7.9}{4.4.8.8}\left(\tfrac{1.2}{3.4} + \tfrac{7.9}{12.12}.\tfrac{2.3}{4.5} ss + \tfrac{7.9.11.13}{12.12.16.16}.\tfrac{3.4}{5.6} s^4 + \&c.\right)$$
$$\tfrac{1}{2} V(1-ss) = \tfrac{3.5.7.9.11}{4.4.8.8.12}\left(\tfrac{1.2}{4.5} + \tfrac{9.11}{12.16}.\tfrac{2.3}{5.6} ss + \tfrac{9.11.13.15}{12.16.16.20}.\tfrac{3.4}{6.7} s^4 + \&c.\right)$$
$$\tfrac{1}{2} X(1-ss) = \tfrac{3.5.7.9.11.13}{4.4.8.8.12.12}\left(\tfrac{1.2.3}{4.5.6} + \tfrac{11.13}{16.16}.\tfrac{2.3.4}{5.6.7} ss + \&c.\right)$$
$$\tfrac{1}{2} Y(1-ss) = \tfrac{3.5.7.9.11.13.15}{4.4.8.8.12.12.16}\left(\tfrac{1.2.3}{5.6.7} + \tfrac{13.15}{16.20}.\tfrac{2.3.4}{6.7.8} ss + \&c.\right)$$
$$\tfrac{1}{2} Z(1-ss) = \tfrac{3.5.7.9.11.13.15.17}{4.4.8.8.12.12.16.16}\left(\tfrac{1.2.3.4}{5.6.7.8} + \tfrac{15.17}{20.20}.\tfrac{2.3.4.5}{6.7.8.9} ss + \&c.\right)$$

&c.

§. LXX. At vero non opus est ut singuli isti valores evolvantur, sufficit enim duos priores collegisse, ex quibus reliqui per sequentes formulas facile formari poterunt.

$$R = \frac{4Q - 3.2P}{ss};\quad S = \frac{8R - sQ}{3ss};$$
$$T = \frac{12S - 7R}{5ss};\quad V = \frac{16T - 9S}{7ss}, \ \&c.$$

Quin etiam ex prima derivari potest secunda, si integrationem in subsidium vocare velimus, est enim

$$Q = 2P - \tfrac{1}{ss}\int P s\, ds.$$

Sin autem quæratur valor primæ seriei, dico eum per integrationem inveniri posse, sumendo s constans, & introducendo variabilem z foreque:

$$P(1-ss) = \frac{\int \frac{dz}{\sqrt{z}}\left(\frac{1-zz}{ss-zz}\right)^{\frac{1}{4}}}{\int \frac{dz}{\sqrt{z}}\left(\frac{1}{ss-zz}\right)^{\frac{1}{4}}}.$$

Si post integrationem ponatur $z = ss$ utrumque autem integrale ita capi sumo, ut evanescat posito $z = 0$, quo casu quidem ſit denominator $\int \frac{dz}{\sqrt{z}} \left(\frac{1}{ss-zz}\right)^{\frac{1}{4}} = \frac{\pi}{\sqrt{2}}$, denotante hic π peripheriam, cujus diameter $= 1$.

§. LXXI. Si ergo eſſet $s = 1$, quo caſu hæ ſeries minime convergerent, foret ob numeratorem noſtræ expreſſionis integralis $= \int \frac{dz}{\sqrt{z}} = 2\sqrt{z} = 2\sqrt{s} = 2$, prima ſeries $P(1-ss) = \frac{2\sqrt{2}}{\pi} = 0{,}9003162$, quæ ſumma per conſuetas approximandi methodos non difficulter erueretur, unde patet ejus ſummam multo facilius obtineri ſi valor ipſius s uti ſemper evenit ſit unitate minor. In genere autem erit

$$P(1-ss) = \frac{\sqrt{2}}{\pi} \int \frac{dz}{\sqrt{z}} \left(\frac{1-zz}{ss-zz}\right)^{\frac{1}{4}}$$

poſito poſt integrationem $z = s$: quæ expreſſio eo magis eſt notatu digna, quod ejus veritas inveſtiganti non tam cito occurrit. Interim tamen expediet quovis caſu, quo valor ipſius s datur, aliquot terminis harum ſerierum actu addendis earum ſummas prope veras colligere, quod negotium eo promtius ſuccedet quo minor fuerit numerus s. Hoc igitur modo inventis quantitatibus P, Q, R, S, &c. erit

$$\frac{1}{z^3} = \frac{1}{r^3}(P + Qs\,cos.\eta + Rss\,cos.2\eta + Ss^3\,cos.3\eta + Ts^4\,cos.4\eta + \&c.)$$

brevitatis gratia autem poſuimus:

$$r = \sqrt{(xx+yy)}, \quad \& \; s = \frac{2xy}{xx+yy}.$$

§. LXXII. Pro faciliori autem quantitatum P, Q, R, S, &c. computo conveniet ſerierum illarum coëfficientes infractiones decimales transformari, unde obtinebitur, ſubſcriptis logarithmis

$P(1-ss)$	$\frac{1}{2}Q(1-ss)$	$\frac{1}{2}R(1-ss)$	$\frac{1}{2}S(1-ss)$
$+1{,}000000$	$+0{,}750000$	$+0{,}468750$	$+0{,}273437$
$0{,}0000000$	$9{,}8750613$	$9{,}6709413$	$9{,}4368580$
$-0{,}062500.ss$	$+0{,}070313.ss$	$+0{,}146490.ss$	$+0{,}149536.ss$
$8{,}7958800$	$8{,}8470326$	$9{,}1657913$	$9{,}1747461$
$-0{,}014648.s^4$	$+0{,}025635.s^4$	$+0{,}072098.s^4$	$+0{,}092525.s^4$
$8{,}1657913$	$8{,}4088294$	$8{,}8579220$	$8{,}9662614$
$-0{,}006409.s^6$	$+0{,}013218.s^6$	$+0{,}044983.s^6$	$+0{,}061647.s^6$
$7{,}8067694$	$8{,}1211634$	$8{,}6530468$	$8{,}7969035$
$-0{,}003580.s^8$	$+0{,}008055.s^8$	$+0{,}028527.s^8$	$+0{,}045167.s^8$
$7{,}5538655$	$7{,}9060481$	$8{,}4552557$	$8{,}6548281$
$-0{,}002282.s^{10}$	$+0{,}005420.s^{10}$	$+0{,}020325.s^{10}$	$+0{,}034088.s^{10}$
$7{,}3583457$	$7{,}7340094$	$8{,}3080408$	$8{,}5325953$
$-0{,}001581.s^{12}$	$+0{,}003896.s^{12}$	$+0{,}015217.s^{12}$	$+0{,}026631.s^{12}$
$7{,}1988962$	$7{,}5905873$	$8{,}1823473$	$8{,}4253853$
$-0{,}001159.s^{14}$	$+0{,}002935.s^{14}$	$+0{,}011821.s^{14}$	
$7{,}0642480$	$7{,}4675831$	$8{,}0726486$	
$-0{,}000887.s^{16}$	$+0{,}002290.s^{16}$		
$6{,}9477098$	$7{,}3598899$		
$-0{,}000700.s^{12}$			
$6{,}8449800$			

SECTIO V.

Evolutio formularum differentialium in series secundum sinus cosinusve angulorum simpliciter progredientes.

§. LXXIII. QUONIAM defectus Analyseos necessitatem nobis imponit omnia integralia, quibus opus est, per series secundum sinus cosinusve angulorum simpliciter progredientes exprimendi, similem formam singulis formulis differentialibus induci oportet. Cum igitur primum pro planeta perturbante sit $y = \frac{c}{1 - e\,cos.\,u}$, erit terminos qui quadratum excentricitatis e altioresque potestates involvunt omittendo:

$$y = c\,(1 + e\,cos.\,u)\,;\ \&\ \frac{1}{yy} = \frac{1}{cc}\,(1 - 2\,e\,cos.\,u).$$

Deinde cum sit $d\theta = du = \frac{a\,d\omega\sqrt{ac}}{yy}$, habebimus

$$d\theta = du = \frac{a\sqrt{a}}{c\sqrt{c}}\,d\omega\,(1 - 2\,e\,cos.\,u).$$

Quia enim hujus motus ratio tantum in perturbationes ingreditur non opus est has formulas accuratius evolvere.

§. LXXIV. Simili modo cum pro planeta perturbato sit $x = \frac{p}{1 - q\,cos.\,v}$, erit:

$$\frac{1}{xx} = \frac{1}{pp}\,(1 + \tfrac{1}{2}\,qq - 2\,q\,cos.\,v + \tfrac{1}{2}\,qq\,cos.\,2\,v),$$

in qua nihil est neglectum. Hac igitur erit utendum, ubi non proprie quæstio circa perturbationes versatur, ex hac enim formula, etiamsi nulla contingeret perturbatio, motus planetæ regularis deduci deberet. Id quod evenit in definiendo elemento motus veri $d\varphi$, quod partim sequitur leges Kepplerianas, partim vero minimis inæqualitatibus perturbatur. Illo respectu verus ejus valor capi debebit, qui erit

$$d\varphi = \frac{a\sqrt{a}}{p\sqrt{p}}\,d\omega\,(1 + \tfrac{1}{2}\,qq - 2\,q\,cos.\,v + \tfrac{1}{2}\,qq\,cos.\,2\,v).$$

Quatenus vero idem valor $d\varphi$ ad solas perturbationes investigandas adhibetur, sufficiet tam pro semiparametro p quam pro excentricitate q valores medios constantes b & k usurpare, atque adeo quadratum ipsius k rejicere, ita ut pro hoc usu habeamus:

$$d\varphi = \frac{a\sqrt{a}}{b\sqrt{b}}\,d\omega\,(1 - 2\,k\,cos.\,v).$$

§. LXXV. Quia formulæ $\frac{a\sqrt{a}}{b\sqrt{b}}$ & $\frac{a\sqrt{a}}{c\sqrt{c}}$ calculum maxime ingrediuntur ponamus brevitatis gratia:

$$\frac{a\sqrt{a}}{b\sqrt{b}} = i,\ \&\ \frac{a\sqrt{a}}{c\sqrt{c}} = m\,;$$

sicque erit pro formulis perturbationes implicantibus:

$$d\theta = du = m\,d\omega\,(1 - 2\,e\,cos.\,u),\ \&$$
$$d\varphi = i\,d\omega\,(1 - 2\,k\,cos.\,v).$$

Unde cum angulus $\varphi - \theta$ nonnisi in hoc negotio occurrat, quoniam posuimus $\varphi - \theta = \eta$, & differentiale anguli η hinc ad differentiale $d\omega$ revocatur. Fiet enim

$$d\eta = (i - m)\,d\omega - 2\,i\,k\,d\omega\,cos.\,v + 2\,m\,e\,d\omega\,cos.\,u,$$

ubi notandum est esse:

$i : 1$ ut motus medius planetæ perturbati ad motum medium Solis vel Terræ;

$m : 1$ ut motus medius planetæ perturbantis ad motum medium Solis vel Terræ.

Quare si perturbationes in motu terræ investigentur erit $i = 1$. Verum pro toto motu planetæ perturbati erit:

$$d\varphi = \frac{b\sqrt{b}}{p\sqrt{p}}\, i\, d\omega\, (1 + \tfrac{1}{2}qq - 2q\, cos.\, \nu + \tfrac{1}{2}qq\, cos.\, 2\nu).$$

§. LXXVI. Nunc agrediamur valorem $\frac{1}{r^3}$, qui quoniam tantum in perturbationibus inest, minores ejus particulas negligere licebit. Quia igitur pro eo posuimus $\sqrt{(xx + yy)} = r$, erit $\frac{1}{r^3} = (xx + yy)^{-\frac{3}{2}}$; $(bb(1 + 2k\, cos.\, \nu) + cc(1 + 2e\, cos.\, u))^{-\frac{3}{2}}$, ideoque

$$\frac{1}{r^3} = \frac{1}{(bb + cc)^{\frac{3}{2}}} - \frac{3bbk\, cos.\, \nu - 3cce\, cos.\, u}{(bb + cc)^{\frac{5}{2}}}.$$

Ponamus brevitatis gratia $\sqrt{(bb + cc)} = f$, ut sit:

$$\frac{1}{r^3} = \frac{1}{f^3} - \frac{3bbk\, cos.\, \nu}{f^5} - \frac{3cce\, cos.\, u}{f^5}.$$

Deinde cum itidem posuerimus $s = \frac{2xy}{xx + yy}$; fiet per easdem positiones:

$$s = \frac{2bc(1 + k\, cos.\, \nu)(1 + e\, cos.\, u)}{ff + 2bbk\, cos.\, \nu + 2cce\, cos.\, u},$$

quæ expressio pari modo evoluta evadet:

$$s = \frac{2bc}{ff} + \frac{2bc(cc - bb)k\, cos.\, \nu}{f^4} - \frac{2bc(cc - bb)e\, cos.\, u}{f^4}.$$

§. LXXVII. Quo etiam hanc formulam commodiorem reddamus, statuamus:

$\frac{2bc}{ff} = \frac{2bc}{bb + cc} = \mu$ & $\frac{cc - bb}{ff} = \frac{cc - bb}{bb + cc} = \nu$, ut $\mu\mu + \nu\nu = 1$; erit $\frac{bb}{ff} = \frac{1 - \nu}{2}$ & $\frac{cc}{ff} = \frac{1 + \nu}{2}$. His autem valoribus introductis, habebimus

$\frac{1}{r^3} = \frac{1}{f^3}(1 - \frac{3}{2}(1 - \nu)\, k\, cos.\, \nu - \frac{3}{2}(1 + \nu)\, e\, cos.\, u)$, &

$s = \mu + \mu\nu k\, cos.\, \nu - \mu\nu e\, cos.\, u = \mu(1 + \nu k\, cos.\, \nu - \nu e\, cos.\, u)$,

unde pro litteris P, Q, R, &c. fit

$$s^2 = \mu^2(1 + 2\nu k\, cos.\, \nu - 2\nu e\, cos.\, u);$$
$$s^3 = \mu^3(1 + 3\nu k\, cos.\, \nu - 3\nu e\, cos.\, u);$$
$$s^4 = \mu^4(1 + 4\nu k\, cos.\, \nu - 4\nu e\, cos.\, u);$$
$$s^5 = \mu^5(1 + 5\nu k\, cos.\, \nu - 5\nu e\, cos.\, u);$$
&c.

tam vero porro:

$1 - ss = \nu\nu - 2\mu^2\nu k\, cos.\, \nu + 2\mu^2\nu e\, cos.\, u$, hincque

$$\frac{1}{1 - ss} = \frac{1}{\nu\nu} + \frac{2\mu^2 k\, cos.\, u}{\nu^3} - \frac{2\mu^2 e\, cos.\, u}{\nu^3}.$$

§. LXXVIII. Si his valoribus adhibitis formulas §. LXXII evolutas ad calculum revocemus, & quantitates P, Q, R, S, &c. investigemus, eæ sequenti modo expressæ reperientur:

$$\frac{P}{r^3} = \frac{1}{f^3}(g + hk\, cos.\, \nu + le\, cos.\, u);$$
$$\frac{Qs}{r^3} = \frac{1}{f^3}(g' + h'k\, cos.\, \nu + l'e\, cos.\, u);$$
$$\frac{Rss}{r^3} = \frac{1}{f^3}(g'' + h''k\, cos.\, \nu + l''e\, cos.\, u);$$
$$\frac{Ss^3}{r^3} = \frac{1}{f^3}(g''' + h'''k\, cos.\, \nu + l'''e\, cos.\, u);$$
&c.

quovis enim casu valores idonei pro g, h, l, g', h', l', &c. per merum calculum numericum reperientur, quos ergo numeros tanquam cognitos spectare licebit. Hinc itaque elicimus $\frac{1}{z^3} = \frac{1}{f^3} \times$

$$\left\{\begin{array}{l} +g \quad + g'\, cos.\, \eta \quad + g''\, cos.\, 2\eta \quad + g'''\, cos.\, 3\eta \\ +hk\, cos.\, \nu + \tfrac{1}{2}h'k\, cos.(\eta - \nu) + \tfrac{1}{2}h''k\, cos.(2\eta - \nu) + \tfrac{1}{2}h'''k\, cos.(3\eta - \nu), \\ \quad + \tfrac{1}{2}h'k\, cos.(\eta + \nu) + \tfrac{1}{2}h''k\, cos.(2\eta + \nu) + \tfrac{1}{2}h'''k\, cos.(3\eta + \nu), \\ +le\, cos.\, u + \tfrac{1}{2}l'e\, cos.(\eta - u) + \tfrac{1}{2}l''e\, cos.(2\eta - u) + \tfrac{1}{2}l'''e\, cos.(3\eta - u), \\ \quad + \tfrac{1}{2}l'e\, cos.(\eta + u) + \tfrac{1}{2}l''e\, cos.(2\eta + u) + \tfrac{1}{2}l'''e\, cos.(3\eta + u), \\ \quad\quad \&c. \end{array}\right\}$$

§. LXXIX. Nunc paulatim ad valores litterarum M & N definiendos procedere possumus. Cum enim ſit $M = \frac{y\,\text{ſin.}\,\eta}{z^3} - \frac{\text{ſin.}\,\eta}{yy}$, primo habebimus

$$\frac{y\,\text{ſin.}\,\eta}{z^3} = \frac{1}{cc}\left(\text{ſin.}\,\eta - e\,\text{ſin.}\,(\eta - u) - e\,\text{ſin.}\,(\eta + u)\right).$$

Deinde ob $y\,\text{ſin.}\,\eta = c\,(\text{ſin.}\,\eta + \tfrac{1}{2}e\,\text{ſin.}\,(\eta - u) + \tfrac{1}{2}e\,\text{ſin.}\,(\eta + u))$, erit $\frac{y\,\text{ſin.}\,\eta}{z^3} = \frac{c}{f^3} \times$

$$\left\{\begin{array}{llll}
+g\,\text{ſin.}\,\eta & +\tfrac{1}{2}g'\,\text{ſin.}\,2\eta & +\tfrac{1}{2}g''\,\text{ſin.}\,3\eta & +\tfrac{1}{2}g'''\,\text{ſin.}\,4\eta \\
+\tfrac{1}{2}ge\,\text{ſin.}\,(\eta-u) & +\tfrac{1}{4}g'e\,\text{ſin.}\,(2\eta-u) & -\tfrac{1}{2}g''\,\text{ſin.}\,\eta & -\tfrac{1}{2}g'''\,\text{ſin.}\,2\eta \\
+\tfrac{1}{2}ge\,\text{ſin.}\,(\eta+u) & +\tfrac{1}{4}g'e\,\text{ſin.}\,(2\eta+u) & -\tfrac{1}{4}g''e\,\text{ſin.}\,(\eta+u) & -\tfrac{1}{4}g'''e\,\text{ſin.}\,(2\eta+u) \\
+\tfrac{1}{2}hk\,\text{ſin.}\,(\eta-v) & +\tfrac{1}{4}h'k\,\text{ſin.}\,(2\eta-v) & +\tfrac{1}{4}g''e\,\text{ſin.}\,(3\eta-u) & +\tfrac{1}{4}g'''e\,\text{ſin.}\,(4\eta-u) \\
+\tfrac{1}{2}hk\,\text{ſin.}\,(\eta+v) & +\tfrac{1}{4}h'k\,\text{ſin.}\,(2\eta+v) & -\tfrac{1}{4}g''e\,\text{ſin.}\,(\eta-u) & -\tfrac{1}{4}g'''e\,\text{ſin.}\,(2\eta-u) \\
+\tfrac{1}{2}le\,\text{ſin.}\,(\eta-u) & +\tfrac{1}{4}l'e\,\text{ſin.}\,(2\eta-u) & +\tfrac{1}{4}g''e\,\text{ſin.}\,(3\eta+u) & +\tfrac{1}{4}g'''e\,\text{ſin.}\,(4\eta+u) \\
+\tfrac{1}{2}le\,\text{ſin.}\,(\eta+u) & +\tfrac{1}{4}l'e\,\text{ſin.}\,(2\eta+u) & -\tfrac{1}{4}h''k\,\text{ſin.}\,(\eta-v) & -\tfrac{1}{4}h'''k\,\text{ſin.}\,(2\eta-v) \\
 & & +\tfrac{1}{4}h''k\,\text{ſin.}\,(3\eta-v) & +\tfrac{1}{4}h'''k\,\text{ſin.}\,(4\eta-v) \\
 & & -\tfrac{1}{4}h''k\,\text{ſin.}\,(\eta+v) & -\tfrac{1}{4}h'''k\,\text{ſin.}\,(2\eta+v) \\
 & & +\tfrac{1}{4}h''k\,\text{ſin.}\,(3\eta+v) & +\tfrac{1}{4}h'''k\,\text{ſin.}\,(4\eta+v) \\
 & & -\tfrac{1}{4}l''e\,\text{ſin.}\,(\eta-u) & -\tfrac{1}{4}l'''e\,\text{ſin.}\,(2\eta-u) \\
 & & +\tfrac{1}{4}l''e\,\text{ſin.}\,(3\eta-u) & +\tfrac{1}{4}l'''e\,\text{ſin.}\,(4\eta-u) \\
 & & -\tfrac{1}{4}l''e\,\text{ſin.}\,(\eta+u) & -\tfrac{1}{4}l'''e\,\text{ſin.}\,(2\eta+u) \\
 & & +\tfrac{1}{4}l''e\,\text{ſin.}\,(3\eta+u) & +\tfrac{1}{4}l'''e\,\text{ſin.}\,(4\eta+u)
\end{array}\right\}$$

&c.

§. LXXX. Omittendis jam terminis, qui pluſquam triplum anguli η involvunt colligemus valorem ipſius $M = +\frac{c}{f^3}\times$

$$\left\{\begin{array}{llll}
+(g-\tfrac{1}{2}g'')\,\text{ſin.}\,\eta & +\tfrac{1}{2}e(g-\tfrac{1}{2}g'')\,\text{ſin.}\,(\eta-u) & +e(g'-g''')\,\text{ſin.}\,(2\eta-u) & +\tfrac{1}{4}e(g''-g'''')\,\text{ſin.}\,(3\eta-u) \\
 & +\tfrac{1}{2}e(g-\tfrac{1}{2}g'')\,\text{ſin.}\,(\eta+u) & +e(g'-g''')\,\text{ſin.}\,(2\eta+u) & +\tfrac{1}{4}e(g''-g'''')\,\text{ſin.}\,(3\eta+u) \\
+\tfrac{1}{2}(g'-g''')\,\text{ſin.}\,2\eta & +\tfrac{1}{2}e(l-\tfrac{1}{2}l'')\,\text{ſin.}\,(\eta-u) & +e(l'-l''')\,\text{ſin.}\,(2\eta-u) & +\tfrac{1}{4}e(l''-l'''')\,\text{ſin.}\,(3\eta-u) \\
 & +\tfrac{1}{2}e(l-\tfrac{1}{2}l'')\,\text{ſin.}\,(\eta+u) & +e(l'-l''')\,\text{ſin.}\,(2\eta+u) & +\tfrac{1}{4}e(l''-l'''')\,\text{ſin.}\,(3\eta+u) \\
+\tfrac{1}{2}(g''-g''')\,\text{ſin.}\,3\eta & +\tfrac{1}{2}k(h-\tfrac{1}{2}h'')\,\text{ſin.}\,(\eta-v) & +k(h'-h''')\,\text{ſin.}\,(2\eta-v) & +\tfrac{1}{4}k(h''-h'''')\,\text{ſin.}\,(3\eta-v) \\
\&c. & +\tfrac{1}{2}k(h-\tfrac{1}{2}h'')\,\text{ſin.}\,(\eta+v) & +k(h'-h''')\,\text{ſin.}\,(2\eta-v) & +\tfrac{1}{4}k(h''-h'''')\,\text{ſin.}\,(3\eta+v)
\end{array}\right\}$$

$$-\frac{1}{cc}\left(\text{ſin.}\,\eta - e\,\text{ſin.}\,(\eta-u) - e\,\text{ſin.}\,(\eta+u)\right).$$

&c.

In qua expreſſione lex eſt manifeſta, cujus ope plures termini, ſi quis laborem ſuſcipere velit, formari poſ-

ſunt. Si quidem numerus $\mu = \frac{2bc}{bb+cc}$ eſt valde parvus, quod evenit, ſi quantitates b & c multum à ratione æqualitatis recedunt, vix ultra litteras g, h, l una virgula notatas progredi eſt opus; etſi autem eæ quantitates propius ad æqualitatem accedunt, tamen calculum vix ultra binas virgulas continuari eſt opus, quia per integrationem hæ ſeries admodum redduntur convergentes. Ob eandem cauſam multo magis terminos per ee, kk & ek affectos rejicere licuerat, præterquam quod excentricitatem utramque e & k valde parvam aſſuminius.

§. LXXXI. Deinde cum ſit $N = \frac{x - y\,\text{coſ.}\,\eta}{z^3} + \frac{\text{coſ.}\,\eta}{yy}$ habebimus primo

$$\frac{\text{coſ.}\,\eta}{yy} = \frac{1}{cc}\left(\text{coſ.}\,\eta - e\,\text{coſ.}\,(\eta-u) - e\,\text{coſ.}\,(\eta+u)\right),$$

porro vero, ob $x = b\,(1 + k\,\text{coſ.}\,v)$, &
$y\,\text{coſ.}\,\eta = c\,(\text{coſ.}\,\eta + \tfrac{1}{2}e\,\text{coſ.}\,(\eta-u) + \tfrac{1}{2}e\,\text{coſ.}\,(\eta+u))$,
erit valor ipſius $N = \frac{1}{cc}(\text{coſ.}\,\eta - e\,\text{coſ.}\,(\eta-u) - e\,\text{coſ.}\,(\eta+u))$.

$$+\frac{b}{f^3}\left\{\begin{array}{lll}
+g & +g'\,\text{coſ.}\,\eta & +g''\,\text{coſ.}\,2\eta \\
+k(g+h)\,\text{coſ.}\,v & +\tfrac{1}{2}k(g'+h')\,\text{coſ.}\,(\eta-v) & +\tfrac{1}{2}k(g''+h'')\,\text{coſ.}\,(2\eta-v) \\
+le\,\text{coſ.}\,u & +\tfrac{1}{2}k(g'+h')\,\text{coſ.}\,(\eta+v) & +\tfrac{1}{2}k(g''+h'')\,\text{coſ.}\,(2\eta+v) \\
 & +\tfrac{1}{2}l'e\,\text{coſ.}\,(\eta-u) & +\tfrac{1}{2}l''e\,\text{coſ.}\,(\eta-u) \\
 & +\tfrac{1}{2}l'e\,\text{coſ.}\,(\eta+u) & +\tfrac{1}{2}l''e\,\text{coſ.}\,(\eta+u)
\end{array}\right\}$$

$$-\frac{c}{f^3}\left\{\begin{array}{lll}
 & +(g+\tfrac{1}{2}g'')\,\text{coſ.}\,\eta & +\tfrac{1}{2}(g'+g''')\,\text{coſ.}\,2\eta \\
+\tfrac{1}{2}g' & +\tfrac{1}{2}k(h+\tfrac{1}{2}h'')\,\text{coſ.}\,(\eta-v) & +\tfrac{1}{4}k(h'+h''')\,\text{coſ.}\,(2\eta-v) \\
 & +\tfrac{1}{2}k(h+\tfrac{1}{2}h'')\,\text{coſ.}\,(\eta+v) & +\tfrac{1}{4}k(h'+h''')\,\text{coſ.}\,(2\eta+v) \\
+\tfrac{1}{2}h'k\,\text{coſ.}\,v & +\tfrac{1}{2}e(g+\tfrac{1}{2}g'')\,\text{coſ.}\,(\eta-u) & +\tfrac{1}{4}e(g'+g''')\,\text{coſ.}\,(2\eta-u) \\
 & +\tfrac{1}{2}e(g+\tfrac{1}{2}g'')\,\text{coſ.}\,(\eta+u) & +\tfrac{1}{4}e(g'+g''')\,\text{coſ.}\,(2\eta+u) \\
+\tfrac{1}{2}e(g'+l')\,\text{coſ.}\,u & +\tfrac{1}{2}e(l+\tfrac{1}{2}l'')\,\text{coſ.}\,(\eta-u) & +\tfrac{1}{4}e(l'+l''')\,\text{coſ.}\,(2\eta-u) \\
 & +\tfrac{1}{2}e(l+\tfrac{1}{2}l'')\,\text{coſ.}\,(\eta+u) & +\tfrac{1}{4}e(l'+l''')\,\text{coſ.}\,(2\eta+u)
\end{array}\right\}$$

§. LXXXII. Excentricitas planetæ perturbantis e has formulas imprimis tantopere reddit prolixas, quæ ſi evanesceret, hi valores commodius & succinctius ita exprimerentur

$$M = -\tfrac{1}{ii}\,\mathit{sin}.\eta + \tfrac{c}{f^3}\left\{\begin{matrix} +(g-\tfrac{1}{2}g'')\,\mathit{sin}.\eta + \tfrac{1}{2}k(h-\tfrac{1}{2}h'').\left\{\begin{matrix}\mathit{sin}.(\eta-\nu)\\ \mathit{sin}.(\eta+\nu)\end{matrix}\right. \\ +\tfrac{1}{2}(g'-g''')\,\mathit{sin}.2\eta + \tfrac{1}{4}k(h'-h''').\left\{\begin{matrix}\mathit{sin}.(2\eta-\nu)\\ \mathit{sin}.(2\eta+\nu)\end{matrix}\right.\end{matrix}\right\}$$

Deinde vero est:

$$N = \tfrac{1}{ii}\,\mathit{cos}.\eta + \tfrac{b}{f^3}\left\{\begin{matrix} +g & +k(g+h)\,\mathit{cos}.\nu \\ +g'\,\mathit{cos}.\eta & +\tfrac{1}{2}k(g'+h').\left\{\begin{matrix}\mathit{cos}.(\eta-\nu)\\ \mathit{cos}.(\eta+\nu)\end{matrix}\right. \\ +g''\,\mathit{cos}.2\eta & +\tfrac{1}{2}k(g''+h'').\left\{\begin{matrix}\mathit{cos}.(2\eta-\nu)\\ \mathit{cos}.(2\eta+\nu)\end{matrix}\right.\end{matrix}\right\}$$

$$-\tfrac{c}{f^3}\left\{\begin{matrix} +\tfrac{1}{2}g' & +\tfrac{1}{2}h'k\,\mathit{cos}.\nu \\ +(g+\tfrac{1}{2}g'')\,\mathit{cos}.\eta & +\tfrac{1}{2}k(h+\tfrac{1}{2}h'').\left\{\begin{matrix}\mathit{cos}.(\eta-\nu)\\ \mathit{cos}.(\eta+\nu)\end{matrix}\right. \\ +\tfrac{1}{2}(g+g''')\,\mathit{cos}.2\eta & +\tfrac{1}{4}k(h'+h''').\left\{\begin{matrix}\mathit{cos}.2\eta-\nu)\\ \mathit{cos}.2\eta+\nu)\end{matrix}\right.\end{matrix}\right\}$$

Attendenti autem mox patebit hanc excentricitatem e sine errore sensibili negligi posse, unde his postremis formulis utemur.

§. LXXXIII. His jam valoribus pro M & N inventis ipsa differentialia quibus perturbationes continentur ad formam desideratam reducere poterimus. Quod igitur primum ad variabilitatem semiparametri p attinet, quoniam invenimus $dp = -2nMax\,d\omega\sqrt{ap}$ in hac expressione ob n minima loco p ejus valorem medium b & loco x valorem $b(1+k\,\mathit{cos}.\nu)$ scribere licebit, unde fit:

$$dp = -2nab\,d\omega.\ M(1+k\,\mathit{cos}.\nu)\sqrt{ab},\ \text{seu}$$
$$\tfrac{dp}{b} = -2na\sqrt{ab}.\ M(1+k\,\mathit{cos}.\nu)\,d\omega.$$

Cum autem posuerimus $a\sqrt{a} = i\,b\sqrt{b}$, erit $\frac{dp}{b} = -2nibbM\,d\omega(1+k\,\mathit{cos}.\nu)$; hincque pro M valorem inventum substituendo

$$\tfrac{dp}{b} = +\tfrac{2nibb}{cc}\,d\omega\left(\mathit{sin}.\eta + \tfrac{1}{2}k\,\mathit{sin}.(\eta-\nu) + \tfrac{1}{2}k\,\mathit{sin}.(\eta+\nu)\right);$$

$$-\tfrac{2nibbc}{f^3}\,d\omega.\left\{\begin{matrix} +(g-\tfrac{3}{2}g'')\,\mathit{sin}.\eta + \tfrac{1}{2}k(g-\tfrac{1}{2}g''+h-\tfrac{1}{2}h'').\left\{\begin{matrix}\mathit{sin}.(\eta-\nu)\\ \mathit{sin}.(\eta+\nu)\end{matrix}\right. \\ +\tfrac{1}{2}(g'-g''')\,\mathit{sin}.2\eta + \tfrac{1}{4}k(g'-g'''+h'-h''').\left\{\begin{matrix}\mathit{sin}.(2\eta-\nu)\\ \mathit{sin}.(2\eta+\nu)\end{matrix}\right.\end{matrix}\right\}$$

ubi ex denominationibus factis est $\frac{bb}{cc} = \frac{m}{i}\sqrt[3]{\frac{m}{i}} = \frac{1-\nu}{1+\nu}$, & $\frac{2bc}{ff} = \mu$, hincque ob $\frac{b}{f} = \sqrt{\frac{1-\nu}{2}}$, erit $\frac{2bbc}{f^3} = \mu\sqrt{\frac{1-\nu}{2}}$. Quare ex cognita ratione motuum mediorum habebitur

$$\mu = \frac{2\sqrt[3]{iimm}}{\sqrt[3]{i^4}+\sqrt[3]{m^4}};\quad \nu = \frac{\sqrt[3]{i^4}-\sqrt[3]{m^4}}{\sqrt[3]{i^4}+\sqrt[3]{m^4}}.$$

§. LXXXIV. Pro variabilitate autem excentricitatis q, quia ea quoque est minima, in ejus expressione ponamus itidem $p = b$ & $q = k$, & quoniam terminos qui quadratum kk continerent, negligimus erit ob $a\sqrt{a} = i\,b\sqrt{b}$; $dq = nibb\,d\omega(M(2\,\mathit{cos}.\nu - \tfrac{1}{2}k + \tfrac{1}{2}k\,\mathit{cos}.2\nu) + N\,\mathit{sin}.\nu)$; unde pro M & N substitutis valoribus obtinebitur:

$$dq = \tfrac{-nibb}{cc}\,d\omega\left(\tfrac{1}{2}\mathit{sin}.(\eta-\nu) + \tfrac{1}{2}\mathit{sin}.(\eta+\nu) - \tfrac{1}{2}k\,\mathit{sin}.\eta + \tfrac{1}{4}k\,\mathit{sin}.(\eta-2\nu) + \tfrac{1}{4}k\,\mathit{sin}.(\eta+2\nu)\right)$$

$$+\tfrac{nib^3}{f^3}\,d\omega\left\{\begin{matrix} +g\,\mathit{sin}.\nu + \tfrac{1}{2}g'\,\mathit{sin}.(\eta+\nu) + \tfrac{1}{2}g''\,\mathit{sin}.(2\eta+\nu) \\ -\tfrac{1}{2}g'\,\mathit{sin}.(\eta-\nu) - \tfrac{1}{2}g''\,\mathit{sin}.(2\eta-\nu) \\ +\tfrac{1}{2}k(g+h)\,\mathit{sin}.2\nu + \tfrac{1}{4}k(g'+h')\,\mathit{sin}.(\eta+2\nu) + \tfrac{1}{4}k(g''+h'')\,\mathit{sin}.(2\eta+2\nu) \\ -\tfrac{1}{4}k(g'+h')\,\mathit{sin}.(\eta-2\nu) - \tfrac{1}{4}k(g''+h'')\,\mathit{sin}.(2\eta-2\nu)\end{matrix}\right\}$$