OBSERVATIONES CIRCA RADICES AEQVATIONVM.

Auctore

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I.

Si habeatur aequatio algebraica cuiusuis gradus ad rationalitatem perducta:

$$x^m = Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + Dx^{m-4} + Ex^{m-5} + \text{etc.}$$

quam etiam hac forma exhibere licet

$$1 = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x^5} + \text{etc.}$$

ac ponatur

$\int x =$ summae omnium radicum

$\int x^2 =$ summae quadratorum earundem radicum

$\int x^3 =$ summae cuborum

$\int x^4 =$ summae biquadratorum

et ita porro;

notum est has summas ita a se inuicem et a litteris A, B, C, D, E etc. pendere vt sit:

$$\int x = A$$
$$\int x^2 = A\int x + 2B$$
$$\int x^3 = A\int x^2 + B\int x + 3C$$
$$\int x^4 = A\int x^3 + B\int x^2 + C\int x + 4D$$
$$\int x^5 = A\int x^4 + B\int x^3 + C\int x^2 + D\int x + 5E$$

etc.

 II.

II.

Ex hac ergo progressionis lege singulae hae summae potestatum ita se habebunt euolutae:

$$\int x = A$$

$$\int x^2 = A^2 + 2B$$

$$\int x^3 = A^3 + 3AB + 3C$$

$$\int x^4 = A^4 + 4A^2B + 4AC + 4D + 2B^2$$

$$\int x^5 = A^5 + 5A^3B + 5A^2C + 5AD + 5E + 5AB^2 + 5BC$$

$$\int x^6 = A^6 + 6A^4B + 6A^3C + 6A^2D + 6AE + 6F + 9A^2B^2 + 12ABC + 6BD + 2B^3 + 3CC$$

$$\int x^7 = A^7 + 7A^5B + 7A^4C + 7A^3D + 7A^2E + 7AF + 7G + 14A^3B^2 + 21A^2BC + 14ABD + 7BE + 7AB^3 + 7AC^2 + 7CD + 7B^2C$$

vlterius has formas non continuandas esse arbitror, cum harum contemplatio sufficiat, ad legem qua singulae formantur explorandam.

III.

Vt ordinem quo in his formis singulae litterae A, B, C, D, E etc. inter se componuntur, facilius perspiciamus, litterae A tri-

tribuamus vnam dimensionem, litterae B duas, litterae C tres, litterae D quatuor et ita porro, atque manifesum est in qualibet forma nonnisi eiusmodi occurrere terminos in quibus dimensionum numerus sit exponenti potestatum radicum, quarum summa exhibetur, aequalis. Ita in forma $\int x^7$ singuli termini continent septem dimensiones, atque adeo omnes termini per mutuam combinationem septem dimensiones adimplentes in ea reperiuntur, quod etiam de omnibus formis est tenendum. Imprimis autem obseruari conuenit, alias litterarum A, B, C, D etc. potestates in has formas non ingredi, nisi quarum exponentes sint numeri integri et positiui, vnde pro quauis potestate summatoria omnes termini eam constituentes ex litterarum A, B, C, D etc. combinatione assignantur, quorum quidem numerus semper est finitus etiamsi ipsa aequatio proposita in infinitum excurrat.

IV.

Cum igitur pro quauis potestate ipsi termini, quatenus ex litteris A, B, C, D etc. conflantur, nullam inuoluant difficultatem, totum negotium ad vncias numericas quibus singuli termini sunt affecti, reducitur. Ad indolem autem harum vnciarum explorandam, seposita prima littera A terminos secundum reliquas litteras B, C, D, E etc. ita in ordines disponi conueniet, vt in primo harum litterarum nulla, in secundo ordine singulae tantum, in tertio vero binae, in quarto ternae et ita porro reperiantur, hoc modo:

$\int x$

$$\int x = A$$

$$\int x^2 = A^2 + 2B$$

$$\begin{aligned}\int x^3 = A^3 &+ 3AB \\ &+ 3C\end{aligned}$$

$$\begin{aligned}\int x^4 = A^4 &+ 4A^2B + 2BB \\ &+ 4AC \\ &+ 4D\end{aligned}$$

$$\begin{aligned}\int x^5 = A^5 &+ 5A^3B + 5ABB \\ &+ 5A^2C + 5BC \\ &+ 5A^2D \\ &+ 5E\end{aligned}$$

$$\begin{aligned}\int x^6 = A^6 &+ 6A^4B + 9A^2BB + 2B^3 \\ &+ 6A^3C + 12ABC \\ &+ 6A^2D + 6BD \\ &+ 6AE + 3CC \\ &+ 6F\end{aligned}$$

$$\begin{aligned}\int x^7 = A^7 &+ 7A^5B + 14A^3BB + 7AB^3 \\ &+ 7A^4C + 21A^2BC + 7B^2C \\ &+ 7A^3D + 14ABD \\ &+ 7A^2E + 7ACC \\ &+ 7AF + 7BE \\ &+ 7G + 7CD\end{aligned}$$

$\int x^8 =$

$$\begin{aligned}\int x^8 = A^8 &+ 8A^6B + 20A^4BB + 16A^2B^3 + 2B^4 \\ &+ 8A^5C + 32A^3BC + 24AB^2C \\ &+ 8A^4D + 24A^2BD + 8B^2D \\ &+ 8A^3E + 12A^2CC + 8BC^2 \\ &+ 8A^2F + 16ABE \\ &+ 8AG + 16ACD \\ &+ 8H + 8BF \\ &\qquad + 8CE \\ &\qquad + 4DD.\end{aligned}$$

V.

In cuiusque formae ordine primo et secundo nulla plane occurrit difficultas, nullumque est dubium, quin pro forma $\int x^n$ sit primus terminus A^n, secundus vero ordo ex his constet terminis

$$nA^{n-2}B + nA^{n-3}C + nA^{n-4}D + nA^{n-5}E + \text{etc.}$$

sequentium vero ordinum ratio minus est manifesta. Hanc autem circumstantiam perpendentes, quod exponens n in omnes quoque sequentes vncias tanquam factor ingrediatur: deinde etiam quod quaelibet litterarum B, C, D, E etc. combinationes, simul permutationum numerum inuoluant, prouti in polynomii potestatibus occurrunt; si in singulis terminis hos binos factores seorsim exhibeamus leui adhibita attentione deprehendemus, in genere has formas ita expressum iri.

Ordo

Ordo I. *Ordo II.* *Ordo III.*

$$\int x^n = A^n + nA^{n-2}B + \frac{n(n-3)}{1\cdot 2}A^{n-4}BB$$
$$+ nA^{n-3}C + \frac{n(n-4)}{1\cdot 2}A^{n-5}\,2BC$$
$$+ nA^{n-4}D + \frac{n(n-5)}{1\cdot 2}A^{n-6}(2BD + CC)$$
$$+ nA^{n-5}E + \frac{n(n-6)}{1\cdot 2}A^{n-7}(2BE + 2CD)$$
$$+ nA^{n-6}F + \frac{n(n-7)}{1\cdot 2}A^{n-8}(2BF + 2CE + DD)$$
$$+ nA^{n-7}G + \frac{n(n-8)}{1\cdot 2}A^{n-9}(2BG + 2CF + 2DE)$$

etc. etc.

Ordo IV.

$$+ \frac{n(n-4)(n-5)}{1\cdot 2\cdot 3}A^{n-6}B^3$$
$$+ \frac{n(n-5)(n-6)}{1\cdot 2\cdot 3}A^{n-7}\,3B^2C$$
$$+ \frac{n(n-6)(n-7)}{1\cdot 2\cdot 3}A^{n-8}(3B^2D + 3BC^2)$$
$$+ \frac{n(n-7)(n-8)}{1\cdot 2\cdot 3}A^{n-9}(3B^2E + 6BCD + C^3)$$
$$+ \frac{n(n-8)(n-9)}{1\cdot 2\cdot 3}A^{n-10}(3B^2F + 6BCE + 3BDD + 3CCD)$$

etc.

Ordo V.

$$+ \frac{n(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3\cdot 4}A^{n-8}B^4$$
$$+ \frac{n(n-6)(n-7)(n-8)}{1\cdot 2\cdot 3\cdot 4}A^{n-9}\,4B^3C$$
$$+ \frac{n(n-7)(n-8)(n-9)}{1\cdot 2\cdot 3\cdot 4}A^{n-10}(4B^3D + 6B^2C^2)$$
$$+ \frac{n(n-8)(n-9)(n-10)}{1\cdot 2\cdot 3\cdot 4}A^{n-11}(4B^3E + 12B^2CD + 4BC^3)$$
$$+ \frac{n(n-9)(n-10)(n-11)}{1\cdot 2\cdot 3\cdot 4}A^{n-12}(4B^3F + 12B^2CE + 6B^2D^2 + 12BC^2D + C^4)$$

etc.

Ordo

Ordo VI.

$$+\frac{n(n-6)(n-7)(n-8)(n-9)}{1\cdot2\cdot3\cdot4\cdot5}A^{n-10}B^5$$
$$+\frac{n(n-7)(n-8)(n-9)(n-10)}{1\cdot2\cdot3\cdot4\cdot5}A^{n-11}\,5B^4C$$
$$+\frac{n(n-8)(n-9)(n-10)(n-11)}{1\cdot2\cdot3\cdot4\cdot5}A^{n-12}(5B^4D+10B^3C^2)$$
$$+\frac{n(n-9)(n-10)(n-11)(n-12)}{1\cdot2\cdot3\cdot4\cdot5}A^{n-13}(5B^4E+20B^3CD+10B^2C^3)$$
$$+\frac{n(n-10)(n-11)(n-12)(n-13)}{1\cdot2\cdot3\cdot4\cdot5}A^{n-14}(5B^4F+20B^3CE+10B^3D^2+30B^2C^2D+5BC^4)$$

etc.

VI.

Hinc ordinem quemcunque in genere euoluere licebit, sit enim index ordinis $\lambda+1$ statuanturque membra huius ordinis;

$$+\frac{n(n-\lambda-1)(n-\lambda-2)(n-\lambda-3)\ldots(n-2\lambda+1)}{1\cdot2\cdot3\cdot4\ldots\ldots\lambda}A^{n-2\lambda}\,.\,O$$
$$+\frac{n(n-\lambda-2)(n-\lambda-3)(n-\lambda-4)\ldots(n-2\lambda)}{1\cdot2\cdot3\cdot4\ldots\ldots\lambda}A^{n-2\lambda-1}\,.\,P$$
$$+\frac{n(n-\lambda-3)(n-\lambda-4)(n-\lambda-5)\ldots(n-2\lambda-1)}{1\cdot2\cdot3\cdot4\ldots\ldots\lambda}A^{n-2\lambda-2}\,.\,Q$$
$$+\frac{n(n-\lambda-4)(n-\lambda-5)(n-\lambda-6)\ldots(n-2\lambda-2)}{1\cdot2\cdot3\cdot4\ldots\ldots\lambda}A^{n-2\lambda-3}\,.\,R$$

etc.

atque valores litterarum O, P, Q, R etc. ita se habebunt, ut sit

$$O+Pz+Qz^2+Rz^3+Sz^4+\text{etc.}=(B+Cz+Dz^2+Ez^3\text{ etc.})^\lambda$$

vnde euolutione facta colligimus:

$$O = B^{\lambda}$$
$$P = \frac{\lambda}{1} \cdot \frac{OC}{B}$$
$$Q = \frac{2\lambda}{2} \cdot \frac{OD}{B} + \frac{\lambda - 1}{2} \cdot \frac{PC}{B}$$
$$R = \frac{3\lambda}{3} \cdot \frac{OE}{B} + \frac{2\lambda - 1}{3} \cdot \frac{PD}{B} + \frac{\lambda - 2}{3} \cdot \frac{QC}{B}$$
$$S = \frac{4\lambda}{4} \cdot \frac{OF}{B} + \frac{3\lambda - 1}{4} \cdot \frac{PE}{B} + \frac{2\lambda - 2}{4} \cdot \frac{QD}{B} + \frac{\lambda - 3}{4} \cdot \frac{RC}{B}$$
$$T = \frac{5\lambda}{5} \cdot \frac{OG}{B} + \frac{4\lambda - 1}{5} \cdot \frac{PF}{B} + \frac{3\lambda - 2}{5} \cdot \frac{QE}{B} + \frac{2\lambda - 3}{5} \cdot \frac{RD}{B} + \frac{\lambda - 4}{5} \cdot \frac{SC}{B}$$

siue valoribus iam inuentis substituendis

$$O = B^{\lambda}$$
$$P = \lambda B^{\lambda-1} C$$
$$Q = \lambda B^{\lambda-1} D + \frac{\lambda(\lambda-1)}{1 \cdot 2} B^{\lambda-2} C^2$$
$$R = \lambda B^{\lambda-1} E + \frac{2\lambda(\lambda-1)}{1 \cdot 2} B^{\lambda-2} CD + \frac{\lambda(\lambda-1)(\lambda-2)}{1 \cdot 2 \cdot 3} B^{\lambda-3} C^3$$
$$S = \lambda B^{\lambda-1} F + \frac{\lambda(\lambda-1)}{1 \cdot 2} B^{\lambda-2}(2CE + DD) + \frac{3\lambda(\lambda-1)(\lambda-2)}{1 \cdot 2 \cdot 3} B^{\lambda-3} C^2 D + \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{1 \cdot 2 \cdot 3 \cdot 4} B^{\lambda-4} C^4$$

etc.

VII.

De hac autem forma generali probe est tenendum, ea summam singularum radicum ad dignitatem n eleuatarum neutiquam exprimi, nisi primo exponens n sit numerus integer positiuus, tum vero ex forma generali quae in infinitum excurrit, omnes termini excludantur in quibus littera A exponentem negatiuum esset adeptura. Hinc quaetsio oritur maximi momenti, quinam futurus sit valor huius formae generalis, si omnes termini in infinitum retineantur?

tur?

tur? idque siue exponens n fuerit siue positiuus, siue negatiuus, siue integer siue fractus? Hanc igitur quaestionem quoniam inde speculationes maxime notatu dignae et in doctrina serierum nouam quandam lucem accendentes oriuntur, hic accuratius euoluendam suscepi. Ostendam autem hac forma generali non summam potestatum exponentis n, quae ex singulis radicibus formantur, sed potius potestatem similem vnius duntaxat radicis eiusque maximae exprimi.

VIII.

Quo hanc inuestigationem simpliciorem reddam a casu huius aequationis $1 = \frac{A}{x} + \frac{B}{xx}$ inchoabo, ita vt litterae C, D, E etc. omnes euanescant: pro hoc ergo casu forma nostra generalis, in cuius valorem inquirimus, erit

$$A^n + nA^{n-2}B + \frac{n(n-3)}{1.2}A^{n-4}B^2 + \frac{n(n-4)(n-5)}{1.2.3}A^{n-6}B^3 + \frac{n(n-5)(n-6)(n-7)}{1.2.3.4}A^{n-8}B^4 + \text{etc.}$$

Ponamus primo $n = 1$, et sit valor seriei $= s$ vt sit

$$s = A + \frac{B}{A^1} - \frac{2}{2}\cdot\frac{B^2}{A^3} + \frac{3.4}{2.3}\cdot\frac{B^3}{A^5} - \frac{4.5.6}{2.3.4}\cdot\frac{B^4}{A^7} + \text{etc.}$$

quae revocatur ad hanc formam;

$$s = \tfrac{1}{2}A + \tfrac{1}{2}A + \tfrac{1}{2}\cdot\frac{2B}{A} - \frac{1.1}{2.4}\cdot\frac{8B^2}{A^3} + \frac{1.1.3}{2.4.6}\cdot\frac{32B^3}{A^5} - \text{etc.}$$

cuius seriei summa manifesto est

$$s = \tfrac{1}{2}A + \sqrt{(\tfrac{1}{4}AA + B)},$$

quae est aequationis propositae radix maior. Tum vero iam

 con-

constat illius seriei generalis valorem esse $= (\frac{1}{2}A + \sqrt{(\frac{1}{4}AA + B)})^n$: ex quo nullum amplius superest dubium, quin illa forma generalis potestatem exponentis n vnius tantum radicis aequationis, eiusque maioris exprimat, hoc saltem casu.

IX.

In genere autem eadem conclusio hoc modo confici poterit. Denotet $s^{(n)}$ totam illam expressionem generalem §. V. exhibitam et in infinitum extensam, sintque $s^{(n-1)}$, $s^{(n-2)}$, $s^{(n-3)}$, etc. eiusdem valores, si loco n scribatur $n-1$, $n-2$, $n-3$ etc. atque ex genesi illius expressionis intelligitur fore

$$s^{(n)} = A s^{(n-1)} + B s^{(n-2)} + C s^{(n-3)} + D s^{(n-4)} + \text{etc.}$$

verum ex ipsa aequatione proposita est quoque

$$x^m = A x^{m-1} + B x^{m-2} + C x^{m-3} + D x^{m-4} + \text{etc.}$$

vnde si hae duae aequationes sequenti modo repraesententur:

$$1 = \frac{A s^{(n-1)}}{s^{(n)}} + \frac{B s^{(n-2)}}{s^{(n)}} + \frac{C s^{(n-3)}}{s^{(n)}} + \frac{D s^{(n-4)}}{s^{(n)}} + \text{etc. et}$$

$$1 = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \text{etc.}$$

quoniam hoc valet pro omnibus numeris n, sequitur fore:

$$s^{(n)} = x s^{(n-1)} = x^2 s^{(n-2)} = x^3 s^{(n-3)} = x^4 s^{(n-4)} \text{ etc.}$$

Cum igitur posito $n = 0$, sit $s^{(0)} = A^0 = 1$, erit pro n scribendo successiue numeros 1, 2, 3, 4 etc.

$$s^{(1)} = x;\ s^{(2)} = x^2;\ s^{(3)} = x^3;\ s^{(4)} = x^4;$$

Quare

Quare euictum est in genere fore $s^{(n)} = x^n$; hic autem pro x sumi debere aequationis propositae radicem maximam, inde patet, quod sumto exponente n infinito, quo casu formae nostrae pars integra ab vniuersa non est censenda discrepare, summa potestatum infinitesimarum ad potestatem infinitesimam radicis maximae solam reducitur.

X.

En ergo theorema notatu dignissimum, vsumque habiturum amplissimum, quod proposita aequatione quacunque huius formae:

$$1 = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x^5} + \text{etc.}$$

cuius radix maxima sit $x = m$, expressionis supra §. V. exhibitae et in infinitum continuatae valor sit m^n. Quare si sumatur $n = 1$, eadem expressio ipsam radicem maximam exprimet; vbi imprimis omni attentione dignum occurrit, quod omnes potestates eiusdem radicis per similes expressiones infinitas exprimantur; quin etiam ponendo $n = 0$ ob $\frac{m^0 - A^0}{0} = l\frac{m}{A}$, logarithmus hyperbolicus maximae radicis m hoc modo exprimetur:

$$\begin{aligned} lm = lA &+ \frac{B}{A^2} - \frac{3B^2}{2A^4} &&+ \frac{4 \cdot 5 B^3}{2 \cdot 3 A^6} \\ &+ \frac{C}{A^3} - \frac{4 \cdot 2BC}{2A^5} &&+ \frac{5 \cdot 6 \cdot 3 B^2 C}{2 \cdot 3 A^7} \\ &+ \frac{D}{A^4} - \frac{5(2BD + CC)}{2A^6} &&+ \frac{6 \cdot 7 (3B^2 D + 3BC^2)}{2 \cdot 3 A^8} \\ &+ \frac{E}{A^5} - \frac{6(2BE + 2CD)}{2A^7} &&+ \frac{7 \cdot 8 (3B^2 E + 6BCD + C^3)}{2 \cdot 3 A^9} \\ &\qquad\quad \text{etc.} &&\qquad\quad \text{etc.} \end{aligned}$$

XI.

XI.

Quoniam ergo hinc cuiusque aequationis radicem maximam non solum ipsam, sed etiam eius quamcunque potestatem per series infinitas commode exprimere licet, hinc primum pulcherrimam illam seriem, quam sagacissimi ingenii vir *Lambertus*, in Actorum Helueticorum volumine IV. pro resolutione aequationum ex tribus tantum terminis constantium tradidit, deducere licet. Quemadmodum enim supra aequatio haec $1 = \frac{A}{x} + \frac{B}{x^2}$ dederat

$$x^n = A^n + nA^{n-2}B + \frac{n(n-3)}{1 \cdot 2}A^{n-4}B^2 + \frac{n(n-4)(n-5)}{1 \cdot 2 \cdot 3}A^{n-6}B^3 + \text{etc.};$$

ita haec aequatio $1 = \frac{A}{x} + \frac{C}{x^3}$ dabit

$$x^n = A^n + nA^{n-3}C + \frac{n(n-5)}{1 \cdot 2}A^{n-6}C^2 + \frac{n(n-7)(n-8)}{1 \cdot 2 \cdot 3}A^{n-9}C^3 + \text{etc.}$$

haecque aequatio $1 = \frac{A}{x} + \frac{D}{x^4}$

$$x^n = A^n + nA^{n-4}D + \frac{n(n-7)}{1 \cdot 2}A^{n-8}D^2 + \frac{n(n-10)(n-11)}{1 \cdot 2 \cdot 3}A^{n-12}D^3 + \text{etc.}$$

ita concludimus pro hac aequatione $1 = \frac{A}{x} + \frac{M}{x^m}$ fore

$$x^n = A^n + nA^{n-m}M + \frac{n(n-2m+1)}{1 \cdot 2}A^{n-2m}M^2 + \frac{n(n-3m+2)(n-3m+1)}{1 \cdot 2 \cdot 3}A^{n-3m}M^3 + \text{etc.}$$

Statuamus nunc $x = y^\lambda$ et $x^m = y^\mu$, tum vero pro M scribamus B et $\frac{n}{\lambda}$ loco n, atque ob $m = \frac{\mu}{\lambda}$ pro resolutione huius aequationis generalis $1 = \frac{A}{y^\lambda} + \frac{B}{y^\mu}$ habebimus:

$y^n = A$

$$y^n = A^{\frac{n}{\lambda}} + \frac{n}{\lambda} A^{\frac{n-\mu}{\lambda}} B + \frac{n(n+\lambda-2\mu)}{1 \cdot 2\lambda^2} A^{\frac{n-2\mu}{\lambda}} B^2 + \frac{n(n+2\lambda-3\mu)(n+\lambda-3\mu)}{1 \cdot 2 \cdot 3\lambda^3} A^{\frac{n-3\mu}{\lambda}} B^3$$
$$+ \frac{n(n+3\lambda-4\mu)(n+2\lambda-4\mu)(n+\lambda-4\mu)}{1 \cdot 2 \cdot 3 \cdot 4\lambda^4} A^{\frac{n-4\mu}{\lambda}} + B^4 + \text{etc.}$$

XII.

Si igitur aequationis $1 = \frac{A}{y^\lambda} + \frac{B}{y^\mu}$ radix ipsa desideretur y, poni oportet $n = 1$, ac fiet:

$$y = A^{\frac{1}{\lambda}} + \frac{1}{\lambda} A^{\frac{1-\mu}{\lambda}} B + \frac{1+\lambda-2\mu}{2\lambda^2} A^{\frac{1-2\mu}{\lambda}} B^2 + \frac{(1+2\lambda-3\mu)(1+\lambda-3\mu)}{2 \cdot 3\lambda^3} A^{\frac{1-3\mu}{\lambda}} B^3$$
$$+ \frac{(1+3\lambda-4\mu)(1+2\lambda-4\mu)(1+\lambda-4\mu)}{2 \cdot 3 \cdot 4\lambda^4} A^{\frac{1-4\mu}{\lambda}} B^4 + \text{etc.}$$

quae est ipsa series *Lamberti* loco allegato exhibita eoque magis notatu digna videtur, quod coëfficientium lex satis quidem est regularis, verumtamen ita comparata, vt si series ipsa proponatur, nulla pateat via eius summam inuestigandi; quod eo magis est mirum, quod nihilominus huius seriei summa non solum constat, sed adeo algebraice exhiberi potest, cum sit vna radicum huius aequationis $1 = \frac{A}{y^\lambda} + \frac{B}{y^\mu}$, eaque maxima. Deinde vero huius seriei proprietas maximi sine dubio est momenti, quod omnes eius potestates similibus seriebus exprimantur.

XIII.

Indolem harum singularium serierum e re erit in aliquot exemplis perspexisse. Sumamus ergo $\lambda = 3$ et $\mu = 2$

vt

vt habeamus hanc aequationem cubicam $y^3 = A + By$, cuius propterea vna radicum erit:

$$y = A^{\frac{1}{3}} + \frac{1}{3} . A^{-\frac{1}{3}} B + 0 A^{-1} B^2 - \frac{2 . 1}{2 . 3} A^{-\frac{5}{3}} \left(\frac{B}{3}\right)^3 + \frac{4 . 1 . 2}{2 . 3 . 4} A^{-\frac{7}{3}} \left(\frac{B}{3}\right)^4$$
$$- \frac{6}{2} . -\frac{3}{3} . \frac{0}{4} . \frac{3}{5} A^{-3} \left(\frac{B}{3}\right)^5 - \frac{8}{2} . -\frac{5}{3} . -\frac{2}{4} . \frac{1}{5} . \frac{4}{6} A^{-\frac{11}{3}} \left(\frac{B}{3}\right)^6 \text{ etc.}$$

quae expressio quo clarior reddatur sumamus $A = a^3$ et $B = 3b$ vt prodeat huius aequationis $y^3 = 3by + a^3$ radix

$$y = a + \frac{b}{a} + \frac{4.1.2}{2.3.4} . \frac{b^4}{a^7} + \frac{10.7.4.1.2.5}{2.3.4.5.6.7} . \frac{b^7}{a^{13}} + \frac{16.13.10.7.4.1.2.5.8}{2.3.4.5.6.7.8.9.10} . \frac{b^{10}}{a^{19}} \text{ etc.}$$
$$- \frac{2.1}{2.3} . \frac{b^3}{a^5} - \frac{8.5.2.1.4}{2.3.4.5.6} . \frac{b^6}{a^{11}} - \frac{14.11.8.5.2.1.4.7}{2.3.4.5.6.7.8.9} . \frac{b^9}{a^{17}} - \frac{20.17.14.11.8.5.2.1.4.7.10}{2.3.4.5.6.7.8.9.10.11.12} . \frac{b^{12}}{a^{23}} - \text{etc.}$$

quae ita concinnius repraesentatur:

$$y = a + \frac{b}{a} + \frac{1}{3} . \frac{b^4}{a^7} + \frac{1}{3} . \frac{7.10}{6.7} . \frac{b^7}{a^{13}} + \frac{1}{3} . \frac{7.10}{6.7} . \frac{13.16}{9.10} . \frac{b^{10}}{a^{19}} + \frac{1}{3} . \frac{7.10}{6.7} . \frac{13.16}{9.10} . \frac{19.22}{12.13} . \frac{b^{13}}{a^{25}}$$
$$- \frac{1}{3} . \frac{b^3}{a^5} - \frac{1}{3} . \frac{5.8}{5.6} . \frac{b^6}{a^{11}} - \frac{1}{3} . \frac{5.8}{5.6} . \frac{11.14}{8.9} . \frac{b^9}{a^{17}} - \frac{1}{3} . \frac{5.8}{5.6} . \frac{11.14}{8.9} . \frac{17.20}{11.12} . \frac{b^{12}}{a^{23}} \text{ etc.}$$

XIV.

Hae series accuratiorem euolutionem merentur, ponamus ergo pro priore

$$s = x + \frac{1}{3} x^4 + \frac{1}{3} . \frac{7.10}{6.7} x^7 + \dots M x^{3n+1} + N x^{3n+4} + \text{etc.}$$

et cum esse debeat $\frac{N}{M} = \frac{6n+1}{3n+3} . \frac{6n+4}{3n+4}$ haec conditio adimpletur hac aequatione differentiali secundi gradus

$$dds = 4x^3 dds + 6xxdxds - 2xsdx^2$$

quae commode per $2xds - sdx$ multiplicata integrabilis euadit; reperitur enim integrando:

$$xds^2 - sdxds + Cdx^2 = 4x^4ds^2 - 4x^3sdxds + xxssdx^2$$

vbi cum sumto x infinite paruo fiat $s = x$ et $\frac{ds}{dx} = 1$ euidens est capi debere $C = 0$ ita vt sit

(xds

$(xds - sdx)\, ds = 4x^3 (xds - sdx)\, ds + xxss\, dx^2 = xx\, (2xds - sdx)^2$;
seu $\frac{ds^2}{ss\,dx^2} = \frac{ds}{xs\,dx} + \frac{x}{1 - 4x^3}$, vnde radicem extrahendo fit

$$\frac{ds}{s\,dx} = \frac{1}{2x} + \frac{1}{2x} \sqrt{\frac{1}{1 - 4x^3}}$$ ita vt habeamus:

$$ls = \tfrac{1}{2} lx + \tfrac{1}{2} \int \frac{dx}{x\sqrt{(1 - 4x^3)}} = \tfrac{1}{2} lx + \tfrac{1}{3} l \frac{2x\sqrt{x}}{1 + \sqrt{(1 - 4x^3)}}.$$

Hinc ergo erit $s = x \sqrt[3]{\frac{2}{1 + \sqrt{(1 - 4x^3)}}}$.

XV.

Ponamus ergo $\frac{b}{aa} = x$, vt habeamus

$$\frac{y}{a} = 1 + x + \tfrac{1}{3}x^4 + \tfrac{1}{3} \cdot \tfrac{7 \cdot 10}{6 \cdot 7} x^7 + \tfrac{1}{3} \cdot \tfrac{7 \cdot 10}{6 \cdot 7} \cdot \tfrac{13 \cdot 16}{9 \cdot 10} \cdot x^{10} + \text{etc.}$$
$$- \tfrac{1}{3}x^3 - \tfrac{1}{3} \cdot \tfrac{5 \cdot 8}{5 \cdot 6} x^6 - \tfrac{1}{3} \cdot \tfrac{5 \cdot 8}{5 \cdot 6} \cdot \tfrac{11}{8} \cdot \tfrac{14}{9} x^9 - \text{etc.}$$

seu $\frac{y}{a} = s + 1 - \frac{1}{3}x^3 - \frac{1}{3} \cdot \frac{5 \cdot 8}{5 \cdot 6} x^6 - \frac{1}{3} \cdot \frac{5 \cdot 8}{5 \cdot 6} \cdot \frac{11 \cdot 14}{8 \cdot 9} x^9 - \text{etc.}$
Ponamus summam seriei $1 - \frac{1}{3}x^3 - \frac{1}{3} \cdot \frac{5}{5} \cdot \frac{8}{6} \cdot x^6 - \text{etc.} = t$ ac reperiemus vt ante, quoniam lex progressionis est eadem:

$$ddt = 4x^3\, ddt + 6xx\, dx\, dt - 2xt\, dx^2$$

cuius integrale propterea est quoque

$$x\,dt^2 - t\,dx\,dt = 4x^4\, dt^2 - 4x^3 t\, dx\, dt + xxtt\, dx^2$$

quia enim sumto x infinite paruo fit $t = 1$ et $\frac{dt}{dx} = 0$ constans addenda etiam euanescit. Porro ergo integrando adipiscimur:

$$t = x \sqrt[3]{\frac{2}{1 - \sqrt{(1 - 4x^3)}}}$$ fietque $t = 1$ si $x = 0$.

Quocirca pro radice aequationis $y^3 = 3by + a^3$ habebimus:

$$\frac{y}{a} = s + t = x\sqrt[3]{\frac{2}{1+\sqrt{(1-4x^3)}}} + x\sqrt[3]{\frac{2}{1-\sqrt{(1-4x^3)}}} = \sqrt[3]{\frac{1-\sqrt{(1-4x^3)}}{2}} + \sqrt[3]{\frac{1+\sqrt{(1-4x^3)}}{2}}$$

existente $x = \frac{b}{aa}$, ideoque

$$y = \sqrt[3]{\frac{a^3-\sqrt{(a^6-4b^3)}}{2}} + \sqrt[3]{\frac{a^3+\sqrt{(a^6-4b^3)}}{2}}$$

quam eandem expressionem regula Cardani suppeditat.

XVI.

Euoluamus aliud exemplum aequationis cubicae, ponendo $\lambda = 1$ et $\mu = 3$, vt sit $y^3 = Ayy + B$ ac posito $\frac{B}{A^3} = x$, nostra forma dat

$$\frac{y}{A} = 1 + x - \frac{4}{2}x^2 + \frac{6.7}{2.3}x^3 - \frac{8.9.10}{2.3.4}x^4 + \frac{10.11.12.13}{2.3.4.5}x^5 - \text{etc.}$$

quae ad hanc legem reducitur continuitatis

$$\frac{y}{A} - \frac{1}{3} = \frac{2}{3} + x - \frac{4}{2}x^2 + \frac{6.7}{2.3}x^3 \ldots \pm Mx^n \mp Nx^{n+1}$$

vt sit $N = \frac{3(3n-1)(3n+1)}{(2n+1)(2n+2)}M = \frac{27nn-3}{4nn+6n+2}M$. Ponamus $\frac{y}{A} - \frac{1}{3} = s$, et relatio inter s et x exprimetur per hanc aequationem differentialem secundi gradus:

$$4xxdds + 2xdxds + 27x^3dds + 27x^2dxds - 3xsdx^2 = 0$$

quae per $\frac{2ds}{x}$ multiplicata et integrata praebet:

$4xds^2 + 27xxds^2 - 3ssdx^2 = Cdx^2$ vnde colligitur

$$\frac{ds}{\sqrt{(C+3ss)}} = \frac{dx}{\sqrt{(4x+27xx)}}$$

cuius

cuius integratio dat

$$\frac{1}{\sqrt{3}}l(s\sqrt{3}+\sqrt{(C+3ss)})=\frac{1}{3\sqrt{3}}l(\frac{2}{3\sqrt{3}}+3x\sqrt{3}+\sqrt{(4x+27xx)})$$

vnde porro elicitur haec aequatio algebraica:

$$s=A(1+\frac{27x}{2}+3\sqrt{(3x+\frac{81xx}{4})})^{\frac{1}{3}}$$
$$+B(1+\frac{27}{2}x-3\sqrt{(3x+\frac{81xx}{4})})^{\frac{1}{3}}$$

quae euoluta vtique praebet

$$s^3=3ABs+(A^3+B^3)(1+\frac{27x}{2})+3(A^3-B^3)\sqrt{(3x+\frac{81xx}{4})}$$

aequatio autem assumta inter s et x erat

$$s^3=\frac{1}{3}s+\frac{2}{27}+x$$

quae in integrali illo completo continetur sumendo

$$A=B=\frac{1}{3}.$$

XVII.

Euolutio haec elegantissima aequationum tribus tantum terminis constantium $1=\frac{A}{y^\lambda}+\frac{B}{y^\mu}$ eo maiorem attentionem meretur, quod nulla via patet directa, ex serie inuenta in genere valorem summae y inuestigandi, etiamsi tandem haec summa maxime concinna aequatione algebraica exhiberi possit. Quod enim casus hic pro aequationibus quadraticis et cubicis expedire licuit, successus huic circumstantiae soli acceptus est referendus, quod harum aequationum resolutio est in potestate; vnde non immerito suspicari licet, si methodus detegeretur huiusmodi series summandi inde eximia subsidia ad resolutionem aequationum cuiuscunque gradus esse redundatura. Simili autem modo euolutio aequationum quaternis terminis constantium exhiberi potest latissime patens, quae autem ita est comparata, vt singuli termini continuo plura membra contineant, quorum tamen ordo satis est perspicuus.

XVIII.

Si enim in genere haec fuerit proposita aequatio quatuor constans terminis:

$$1 = \frac{A}{y^{\lambda}} + \frac{B}{y^{\mu}} + \frac{C}{y^{\nu}}$$

atque ponamus $y^n = P + Q + R + S + T$ etc. hae partes P, Q, R, S, T etc. sequenti modo determinantur:

$$P = A^{\frac{n}{\lambda}}$$

$$Q = \frac{n}{\lambda} A^{\frac{n-\mu}{\lambda}} B + \frac{n}{\lambda} A^{\frac{n-\nu}{\lambda}} C$$

$$R = \begin{cases} + \frac{n(n+\lambda-2\mu)}{1.2\lambda^2} A^{\frac{n-2\mu}{\lambda}} BB \\ + \frac{2n(n+\lambda-\mu-\nu)}{1.2\lambda^2} A^{\frac{n-\mu-\nu}{\lambda}} BC \\ + \frac{n(n+\lambda-2\nu)}{1.2\lambda^2} A^{\frac{n-2\nu}{\lambda}} CC \end{cases}$$

+

$$S = \begin{cases} + \frac{n(n+\lambda-3\mu)(n+2\lambda-3\mu)}{1.2.3\lambda^3} A^{\frac{n-3\mu}{\lambda}} B^3 \\ + \frac{3n(n+\lambda-2\mu-\nu)(n+2\lambda-2\mu-\nu)}{1.2.3\lambda^3} A^{\frac{n-2\mu-\nu}{\lambda}} B^2 C \\ + \frac{3n(n+\lambda-\mu-2\nu)(n+2\lambda-\mu-2\nu)}{1.2.3\lambda^3} A^{\frac{n-\mu-2\nu}{\lambda}} B C^2 \\ + \frac{n(n+\lambda-3\nu)(n+2\lambda-3\nu)}{1.2.3\lambda^3} A^{\frac{n-3\nu}{\lambda}} C^3 \end{cases}$$

$$T = \begin{cases} + \frac{n(n+\lambda-4\mu)(n+2\lambda-4\mu)(n+3\lambda-4\mu)}{1.2.3.4\lambda^4} A^{\frac{n-4\mu}{\lambda}} B^4 \\ + \frac{4n(n+\lambda-3\mu-\nu)(n+2\lambda-3\mu-\nu)(n+3\lambda-3\mu-\nu)}{1.2.3.4\lambda^4} A^{\frac{n-3\mu-\nu}{\lambda}} B^3 C \\ + \frac{6n(n+\lambda-2\mu-2\nu)(n+2\lambda-2\mu-2\nu)(n+3\lambda-2\mu-2\nu)}{1.2.3.4\lambda^4} A^{\frac{n-2\mu-2\nu}{\lambda}} B^2 C^2 \\ + \frac{4n(n+\lambda-\mu-3\nu)(n+2\lambda-\mu-3\nu)(n+3\lambda-\mu-3\nu)}{1.2.3.4\lambda^4} A^{\frac{n-\mu-3\nu}{\lambda}} B C^3 \\ + \frac{n(n+\lambda-4\nu)(n+2\lambda-4\nu)(n+3\lambda-4\nu)}{1.2.3.4\lambda^4} A^{\frac{n-4\nu}{\lambda}} C^4. \end{cases}$$

XIX.

Hinc iam quotcunque aequatio contineat terminos

$$1 = \frac{A}{y^\lambda} + \frac{B}{y^\mu} + \frac{C}{y^\nu} + \frac{D}{y^\xi} + \text{etc.}$$

in genere valor potestatis indefinitae y^n assignari poterit, aequabitur enim seriei ex infinito terminorum numero conflatae, qui ex omnibus quantitatum B, C, D etc. combinationibus nascuntur. Sufficiet igitur in genere terminum huic combinationi $B^\beta C^\gamma D^\delta$ etc. respondentem definiuisse, vbi pro β, γ, δ etc. successiue omnes numeri integri positiui

sitiui a cyphra 0, 1, 2, 3 etc. in infinitum substitui sunt intelligendi. Ad hunc autem terminum inueniendum primo indagari debet numerus combinationum formae $B^\beta C^\gamma D^\delta$ etc. quem statuamus $= N$ et posita exponentium summa $\beta + \gamma + \delta +$ etc. $= p$ notum est fore

$$N = \frac{1 \cdot 2 \cdot 3 \cdots\cdots\cdots p}{1 \cdot 2 \ldots \beta \cdot 1 \cdot 2 \ldots \gamma \cdot 1 \cdot 2 \ldots \delta \text{ etc.}}$$

deinde ponamus breuitatis gratia $\beta\mu + \gamma\nu + \delta\xi +$ etc. $= q$ atque terminus quaesitus formae $B^\beta C^\gamma D^\delta$ etc. conueniens erit $N \cdot \frac{n}{\lambda} \cdot \frac{n+\lambda-q}{2\lambda} \cdot \frac{n+2\lambda-q}{3\lambda} \cdot \frac{n+3\lambda-q}{4\lambda} \cdots \frac{n+(p-1)\lambda-q}{p\lambda} A^{\frac{n-q}{\lambda}} B^\beta C^\gamma D^\delta$ etc. Omnes ergo hi termini iunctim sumti verum valorem potestatis y^n determinabunt.

XX.

Euolutio aequationis $1 = \frac{A}{y} + By^3$.

Vt exemplum aequationis biquadraticae proferam, hanc aequationem, quae istam formam dat $By^4 = y - A$ euoluendam suscipio. Cum igitur sit $\lambda = 1$ et $\mu = -3$ hanc adipiscimur seriem,

$$y = A + A^4B + \tfrac{8}{2} A^7 B^2 + \frac{11 \cdot 12}{2 \cdot 3} A^{10} B^3 + \frac{14 \cdot 15 \cdot 16}{2 \cdot 3 \cdot 4} A^{13} B^4 + \frac{17 \cdot 18 \cdot 19 \cdot 20}{2 \cdot 3 \cdot 4 \cdot 5} A^{16} B^5 \text{ etc.}$$

In hac serie quilibet terminus ita pendet a praecedente, vt quisque terminus per praecedentem diuisus praebeat quotum huius formae $4\frac{(4n-3)(4n-2)(4n-1)}{3n(3n-1)(3n+1)} A^3 B$, ex quo summatio huius seriei perducitur ad aequationem differentialem

lem tertii gradus, quae facto $A = \frac{3}{4}u$ et $B = \frac{1}{4}$ vt aequatio proposita sit $y^4 = 4y - 3u$ ita se habebit

$$32(1-u^3)\,d^3y - 144\,uududdy - 86\,udu^2dy + 5\,ydu^3 = 0$$

sumto scilicet elemento du constante. Quemadmodum autem illa aequatio in hac contineatur, non perspicitur.

XXI.

Obseruo autem hanc aequationem integrabilem reddi si multiplicetur per y, singuli enim termini quatenus fieri potest integrati praebent vt sequitur:

$$\int y\,d^3y = y\,ddy - \tfrac{1}{2}dy^2 \quad \text{[per 32]}$$

$$\int u^3y\,d^3y = u^3yddy - \tfrac{1}{2}u^3dy^2 - 3uuydudy + 3uy^2du^2 + \tfrac{9}{2}\int uu\,dudy^2 - 3\int yy\,du^2 \quad \text{[per } -32]$$

$$\int uuyduddy = uuydudy - uyydu^2 - \int uudady^2 - \int yydu^3 \quad \text{[per } -144]$$

$$\int udu^2y\,dy = \tfrac{1}{2}uy^2du^2 - \tfrac{1}{2}\int yy\,du^3 \quad \text{[per } -86]$$

$$\int yy\,du^3 = \int yy\,du^3 \quad \text{[per 5]}$$

vnde nascitur haec forma integrata

$$16(1-u^3)(2yddy - dy^2) - 48\,uuydudy + 5\,uy^2du = Cdu^2$$

quae ponendo $y = zz$, ob $yy = z^4$, $ydy = 2z^3dz$ et $yddy + dy^2 = yddy + 4zzdz^2 = 2z^3ddz + 6zzdz^2$ ideoque $2yddy = 4z^3ddz + 4zzdz^2$ seu $2yddy - dy^2 = 4z^3ddz$ induit hanc formam

$$64(1-u^3)\,z^3ddz - 96uuz^3dudz + 5uz^4du^2 = Cdu^2,$$

$$\text{vel } 64(1-u^3)\,ddz - 96\,uu\,du\,dz + 5\,uz\,du^2 = \frac{Cdu^2}{z^3}$$

quae ergo hanc aequationem integralem $z^8 = 3zz - 3u$ in in se complectitur; idque casu quo constans $C = -9$, propterea quod est $y = \frac{3}{4}u + \frac{3^4}{4^5}u^4 + \frac{3^7}{4^8}u^7 +$ etc. ideoque sumto u infinite paruo $z = \frac{1}{2}\sqrt{3u}$.

XXII.

Cum nulla via pateat, hanc aequationem differentialem secundi gradus vlterius reducendi, operae pretium erit inuestigare, quomodo et quatenus ea cum aequatione finita $z^8 = 4zz - 3u$ conueniat. In hunc finem repraesentemus aequationem differentialem hac forma:

$$Lz^3 ddz + Mz^3 du dz + Nz^4 du^2 = C du^2$$

vt sit $L = 64(1 - u^3)$; $M = -96uu$; et $N = 5u$ at aequatio finita differentiata dat

$$8z^7 dz = 8z dz - 3du \text{ seu } 8dz(u - zz) = z du$$

vnde fit porro differentiando:

$$8ddz(u - zz) = 16z dz^2 - 7du dz = \frac{9z^3 - 7uz}{8(u - zz)^2} du^2.$$

Cum ergo sit

$$\frac{dz}{du} = \frac{z}{8(u - zz)} \text{ et } \frac{ddz}{du^2} = \frac{9z^3 - 7uz}{64(u - zz)^3}$$

prodibit facta substitutione haec aequatio

$$\frac{(1 - u^3)z^4(9zz - 7u)}{(u - zz)^3} - \frac{12uuz^4}{u - zz} + 5uz^4 = C \text{ seu}$$

$$(1 - u^3)z^4(9zz - 7u) - (7uu + 5uzz)z^4(u - zz)^2 + C(u - zz)^3 = 0$$

quae euoluta et ope aequationis $z^8 = 4zz - 3u$ ad potestates ipsius z octaua minores depressa perducit ad hanc:

$$(9 + C)z^6 - 3(9 + C)uz^4 + 3(9 + Cuuzz - (9 + C)u^3 = 0$$

cui valor $C = -9$ manifesto satisfacit.

XXIII.

XXIII.

Plus autem hinc concludere non licet, quam aequationem hanc $z^3 = 4zz - 3u$ contineri in hac aequatione differentio - differentiali:

$$64(1-u^3)z^3 ddz - 96uuz^3 du dz + 5uz^4 du^2 = C du^2$$

casu quo $C = -9$, interim tamen ne hoc quidem casu integrale completum exhibere licet, in quod praeterea duae quantitates constantes ingrediantur. Multo minus autem in genere quicunque valor ipsi C tribuatur, integrationem sperare poterimus cum ne casu quidem $C = 0$, methodis cognitis integrationem admittat. Ex quo intelligimus si aequationes algebraicae, quarum radices hic ad series infinitas perduximus, tertium gradum superent, serierum inde natarum summas nullius methodi adhuc cognitae ope inuestigari posse.

XXIV.

Coronidis loco adiungam problema inuersum, quo proposita huiusmodi aequatione cubica $y^3 + py + q = 0$, inuestigari oporteat aequationem differentialem secundi ordinis huius formae $ddy + Qdy + Ry = 0$, in qua illa contineatur: quae inuestigatio semper succedit; differentiatione enim bis instituta, indeque hic loco dy et ddy valoribus substitutis, vt termini prodeant solam quantitatem y eiusque potestates continentes, quas ope aequationis $y^3 + py + q = 0$ infra tertiam deprimere licebit: quo facto seorsim ad nihilum redigantur partes cum ab

y liberae, tum vero ipsam y eiusque quadratum yy continentes, vbi commode eueniet, vt simul ac binis conditionibus fuerit satisfactum; tertia sponte adimpleatur. Hoc autem modo calculum instituendo reperietur:

$$Q = \frac{18ppqdp^2 - 2(8p^3 - 27qq)dpdq - 54pqdq^2}{(3qdp - 2pdq)(4p^3 + 27qq)} + \frac{2pddq - 3qddp}{3qdp - 2pdq}$$

$$R = \frac{6p(dq^3 + pdp^2dq - qdp^3)}{(3qdp - 2pdq)(4p^3 + 27qq)} + \frac{dqddp - dpddq}{3qdp - 2pdq}$$

Haec autem aequatio per $\frac{4p^3 + 27qq}{(3qdp - 2pdq)^2}(2pdy - ydp)$ multiplicata integrabilis redditur, indeque porro pro y aequatio cubica latius patens quam proposita elicietur.

XXV.

Aequatio differentialis secundi gradus magis fit concinna si ponatur $qq = \frac{4p^3x}{17}$, fiet enim

$$ddy - dy\left(\frac{ddx}{dx} + \frac{dp}{p} - \frac{dx}{2x} - \frac{dx}{2(1+x)}\right) + y\left(\frac{dpddx}{2pdx} - \frac{ddp}{2p} + \frac{3dp^2}{4pp} - \frac{dpdx}{4px} - \frac{dpdx}{4p(1+x)} - \frac{-dx^2}{36x(1+x)}\right) = 0$$

quae per $\frac{x(1+x)}{ppdx^2}(2pdy - ydp)$ multiplicata et integrata praebet

$$\frac{x(1+x)}{pdx^2}\left(dy - \frac{ydp}{2p}\right)^2 = \frac{C}{18} + \frac{yy}{18p}$$

et ponendo $y = z\sqrt{p}$ hinc reperitur

$$\frac{3dz\sqrt{2}}{\sqrt{(C + zz)}} = \frac{dx}{\sqrt{x(1+x)}}$$

quae denuo integrata dat:

$$(z + \sqrt{(C + zz)})^3 = D\left(\tfrac{1}{2} + x + \sqrt{x(1+x)}\right)$$

unde tandem eruitur:

$$z = \frac{y}{\sqrt{p}} = A\left(\tfrac{1}{2} + x + \sqrt{x(1+x)}\right)^{\frac{1}{3}} + B\left(\tfrac{1}{2} + x - \sqrt{x(1+x)}\right)^{\frac{1}{3}}$$

ac cubo sumendo

$$z^3 = \tfrac{3}{4}ABz + (A^3 + B^3)\left(\tfrac{1}{2} + x\right) + (A^3 - B^3)\sqrt{x(1+x)}.$$

PRO-