SECTION IV.

Of Algebraic Equations, *and of the* Resolution *of those* Equations.

CHAP. I.

Of the Solution *of* Problems *in general.*

563. The principal object of Algebra, as well as of all the other branches of Mathematics, is to determine the value of quantities that were before unknown; and this is obtained by considering attentively the conditions given, which are always expressed in known numbers. For this reason, Algebra has been defined, *The science which teaches how to determine unknown quantities by means of those that are known.*

564. The above definition agrees with all that has been hitherto laid down: for we have always seen that the knowledge of certain quantities leads to that of other quantities, which before might have been considered as unknown.

Of this, Addition will readily furnish an example; for, in order to find the sum of two or more given numbers, we had to seek for an unknown number, which should be equal to those known numbers taken together. In Subtraction, we sought for a number which should be equal to the difference of two known numbers. A multitude of other examples are presented by Multiplication, Division, the Involution of powers, and the Extraction of roots; the question being always reduced to finding, by means of known quantities, other quantities which are unknown.

565. In the last section, also, different questions were resolved, in which it was required to determine a number that could not be deduced from the knowledge of other given numbers, except under certain conditions. All those questions were reduced to finding, by the aid of some given numbers, a new number, which should have a certain connexion with them; and this connexion was determined by

certain conditions, or properties, which were to agree with the quantity sought.

566. In Algebra, when we have a question to resolve, we represent the number sought by one of the last letters of the alphabet, and then consider in what manner the given conditions can form an equality between two quantities. This equality is represented by a kind of formula, called an *equation*, which enables us finally to determine the value of the number sought, and consequently to resolve the question. Sometimes several numbers are sought; but they are found in the same manner by equations.

567. Let us endeavour to explain this farther by an example. Suppose the following question, or *problem*, was proposed:

Twenty persons, men and women, dine at a tavern; the share of the reckoning for one man is 8 shillings, for one woman 7 shillings, and the whole reckoning amounts to 7*l.* 5*s.* Required the number of men and women separately?

In order to resolve this question, let us suppose that the number of men is $= x$; and, considering this number as known, we shall proceed in the same manner as if we wished to try whether it corresponded with the conditions of the question. Now, the number of men being $= x$, and the men and women making together twenty persons, it is easy to determine the number of the women, having only to subtract that of the men from 20, that is to say, the number of women must be $20 - x$.

But each man spends 8 shillings; therefore x men must spend $8x$ shillings. And since each woman spends 7 shillings, $20 - x$ women must spend $140 - 7x$ shillings. So that adding together $8x$ and $140 - 7x$, we see that the whole 20 persons must spend $140 + x$ shillings. Now, we know already how much they have spent; namely, 7*l.* 5*s.* or 145*s.*; there must be an equality, therefore, between $140 + x$ and 145; that is to say, we have the equation $140 + x = 145$, and thence we easily deduce $x = 5$, and consequently $20 - x = 20 - 5 = 15$; so that the company consisted of 5 men, and 15 women.

568. Again, Suppose twenty persons, men and women, go to a tavern; the men spend 24 shillings, and the women as much: but it is found that the men have spent 1 shilling each more than the women. Required the number of men and women separately?

Let the number of men be represented by x.

Then the women will be $20 - x$.

Now, the x men having spent 24 shillings, the share of each man is $\frac{24}{x}$. The $20 - x$ women having also spent 24 shillings, the share of each woman is $\frac{24}{20-x}$.

But we know that the share of each woman is one shilling less than that of each man; if, therefore, we subtract 1 from the share of a man, we must obtain that of a woman; and consequently $\frac{24}{x} - 1 = \frac{24}{20-x}$. This, therefore, is the equation, from which we are to deduce the value of x. This value is not found with the same ease as in the preceding question; but we shall afterwards see that $x = 8$, which value answers to the equation; for $\frac{24}{8} - 1 = \frac{24}{12}$ includes the equality $2 = 2$.

569. It is evident therefore how essential it is, in all problems, to consider the circumstances of the question attentively, in order to deduce from it an equation that shall express by letters the numbers sought, or unknown. After that, the whole art consists in resolving those equations, or deriving from them the values of the unknown numbers; and this shall be the subject of the present section.

570. We must remark, in the first place, the diversity which subsists among the questions themselves. In some, we seek only for one unknown quantity; in others, we have to find two, or more; and, it is to be observed, with regard to this last case, that, in order to determine them all, we must deduce from the circumstances, or the conditions of the problem, as many equations as there are unknown quantities.

571. It must have already been perceived, that an equation consists of two parts separated by the sign of equality, $=$, to shew that those two quantities are equal to one another; and we are often obliged to perform a great number of transformations on those two parts, in order to deduce from them the value of the unknown quantity: but these transformations must be all founded on the following principles, namely, That two equal quantities remain equal, whether we add to them, or subtract from them, equal quantities; whether we multiply them, or divide them, by the same number; whether we raise them both to the same power, or extract their roots of the same degree; or lastly,

whether we take the logarithms of those quantities, as we have already done in the preceding section.

572. The equations which are most easily resolved, are those in which the unknown quantity does not exceed the first power, after the terms of the equation have been properly arranged; and these are called *simple equations*, or *equations of the first degree*. But if, after having reduced an equation, we find in it the square, or the second power, of the unknown quantity, it is called an *equation of the second degree*, which is more difficult to resolve. *Equations of the third degree* are those which contain the cube of the unknown quantity, and so on. We shall treat of all these in the present section.

CHAP. II.

Of the Resolution *of* Simple Equations, *or* Equations *of the* First Degree.

573. When the number sought, or the unknown quantity, is represented by the letter x, and the equation we have obtained is such, that one side contains only that x, and the other simply a known number, as, for example, $x = 25$, the value of x is already known. We must always endeavour, therefore, to arrive at such a form, however complicated the equation may be when first obtained: and, in the course of this section, the rules shall be given, and explained, which serve to facilitate these reductions.

574. Let us begin with the simplest cases, and suppose, first, that we have arrived at the equation $x + 9 = 16$. Here we see immediately that $x = 7$: and, in general, if we have found $x + a = b$, where a and b express any known numbers, we have only to subtract a from both sides, to obtain the equation $x = b - a$, which indicates the value of x.

575 If we have the equation $x - a = b$, we must add a to both sides, and shall obtain the value of $x = b + a$. We must proceed in the same manner, if the equation have this form, $x - a = a^2 + 1$: for we shall find immediately $x = a^2 + a + 1$.

In the equation $x - 8a = 20 - 6a$, we find

$$x = 20 - 6a + 8a, \text{ or } x = 20 + 2a.$$

And in this, $x + 6a = 20 + 3a$, we have

$$x = 20 + 3a - 6a, \text{ or } x = 20 - 3a.$$

576. If the original equation have this form, $x - a + b = c$, we may begin by adding a to both sides, which will give $x + b = c + a$; and then subtracting b from both sides, we shall find $x = c + a - b$. But we might also add $+ a - b$ at once to both sides; and thus obtain immediately $x = c + a - b$.

So likewise in the following examples:

If $x - 2a + 3b = 0$, we have $x = 2a - 3b$.

If $x - 3a + 2b = 25 + a + 2b$, we have $x = 25 + 4a$.

If $x - 9 + 6a = 25 + 2a$, we have $x = 34 - 4a$.

577. When the given equation has the form $ax = b$, we only divide the two sides by a, to obtain $x = \frac{b}{a}$. But if the equation has the form $ax + b - c = d$, we must first make the terms that accompany ax vanish, by adding to both sides $-b + c$; and then dividing the new equation $ax = d - b + c$ by a, we shall have $x = \frac{d-b+c}{a}$.

The same value of x would have been found by subtracting $+b - c$ from the given equation; that is, we should have had, in the same form,

$ax = d - b + c$, and $x = \frac{d-b+c}{a}$. Hence,

If $2x + 5 = 17$, we have $2x = 12$, and $x = 6$.

If $3x - 8 = 7$, we have $3x = 15$, and $x = 5$.

If $4x - 5 - 3a = 15 + 9a$, we have $4x = 20 + 12a$, and consequently $x = 5 + 3a$.

578. When the first equation has the form $\frac{x}{a} = b$, we multiply both sides by a, in order to have $x = ab$.

But if it is $\frac{x}{a} + b - c = d$, we must first make $\frac{x}{a} = d - b + c$; after which we find

$$x = (d - b + c)\,a = ad - ab + ac.$$

Let $\frac{1}{2}x - 3 = 4$, then $\frac{1}{2}x = 7$, and $x = 14$.

Let $\frac{1}{3}x - 1 + 2a = 3 + a$, then $\frac{1}{3}x = 4 - a$, and $x = 12 - 3a$.

Let $\frac{x}{a-1} - 1 = a$, then $\frac{x}{a-1} = a + 1$, and $x = a^2 - 1$.

579. When we have arrived at such an equation as

$\frac{ax}{b} = c$, we first multiply by b, in order to have $ax = bc$, and then dividing by a, we find $x = \frac{bc}{a}$.

If $\frac{ax}{b} - c = d$, we begin by giving the equation this form $\frac{ax}{b} = d + c$; after which we obtain the value of $ax = bd + bc$, and then that of $x = \frac{bd+bc}{a}$.

Let $\frac{2}{3}x - 4 = 1$, then $\frac{2}{3}x = 5$, and $2x = 15$; whence $x = \frac{15}{2} = 7\frac{1}{2}$.

If $\frac{3}{4}x + \frac{1}{2} = 5$, we have $\frac{3}{4}x = 5 - \frac{1}{2} = \frac{9}{2}$; whence $3x = 18$, and $x = 6$.

580. Let us now consider a case, which may frequently occur; that is, when two or more terms contain the letter x, either on one side of the equation, or on both.

If those terms are all on the same side, as in the equation $x + \frac{1}{2}x + 5 = 11$, we have $x + \frac{1}{2}x = 6$; or $3x = 12$; and lastly, $x = 4$.

Let $x + \frac{1}{2}x + \frac{1}{3}x = 44$, be an equation, in which the value of x is required. If we first multiply by 3, we have $4x + \frac{3}{2}x = 132$; then multiplying by 2, we have $11x = 264$; wherefore $x = 24$. We might have proceeded in a more concise manner, by beginning with the reduction of the three terms which contain x to the single term $\frac{11}{6}x$; and then dividing the equation $\frac{11}{6}x = 44$ by 11. This would have given $\frac{1}{6}x = 4$, and $x = 24$, as before.

Let $\frac{2}{3}x - \frac{3}{4}x + \frac{1}{2}x = 1$. We shall have, by reduction, $\frac{5}{12}x = 1$, and $x = 2\frac{2}{5}$.

And, generally, let $ax - bx + cx = d$; which is the same as $(a - b + c)x = d$, and, by division, we derive $x = \frac{d}{a-b+c}$.

581. When there are terms containing x on both sides of the equation, we begin by making such terms vanish from that side from which it is most easily expunged; that is to say, in which there are the fewest terms so involved.

If we have, for example, the equation $3x + 2 = x + 10$, we must first subtract x from both sides, which gives $2x + 2 = 10$; wherefore $2x = 8$, and $x = 4$.

Let $x + 4 = 20 - x$; here it is evident that $2x + 4 = 20$; and consequently $2x = 16$, and $x = 8$.

Let $x + 8 = 32 - 3x$, this gives us $4x + 8 = 32$; or $4x = 24$, whence $x = 6$.

Let $15 - x = 20 - 2x$, here we shall have

$$15 + x = 20, \text{ and } x = 5.$$

Let $1 + x = 5 - \frac{1}{2}x$; this becomes $1 + \frac{3}{2}x = 5$, or $\frac{3}{2}x = 4$; therefore $3x = 8$; and lastly, $x = \frac{8}{3} = 2\frac{2}{3}$.

If $\frac{1}{2} - \frac{1}{3}x = \frac{1}{3} - \frac{1}{4}x$, we must add $\frac{1}{3}x$, which gives $\frac{1}{2} = \frac{1}{3} + \frac{1}{12}x$; subtracting $\frac{1}{3}$, and transposing the terms, there remains $\frac{1}{12}x = \frac{1}{6}$; then multiplying by 12, we obtain $x = 2$.

If $1\frac{1}{2} - \frac{2}{3}x = \frac{1}{4} + \frac{1}{2}x$, we add $\frac{2}{3}x$, which gives $1\frac{1}{2} = \frac{1}{4} + \frac{7}{6}x$; then subtracting $\frac{1}{4}$, and transposing, we have $\frac{7}{6}x = 1\frac{1}{4}$, whence we deduce $x = 1\frac{1}{14} = \frac{15}{14}$ by multiplying by 6 and dividing by 7.

582. If we have an equation in which the unknown number x is a denominator, we must make the fraction vanish by multiplying the whole equation by that denominator.

Suppose that we have found $\frac{100}{x} - 8 = 12$, then, adding 8, we have $\frac{100}{x} = 20$; and multiplying by x, it becomes $100 = 20x$; lastly, dividing by 20, we find $x = 5$.

Let now $\frac{5x+3}{x-1} = 7$; here multiplying by $x - 1$, we have $5x + 3 = 7x - 7$; and subtracting $5x$, there remains $3 = 2x - 7$; then adding 7, we have $2x = 10$; whence $x = 5$.

583. Sometimes, also, radical signs are found in equations of the first degree. For example: A number x, below 100, is required, such, that the square root of $100 - x$ may be equal to 8; or $\sqrt{(100 - x)} = 8$. The square of both sides will give $100 - x = 64$, and adding x, we have $100 = 64 + x$; whence we obtain $x = 100 - 64 = 36$.

Or, since $100 - x = 64$, we might have subtracted 100 from both sides; which would have given $-x = -36$; or, multiplying by -1, $x = 36$.

584. Lastly, the unknown number x is sometimes found as an exponent, of which we have already seen some examples; and, in this case, we must have recourse to logarithms.

Thus, when we have $2^x = 512$, we take the logarithms of both sides; whence we obtain $x \; log.\, 2 = log.\, 512$; and dividing by $log.\, 2$, we find $x = \frac{log.\, 512}{log.\, 2}$. The Tables then

give $x = \frac{2{\cdot}7092700}{0{\cdot}3010300} = \frac{27092700}{3010300}$, or $x = 9$.

Let $5 \times 3^{2x} - 100 = 305$; we add 100, which gives $5 \times 3^{2x} = 405$; dividing by 5, we have $3^{2x} = 81$; and taking the logarithms, $2x \; log.\, 3 = log.\, 81$, and dividing by $2 \; log.\, 3$, we have $x = \frac{log.\, 81}{2 log.\, 3}$, or $x = \frac{log.\, 81}{log.\, 9}$; whence

$$x = \frac{1{\cdot}9084850}{0{\cdot}9542425} = \frac{19084850}{9542425} = 2.$$

QUESTIONS FOR PRACTICE.

1. If $x - 4 + 6 = 8$, then will $x = 6$.
2. If $4x - 8 = 3x + 20$, then will $x = 28$.
3. If $ax = ab - a$, then will $x = b - 1$.
4. If $2x + 4 = 16$, then will $x = 6$.
5. If $ax + 2ba = 3c^2$, then will $x = \frac{3c^2}{a} - 2b$.
6. If $\frac{x}{2} = 5 + 3$, then will $x = 16$.
7. If $\frac{2x}{3} - 2 = 6 + 4$, then will $2x - 6 = 18$.
8. If $a - \frac{b}{x} = c$, then will $x = \frac{b}{a-c}$.
9. If $5x - 15 = 2x + 6$, then will $x = 7$.
10. If $40 - 6x - 16 = 120 - 14x$, then will $x = 12$.
11. If $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 10$, then will $x = 24$.
12. $\frac{x-3}{2} + \frac{x}{3} = 20 - \frac{x-19}{2}$, then will $x = 23\frac{1}{4}$.
13. If $\sqrt{\frac{2}{3}x} + 5 = 7$, then will $x = 6$.
14. If $x + \sqrt{(a^2 + x^2)} = \frac{2a^2}{\sqrt{(a^2+x^2)}}$, then will $x = a\sqrt{\frac{1}{3}}$.
15. If $3ax + \frac{a}{2} - 3 = bx - a$, then will $x = \frac{6-3a}{6a-2b}$.
16. If $\sqrt{(12 + x)} = 2 + \sqrt{x}$, then will $x = 4$.
17. If $y + \sqrt{(a^2 + y^2)} = \frac{2a^2}{(a^2+y^2)^{\frac{1}{2}}}$, then will $y = \frac{1}{3}a\sqrt{3}$.
18. If $\frac{y+1}{2} + \frac{y+2}{3} = 16 - \frac{y+3}{4}$, then will $y = 13$.

19. If $\sqrt{x}+\sqrt{(a+x)}=\frac{2a}{\sqrt{(a+x)}}$, then will $x=\frac{a}{3}$.

20. If $\sqrt{(aa+xx)}=\sqrt[4]{(b^4+x^4)}$, then will $x=\frac{\sqrt{(b4-a^4)}}{2a^2}$.

21. If $y=\sqrt{(a^2+\sqrt{(b^2+x^2)})}-a$, then will $x=\frac{bb}{4a}-a$.

22. If $\frac{128}{3x-4}=\frac{216}{5x-6}$, then will $x=12$.

23. If $\frac{42x}{x-2}=\frac{35x}{x-3}$, then will $x=8$.

24. If $\frac{45}{2x+3}=\frac{57}{4x-5}$, then will $x=6$.

25. If $\frac{x^2-12}{3}=\frac{x^2-4}{4}$, then will $x=6$.

26. If $615x-7x^3=48x$, then will $x=9$.

CHAP. III.

Of the Solution *of* Questions *relating to the preceding* Chapter.

585. *Question* 1. To divide 7 into two such parts that the greater may exceed the less by 3.

Let the greater part be x, then the less will be $7-x$; so that $x=7-x+3$, or $x=10-x$. Adding x, we have $2x=10$; and dividing by 2, $x=5$.

The two parts therefore are 5 and 2.

Question 2. It is required to divide a into two parts, so that the greater may exceed the less by b.

Let the greater part be x, then the other will be $a-x$; so that $x=a-x+b$. Adding x, we have $2x=a+b$; and dividing by 2, $x=\frac{a+b}{2}$.

Another method of solution. Let the greater part $=x$; which as it exceeds the less by b, it is evident that this is less than the other by b, and therefore must be $=x-b$. Now, these two parts, taken together, ought to make a; so that $2x-b=a$; adding b, we have $2x=a+b$, wherefore $x=\frac{a+b}{2}$, which is the value of the greater part; and that of the less will be $\frac{a+b}{2}-b$, or $\frac{a+b}{2}-\frac{2b}{2}$, or $\frac{a-b}{2}$.

586. *Question* 3. A father leaves 1600 pounds to be divided among his three sons in the following manner; viz. the eldest is to have 200 pounds more than the second, and the second 100 pounds more than the youngest. Required the share of each.

Let the share of the third son be x
Then the second's will be - - $x+100$; and
The first son's share - - - $x+300$.

Now, these three sums make up together 1600*l.*; we have, therefore,

$3x+400=1600$
$3x=1200$
and $x=400$

The share of the youngest is 400*l.*
That of the second is - - 500*l.*
That of the eldest is - - 700*l.*

587. *Question* 4. A father leaves to his four sons 8600*l.* and, according to the will, the share of the eldest is to be double that of the second, minus 100*l.*; the second is to receive three times as much as the third, minus 200*l.*; and the third is to receive four times as much as the fourth, minus 300*l.* What are the respective portions of these four sons?

Call the youngest son's share x
Then the third son's is - $4x-300$
The second son's is - - $12x-1100$
And the eldest's - - - $24x-2300$

Now, the sum of these four shares must make 8600*l.* We have, therefore, $41x-3700=8600$, or
$41x=12300$, and $x=300$.

Therefore the youngest's share is 300*l.*
The third son's - - - - - - 900*l.*
The second's - - - - - - 2500*l.*
The eldest's - - - - - - - 4900*l.*

588. *Question* 5. A man leaves 11000 crowns to be divided between his widow, two sons, and three daughters. He intends that the mother should receive twice the share of a son, and that each son should receive twice as much

as a daughter. Required how much each of them is to receive.

Suppose the share of each daughter to be x
Then each son's is consequently - - - $2x$
And the widow's - - - - - - - - $4x$

The whole inheritance, therefore, is $3x + 4x + 4x$; or $11x = 11000$, and, consequently, $x = 1000$.

Each daughter, therefore, is to receive 1000 crowns;
So that the three receive in all 3000
Each son receives 2000;
So that the two sons receive - 4000
And the mother receives - - 4000

Sum 11000 crowns

589. *Question* 6. A father intends by his will, that his three sons should share his property in the following manner: the eldest is to receive 1000 crowns less than half the whole fortune; the second is to receive 800 crowns less than the third of the whole; and the third is to have 600 crowns less than the fourth of the whole. Required the sum of the whole fortune, and the portion of each son.

Let the fortune be expressed by x:
The share of the first son is $\frac{1}{2}x - 1000$
That of the second - - - $\frac{1}{3}x - 800$
That of the third - - - $\frac{1}{4}x - 600$

So that the three sons receive in all $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x - 2400$, and this sum must be equal to x. We have, therefore, the equation $\frac{13}{12}x - 2400 = x$; and subtracting x, there remains $\frac{1}{12}x - 2400 = 0$; then adding 2400, we have $\frac{1}{12}x = 2400$; and, lastly, multiplying by 12, we obtain $x = 28800$.

The fortune, therefore, consists of 28800 crowns; and
The eldest son receives 13400 crowns
The second - - - - 8800
The youngest - - - 6600

28800 crowns.

590. *Question* 7. A father leaves four sons, who share his property in the following manner: the first takes the half of the fortune, minus 3000*l.*; the second takes the third, minus 1000*l.*; the third takes exactly the fourth of the property; and the fourth takes 600*l.* and the fifth part of the property. What was the whole fortune, and how much did each son receive?

Let the whole fortune be represented by x:
Then the eldest son will have $\frac{1}{2}x - 3000$
The second - - - - - $\frac{1}{3}x - 1000$
The third - - - - - $\frac{1}{4}x$
The youngest - - - - $\frac{1}{5}x + 600$.

And the four will have received in all $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x + \frac{1}{5}x - 3400$, which must be equal to x.

Whence results the equation $\frac{77}{60}x - 3400 = x$; then subtracting x, we have $\frac{17}{60}x - 3400 = 0$; adding 3400, we obtain $\frac{17}{60}x = 3400$; then dividing by 17, we have $\frac{1}{60}x = 200$; and multiplying by 60, gives $x = 12000$.

The fortune therefore consisted of 12000*l.*
The first son received 3000
The second - - - 3000
The third - - - 3000
The fourth - - - 3000

591. *Question* 8. To find a number such, that if we add to it its half, the sum exceeds 60 by as much as the number itself is less than 65.

Let the number be represented by x:
Then, $x + \frac{1}{2}x - 60 = 65 - x$, or $\frac{3}{2}x - 60 = 65 - x$. Now, by adding x, we have $\frac{5}{2}x - 60 = 65$; adding 60, we have $\frac{5}{2}x = 125$; dividing by 5, gives $\frac{1}{2}x = 25$; and multiplying by 2, we have $x = 50$.

Consequently, the number sought is 50.

592. *Question* 9. To divide 32 into two such parts, that if the less be divided by 6, and the greater by 5, the two quotients taken together may make 6.

Let the less of the two parts sought be x; then the greater will be $32 - x$. The first, divided by 6, gives $\frac{x}{6}$; and the second, divided by 5, gives $\frac{32-x}{5}$. Now $\frac{x}{6} + \frac{32-x}{5} = 6$: so that multiplying by 5, we have $\frac{5}{6}x + 32 - x = 30$, or $-\frac{1}{6}x + 32 = 30$; adding $\frac{1}{6}x$, we have $32 = 30 + \frac{1}{6}x$; subtracting 30, there remains $2 = \frac{1}{6}x$; and lastly, multiplying by 6, we have $x = 12$.

So that the less part is 12, and the greater part is 20.

593. *Question* 10. To find such a number, that if multiplied by 5, the product shall be as much less than 40 as the number itself is less than 12.

Let the number be x; which is less than 12 by $12 - x$; then taking the number x five times, we have $5x$, which is less than 40 by 40 — $5x$, and this quantity must be equal to $12 - x$.

We have, therefore, $40 - 5x = 12 - x$. Adding $5x$, we have $40 = 12 + 4x$; and subtracting 12, we obtain $28 = 4x$; lastly, dividing by 4, we have $x = 7$, the number sought.

594. *Question* 11. To divide 25 into two such parts, that the greater may be equal to 49 times the less.

Let the less part be x, than the greater will be $25 - x$; and the latter divided by the former ought to give the quotient 49: we have therefore $\frac{25-x}{x} = 49$. Multiplying by x, we have $25 - x = 49x$; adding x, we have $25 = 50x$; and dividing by 50, gives $x = \frac{1}{2}$.

The less of the two numbers is $\frac{1}{2}$, and the greater is $24\frac{1}{2}$; dividing therefore the latter by $\frac{1}{2}$, or multiplying by 2, we obtain 49.

595. *Question* 12. To divide 48 into nine parts, so that every part may be always $\frac{1}{2}$ greater than the part which precedes it.

Let the first, or least part be x, then the second will be $x + \frac{1}{2}$, the third $x + 1$, &c.

Now, these parts form an arithmetical progression, whose first term is x; therefore the ninth and last term will be $x + 4$. Adding those two terms together, we have $2x + 4$; multiplying this quantity by the number of terms, or by 9, we have $18x + 36$; and dividing this product by 2, we obtain the sum of all the nine parts $= 9x + 18$; which ought to be equal to 48. We have, therefore, $9x + 18 = 48$; subtracting 18, there remains $9x = 30$; and dividing by 9, we have $x = 3\frac{1}{3}$.

The first part, therefore, is $3\frac{1}{3}$, and the nine parts will succeed in the following order:

$$\overset{1}{3\tfrac{1}{3}} + \overset{2}{3\tfrac{5}{6}} + \overset{3}{4\tfrac{1}{3}} + \overset{4}{4\tfrac{5}{6}} + \overset{5}{5\tfrac{1}{3}} + \overset{6}{5\tfrac{5}{6}} + \overset{7}{6\tfrac{1}{3}} + \overset{8}{6\tfrac{5}{6}} + \overset{9}{7\tfrac{1}{3}}.$$

Which together make 48.

596. *Question* 13. To find an arithmetical progression, whose first term is 5, the last term 10, and the entire sum 60.

Here we know neither the difference nor the number of terms; but we know that the first and the last term would enable us to express the sum of the progression, provided only the number of terms were given. We shall therefore suppose this number to be x, and express the sum of the

progression by $\frac{15x}{2}$. We know also, that this sum is 60; so that $\frac{15x}{2} = 60$; or $\frac{1}{2}x = 4$, and $x = 8$.

Now, since the number of terms is 8, if we suppose the difference to be z, we have only to seek for the eighth term upon this supposition, and to make it equal to 10. The second term is $5 + z$, the third is $5 + 2z$, and the eighth is $5 + 7z$; so that

$$5 + 7z = 10$$
$$7z = 5$$
$$\text{and } z = \tfrac{5}{7}.$$

The difference of the progression, therefore, is $\frac{5}{7}$, and the number of terms is 8; consequently, the progression is

$$\overset{1}{5} + \overset{2}{5\tfrac{5}{7}} + \overset{3}{6\tfrac{3}{7}} + \overset{4}{7\tfrac{1}{7}} + \overset{5}{7\tfrac{6}{7}} + \overset{6}{8\tfrac{4}{7}} + \overset{7}{9\tfrac{2}{7}} + \overset{8}{10},$$

the sum of which is 60.

597. *Question* 14. To find such a number, that if 1 be subtracted from its double, and the remainder be doubled, from which if 2 be subtracted, and the remainder divided by 4, the number resulting from these operations shall be 1 less than the number sought.

Suppose this number to be x; the double is $2x$; subtracting 1, there remains $2x - 1$; doubling this, we have $4x - 2$; subtracting 2, there remains $4x - 4$; dividing by 4, we have $x - 1$; and this must be 1 less than x; so that

$$x - 1 = x - 1.$$

But this is what is called an *identical equation;* and shews that x is indeterminate; or that any number whatever may be substituted for it.

598. *Question* 15. I bought some ells of cloth at the rate of 7 crowns for 5 ells, which I sold again at the rate of 11 crowns for 7 ells, and I gained 100 crowns by the transaction. How much cloth was there?

Supposing the number of ells to be x, we must first see how much the cloth cost; which is found by the following proportion:

$$\text{As } 5 : x :: 7 : \frac{7x}{5} \text{ the price of the ells.}$$

This being the expenditure; let us now see the receipt: in order to which, we must make the following proportion:

E. C. E.

As $7 : 11 :: x : \frac{11}{7}x$ crowns;

and this receipt ought to exceed the expenditure by 100 crowns. We have, therefore, this equation:

$$\frac{11}{7}x = \frac{7}{5}x + 100.$$

Subtracting $\frac{7}{5}x$, there remains $\frac{6}{35}x = 100$; therefore $6x = 3500$, and $x = 583\frac{1}{3}$.

There were, therefore, $583\frac{1}{3}$ ells bought for $816\frac{2}{3}$ crowns, and sold again for $916\frac{2}{3}$ crowns; by which means the profit was 100 crowns.

599. *Question* 16. A person buys 12 pieces of cloth for 140*l.*; of which two are white, three are black, and seven are blue: also, a piece of the black cloth costs two pounds more than a piece of the white, and a piece of the blue cloth costs three pounds more than a piece of the black. Required the price of each kind.

Let the price of a white piece be x pounds; then the two pieces of this kind will cost $2x$; also, a black piece costing $x + 2$, the three pieces of this color will cost $3x + 6$; and lastly, as a blue piece costs $x + 5$, the seven blue pieces will cost $7x + 35$: so that the twelve pieces amount in all to $12x + 41$.

Now, the known price of these twelve pieces is 140 pounds; we have, therefore, $12x + 41 = 140$, and $12x = 99$; wherefore $x = 8\frac{1}{4}$. So that

A piece of white cloth costs $8\frac{1}{4}l.$
A piece of black cloth costs $10\frac{1}{4}l.$
A piece of blue cloth costs $13\frac{1}{4}l.$

600. *Question* 17. A man having bought some nutmegs, says that three of them cost as much more than one penny, as four cost him more than two pence halfpenny. Required the price of the nutmegs?

Let x be the excess of the price of three nuts above one penny, or four farthings. Now, if three nutmegs cost $x + 4$ farthings, four will cost, by the condition of the question, $x + 10$ farthings; but the price of three nutmegs gives that of four in another way, namely, by the Rule of Three. Thus,

$$3 : x + 4 :: 4 : \frac{4x + 16}{3}.$$

So that $\frac{4x+16}{3} = x + 10$; or, $4x + 16 = 3x + 30$; therefore $x + 16 = 30$, and $x = 14$.

Three nutmegs, therefore, cost $4\frac{1}{2}d.$, and four cost $6d.$: wherefore each costs $1\frac{1}{2}d.$

601. *Question* 18. A certain person has two silver cups, and only one cover for both. The first cup weighs 12 ounces; and if the cover be put on it, it weighs twice as much as the other cup: but when the other cup has the cover, it weighs three times as much as the first. Required the weight of the second cup, and that of the cover.

Suppose the weight of the cover to be x ounces; then the first cup being covered it will weigh $x + 12$; and this weight being double that of the second, the second cup must weigh $\frac{1}{2}x + 6$; and, with the cover, it will weigh $x + \frac{1}{2}x + 6$, $\frac{3}{2}x + 6$; which weight ought to be the triple of 12; that is, three times the weight of the first cup. We shall therefore have the equation $\frac{3}{2}x + 6 = 36$, or $\frac{3}{2}x = 30$; so that $\frac{1}{2}x = 10$ and $x = 20$.

The cover, therefore, weighs 20 ounces, and the second cup weighs 16 ounces.

602. *Question* 19. A banker has two kinds of change: there must be a pieces of the first to make a crown; and b pieces of the second to make the same. Now, a person wishes to have c pieces for a crown. How many pieces of each kind must the banker give him?

Suppose the banker gives x pieces of the first kind; it is evident that he will give $c - x$ pieces of the other kind; but the x pieces of the first are worth $\frac{x}{a}$ crown, by the proportion $a : x :: 1 : \frac{x}{a}$; and the $c - x$ pieces of the second kind are worth $\frac{c-x}{b}$ crown, because we have $b : c - x :: 1 : \frac{c-x}{b}$. So that, $\frac{x}{a} + \frac{c-x}{b} = 1$;

or $\frac{bx}{a} + c - x = b$; or $bx + ac - ax = ab$;

or, rather $bx - ax = ab - ac$;

whence we have $x = \frac{ab-ac}{b-a}$, or $x = \frac{a(b-c)}{b-a}$;

consequently, $c - x$, the pieces of the second kind, must be

$$= \frac{bc-ab}{b-a} = \frac{b(c-a)}{b-a}.$$

The banker must therefore give $\frac{a(b-c)}{b-a}$ pieces of the first kind, and $\frac{b(c-a)}{b-a}$ pieces of the second kind.

Remark. These two numbers are easily found by the Rule of Three, when it is required to apply the results which we have obtained. Thus, to find the first we say, $b-a:a::b-c:\frac{a(b-c)}{b-a}$; and the second number is found thus; $b-a:b::c-a:\frac{b(c-a)}{b-a}$.

It ought to be observed also, that a is less than b, and that c is less than b; but at the same time greater than a, as the nature of the thing requires.

603. *Question* 20. A banker has two kinds of change; 10 pieces of one make a crown, and 20 pieces of the other make a crown; and a person wishes to change a crown into 17 pieces of money: how many of each sort must he have?

We have here $a=10$, $b=20$, and $c=17$, which furnishes the following proportions:

First, $10:10::3:3$, so that the number of pieces of the first kind is 3.

Secondly, $10:20::7:14$, and the number of the second kind is 14.

604. *Question* 21. A father leaves at his death several children, who share his property in the following manner: namely, the first receives a hundred pounds, and the tenth part of the remainder; the second receives two hundred pounds, and the tenth part of the remainder; the third takes three hundred pounds, and the tenth part of what remains; and the fourth takes four hundred pounds, and the tenth part of what then remains; and so on. And it is found that the property has thus been divided equally among all the children. Required how much it was, how many children there were, and how much each received?

This question is rather of a singular nature, and therefore deserves particular attention. In order to resolve it more easily, we shall suppose the whole fortune to be z pounds; and since all the children receive the same sum, let the share of each be x, by which means the number of children will be expressed by $\frac{z}{x}$. Now, this being laid down, we may proceed to the solution of the question, as follows:

Sum or property to be divided.	Order of the children.	Portion of each.	Differences.
z	1st.	$x=100+\frac{z-100}{10}$	
$z-x$	2d.	$x=200+\frac{z-x-200}{10}$	$100-\frac{x-100}{10}=0$
$z-2x$	3d.	$x=300+\frac{z-2x-300}{10}$	$100-\frac{x-100}{10}=0$
$z-3x$	4th.	$x=400+\frac{z-3x-400}{10}$	$100-\frac{x-100}{10}=0$
$z-4x$	5th.	$x=500+\frac{z-4x-500}{10}$	$100-\frac{x-100}{10}=0$
$z-5x$	6th.	$x=600+\frac{z-5x-600}{10}$	and so on.

We have inserted, in the last column, the differences which we obtain by subtracting each portion from that which follows; but all the portions being equal, each of the differences must be $=0$. As it happens also, that all these differences are expressed exactly alike, it will be sufficient to make one of them equal to nothing, and we shall have the equation $100-\frac{x-100}{10}=0$. Here, multiplying by 10, we have $1000-x-100=0$, or $900-x=0$; and, consequently, $x=900$.

We know now, therefore, that the share of each child was 900; so that taking any one of the equations of the third column, the first, for example, it becomes, by substituting the value of x, $900=100+\frac{z-100}{10}$, whence we immediately obtain the value of z; for we have

$$9000=1000+z-100, \text{ or } 9000=900+z;$$

therefore $z=8100$; and consequently $\frac{z}{x}=9$.

So that the number of children was 9; the fortune left by the father was 8100 pounds; and the share of each child was 900 pounds.

QUESTIONS FOR PRACTICE.

1. To find a number, to which if there be added a half, a third, and a fourth of itself, the sum will be 50. *Ans.* 24.

2. A person being asked what his age was, replied that $\frac{3}{4}$ of his age multiplied by $\frac{1}{12}$ of his age gives a product equal to his age. What was his age? *Ans.* 16.

3. The sum of 660*l.* was raised for a particular purpose by four persons, A, B, C, and D; B advanced twice as much as A; C as much as A and B together; and D as much as B and C. What did each contribute?

Ans. 60*l.*, 120*l.*, 180*l.*, and 300*l.*

4. To find that number whose $\frac{1}{3}$ part exceeds its $\frac{1}{4}$ part by 12. *Ans.* 144.

5. What sum of money is that whose $\frac{1}{3}$ part, $\frac{1}{4}$ part, and $\frac{1}{5}$ part, added together, shall amount to 94 pounds?

Ans. 120*l.*

6. In a mixture of copper, tin, and lead, one half of the whole −16*lb.* was copper; one-third of the whole −12*lb.* tin; and one-fourth of the whole +4*lb.* lead: what quantity of each was there in the composition?

Ans. 128*lb.* of copper, 84*lb.* of tin, and 76*lb.* of lead.

7. A bill of 120*l.* was paid in guineas and moidores, and the number of pieces of both sorts were just 100; to find how many there were of each. *Ans.* 50.

8. To find two numbers in the proportion of 2 to 1, so that if 4 be added to each, the two sums shall be in the proportion of 3 to 2. *Ans.* 4 and 8.

9. A trader allows 100*l.* per annum for the expenses of his family, and yearly augments that part of his stock which is not so expended, by a third part of it; at the end of three years, his original stock was doubled: what had he at first?

Ans. 1480*l.*

10. A fish was caught whose tail weighed 9*lb.* His head weighed as much as his tail and $\frac{1}{2}$ his body; and his body weighed as much as his head and tail: what did the whole fish weigh? *Ans.* 72*lb.*

11. One has a lease for 99 years; and being asked how much of it was already expired, answered, that two-thirds of the time past was equal to four-fifths of the time to come: required the time past. *Ans.* 54 years.

12. It is required to divide the number 48 into two such parts, that the one part may be three times as much above 20, as the other wants of 20. *Ans.* 32 and 16.

13. One rents 25 acres of land at 7 pounds 12 shillings per annum; this land consisting of two sorts, he rents the better sort at 8 shillings per acre, and the worse at 5: required the number of acres of the better sort.

Ans. 9 of the better.

14. A certain cistern, which would be filled in 12 minutes

by two pipes running into it, would be filled in 20 minutes by one alone. Required in what time it would be filled by the other alone. *Ans.* 30 minutes.

15. Required two numbers, whose sum may be s, and their proportion as a to b. *Ans.* $\frac{as}{a+b}$, and $\frac{bs}{a+b}$.

16. A privateer, running at the rate of 10 miles an hour, discovers a ship 18 miles off making way at the rate of 8 miles an hour: it is demanded how many miles the ship can run before she will be overtaken? *Ans.* 72.

17. A gentleman distributing money among some poor people, found that he wanted 10*s.* to be able to give 5*s.* to each; therefore he gives 4*s.* only, and finds that he has 5*s.* left: required the number of shillings and of poor people.

Ans. 15 poor, and 65 shillings.

18. There are two numbers whose sum is the 6th part of their product, and the greater is to the less as 3 to 2. Required those numbers. *Ans.* 15 and 10.

N. B. This question may be solved by means of one unknown letter.

19. To find three numbers, so that the first, with half the other two, the second with one-third of the other two, and the third with one fourth of the other two, may be equal to 34. *Ans.* 26, 22, and 10.

20. To find a number consisting of three places, whose digits are in arithmetical progression: if this number be divided by the sum of its digits, the quotients will be 48; and if from the number 198 be subtracted, the digits will be inverted. *Ans.* 432.

21. To find three numbers, so that $\frac{1}{2}$ the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall be equal to 62: $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third, equal to 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third, equal to 38. *Ans.* 24, 60, 120.

22. If A and B, together, can perform a piece of work in 8 days; A and C together in 9 days; and B and C in 10 days; how many days will it take each person, alone, to perform the same work? *Ans.* $14\frac{34}{49}$, $17\frac{23}{41}$, $23\frac{7}{31}$.

23. What is that fraction which will become equal to $\frac{1}{3}$, if an unit be added to the numerator; but on the contrary, if an unit be added to the denominator, it will be equal to $\frac{1}{4}$?

Ans. $\frac{4}{15}$.

24. The dimensions of a certain rectangular floor are such, that if it had been 2 feet broader, and 3 feet longer, it would have been 64 square feet larger; but if it had been 3

feet broader and 2 feet longer, it would then have been 68 square feet larger: required the length and breadth of the floor. *Ans.* Length 14 feet, and breadth 10 feet.

25. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but two of the greyhound's leaps are as much as three of the hare's: how many leaps must the greyhound take to catch the hare? *Ans.* 300.

CHAP. IV.

Of the Resolution *of two or more* Equations *of the* First Degree.

605. It frequently happens that we are obliged to introduce into algebraic calculations two or more unknown quantities, represented by the letters x, y, z: and if the question is determinate, we are brought to the same number of equations as there are unknown quantities; from which it is then required to deduce those quantities. As we consider, at present, those equations only, which contain no powers of an unknown quantity higher than the first, and no products of two or more unknown quantities, it is evident that all those equations have the form

$$az + by + cx = d.$$

606. Beginning therefore with two equations, we shall endeavour to find from them the value of x and y: and, in order that we may consider this case in a general manner, let the two equations be,

$$ax + by = c; \text{ and } fx + gy = h;$$

in which, a, b, c, and f, g, h, are known numbers. It is required, therefore, to obtain, from these two equations, the two unknown quantities x and y.

607. The most natural method of proceeding will readily present itself to the mind; which is, to determine, from both equations, the value of one of the unknown quantities, as for example x, and to consider the equality of those two values; for then we shall have an equation, in which the unknown quantity y will be found by itself, and may be determined by the rules already given. Then, knowing y, we shall have only to substitute its value in one of the quantities that express x.

608. According to this rule, we obtain from the first equation, $x = \frac{c-by}{a}$, and from the second, $x = \frac{h-gy}{f}$: then putting these values equal to each other, we have this new equation:

$$\frac{c-by}{a} = \frac{h-gy}{f};$$

multiplying by a, the product is $c - by = \frac{ah-agy}{f}$; and then by f, the product is $fc - fby = ah - agy$; adding agy, we have $fc - fby + agy = ah$; subtracting fc, gives $-fby + agy = ah - fc$; or $(ag - bf)y = ah - fc$; lastly, dividing by $ag - bf$, we have

$$y = \frac{ah-fc}{ag-bf}.$$

In order now to substitute this value of y in one of the two values which we have found of x, as in the first, where $x = \frac{c-by}{a}$, we shall first have

$$by = \frac{abh-bcf}{ag-bf}; \text{ whence } c - by = c - \frac{abh-bcf}{ag-bf},$$

$$= \frac{acg-bcf-abh+bcf}{ag-bf} = \frac{acg-abh}{ag-bf}; \text{ and dividing by } a,$$

$$x = \frac{c-by}{a} = \frac{cg-bh}{ag-bf}.$$

609. *Question* 1. To illustrate this method by examples, let it be proposed to find two numbers, whose sum may be 15, and difference 7.

Let us call the greater number x, and the less y: then we shall have

$$x + y = 15, \text{ and } x - y = 7.$$

The first equation gives

$$x = 15 - y$$

and the second, $x = 7 + y$;

whence results this equation, $15 - y = 7 + y$. So that $15 = 7 + 2y$; $2y = 8$, and $y = 4$; by which means we find $x = 11$.

So that the less number is 4, and the greater is 11.

610. *Question* 2. We may also generalise the preceding

question, by requiring two numbers, whose sum may be a, and the difference b.

Let the greater of the two numbers be expressed by x, and the less by y; we shall then have $x + y = a$, and $x - y = b$.

Here the first equation gives $x = a - y$, and the second $x = b + y$.

Therefore, $a - y = b + y$; $a = b + 2y$; $2y = a - b$; lastly, $y = \frac{a-b}{2}$, and, consequently,

$$x = a - y = a - \frac{a-b}{2} = \frac{a+b}{2}.$$

Thus, we find the greater number, or x, is $\frac{a+b}{2}$, and the less, or y, is $\frac{a-b}{2}$; or, which comes to the same, $x = \frac{1}{2}a + \frac{1}{2}b$, and $y = \frac{1}{2}a - \frac{1}{2}b$. Hence we derive the following theorem: When the sum of any two numbers is a, and their difference is b, the greater of the two numbers will be equal to half the sum *plus* half the difference; and the less of the two numbers will be equal to half the sum *minus* half the difference.

611. We may resolve the same question in the following manner:

Since the two equations are,

$$x + y = a, \text{ and}$$
$$x - y = b;$$

if we add the one to the other, we have $2x = a + b$.

$$\text{Therefore } x = \frac{a+b}{2}.$$

Lastly, subtracting the same equations from each other, we have $2y = a - b$; and therefore

$$y = \frac{a-b}{2}.$$

612. *Question 3.* A mule and an ass were carrying burdens amounting to several hundred weight. The ass complained of his, and said to the mule, I need only one hundred weight of your load, to make mine twice as heavy as yours; to which the mule answered, But if you give me a hundred weight of yours, I shall be loaded three times as much you will be. How many hundred weight did each carry?

Suppose the mule's load to be x hundred weight, and that of the ass to be y hundred weight. If the mule gives one hundred weight to the ass, the one will have $y + 1$, and there will remain for the other $x - 1$; and since, in this case, the ass is loaded twice as much as the mule, we have $y + 1 = 2x - 2$.

Farther, if the ass gives a hundred weight to the mule, the latter has $x + 1$, and the ass retains $y - 1$; but the burden of the former being now three times that of the latter, we have $x + 1 = 3y - 3$.

Consequently our two equations will be,

$$y + 1 = 2x - 2, \text{ and } x + 1 = 3y - 3.$$

From the first, $x = \frac{y+3}{2}$, and the second gives $x = 3y - 4$; whence we have the new equation $\frac{y+3}{2} = 3y - 4$, which gives $y = \frac{11}{5}$; this also determines the value of x, which becomes $2\frac{3}{5}$.

The mule therefore carried $2\frac{3}{5}$ hundred weight, and the ass $2\frac{1}{5}$ hundred weight.

613. When there are three unknown numbers, and as many equations; as, for example,

$$x + y - z = 8,$$
$$x + z - y = 9,$$
$$y + z - x = 10;$$

we begin, as before, by deducing a value of x from each, and have, from the

$$\text{1st } x = 8 + z - y;$$
$$\text{2d } x = 9 + y - z;$$
$$\text{3d } x = y + z - 10.$$

Comparing the first of these values with the second, and after that with the third, we have the following equations:

$$8 + z - y = 9 + y - z,$$
$$8 + z - y = y + z - 10.$$

Now, the first gives $2z - 2y = 1$, and, by the second, $2y = 18$, or $y = 9$; if therefore we substitute this value of y in $2z - 2y = 1$, we have $2z - 18 = 1$, or $2z = 19$, so that $z = 9\frac{1}{2}$; it remains, therefore, only to determine x, which is easily found $= 8\frac{1}{2}$.

Here it happens, that the letter z vanishes in the last equation, and that the value of y is found immediately; but if this had not been the case, we should have had two equations between z and y, to be resolved by the preceding rule.

614. Suppose we had found the three following equations:

$$3x + 5y - 4z = 25,$$
$$5x - 2y + 3z = 46,$$
$$3y + 5z - x = 62.$$

If we deduce from each the value of x, we shall have from the

$$\text{1st } x = \frac{25-5y+4z}{3},$$
$$\text{2d } x = \frac{46+2y-3z}{5},$$
$$\text{3d } x = 3y + 5z - 62.$$

Comparing these three values together, and first the third with the first,

we have $3y + 5z - 62 = \frac{25-5y+4z}{3}$;

multiplying by 3, gives $9y + 15z - 186 = 25 - 5y + 4z$;
so that $9y + 15z = 211 - 5y + 4z$,
and $14y + 11z = 211$.

Comparing also the third with the second,

we have $3y + 5z - 62 = \frac{46+2y-3z}{5}$,

or $46 + 2y - 3z = 15y + 25z - 310$,
which, when reduced, becomes $356 = 13y + 28z$.

We shall now deduce, from these two new equations, the value of y:

1st $14y + 11z = 211$; or $14y = 211 - 11z$,

$$\text{and } y = \frac{211-11z}{14}.$$

2d $13y + 28z = 356$; or $13y = 356 - 28z$,

$$\text{and } y = \frac{356-28z}{13}.$$

These two values form the new equation

$$\frac{211-11z}{14} = \frac{356-28z}{13}, \text{ whence,}$$

$2743 - 143z = 4984 - 392z$, or $249z = 2241$, and $z = 9$.

This value being substituted in one of the two equations of y and z, we find $y = 8$; and, lastly, a similar substitution in one of the three values of x, will give $x = 7$.

615. If there were more than three unknown quantities to determine, and as many equations to resolve, we should proceed in the same manner; but the calculations would often prove very tedious.

It is proper, therefore, to remark, that, in each particular case, means may always be discovered of greatly facilitating the solution; which consist in introducing into the calculation, beside the principal unknown quantities, a new unknown quantity arbitrarily assumed, such as, for example, the sum of all the rest; and when a person is a little accustomed to such calculations, he easily perceives what is most proper to be done *. The following examples may serve to facilitate the application of these artifices.

616. *Question* 4. Three persons, A, B, and C, play together; and, in the first game, A loses to each of the other two, as much money as each of them has. In the next game, B loses to each of the other two, as much money as they then had. Lastly, in the third game, A and B gain each, from C, as much money as they had before. On leaving off, they find that each has an equal sum, namely, 24 guineas. Required, with how much money each sat down to play?

Suppose that the stake of the first person was x, that of the second y, and that of the third z: also, let us make the sum of all the stakes, or $x + y + z, = s$. Now, A losing in the first game as much money as the other two have, he loses $s - x$ (for he himself having had x, the two others must have had $s - x$); therefore there will remain to him $2x - s$; also B will have $2y$, and C will have $2z$.

So that, after the first game, each will have as follows: A $= 2x - s$, B $= 2y$, and C $= 2z$.

In the second game, B, who has now $2y$, loses as much money as the other two have, that is to say, $s - 2y$; so that he has left $4y - s$. With regard to the other two, they will each have double what they had; so that after the second game, the three persons have as follows: A $= 4x - 2s$, B $= 4y - s$, and C $= 4z$.

In the third game, C, who has now $4z$, is the loser; he loses to A, $4x - 2s$, and to B, $4y - s$; consequently, after this game, they will have:

* M. Cramer has given, at the end of his Introduction to the Analysis of Curve Lines, a very excellent rule for determining immediately, and without the necessity of passing through the ordinary operations, the value of the unknown quantities of such equations, to any number. F. T.

$$A = 8x - 4s,\ B = 8y - 2s,\ \text{and}\ C = 8z - s.$$

Now, each having at the end of this game 24 guineas, we have three equations, the first of which immediately gives x, the second y, and the third z; farther, s is known to be 72, since the three persons have in all 72 guineas at the end of the last game; but it is not necessary to attend to this at first; since we have,

1st $8x - 4s = 24$, or $8x = 24 + 4s$, or $x = 3 + \frac{1}{2}s$;
2d $8y - 2s = 24$, or $8y = 24 + 2s$, or $y = 3 + \frac{1}{4}s$;
3d $8z - s = 24$, or $8z = 24 + s$, or $z = 3 + \frac{1}{8}s$;

and adding these three values, we have

$$x + y + z = 9 + \tfrac{7}{8}s.$$

So that, since $x + y + z = s$, we have $s = 9 + \frac{7}{8}s$; and, consequently, $\frac{1}{8}s = 9$, and $s = 72$.

If we now substitute this value of s in the expressions which we have found for x, y, and z, we shall find that, before they began to play, A had 39 guineas, B 21, and C 12.

This solution shews, that, by means of an expression for the sum of the three unknown quantities, we may overcome the difficulties which occur in the ordinary method.

617. Although the preceding question appears difficult at first, it may be resolved even without algebra, by proceeding inversely. For since the players, when they left off, had each 24 guineas, and, in the third game, A and B doubled their money, they must have had before that last game, as follows:

$$A = 12,\ B = 12,\ \text{and}\ C = 48.$$

In the second game, A and C doubled their money; so that before that game they had;

$$A = 6,\ B = 42,\ \text{and}\ C = 24.$$

Lastly, in the first game, A and C gained each as much money as they began with; so that at first the three persons had:

$$A = 39,\ B = 21,\ C = 12.$$

The same result as we obtained by the former solution.

618. *Question 5.* Two persons owe conjointly 29 pistoles; they have both money, but neither of them enough to enable him, singly, to discharge this common debt: the first debtor says therefore to the second, If you give me $\frac{2}{3}$ of your money, I can immediately pay the debt; and the second answers, that he also could discharge the debt, if the other would give him $\frac{3}{4}$ of his money. Required, how many pistoles each had?

Suppose that the first has x pistoles, and that the second has y pistoles.

Then we shall first have, $x + \frac{2}{3}y = 29$;

and also, $y + \frac{3}{4}x = 29$.

The first equation gives $x = 29 - \frac{2}{3}y$,

and the second $x = \dfrac{116-4y}{3}$;

so that $29 - \frac{2}{3}y = \dfrac{116-4y}{3}$.

From which equation, we obtain $y = 14\frac{1}{2}$;

Therefore $x = 19\frac{1}{3}$.

Hence the first person had $19\frac{1}{3}$ pistoles, and the second had $14\frac{1}{2}$ pistoles.

619. *Question 6.* Three brothers bought a vineyard for a hundred guineas. The youngest says, that he could pay for it alone, if the second gave him half the money which he had; the second says, that if the eldest would give him only the third of his money, he could pay for the vineyard singly; lastly, the eldest asks only a fourth part of the money of the youngest, to pay for the vineyard himself. How much money had each?

Suppose the first had x guineas; the second, y guineas; the third, z guineas; we shall then have the three following equations:

$$x + \tfrac{1}{2}y = 100;$$
$$y + \tfrac{1}{3}z = 100;$$
$$z + \tfrac{1}{4}x = 100;$$

two of which only give the value of x, namely,

1st $x = 100 - \frac{1}{2}y$,
3d $x = 400 - 4z$.

So that we have the equation,

$100 - \frac{1}{2}y = 400 - 4z$, or $4z - \frac{1}{2}y = 300$, which must be combined with the second, in order to determine y and z. Now, the second equation was, $y + \frac{1}{3}z = 100$: we therefore deduce from it $y = 100 - \frac{1}{3}z$; and the equation found last being $4z - \frac{1}{2}y = 300$, we have $y = 8z - 600$. The final equation, therefore, becomes

$100 - \frac{1}{3}z = 8z - 600$; so that $8\frac{1}{3}z = 700$, or $\frac{25}{3}z = 700$, and $z = 84$. Consequently,

$$y = 100 - 28 = 72,\ \text{and}\ x = 64.$$

The youngest therefore had 64 guineas, the second had 72 guineas, and the eldest had 84 guineas.

620. As, in this example, each equation contains only two unknown quantities, we may obtain the solution required in an easier way.

The first equation gives $y = 200 - 2x$, so that y is determined by x; and if we substitute this value in the second equation, we have

$$200 - 2x + \tfrac{1}{3}z = 100;\ \text{therefore}\ \tfrac{1}{3}z = 2x - 100,$$
$$\text{and}\ z = 6x - 300.$$

So that z is also determined by x; and if we introduce this value into the third equation, we obtain $6x - 300 + \frac{1}{4}x = 100$, in which x stands alone, and which, when reduced to $25x - 1600 = 0$, gives $x = 64$. Consequently,

$$y = 200 - 128 = 72,\ \text{and}\ z = 384 - 300 = 84.$$

621. We may follow the same method, when we have a greater number of equations. Suppose, for example, that we have in general;

$$1.\ u + \frac{x}{a} = n, \qquad 2.\ x + \frac{y}{b} = n,$$
$$3.\ y + \frac{z}{c} = n, \qquad 4.\ z + \frac{u}{d} = n;$$

or, destroying the fractions, these equations become,

$$1.\ au + x = an, \qquad 2.\ bx + y = bn,$$
$$3.\ cy + z = cn, \qquad 4.\ dz + u = dn.$$

Here, the first gives immediately $x = an - au$, and, this value being substituted in the second, we have $abn - abu + y = bn$; so that $y = bn - abn + abu$; and the substitution of this value, in the third equation, gives $bcn - abcn + abcu + z = cn$; therefore

$$z = cn - bcn + abcn - abcu.$$

Substituting this in the fourth equation, we have

$$cdn - bcdn + abcdn - abcdu + u = dn.$$

So that $dn - cdn + bcdn - abcdn = abcdu - u$, or $(abcd - 1) \cdot u = abcdn - bcdn + cdn - dn$; whence we have

$$u = \frac{abcdn - bcdn + cdn - dn}{abcd - 1} = \frac{n \cdot (abcd - bcd + cd - d)}{abcd - 1}.$$

And, consequently, by substituting this value of u in the equation, $x = an - au$, we have

$$x = \frac{abcdn - acdn + adn - an}{abcd - 1} = \frac{n \cdot (abcd - acd + ad - a)}{abcd - 1}.$$

$$y = \frac{abcdn - abdn + abn - bn}{abcd - 1} = \frac{n \cdot (abcd - abd + ab - b)}{abcd - 1}.$$
$$z = \frac{abcdn - abcn + bcn - cn}{abcd - 1} = \frac{n \cdot (abcd - abc + bc - c)}{abcd - 1}.$$
$$u = \frac{abcdn - bcdn + cdn - dn}{abcd - 1} = \frac{n \cdot (abcd - bcd + cd - d)}{abcd - 1}.$$

622. *Question* 7. A captain has three companies, one of Swiss, another of Swabians, and a third of Saxons. He wishes to storm with part of these troops, and he promises a reward of 901 crowns, on the following condition; namely, that each soldier of the company, which assaults, shall receive 1 crown, and that the rest of the money shall be equally distributed among the two other companies. Now, it is found, that if the Swiss make the assault, each soldier of the other companies will receive half-a-crown; that, if the Swabians assault, each of the others will receive $\frac{1}{3}$ of a crown; and, lastly, if the Saxons make the assault, each of the others will receive $\frac{1}{4}$ of a crown. Required the number of men in each company?

Let us suppose the number of Swiss to be x, that of Swabians y, and that of Saxons z. And let us also make $x + y + z = s$, because it is easy to see, that, by this, we abridge the calculation considerably. If, therefore, the Swiss make the assault, their number being x, that of the other will be $s - x$: now, the former receive 1 crown, and the latter half-a-crown; so that we shall have,

$$x + \tfrac{1}{2}s - \tfrac{1}{2}x = 901.$$

In the same manner, if the Swabians make the assault, we have

$$y + \tfrac{1}{3}s - \tfrac{1}{3}y = 901.$$

And lastly, if the Saxons make the assault, we have,

$$z + \tfrac{1}{4}s - \tfrac{1}{4}z = 901.$$

Each of these three equations will enable us to determine one of the unknown quantities, x, y, and z;

For the first gives $x = 1802 - s$,
the second $2y = 2703 - s$,
the third $3z = 3604 - s$.

And if we now take the values of $6x$, $6y$, and $6z$, and write those values one above the other, we shall have

$$6x = 10812 - 6s,$$
$$6y = 8109 - 3s,$$
$$6z = 7208 - 2s,$$

and, by addition, $6s = 26129 - 11s$; or $17s = 26129$;

so that $s = 1537$; which is the whole number of soldiers. By this means we find,

$$x = 1802 - 1537 = 265;$$
$$2y = 2703 - 1537 = 1166, \text{ or } y = 583;$$
$$3x = 3604 - 1537 = 2067, \text{ or } z = 689.$$

The company of Swiss therefore has 265 men; that of Swabians 583; and that of Saxons 689.

CHAP. V.

Of the Resolution *of* Pure Quadratic Equations.

623. An equation is said to be of the second degree, when it contains the square, or the second power, of the unknown quantity, without any of its higher powers; and an equation, containing likewise the third power of the unknown quantity, belongs to cubic equations, and its resolution requires particular rules.

624. There are, therefore, only three kinds of terms in an equation of the second degree:

1. The terms in which the unknown quantity is not found at all, or which is composed only of known numbers.

2. The terms in which we find only the first power of the unknown quantity.

3. The terms which contain the square, or the second power, of the unknown quantity.

So that x representing an unknown quantity, and the letters a, b, c, d, &c. the known quantities, the terms of the first kind will have the form a, the terms of the second kind will have the form bx, and the terms of the third kind will have the form cx^2.

625. We have already seen, how two or more terms of the same kind may be united together, and considered as a single term.

For example, we may consider the formula $ax^2 - bx^2 + cx^2$ as a single term, representing it thus, $(a - b + c)x^2$; since, in fact, $(a - b + c)$ is a known quantity.

And also, when such terms are found on both sides of the sign $=$, we have seen how they may be brought to one side,

and then reduced to a single term. Let us take, for example, the equation,

$$2x^2 - 3x + 4 = 5x^2 - 8x + 11;$$

we first subtract $2x^2$, and there remains

$$-3x + 4 = 3x^2 - 8x + 11;$$

then adding $8x$, we obtain,

$$5x + 4 = 3x^2 + 11;$$

lastly, subtracting 11, there remains $3x^2 = 5x - 7$.

626. We may also bring all the terms to one side of the sign $=$, so as to leave *zero*, or 0, on the other; but it must be remembered, that when terms are transposed from one side to the other, their signs must be changed.

Thus, the above equation will assume this form, $3x^2 - 5x + 7 = 0$; and, for this reason also, the following general formula represents all equations of the second degree;

$$ax^2 \pm bx \pm c = 0;$$

in which the sign $\pm$ is read *plus* or *minus*, and indicates, that such terms as it stands before may be sometimes positive, and sometimes negative.

627. Whatever therefore be the original form of a quadratic equation, it may always be reduced to this formula of three terms. If we have, for example, the equation

$$\frac{ax+b}{cx+d} = \frac{ex+f}{gx+h}$$

we may, first, destroy the fractions; multiplying, for this purpose, by $cx + d$, which gives

$ax + b = \frac{cex^2 + cfx + edx + fd}{gx+h}$, then by $gx + h$, we have

$$agx^2 + bgx + ahx + bh = cex^2 + cfx + edx + fd,$$

which is an equation of the second degree, reducible to the three following terms, which we shall transpose by arranging them in the usual manner:

$$\left.\begin{matrix} ag \\ -ce \end{matrix}\right\} x^2 + \left\{\begin{matrix} +bg \\ +ah \\ -cf \\ -ed \end{matrix}\right\} x + \left\{\begin{matrix} +bh \\ -fd \end{matrix}\right\} = 0.$$

We may exhibit this equation also in the following form, which is still more clear:

$$(ag - ce)x^2 + (bg + ah - cf - ed)x + bh - fd = 0.$$

628. Equations of the second degree, in which all the three kinds of terms are found, are called *complete*, and the resolution of them is attended with greater difficulties; for

which reason we shall first consider those, in which one of the terms is wanting.

Now, if the term x^2 were not found in the equation, it would not be a quadratic, but would belong to those of which we have already treated; and if the term, which contains only known numbers, were wanting, the equation would have this form, $ax^2 \pm bx = 0$, which being divisible by x, may be reduced to $ax \pm b = 0$, which is likewise a simple equation, and belongs not to the present class.

629. But when the middle term, which contains the first power of x, is wanting, the equation assumes this form, $ax^2 \pm c = 0$, or $ax^2 = \mp c$; as the sign of c may be either positive, or negative.

We shall call such an equation a *pure* equation of the second degree, and the resolution of it is attended with no difficulty; for we have only to divide by a, which gives $x^2 = \frac{c}{a}$; and taking the square root of both sides, we find $x = \sqrt{\frac{c}{a}}$; by which means the equation is resolved.

630. But there are three cases to be considered here. In the first, when $\frac{c}{a}$ is a square number (of which we can therefore really assign the root) we obtain for the value of x a rational number, which may be either integral, or fractional. For example, the equation $x^2 = 144$, gives $x = 12$. And $x^2 = \frac{9}{16}$, gives $x = \frac{3}{4}$.

The second case is, when $\frac{c}{a}$ is not a square, in which case we must therefore be contented with the sign $\sqrt{}$. If, for example, $x^2 = 12$, we have $x = \sqrt{12}$, the value of which may be determined by approximation, as we have already shewn.

The third case is that, in which $\frac{c}{a}$ becomes a negative number: the value of x is then altogether impossible and imaginary; and this result proves that the question, which leads to such an equation, is in itself impossible.

631. We shall also observe, before proceeding farther, that whenever it is required to extract the square root of a number, that root, as we have already remarked, has always two values, the one positive and the other negative. Suppose, for example, we have the equation $x^2 = 49$, the value

of x will be not only $+7$, but also -7, which is expressed by $x = \pm 7$. So that all those questions admit of a double answer; but it will be easily perceived that in several cases, as those which relate to a certain number of men, the negative value cannot exist.

632. In such equations, also, as $ax^2 = bx$, where the known quantity c is wanting, there may be two values of x, though we find only one if we divide by x. In the equation $x^2 = 3x$, for example, in which it is required to assign such a value of x, that x^2 may become equal to $3x$, this is done by supposing $x = 3$, a value which is found by dividing the equation by x; but, beside this value, there is also another, which is equally satisfactory, namely, $x = 0$; for then $x^2 = 0$, and $3x = 0$. Equations therefore of the second degree, in general, admit of two solutions, whilst simple equations admit only of one.

We shall now illustrate what we have said with regard to pure equations of the second degree by some examples.

633. *Question* 1. Required a number, the half of which multiplied by the third, may produce 24.

Let this number be x; then by the question $\frac{1}{2}x$, multiplied by $\frac{1}{3}x$, must give 24; we shall therefore have the equation $\frac{1}{6}x^2 = 24$.

Multiplying by 6, we have $x^2 = 144$; and the extraction of the root gives $x = \pm 12$. We put $\pm$; for if $x = +12$, we have $\frac{1}{2}x = 6$, and $\frac{1}{3}x = 4$: now, the product of these two numbers is 24; and if $x = -12$, we have $\frac{1}{2}x = -6$, and $\frac{1}{3}x = -4$, the product of which is likewise 24.

634. *Question* 2. Required a number such, that being increased by 5, and diminished by 5, the product of the sum by the difference may be 96.

Let this number be x, then $x + 5$, multiplied by $x - 5$, must give 96; whence results the equation,

$$x^2 - 25 = 96.$$

Adding 25, we have $x^2 = 121$; and extracting the root, we have $x = 11$. Thus $x + 5 = 16$, also $x - 5 = 6$; and, lastly, $6 \times 16 = 96$.

635. *Question* 3. Required a number such, that by adding it to 10, and subtracting it from 10, the sum, multiplied by the difference, will give 51.

Let x be this number; then $10 + x$, multiplied by $10 - x$, must make 51, so that $100 - x^2 = 51$. Adding x^2, and subtracting 51, we have $x^2 = 49$, the square root of which gives $x = 7$.

636. *Question* 4. Three persons, who had been playing, leave off; the first, with as many times 7 crowns, as the second has three crowns; and the second, with as many

times 17 crowns, as the third has 5 crowns. Farther, if we multiply the money of the first by the money of the second, and the money of the second by the money of the third, and, lastly, the money of the third by that of the first, the sum of these three products will be $3830\frac{2}{3}$. How much money has each?

Suppose that the first player has x crowns; and since he has as many times 7 crowns, as the second has 3 crowns, we know that his money is to that of the second, in the ratio of 7 : 3.

We shall therefore have $7 : 3 :: x : \frac{3}{7}x$, the money of the second player.

Also, as the money of the second player is to that of the third in the ratio of 17 : 5, we shall have $17 : 5 :: \frac{3}{7}x : \frac{15}{119}x$, the money of the third player.

Multiplying x, or the money of the first player, by $\frac{3}{7}x$, the money of the second, we have the product $\frac{3}{7}x^2$: then, $\frac{3}{7}x$, the money of the second, multiplied by the money of the third, or by $\frac{15}{119}x$, gives $\frac{45}{833}x^2$; and, lastly, the money of the third, or $\frac{15}{119}x$, multiplied by x, or the money of the first, gives $\frac{15}{119}x^2$. Now, the sum of these three products is $\frac{3}{7}x^2 + \frac{45}{833}x^2 + \frac{15}{119}x^2$; and reducing these fractions to the same denominator, we find their sum $\frac{507}{833}x^2$, which must be equal to the number $3830\frac{2}{3}$.

We have therefore, $\frac{507}{833}x^2 = 3830\frac{2}{3}$.

So that $\frac{1521}{833}x^2 = 11492$, and $1521x^2$ being equal to 9572836, dividing by 1521, we have $x^2 = \frac{9572836}{1521}$; and taking its root, we find $x = \frac{3094}{39}$. This fraction is reducible to lower terms, if we divide by 13; so that $x = \frac{238}{3} = 79\frac{1}{3}$; and hence we conclude, that $\frac{3}{7}x = 34$, and $\frac{15}{119}x = 10$.

The first player therefore has $79\frac{1}{3}$ crowns, the second has 34 crowns, and the third 10 crowns.

Remark. This calculation may be performed in an easier manner; namely, by taking the factors of the numbers which present themselves, and attending chiefly to the squares of those factors.

It is evident, that $507 = 3 \times 169$, and that 169 is the square of 13; then, that $833 = 7 \times 119$, and $119 = 7 \times 17$: therefore $\frac{3 \times 169}{17 \times 49}x^2 = 3830\frac{2}{3}$, and if we multiply by 3, we have $\frac{9 \times 169}{17 \times 49}x^2 = 11492$. Let us resolve this number also into its factors; and we readily perceive, that the first is 4, that is to say, that $11492 = 4 \times 2873$; farther, 2873 is divisible by 17, so that $2873 = 17 \times 169$.

Consequently, our equation will assume the following form, $\frac{9 \times 169}{17 \times 49}x^2 = 4 \times 17 \times 169$, which, divided by 169, is reduced to $\frac{9}{17 \times 49}x^2 = 4 \times 17$; multiplying also by 17×49, and dividing by 9, we have $x^2 = \frac{4 \times 289 \times 49}{9}$, in which all the factors are squares; whence we have, without any further calculation, the root $x = \frac{2 \times 17 \times 7}{3} = \frac{238}{3} = 79\frac{1}{3}$, as before.

637. *Question* 5. A company of merchants appoint a factor at Archangel. Each of them contributes for the trade, which they have in view, ten times as many crowns as there are partners; and the profit of the factor is fixed at twice as many crowns, *per cent*, as there are partners. Also, if $\frac{1}{100}$ part of his total gain be multiplied by $2\frac{2}{9}$, it will give the number of partners. That number is required.

Let it be x; and since, each partner has contributed $10x$, the whole capital is $10x^2$. Now, for every hundred crowns, the factor gains $2x$, so that with the capital of $10x^2$ his profit will be $\frac{1}{5}x^3$. The $\frac{1}{100}$ part of his gain is $\frac{1}{500}x^3$; multiplying by $2\frac{2}{9}$, or by $\frac{20}{9}$, we have $\frac{20}{4500}x^3$, or $\frac{1}{225}x^3$, and this must be equal to the number of partners, or x.

We have, therefore, the equation $\frac{1}{225}x^3 = x$, or $x^3 = 225x$; which appears, at first, to be of the third degree; but as we may divide by x, it is reduced to the quadratic $x^2 = 225$; whence $x = 15$.

So that there are fifteen partners, and each contributed 150 crowns.

QUESTIONS FOR PRACTICE.

1. To find a number, to which 20 being added, and from which 10 being subtracted, the square of the sum, added to twice the square of the remainder, shall be 17475. *Ans.* 75.

2. What two numbers are those, which are to one another in the ratio of 3 to 5, and whose squares, added together, make 1666? *Ans.* 21 and 35.

3. The sum $2a$, and the sum of the squares $2b$, of two numbers being given; to find the numbers. *Ans.* $a - \sqrt{(b - a^2)}$ and $a + \sqrt{(b - a^2)}$.

4. To divide the number 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

5. To find three such numbers, that the sum of the first and second multiplied into the third, may be equal to 63; and the sum of the second and third, multiplied into the first, may be equal to 28; also, that the sum of the first and third, multiplied into the second, may be equal to 55.
Ans. 2, 5, 9.

6. What two numbers are those, whose sum is to the greater as 11 to 7; the difference of their squares being 132?
Ans. 14 and 8.

CHAP. VI.

Of the Resolution *of* Mixt Equations *of the* Second Degree.

638. An equation of the second degree is said to be *mixt*, or complete, when three terms are found in it, namely, that which contains the square of the unknown quantity, as ax^2; that, in which the unknown quantity is found only in the first power, as bx; and, lastly, the term which is composed of only known quantities. And since we may unite two or more terms of the same kind into one, and bring all the terms to one side of the sign $=$, the general form of a mixt equation of the second degree will be

$$ax^2 \pm bx \pm c = 0.$$

In this chapter, we shall shew how the value of x may be derived from such equations: and it will be seen, that there are two methods of obtaining it.

639. An equation of the kind that we are now considering may be reduced, by division, to such a form, that the first term may contain only the square, x^2, of the unknown quantity x. We shall leave the second term on the same side with x, and transpose the known term to the other side of the sign $=$. By these means our equation will assume the form of $x^2 \pm px = \pm q$, in which p and q represent any known numbers, positive or negative; and the whole is at present reduced to determining the true value of x. We shall begin by remarking, that if $x^2 + px$ were a real square, the resolution would be attended with no difficulty, because it would only be required to take the square root of both sides.

640. But it is evident that $x^2 + px$ cannot be a square; since we have already seen, (Art. 307.) that if a root consists of two terms, for example, $x + n$, its square always contains three terms, namely, twice the product of the two parts, beside the square of each part; that is to say, the square of $x + n$ is $x^2 + 2nx + n^2$. Now, we have already on one side $x^2 + px$; we may, therefore, consider x^2 as the square of the first part of the root, and in this case px must represent twice the product of x, the first part of the root, by the second part: consequently, this second part must be $\frac{1}{2}p$, and in fact the square of $x + \frac{1}{2}p$, is found to be

$$x^2 + px + \tfrac{1}{4}p^2.$$

641. Now, $x^2 + px + \frac{1}{4}p^2$ being a real square, which has for its root $x + \frac{1}{2}p$, if we resume our equation $x^2 + px = q$, we have only to add $\frac{1}{4}p^2$ to both sides, which gives us $x^2 + px + \frac{1}{4}p^2 = q + \frac{1}{4}p^2$, the first side being actually a square, and the other containing only known quantities. If, therefore, we take the square root of both sides, we find $x + \frac{1}{2}p = \sqrt{(\frac{1}{4}p^2 + q)}$; subtracting $\frac{1}{2}p$, we obtain $x = -\frac{1}{2}p + \sqrt{(\frac{1}{4}p^2 + q)}$; and, as every square root may be taken either affirmatively or negatively, we shall have for x two values expressed thus;

$$x = -\tfrac{1}{2}p \pm \sqrt{(\tfrac{1}{4}p^2 + q)}.$$

642. This formula contains the rule by which all quadratic equations may be resolved; and it will be proper to commit it to memory, that it may not be necessary, every time, to repeat the whole operation which we have gone through. We may always arrange the equation in such a manner, that the pure square x^2 may be found on one side, and the above equation have the form $x^2 = -px + q$, where we see immediately that $x = -\frac{1}{2}p \pm \sqrt{(\frac{1}{4}p^2 + q)}$.

643. The general rule, therefore, which we deduce from that, in order to resolve the equation $x^2 = -px + q$, is founded on this consideration;

That the unknown quantity x is equal to half the coefficient, or multiplier of x on the other side of the equation, *plus* or *minus* the square root of the square of this number, and the known quantity which forms the third term of the equation.

Thus, if we had the equation $x^2 = 6x + 7$, we should immediately say, that $x = 3 \pm \sqrt{(9 + 7)} = 3 \pm 4$, whence we have these two values of x, namely, $x = 7$, and $x = -1$. In the same manner, the equation $x^2 = 10x - 9$, would give $x = 5 \pm \sqrt{(25 - 9)} = 5 \pm 4$, that is to say, the two values of x are 9 and 1.

644. This rule will be still better understood, by distin-

guishing the following cases: 1st, When p is an even number; 2d, When p is an odd number; and 3d, When p is a fractional number.

1st, Let p be an even number, and the equation such, that $x^2 = 2px + q$; we shall, in this case, have

$$x = p \pm \sqrt{(p^2 + q)}.$$

2d, Let p be an odd number, and the equation $x^2 = px + q$; we shall here have $x = \frac{1}{2}p \pm \sqrt{(\frac{1}{4}p^2 + q)}$; and since $\frac{1}{4}p^2 + q = \frac{p^2+4q}{4}$, we may extract the square root of the denominator, and write

$$x = \frac{1}{2}p \pm \frac{\sqrt{(p^2+4q)}}{2} = \frac{p \pm \sqrt{(p^2+4q)}}{2}.$$

3d, Lastly, if p be a fraction, the equation may be resolved in the following manner. Let the equation be $ax^2 = bx + c$, or $x^2 = \frac{bx}{a} + \frac{c}{a}$, and we shall have, by the rule, $x = \frac{b}{2a} \pm \sqrt{(\frac{b^2}{4a^2} + \frac{c}{a})}$. Now, $\frac{b^2}{4a^2} + \frac{c}{a} = \frac{b^2+4ac}{4a^2}$, the denominator of which is a square; so that

$$x = \frac{b \pm \sqrt{(b^2+4ac)}}{2a}.$$

645. The other method of resolving mixt quadratic equations is, to transform them into pure equations; which is done by substitution: for example, in the equation $x^2 = px + q$, instead of the unknown quantity x, we may write another unknown quantity, y, such, that $x = y + \frac{1}{2}p$; by which means, when we have determined y, we may immediately find the value of x.

If we make this substitution of $y + \frac{1}{2}p$ instead of x, we have $x^2 = y^2 + py + \frac{1}{4}p^2$, and $px = py + \frac{1}{2}p^2$; consequently, our equation will become

$$y^2 + py + \frac{1}{4}p^2 = py + \frac{1}{2}p^2 + q;$$

which is first reduced, by subtracting py, to

$$y^2 + \frac{1}{4}p^2 = \frac{1}{2}p^2 + q;$$

and then, by subtracting $\frac{1}{4}p^2$, to $y^2 = \frac{1}{4}p^2 + q$. This is a pure quadratic equation, which immediately gives

$$y = \pm \sqrt{(\frac{1}{4}p^2 + q)}.$$

Now, since $x = y + \frac{1}{2}p$, we have

$$x = \frac{1}{2}p \pm \sqrt{(\frac{1}{4}p^2 + q)},$$

as we found it before. It only remains, therefore, to illustrate this rule by some examples.

646. *Question* 1. There are two numbers; the one exceeds the other by 6, and their product is 91: what are those numbers?

If the less be x, the other will be $x + 6$, and their product $x^2 + 6x = 91$. Subtracting $6x$, there remains $x^2 = 91 - 6x$, and the rule gives

$x = -3 \pm \sqrt{(9 + 91)} = -3 \pm 10$; so that $x = 7$, or $x = -13$.

The question therefore admits of two solutions;

By the one, the less number $x = 7$, and the greater $x + 6 = 13$.

By the other, the less number $x = -13$, and the greater $x + 6 = -7$.

647. *Question* 2. To find a number such, that if 9 be taken from its square, the remainder may be a number, as much greater than 100, as the number itself is less than 23.

Let the number sought be x. We know that $x^2 - 9$ exceeds 100 by $x^2 - 109$: and since x is less than 23 by $23 - x$, we have this equation

$$x^2 - 109 = 23 - x.$$

Therefore $x^2 = -x + 132$, and, by the rule, $x = -\frac{1}{2} \pm \sqrt{(\frac{1}{4} + 132)} = -\frac{1}{2} \pm \sqrt{(\frac{529}{4})} = -\frac{1}{2} \pm \frac{23}{2}$. So that $x = 11$, or $x = -12$.

Hence, when only a positive number is required, that number will be 11, the square of which *minus* 9 is 112, and consequently greater than 100 by 12, in the same manner as 11 is less than 23 by 12.

648. *Question* 3. To find a number such, that if we multiply its half by its third, and to the product add half the number required, the result will be 30.

Supposing the number to be x, its half, multiplied by its third, will give $\frac{1}{6}x^2$; so that $\frac{1}{6}x^2 + \frac{1}{2}x = 30$; and multiplying by 6, we have $x^2 + 3x = 180$, or $x^2 = -3x + 180$, which gives $x = -\frac{3}{2} \pm \sqrt{(\frac{9}{4} + 180)} = -\frac{3}{2} \pm \frac{27}{2}$.

Consequently, either $x = 12$, or $x = -15$.

649. *Question* 4. To find two numbers, the one being double the other, and such, that if we add their sum to their product, we may obtain 90.

Let one of the numbers be x, then the other will be $2x$; their product also will be $2x^2$, and if we add to this $3x$, or their sum, the new sum ought to make 90. So that $2x^2 + 3x = 90$; or $2x^2 = 90 - 3x$; whence $x^2 = -\frac{3}{2}x + 45$, and thus we obtain

$$x = -\tfrac{3}{4} \pm \sqrt{(\tfrac{9}{16} + 45)} = -\tfrac{3}{4} \pm \tfrac{27}{4}.$$

Consequently $x = 6$, or $x = -7\tfrac{1}{2}$.

650. *Question* 5. A horse-dealer bought a horse for a certain number of crowns, and sold it again for 119 crowns, by which means his profit was as much per cent as the horse cost him; what was his first purchase?

Suppose the horse cost x crowns; then, as the dealer gains x per cent, we have this proportion:

$$\text{As } 100 : x :: x : \frac{x^2}{100};$$

since therefore he has gained $\frac{x^2}{100}$, and the horse originally cost him x crowns, he must have sold it for $x + \frac{x^2}{100}$; therefore $x + \frac{x^2}{100} = 119$; and subtracting x, we have $\frac{x^2}{100} = -x + 119$; then multiplying by 100, we obtain $x^2 = -100x + 11900$. Whence, by the rule, we find $x = -50 \pm \sqrt{(2500 + 11900)} = -50 \pm \sqrt{14400} = -50 \pm 120 = 70$.

The horse therefore cost 70 crowns, and since the horse-dealer gained 70 per cent when he sold it again, the profit must have been 49 crowns. So that the horse must have been sold again for $70 + 49$, that is to say, for 119 crowns.

651. *Question* 6. A person buys a certain number of pieces of cloth: he pays for the first 2 crowns, for the second 4 crowns, for the third 6 crowns, and in the same manner always 2 crowns more for each following piece. Now, all the pieces together cost him 110: how many pieces had he?

Let the number sought be x; then, by the question, the purchaser paid for the different pieces of cloth in the following manner:

for the 1, 2, 3, 4, 5 . . . . x pieces
he pays 2, 4, 6, 8, 10 . . . . $2x$ crowns.

It is therefore required to find the sum of the arithmetical ... $4 + 6 + 8 + \ldots\ldots 2x$, which consists of ...e may deduce from it the price of all the ...taken together. The rule which we have ...r this operation requires us to add the last ...t; and the sum is $2x + 2$; which must be ...e number of terms x, and the product will

be $2x^2 + 2x$; lastly, if we divide by the difference 2, the quotient will be $x^2 + x$, which is the sum of the progression; so that we have $x^2 + x = 110$; therefore $x^2 = -x + 110$, and $x = -\tfrac{1}{2} + \sqrt{(\tfrac{1}{4} + 110)} = -\tfrac{1}{2} + \tfrac{21}{2} = 10$.

And hence the number of pieces of cloth is 10.

652. *Question* 7. A person bought several pieces of cloth for 180 crowns; and if he had received for the same sum 3 pieces more, he would have paid 3 crowns less for each piece. How many pieces did he buy?

Let us represent the number sought by x; then each piece will have cost him $\frac{180}{x}$ crowns. Now, if the purchaser had had $x + 3$ pieces for 180 crowns, each piece would have cost $\frac{180}{x+3}$ crowns; and, since this price is less than the real price by three crowns, we have this equation,

$$\frac{180}{x+3} = \frac{180}{x} - 3.$$

Multiplying by x, we obtain $\frac{180x}{x+3} = 180 - 3x$; dividing by 3, we have $\frac{60x}{x+3} = 60 - x$; and again, multiplying by $x + 3$, gives $60x = 180 + 57x - x^2$; therefore adding x^2, we shall have $x^2 + 60x = 180 + 57x$; and subtracting $60x$, we shall have $x^2 = -3x + 180$.

The rule consequently gives,

$$x = -\tfrac{3}{2} + \sqrt{(\tfrac{9}{4} + 180)}, \text{ or } x = -\tfrac{3}{2} + \tfrac{27}{2} = 12.$$

He therefore bought for 180 crowns 12 pieces of cloth at 15 crowns the piece; and if he had got 3 pieces more, namely, 15 pieces for 180 crowns, each piece would have cost only 12 crowns, that is to say, 3 crowns less.

653. *Question* 8. Two merchants enter into partnership with a stock of 100 pounds; one leaves his money in the partnership for three months, the other leaves his for two months, and each takes out 99 pounds of capital and profit. What proportion of the stock did they separately furnish?

Suppose the first partner contributed x pounds, the other will have contributed $100 - x$. Now, the former receiving 99*l*., his profit is $99 - x$, which he has gained in three months with the principal x; and since the second receives also 99*l*., his profit is $x - 1$, which he has gained in two months with the principal $100 - x$; it is evident also, that the profit of this second partner would have been

$\frac{3x-3}{2}$, if he had remained three months in the partnership: and as the profits gained in the same time are in proportion to the principals, we have the following proportion,

$$x : 99 - x :: 100 - x : \frac{3x-3}{2}.$$

And the equality of the product of the extremes to that of the means, gives the equation,

$$\frac{3x^2-3x}{2} = 9900 - 199x + x^2;$$

then multiplying this by 2, we have $3x^2 - 3x = 19800 - 398x + 2x^2$; and subtracting $2x^2$, we obtain $x^2 - 3x = 19800 - 398x$. Adding $3x$, gives $x^2 = 19800 - 395x$; then by the rule, $x = -\frac{395}{2} + \sqrt{(\frac{156025}{4} + \frac{79200}{4})} = -\frac{395}{2} + \frac{485}{2} = \frac{90}{2} = 45$.

The first partner therefore contributed 45*l.* and the other 55*l.* The first having gained 54*l.* in three months, would have gained in one month 18*l.*; and the second having gained 44*l.* in two months, would have gained 22*l.* in one month: now these profits agree; for if, with 45*l.*, 18*l.* are gained in one month, 22*l.* will be gained in the same time with 55*l.*

654. *Question* 9. Two girls carry 100 eggs to market; the one had more than the other, and yet the sum which they both received for them was the same. The first says to the second, If I had had your eggs, I should have received 15 pence. The other answers, If I had had yours, I should have received $6\frac{2}{3}$ pence. How many eggs did each carry to market?

Suppose the first had x eggs; then the second must have had $100 - x$.

Since, therefore, the former would have sold $100 - x$ eggs for 15 pence, we have the following proportion:

$$(100 - x) : 15 :: x : \frac{15x}{100-x}.$$

Also, since the second would have sold x eggs for $6\frac{2}{3}$ pence, we readily find how much she got for $100 - x$ eggs, thus:

$$\text{As } x : (100 - x) :: \tfrac{20}{3} : \frac{2000-20x}{3x}.$$

Now, both the girls received the same money; we have

consequently the equation, $\frac{15x}{100-x} = \frac{2000-20x}{3x}$, which becomes $25x^2 = 200000 - 4000x$; and, lastly,

$$x^2 = -160x + 8000;$$

whence we obtain

$$x = -80 + \sqrt{(6400 + 8000)} = -80 + 120 = 40.$$

So that the first girl had 40 eggs, the second had 60, and each received 10 pence.

655. *Question* 10. Two merchants sell each a certain quantity of silk; the second sells 3 ells more than the first, and they received together 35 crowns. Now, the first says to the second, I should have got 24 crowns for your silk; the other answers, And I should have got for yours 12 crowns and a half. How many ells had each?

Suppose the first had x ells; then the second must have had $x + 3$ ells; also, since the first would have sold $x + 3$ ells for 24 crowns, he must have received $\frac{24x}{x+3}$ crowns for his x ells. And, with regard to the second, since he would have sold x ells for $12\frac{1}{2}$ crowns, he must have sold his $x + 3$ ells for $\frac{25x+75}{2x}$; so that the whole sum they received was

$$\frac{24x}{x+3} + \frac{25x+75}{2x} = 35 \text{ crowns}.$$

This equation becomes $x^2 = 20x - 75$; whence we have $x = 10 \pm \sqrt{(100 - 75)} = 10 \pm 5$.

So that the question admits of two solutions: according to the first, the first merchant had 15 ells, and the second had 18; and since the former would have sold 18 ells for 24 crowns, he must have sold his 15 ells for 20 crowns. The second, who would have sold 15 ells for 12 crowns and a half, must have sold his 18 ells for 15 crowns; so that they actually received 35 crowns for their commodity.

According to the second solution, the first merchant had 5 ells, and the other 8 ells; and since the first would have sold 8 ells for 24 crowns, he must have received 15 crowns for his 5 ells; also, since the second would have sold 5 ells for 12 crowns and a half, his 8 ells must have produced him 20 crowns; the sum being, as before, 35 crowns.

CHAP. VII.

Of the Extraction *of the* Roots *of* Polygon Numbers.

656. We have shewn, in a preceding chapter*, how polygonal numbers are to be found; and what we then called *a side*, is also called *a root*. If, therefore, we represent the root by x, we shall find the following expressions for all polygonal numbers:

the IIIgon, or triangle, is $\frac{x^2+x}{2}$,

the IVgon, or square, - x^2,

the Vgon - - - - $\frac{3x^2-x}{2}$,

the VIgon - - - - $2x^2-x$,

the VIIgon - - - - $\frac{5x^2-3x}{2}$,

the VIIIgon - - - $3x^2-2x$,

the IXgon - - - - $\frac{7x^2-5x}{2}$,

the Xgon - - - - - $4x^2-3x$,

the $n$gon - - - - - $\frac{(n-2)x^2-(n-4)x}{2}$.

657. We have already shewn, that it is easy, by means of these formulæ, to find, for any given root, any polygon number required: but when it is required reciprocally to find the side, or the root of a polygon, the number of whose sides is known, the operation is more difficult, and always requires the solution of a quadratic equation; on which account the subject deserves, in this place, to be separately considered. In doing this we shall proceed regularly, beginning with the triangular numbers, and passing from them to those of a greater number of angles.

658. Let therefore 91 be the given triangular number, the side or root of which is required.

If we make this root $=x$, we must have

* Chap. 5, Sect. III.

$\frac{x^2+x}{2} = 91$; or $x^2+x = 182$, and $x^2 = -x + 182$; consequently,

$x = -\frac{1}{2} + \sqrt{(\frac{1}{4} + 182)} = -\frac{1}{2} + \sqrt{(\frac{729}{4})} = -\frac{1}{2} + \frac{27}{2} = 13$;

from which we conclude, that the triangular root required is 13; for the triangle of 13 is 91.

659. But, in general, let a be the given triangular number, and let its root be required.

Here, if we make it $= x$, we have $\frac{x^2+x}{2} = a$; or $x^2 + x = 2a$; therefore, $x^2 = -x + 2a$, and by the rule $x = -\frac{1}{2} + \sqrt{(\frac{1}{4} + 2a)}$, or $x = \frac{-1+\sqrt{(8a+1)}}{2}$.

This result gives the following rule: To find a triangular root, we must multiply the given triangular number by 8; add 1 to the product, extract the root of the sum, subtract 1 from that root, and lastly, divide the remainder by 2.

660. So that all triangular numbers have this property; that if we multiply them by 8, and add unity to the product, the sum is always a square; of which the following small Table furnishes some examples:

Triangles	1,	3,	6,	10,	15,	21,	28,	36,	45,	55,	&c.
8 times + 1 =	9,	25,	49,	81,	121,	169,	225,	289,	361,	441,	&c.

If the given number a does not answer this condition, we conclude, that it is not a real triangular number, or that no rational root of it can be assigned.

661. According to this rule, let the triangular root of 210 be required; we shall have $a = 210$, and $8a + 1 = 1681$, the square root of which is 41; whence we see, that the number 210 is really triangular, and that its root is $\frac{41-1}{2} = 20$. But if 4 were given as the triangular number, and its root were required, we should find it $= \frac{\sqrt{33}}{2} - \frac{1}{2}$, and consequently irrational. However, the triangle of this root, $\frac{\sqrt{33}}{2} - \frac{1}{2}$, may be found in the following manner:

Since $x = \frac{\sqrt{33}-1}{2}$, we have $x^2 = \frac{17-\sqrt{33}}{2}$, and adding

$x = \frac{\sqrt{33}-1}{2}$ to it, the sum is $x^2 + x = \frac{16}{2} = 8$. Consequently, the triangle, or the triangular number, $\frac{x^2+x}{2} = 4$.

662. The quadrangular numbers being the same as squares, they occasion no difficulty. For, supposing the given quadrangular-number to be a, and its required root x, we shall have $x^2 = a$, and consequently, $x = \sqrt{a}$; so that the square root and the quadrangular root are the same thing.

663. Let us now proceed to pentagonal numbers.

Let 22 be a number of this kind, and x its root; then, by the third formula, we shall have $\frac{3x^2-x}{2} = 22$, or $3x^2 - x = 44$, or $x^2 = \frac{1}{3}x + \frac{44}{3}$; from which we obtain,

$x = \frac{1}{6} + \sqrt{(\frac{1}{36} + \frac{44}{3})}$, or $x = \frac{1+\sqrt{(529)}}{6} = \frac{1}{6} + \frac{23}{6} = 4$;

and consequently 4 is the pentagonal root of the number 22.

664. Let the following question be now proposed: the pentagon a being given, to find its root.

Let this root be x, and we have the equation

$\frac{3x^2-x}{2} = a$, or $3x^2 - x = 2a$, or $x^2 = \frac{1}{3}x + \frac{2a}{3}$; by means of which we find $x = \frac{1}{6} + \sqrt{(\frac{1}{36} + \frac{2a}{3})}$, that is,

$x = \frac{1+\sqrt{(24a+1)}}{6}$. Therefore, when a is a real pentagon, $24a + 1$ must be a square.

Let 330, for example, be the given pentagon, the root will be $x = \frac{1+\sqrt{(7921)}}{6} = \frac{1+89}{6} = 15$.

665. Again, let a be a given hexagonal number, the root of which is required.

If we suppose it $= x$, we shall have $2x^2 - x = a$, or $x^2 = \frac{1}{2}x + \frac{1}{2}a$; and this gives

$$x = \frac{1}{4} + \sqrt{(\frac{1}{16} + \frac{1}{2}a)} = \frac{1+\sqrt{(8a+1)}}{4}.$$

So that, in order that a may be really a hexagon, $8a + 1$ must become a square; whence we see, that all hexagonal numbers are contained in triangular numbers; but it is not the same with the roots.

For example, let the hexagonal number be 1225, its root will be $x = \frac{1+\sqrt{9801}}{4} = \frac{1+99}{4} = 25$.

666. Suppose a an heptagonal number, of which the root is required.

Let this root be x, then we shall have $\frac{5x^2-3x}{2} = a$, or

$x^2 = \frac{3}{5}x + \frac{2}{5}a$, which gives

$$x = \frac{3}{10} + \sqrt{(\frac{9}{100} + \frac{2}{5}a)} = \frac{3+\sqrt{(40a+9)}}{10};$$

therefore the heptagonal numbers have this property, that if they be multiplied by 40, and 9 be added to the product, the sum will always be a square.

Let the heptagon, for example, be 2059; its root will be found $= \frac{3+\sqrt{(82369)}}{10} = \frac{3+287}{10} = 29$.

667. Let us suppose a an octagonal number, of which the root x is required.

We shall here have $3x^2 - 2x = a$, or $x^2 = \frac{2}{3}x + \frac{1}{3}a$, whence results $x = \frac{1}{3} + \sqrt{(\frac{1}{9} + \frac{1}{3}a)} = \frac{1+\sqrt{(3a+1)}}{3}$.

Consequently, all octagonal numbers are such, that if multiplied by 3, and unity be added to the product, the sum is constantly a square.

For example, let 3816 be an octagon; its root will be $x = \frac{1+\sqrt{11449}}{3} = \frac{1+107}{3} = 36$.

668. Lastly, let a be a given n-gonal number, the root of which it is required to assign; we shall then, by the last formula, have this equation:

$\frac{(n-2)x^2-(n-4)x}{2} = a$, or $(n-2)x^2 - (n-4)x = 2a$;

consequently, $x^2 = \frac{(n-4)x}{n-2} + \frac{2a}{n-2}$; whence,

$$x = \frac{n-4}{2(n-2)} + \sqrt{(\frac{(n-4)^2}{4(n-2)^2} + \frac{2a}{n-2})}, \text{ or}$$

$$x = \frac{n-4}{2(n-2)} + \sqrt{(\frac{(n-4)^2}{4(n-2)^2} + \frac{8(n-2)a}{4(n-2)^2})}, \text{ or}$$

$$x = \frac{n-4+\sqrt{(8(n-2)a+(n-4)^2)}}{2(n-2)}.$$

This formula contains a general rule for finding all the possible polygonal roots of given numbers.

For example, let there be given the XXIV-gonal number, 3009: since a is here $= 3009$ and $n = 24$, we have $n - 2 = 22$ and $n - 4 = 20$; wherefore the root, or

$$x, = \frac{20 + \sqrt{(529584 + 400)}}{44} = \frac{20 + 728}{44} = 17.$$

CHAP. VIII.

Of the Extraction *of the* Square Roots *of* Binomials.

669. By a *binomial** we mean a quantity composed of two parts, which are either both affected by the sign of the square root, or of which one, at least, contains that sign.

For this reason $3 + \sqrt{5}$ is a binomial, and likewise $\sqrt{8} + \sqrt{3}$; and it is indifferent whether the two terms be joined by the sign $+$ or by the sign $-$. So that $3 - \sqrt{5}$ and $3 + \sqrt{5}$ are both binomials.

670. The reason that these binomials deserve particular attention is, that in the resolution of quadratic equations we are always brought to quantities of this form, when the resolution cannot be performed. For example, the equation $x^2 = 6x - 4$ gives $x = 3 + \sqrt{5}$.

It is evident, therefore, that such quantities must often occur in algebraic calculations; for which reason, we have already carefully shewn how they are to be treated in the ordinary operations of addition, subtraction, multiplication, and division: but we have not been able till now to shew how their square roots are to be extracted; that is, so far as that extraction is possible; for when it is not, we must be satisfied with affixing to the quantity another radical sign. Thus, the square root of $3 + \sqrt{2}$ is written $\sqrt{3 + \sqrt{2}}$; or $\sqrt{(3 + \sqrt{2})}$.

671. It must here be observed, in the first place, that the squares of such binomials are also binomials of the same kind; in which also one of the terms is always rational.

For, if we take the square of $a + \sqrt{b}$, we shall obtain $(a^2 + b) + 2a\sqrt{b}$. If therefore it were required reciprocally to take the root of the quantity $(a^2 + b) + 2a\sqrt{b}$, we should find it to be $a + \sqrt{b}$, and it is undoubtedly much easier to form an idea of it in this manner, than if we had only put the sign $\sqrt{}$ before that quantity. In the same manner, if we take the square of $\sqrt{a} + \sqrt{b}$, we find it $(a + b) + 2\sqrt{ab}$; therefore, reciprocally, the square root of $(a + b) + 2\sqrt{ab}$ will be $\sqrt{a} + \sqrt{b}$, which is likewise more easily understood, than if we had been satisfied with putting the sign $\sqrt{}$ before the quantity.

672. It is chiefly required, therefore, to assign a character, which may, in all cases, point out whether such a square root exists or not; for which purpose we shall begin with an easy quantity, requiring whether we can assign, in the sense that we have explained, the square root of the binomial $5 + 2\sqrt{6}$.

Suppose, therefore, that this root is $\sqrt{x} + \sqrt{y}$; the square of it is $(x + y) + 2\sqrt{xy}$, which must be equal to the quantity $5 + 2\sqrt{6}$. Consequently, the rational part $x + y$ must be equal to 5, and the irrational part $2\sqrt{xy}$ must be equal to $2\sqrt{6}$; which last equality gives $\sqrt{xy} = \sqrt{6}$. Now, since $x + y = 5$, we have $y = 5 - x$, and this value substituted in the equation $xy = 6$, produces $5x - x^2 = 6$, or $x^2 = 5x - 6$; therefore, $x = \frac{5}{2} + \sqrt{(\frac{25}{4} - \frac{24}{4})} = \frac{5}{2} + \frac{1}{2} = 3$. So that $x = 3$ and $y = 2$; whence we conclude, that the square root of $5 + 2\sqrt{6}$ is $\sqrt{3} + \sqrt{2}$.

673. As we have here found the two equations, $x + y = 5$, and $xy = 6$, we shall give a particular method for obtaining the values of x and y.

Since $x + y = 5$, by squaring, $x^2 + 2xy + y^2 = 25$; and as we know that $x^2 - 2xy + y^2$ is the square of $x - y$, let us subtract from $x^2 + 2xy + y^2 = 25$, the equation $xy = 6$, taken four times, or $4xy = 24$, in order to have $x^2 - 2xy + y^2 = 1$; whence by extraction we have $x - y = 1$; and as $x + y = 5$, we shall easily find $x = 3$, and $y = 2$: wherefore, the square root of $5 + 2\sqrt{6}$ is $\sqrt{3} + \sqrt{2}$.

674. Let us now consider the general binomial $a + \sqrt{b}$, and supposing its square root to be $\sqrt{x} + \sqrt{y}$, we shall have the equation $(x + y) + 2\sqrt{xy} = a + \sqrt{b}$; so that $x + y = a$, and $2\sqrt{xy} = \sqrt{b}$, or $4xy = b$; subtracting this square from the square of the equation $x + y = a$, that is, from $x^2 + 2xy + y^2 = a^2$, there remains $x^2 - 2xy + y^2 = a^2 - b$, the square root of which is $x - y = \sqrt{(a^2 - b)}$.

* In algebra we generally give the name *binomial* to any quantity composed of two terms; but Euler has thought proper to confine this appellation to those expressions, which the French analysts call *quantities partly commensurable, and partly incommensurable*. F. T.

Now, $x + y = a$; we have therefore $x = \frac{a + \surd(a^2 - b)}{2}$,

and $y = \frac{a - \surd(a^2 - b)}{2}$; consequently, the square root required of $a + \surd b$ is $\surd\frac{(a + \surd(a^2 - b))}{2} + \surd\frac{(a - \surd(a^2 - b))}{2}$.

675. We admit that this expression is more complicated than if we had simply put the radical sign $\surd$ before the given binomial $a + \surd b$, and written it $\surd(a + \surd b)$: but the above expression may be greatly simplified when the numbers a and b are such, that $a^2 - b$ is a square; since then the sign $\surd$, which is under the radical, disappears. We see also, at the same time, that the square root of the binomial $a + \surd b$ cannot be conveniently extracted, except when $a^2 - b = c^2$; for in this case the square root required is $\surd(\frac{a + c}{2}) + \surd(\frac{a - c}{2})$: but if $a^2 - b$ is not a perfect square, we cannot express the square root of $a + \surd b$ more simply, than by putting the radical sign $\surd$ before it.

676. The condition, therefore, which is requisite, in order that we may express the square root of a binomial $a + \surd b$ in a more convenient form, is, that $a^2 - b$ be a square; and if we represent that square by c^2, we shall have for the square root in question $\surd(\frac{a + c}{2}) + \surd(\frac{a - c}{2})$. We must farther remark, that the square root of $a - \surd b$ will be $\surd(\frac{a + c}{2}) - \surd(\frac{a - c}{2})$; for, by squaring this quantity, we get $a - 2\surd(\frac{a^2 - c^2}{4})$; now, since $c^2 = a^2 - b$, and consequently $a^2 - c^2 = b$, the same square is found

$$= a - 2\surd\frac{b}{4} = a - \frac{2\surd b}{2} = a - \surd b.$$

677. When it is required, therefore, to extract the square root of a binomial, as $a \pm \surd b$, the rule is, to subtract from the square a^2 of the rational part the square b of the irrational part, to take the square root of the remainder, and calling that root c, to write for the root required

$$\surd(\frac{a + c}{2}) \pm \surd(\frac{a - c}{2}).$$

678. If the square root of $2 + \surd 3$ were required, we

should have $a = 2$ and $\surd b = \surd 3$; wherefore $a^2 - b = c^2 = 1$; so that, by the formula just given, the root sought $= \surd\frac{3}{2} + \surd\frac{1}{2}$.

Let it be required to find the square root of the binomial $11 + 6\surd 2$. Here we shall have $a = 11$, and $\surd b = 6\surd 2$; consequently, $b = 36 \times 2 = 72$, and $a^2 - b = 49$, which gives $c = 7$; and hence we conclude, that the square root of $11 + 6\surd 2$ is $\surd 9 + \surd 2$, or $3 + \surd 2$.

Required the square root of $11 + 2\surd 30$. Here $a = 11$, and $\surd b = 2\surd 30$; consequently, $b = 4 \times 30 = 120$, $a^2 - b = 1$, and $c = 1$; therefore the root required is $\surd 6 + \surd 5$.

679. This rule also applies, even when the binomial contains imaginary, or impossible quantities.

Let there be proposed, for example, the binomial $1 + 4\surd -3$. First, we shall have $a = 1$ and $\surd b = 4\surd -3$, that is to say, $b = -48$, and $a^2 - b = 49$; therefore $c = 7$, and consequently the square root required is $\surd 4 + \surd -3 = 2 + \surd -3$.

Again, let there be given $-\frac{1}{2} + \frac{1}{2}\surd -3$. First, we have $a = -\frac{1}{2}$; $\surd b = \frac{1}{2}\surd -3$, and $b = \frac{1}{4} \times -3 = -\frac{3}{4}$; whence $a^2 - b = \frac{1}{4} + \frac{3}{4} = 1$, and $c = 1$; and the result required is $\surd\frac{1}{4} + \surd -\frac{3}{4} = \frac{1}{2} + \frac{\surd -3}{2}$, or $\frac{1}{2} + \frac{1}{2}\surd -3$.

Another remarkable example is that in which it is required to find the square root of $2\surd -1$. As there is here no rational part, we shall have $a = 0$. Now, $\surd b = 2\surd -1$, and $b = -4$; wherefore $a^2 - b = 4$ and $c = 2$; consequently, the square root required is $\surd 1 + \surd -1 = 1 + \surd -1$, and the square of this quantity is found to be $1 + 2\surd -1 - 1 = 2\surd -1$.

680. Suppose now we have such an equation as $x^2 = a \pm \surd b$, and that $a^2 - b = c^2$; we conclude from this, that the value of $x = \surd(\frac{a + c}{2}) \pm \surd(\frac{a - c}{2})$, which may be useful in many cases.

For example, if $x^2 = 17 + 12\surd 2$, we shall have $x = 3 + \surd 8 = 3 + 2\surd 2$.

681. This case occurs most frequently in the resolution of equations of the fourth degree, such as $x^4 = 2ax^2 + d$. For, if we suppose $x^2 = y$, we have $x^4 = y^2$, which reduces the given equation to $y^2 = 2ay + d$, and from this we find $y = a \pm \surd(a^2 + d)$, therefore, $x^2 = a \pm \surd(a^2 + d)$, and consequently we have another evolution to perform. Now,

since $\sqrt{b} = \sqrt{(a^2 + d)}$, we have $b = a^2 + d$, and $a^2 - b = -d$; if, therefore, $-d$ is a square, as c^2, that is to say, $d = -c^2$, we may assign the root required.

Suppose, in reality, that $d = -c^2$; or that the proposed equation of the fourth degree is $x^4 = 2ax^2 - c^2$, we shall then find that $x = \sqrt{(\frac{a+c}{2})} \pm \sqrt{(\frac{a-c}{2})}$.

682. We shall illustrate what we have just said by some examples.

1. Required two numbers, whose product may be 105, and whose squares may together make 274.

Let us represent those two numbers by x and y; we shall then have the two equations,

$$xy = 105$$
$$x^2 + y^2 = 274.$$

The first gives $y = \frac{105}{x}$, and this value of y being substituted in the second equation, we have

$$x^2 + \frac{105^2}{x^2} = 274.$$

Wherefore $x^4 + 105^2 = 274x^2$, or $x^4 = 274x^2 - 105^2$.

If we now compare this equation with that in the preceding article, we have $2a = 274$, and $-c^2 = -105^2$; consequently, $c = 105$, and $a = 137$. We therefore find

$$x = \sqrt{(\frac{137+105}{2})} \pm \sqrt{(\frac{137-105}{2})} = 11 \pm 4.$$

Whence $x = 15$, or $x = 7$. In the first case, $y = 7$, and in the second case, $y = 15$; whence the two numbers sought are 15 and 7.

683. It is proper, however, to observe, that this calculation may be performed much more easily in another way. For, since $x^2 + 2xy + y^2$ and $x^2 - 2xy + y^2$ are squares, and since the values of $x^2 + y^2$ and of xy are given, we have only to take the double of this last quantity, and then to add and subtract it from the first, as follows: $x^2 + y^2 = 274$; to which if we add $2xy = 210$, we have

$x^2 + 2xy + y^2 = 484$, which gives $x + y = 22$.

But subtracting $2xy$, there remains $x^2 - 2xy + y^2 = 64$, whence we find $x - y = 8$.

So that $2x = 30$, and $2y = 14$; consequently, $x = 15$ and $y = 7$.

The following general question is resolved by the same method.

2. Required two numbers, whose product may be m, and the sum of the squares n.

If those numbers are represented by x and y, we have the two following equations:

$$xy = m$$
$$x^2 + y^2 = n.$$

Now, $2xy = 2m$ being added to $x^2 + y^2 = n$, we have $x^2 + 2xy + y^2 = n + 2m$, and consequently,

$$x + y = \sqrt{(n + 2m)}.$$

But subtracting $2xy$, there remains $x^2 - 2xy + y^2 = n - 2m$, whence we get $x - y = \sqrt{(n - 2m)}$; we have, therefore, $x = \frac{1}{2}\sqrt{(n + 2m)} + \frac{1}{2}\sqrt{(n - 2m)}$; and

$$y = \tfrac{1}{2}\sqrt{(n + 2m)} - \tfrac{1}{2}\sqrt{(n - 2m)}.$$

684. 3. Required two numbers, such, that their product may be 35, and the difference of their squares 24.

Let the greater of the two numbers be x, and the less y: then we shall have the two equations

$$xy = 35,$$
$$x^2 - y^2 = 24;$$

and as we have not the same advantages here, we shall proceed in the usual manner. Here, the first equation gives $y = \frac{35}{x}$, and, substituting this value of y in the second, we have $x^2 - \frac{1225}{x^2} = 24$. Multiplying by x^2, we have $x^4 - 1225 = 24x^2$; or $x^4 = 24x^2 + 1225$. Now, the second member of this equation being affected by the sign $+$, we cannot make use of the formula already given, because having $c^2 = -1225$, c would become imaginary.

Let us therefore make $x^2 = z$; we shall then have $z^2 = 24z + 1225$, whence we obtain

$$z = 12 \pm \sqrt{(144 + 1225)} \text{ or } z = 12 \pm 37;$$

consequently, $x^2 = 12 \pm 37$; that is to say, either $= 49$, or $= -25$.

If we adopt the first value, we have $x = 7$ and $y = 5$. The second value gives $x = \sqrt{-25}$; and, since $xy = 35$, we have $y = \frac{35}{\sqrt{-25}} = \sqrt{\frac{1225}{-25}} = \sqrt{-49}$.

685. We shall conclude this chapter with the following question.

4. Required two numbers, such, that their sum, their product, and the difference of their squares, may be all equal.

Let x be the greater of the two numbers, and y the less; then the three following expressions must be equal to one another: namely, the sum, $x+y$; the product, xy; and the difference of the squares, x^2-y^2. If we compare the first with the second, we have $x+y=xy$; which will give a value of x: for $y = xy - x = (y-1)$, and $x = \frac{y}{y-1}$;

Consequently, $x + y = \frac{y}{y-1} + y = \frac{y^2}{y-1}$, and $xy = \frac{y^2}{y-1}$, that is to say, the sum is equal to the product; and to this also the difference of the squares ought to be equal. Now, we have $x^2 - y^2 = \frac{y^2}{y^2-2y+1} - y^2 = \frac{-y^4+2y^3}{y^2-2y+1}$; so that making this equal to the quantity found $\frac{y^2}{y-1}$, we have $\frac{y^2}{y-1} = \frac{-y^4+2y^3}{y^2-2y+1}$; dividing by y^2, we have $\frac{1}{y-1} = \ldots\ldots \frac{-y^2+2y}{y^2-2y+1}$; and multiplying by y^2-2y+1, or $(y-1)^2$, we have $y-1 = -y^2+2y$; consequently, $y^2 = y+1$; which gives $y = \frac{1}{2} \pm \sqrt{(\frac{1}{4}+1)} = \frac{1}{2} \pm \sqrt{\frac{5}{4}}$; or $y = \frac{1 \pm \sqrt{5}}{2}$, and since $x = \frac{y}{y-1}$, we shall have, by substitution, and using the sign $+$, $x = \frac{\sqrt{5}+1}{\sqrt{5}-1}$.

In order to remove the surd quantity from the denominator, multiply both terms by $\sqrt{5}+1$, and we obtain $x = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$.

Therefore the greater of the numbers sought, or x, $= \frac{3+\sqrt{5}}{2}$; and the less, y, $= \frac{1+\sqrt{5}}{2}$.

Hence their sum $x+y = 2+\sqrt{5}$; their product $xy = 2+\sqrt{5}$; and since $x^2 = \frac{7+3\sqrt{5}}{2}$, and $y^2 = \frac{3+\sqrt{5}}{2}$, we have also the difference of the squares $x^2 - y^2 = 2+\sqrt{5}$, being all the same quantity.

686. As this solution is very long, it is proper to remark that it may be abridged. In order to which, let us begin with making the sum $x+y$ equal to the difference of the squares x^2-y^2; we shall then have $x+y = x^2-y^2$; and dividing by $x+y$, because $x^2-y^2 = (x+y)\times(x-y)$, we find $1 = x-y$ and $x = y+1$. Consequently, $x+y = 2y+1$, and $x^2-y^2 = 2y+1$; farther, as the product xy, or y^2+y, must be equal to the same quantity, we have $y^2+y = 2y+1$, or $y^2 = y+1$, which gives, as before, $y = \frac{1+\sqrt{5}}{2}$.

687. The preceding question leads also to the solution of the following.

5. To find two numbers, such, that their sum, their product, and the sum of their squares, may be all equal.

Let the numbers sought be represented by x and y; then there must be an equality between $x+y$, xy, and x^2+y^2.

Comparing the first and second quantities, we have $x+y=xy$, whence $x = \frac{y}{y-1}$; consequently, xy, and $x+y = \frac{y^2}{y-1}$. Now, the same quantity is equal to x^2+y^2; so that we have

$$\frac{y^2}{y^2-2y+1} + y^2 = \frac{y^2}{y-1}.$$

Multiplying by y^2-2y+1, the product is

$$y^4 - 2y^3 + 2y^2 = y^3 - y^2, \text{ or } y^4 = 3y^3 - 3y^2;$$

and dividing by y^2, we have $y^2 = 3y - 3$; which gives $y = \frac{3}{2} \pm \sqrt{(\frac{9}{4}-3)} = \frac{3+\sqrt{-3}}{2}$; consequently, $y - 1 = \frac{1+\sqrt{-3}}{2}$, whence results $x = \frac{3+\sqrt{-3}}{1+\sqrt{-3}}$; and multiplying both terms by $1-\sqrt{-3}$, the result is

$$x = \frac{6-2\sqrt{-3}}{4}, \text{ or } \frac{3-\sqrt{-3}}{2}.$$

Therefore the numbers sought are $x = \frac{3-\sqrt{-3}}{2}$, and $y = \frac{3+\sqrt{-3}}{2}$, the sum of which is $x+y=3$, their product $xy=3$; and lastly, since $x^2 = \frac{3-3\sqrt{-3}}{2}$, and

$y^2 = \frac{3+3\sqrt{-3}}{2}$, the sum of the squares $x^2 + y^2 = 3$, all the same quantity as required.

688. We may greatly abridge this calculation by a particular artifice, which is applicable likewise to other cases; and which consists in expressing the numbers sought by the sum and the difference of two letters, instead of representing them by distinct letters.

In our last question, let us suppose one of the numbers sought to be $p + q$, and the other $p - q$; then their sum will be $2p$, their product will be $p^2 - q^2$, and the sum of their squares will be $2p^2 + 2q^2$, which three quantities must be equal to each other; therefore making the first equal to the second, we have $2p = p^2 - q^2$, which gives $q^2 = p^2 - 2p$.

Substituting this value of q^2 in the third quantity $(2p^2 + 2q^2)$, and comparing the result $4p^2 - 4p$ with the first, we have $2p = 4p^2 - 4p$, whence $p = \frac{3}{2}$.

Consequently, $q^2 = p^2 - 2p = -\frac{3}{4}$, and $q = \frac{\sqrt{-3}}{2}$;

so that the numbers sought are $p + q = \frac{3+\sqrt{-3}}{2}$, and

$p - q = \frac{3-\sqrt{-3}}{2}$, as before.

QUESTIONS FOR PRACTICE.

1. What two numbers are those, whose difference is 15, and half of their product equal to the cube of the less?
Ans. 3 and 18.

2. To find two numbers whose sum is 100, and product 2059.
Ans. 71 and 29.

3. There are three numbers in geometrical progression: the sum of the first and second is 10, and the difference of the second and third is 24. What are they?
Ans. 2, 8 and 32.

4. A merchant having laid out a certain sum of money in goods, sells them again for 24*l.* gaining as much per cent as the goods cost him: required, what they cost him. *Ans.* 20*l.*

5. The sum of two numbers is a, their product b. Required the numbers.

Ans. $\frac{a}{2} \pm \sqrt{(-b + \frac{a^2}{4})}$, and

$\frac{a}{2} \mp \sqrt{(-b + \frac{a^2}{4})}$.

6. The sum of two numbers is a, and the sum of their squares b. Required the numbers.

Ans. $\frac{a}{2} \pm \sqrt{(\frac{2b-a^2}{4})}$, and

$\frac{a}{2} \mp \sqrt{(\frac{2b-a^2}{4})}$.

7. To divide 36 into three such parts, that the second may exceed the first by 4, and that the sum of all their squares may be 464.
Ans. 8, 12, 16.

8. A person buying 120 pounds of pepper, and as many of ginger, finds that for a crown he has one pound more of ginger than of pepper. Now, the whole price of the pepper exceeded that of the ginger by six crowns: how many pounds of each had he for a crown?
Ans. 4 of pepper, and 5 of ginger.

9. Required three numbers in continual proportion, 60 being the middle term, and the sum of the extremes being equal to 125.
Ans. 45, 60, 80.

10. A person bought a certain number of oxen for 80 guineas: if he had received 4 more for the same money, he would have paid one guinea less for each head. What was the number of oxen?
Ans. 16.

11. To divide the number 10 into two such parts, that their product being added to the sum of their squares, may make 76.
Ans. 4 and 6.

12. Two travellers A and B set out from two places, Γ and Δ, at the same time; A from Γ with a design to pass through Δ, and B from Δ to travel the same way: after A had overtaken B, they found on computing their travels, that they had both together travelled 30 miles; that A had passed through Δ four days before, and that B, at his rate of travelling, was a journey of nine days distant from Γ. Required the distance between the places Γ and Δ. *Ans.* 6 miles.

CHAP. IX.

Of the Nature *of* Equations *of the* Second Degree.

689. What we have already said sufficiently shews, that equations of the second degree admit of two solutions; and this property ought to be examined in every point of view, because the nature of equations of a higher degree will be very much illustrated by such an examination. We shall therefore retrace, with more attention, the reasons which render an equation of the second degree capable of a double solution; since they undoubtedly will exhibit an essential property of those equations.

690. We have already seen, indeed, that this double solution arises from the circumstance that the square root of any number may be taken either positively, or negatively; but, as this principle will not easily apply to equations of higher degrees, it may be proper to illustrate it by a distinct analysis. Taking, therefore, for an example, the quadratic equation, $x^2 = 12x - 35$, we shall give a new reason for this equation being resolvible in two ways, by admitting for x the values 5 and 7, both of which will satisfy the terms of the equation.

691. For this purpose it is most convenient to begin with transposing the terms of the equation, so that one of the sides may become 0; the above equation consequently takes the form

$$x^2 - 12x + 35 = 0;$$

and it is now required to find a number such, that, if we substitute it for x, the quantity $x^2 - 12x + 35$ may be really equal to nothing; after which, we shall have to shew how this may be done in two different ways.

692. Now, the whole of this consists in clearly shewing, that a quantity of the form $x^2 - 12x + 35$ may be considered as the product of two factors. Thus, in reality, the quantity of which we speak is composed of the two factors $(x - 5) \times (x - 7)$: and since the above quantity must become 0, we must also have the product $(x - 5) \times (x - 7) = 0$; but a product, of whatever number of factors it is composed, becomes equal to 0, only when one of those factors is reduced to 0. This is a fundamental principle, to which we must pay particular attention, especially when equations of higher degrees are treated of.

693. It is therefore easily understood, that the product $(x - 5) \times (x - 7)$ may become 0 in two ways: first, when the first factor $x - 5 = 0$; and also, when the second factor $x - 7 = 0$. In the first case, $x = 5$, in the second $x = 7$. The reason is therefore very evident, why such an equation $x^2 - 12x + 35 = 0$, admits of two solutions; that is to say, why we can assign two values of x, both of which equally satisfy the terms of the equation; for it depends upon this fundamental principle, that the quantity $x^2 - 12x + 35$ may be represented by the product of two factors.

694. The same circumstances are found in all equations of the second degree: for, after having brought the terms to one side, we find an equation of the following form $x^2 - ax + b = 0$, and this formula may be always considered as the product of two factors, which we shall represent by $(x - p) \times (x - q)$, without concerning ourselves what numbers the letters p and q represent, or whether they be negative or positive. Now, as this product must be $= 0$, from the nature of our equation, it is evident that this may happen in two cases; in the first place, when $x = p$; and in the second place, when $x = q$; and these are the two values of x which satisfy the terms of the equation.

695. Let us here consider the nature of these two factors, in order that the multiplication of the one by the other may exactly produce $x^2 - ax + b$. By actually multiplying them, we obtain $x^2 - (p + q)\,x + pq$; which quantity must be the same as $x^2 - ax + b$, therefore we have evidently $p + q = a$, and $pq = b$. Hence is deduced this very remarkable property; that in every equation of the form $x^2 - ax + b = 0$, the two values of x are such, that their sum is equal to a, and their product equal to b: it therefore necessarily follows, that, if we know one of the values, the other also is easily found.

696. We have at present considered the case, in which the two values of x are positive, and which requires the second term of the equation to have the sign $-$, and the third term to have the sign $+$. Let us also consider the cases, in which either one or both values of x become negative. The first takes place, when the two factors of the equation give a product of this form, $(x - p) \times (x + q)$; for then the two values of x are $x = p$, and $x = -q$; and the equation itself becomes

$$x^2 + (q - p)\,x - pq = 0;$$

the second term having the sign $+$, when q is greater than p, and the sign $-$, when q is less than p; lastly, the third term is always negative.

The second case, in which both values of x are negative, occurs, when the two factors are

$$(x+p) \times (x+q);$$

for we shall then have $x = -p$, and $x = -q$; the equation itself therefore becomes

$$x^2 + (p+q)x + pq = 0,$$

in which both the second and third terms are affected by the sign $+$.

697. The signs of the second and the third terms consequently shew us the nature of the roots of any equation of the second degree. For let the equation be $x^2 \ldots\ldots ax \ldots\ldots b = 0$. If the second and third terms have the sign $+$, the two values of x are both negative; if the second term have the sign $-$, and the third term $+$, both values are positive: lastly, if the third term also have the sign $-$, one of the values in question is positive. But, in all cases whatever, the second term contains the *sum* of the two values, and the third term contains their *product*.

698. After what has been said, it will be easy to form equations of the second degree containing any two given values. Let there be required, for example, an equation such, that one of the values of x may be 7, and the other -3. We first form the simple equations $x = 7$, and $x = -3$; whence, $x - 7 = 0$, and $x + 3 = 0$; these give us the factors of the equation required, which consequently becomes $x^2 - 4x - 21 = 0$. Applying here, also, the above rule, we find the two given values of x; for if $x^2 = 4x + 21$, we have $x = 2 \pm \surd 25 = 2 \pm 5$, that is to say, $x = 7$, or $x = -3$.

699. The values of x may also happen to be equal. Suppose, for example, that an equation is required, in which both values may be 5: here the two factors will be $(x-5) \times (x-5)$, and the equation sought will be $x^2 - 10x + 25 = 0$. In this equation, x appears to have only one value; but it is because x is twice found $= 5$, as the common method of resolution shews; for we have $x^2 = 10x - 25$; wherefore $x = 5 \pm \surd 0 = 5 \pm 0$, that is to say, x is in two ways $= 5$.

700. A very remarkable case sometimes occurs, in which both values of x become imaginary, or impossible; and it is then wholly impossible to assign any value for x, that would satisfy the terms of the equation. Let it be proposed, for example, to divide the number 10 into two parts, such that their product may be 30. If we call one of those parts x, the other will be $10 - x$, and their product will be $10x - x^2 = 30$; wherefore $x^2 = 10x - 30$, and $x = 5 \pm \surd -5$, which, being an imaginary number, shews that the question is impossible.

701. It is very important, therefore, to discover some sign, by means of which we may immediately know whether an equation of the second degree be possible or not.

Let us resume the general equation $x^2 - ax + b = 0$. We shall have $x^2 = ax - b$, and $x = \frac{1}{2}a \pm \surd(\frac{1}{4}a^2 - b)$. This shews, that if b be greater than $\frac{1}{4}a^2$, or $4b$ greater than a^2, the two values of x are always imaginary, since it would be required to extract the square root of a negative quantity; on the contrary, if b be less than $\frac{1}{4}a^2$, or even less than 0, that is to say, if it be a negative number, both values will be possible or real. But, whether they be real or imaginary, it is no less true, that they are still expressible, and always have this property, that their sum is equal to a, and their product equal to b. Thus, in the equation $x^2 - 6x + 10 = 0$, the sum of the two values of x must be 6, and the product of these two values must also be 10; now, we find, 1. $x = 3 + \surd -1$, and 2. $x = 3 - \surd -1$, quantities whose sum is 6, and the product 10.

702. The expression which we have just found may likewise be represented in a manner more general, and so as to be applied to equations of this form, $fx^2 \pm gx + h = 0$; for this equation gives

$$x^2 = \mp \frac{gx}{f} - \frac{h}{f}, \text{ and } x = \mp \frac{g}{2f} \pm \surd\left(\frac{g^2}{4f^2} - \frac{h}{f}\right), \text{ or } \ldots$$

$$x = \frac{\mp g \pm \surd(g^2 - 4fh)}{2f};$$ whence we conclude, that the two values are imaginary, and, consequently, the equation impossible, when $4fh$ is greater than g^2; that is to say, when, in the equation $fx^2 - gx + h = 0$, four times the product of the first and the last term exceeds the square of the second term: for the product of the first and the last term, taken four times, is $4fhx^2$, and the square of the middle term is g^2x^2; now, if $4fhx^2$ be greater than g^2x^2, $4fh$ is also greater than g^2, and, in that case, the equation is evidently impossible; but in all other cases, the equation is possible, and two real values of x may be assigned. It is true, they are often irrational; but we have already seen, that, in such cases, we may always find them by approximation: whereas no approximations can take place with regard to imaginary expressions, such as $\surd -5$; for 100 is as far from being the value of that root, as 1, or any other number.

703. We have farther to observe, that any quantity of

the second degree, $x^2 \pm ax \pm b$, must always be resolvible into two factors, such as $(x \pm p) \times (x \pm q)$. For, if we took three factors, such as these, we should come to a quantity of the third degree; and taking only one such factor, we should not exceed the first degree. It is therefore certain, that every equation of the second degree necessarily contains two values of x, and that it can neither have more nor less.

704. We have already seen, that when the two factors are found, the two values of x are also known, since each factor gives one of those values, by making it equal to 0. The converse also is true, *viz.* that when we have found one value of x, we know also one of the factors of the equation; for if $x = p$ represents one of the values of x, in any equation of the second degree, $x - p$ is one of the factors of that equation; that is to say, all the terms having been brought to one side, the equation is divisible by $x - p$; and farther, the quotient expresses the other factor.

705. In order to illustrate what we have now said, let there be given the equation $x^2 + 4x - 21 = 0$, in which we know that $x = 3$ is one of the values of x, because $(3 \times 3) + (4 \times 3) - 21 = 0$; this shews, that $x - 3$ is one of the factors of the equation, or that $x^2 + 4x - 21$ is divisible by $x - 3$, which the actual division proves. Thus,

$$\begin{array}{r|l} x - 3) \; x^2 + 4x - 21 & (x + 7 \\ x^2 - 3x & \\ \hline 7x - 21 & \\ 7x - 21 & \\ \hline 0. & \end{array}$$

So that the other factor is $x + 7$, and our equation is represented by the product $(x - 3) \times (x + 7) = 0$; whence the two values of x immediately follow, the first factor giving $x = 3$, and the other $x = -7$.

CHAP. X.

Of Pure Equations *of the* Third Degree.

706. An equation of the third degree is said to be *pure*, when the cube of the unknown quantity is equal to a known

quantity, and when neither the square of the unknown quantity, nor the unknown quantity itself, is found in the equation; so that

$$x^3 = 125, \text{ or, more generally, } x^3 = a,\ x^3 = \frac{a}{b}, \text{ \&c.}$$

are equations of this kind.

707. It is evident how we are to deduce the value of x from such an equation, since we have only to extract the cube root of both sides. Thus, the equation $x^3 = 125$ gives $x = 5$, the equation $x^3 = a$ gives $x = \sqrt[3]{a}$, and the equation $x^3 = \frac{a}{b}$ gives $x = \sqrt[3]{\frac{a}{b}}$, or $x = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$. To be able, therefore, to resolve such equations, it is sufficient that we know how to extract the cube root of a given number.

708. But in this manner, we obtain only one value for x: but since every equation of the second degree has two values, there is reason to suppose that an equation of the third degree has also more than one value. It will be deserving our attention to investigate this; and, if we find that in such equations x must have several values, it will be necessary to determine those values.

709. Let us consider, for example, the equation $x^3 = 8$, with a view of deducing from it all the numbers, whose cubes are, respectively, 8. As $x = 2$ is undoubtedly such a number, what has been said in the last chapter shews that the quantity $x^3 - 8 = 0$, must be divisible by $x - 2$: let us therefore perform this division.

$$\begin{array}{r|l} x - 2) \; x^3 - 8 & (x^2 + 2x + 4 \\ x^3 - 2x^2 & \\ \hline 2x^2 - 8 & \\ 2x^2 - 4x & \\ \hline 4x - 8 & \\ 4x - 8 & \\ \hline 0. & \end{array}$$

Hence it follows, that our equation, $x^3 - 8 = 0$, may be represented by these factors;

$$(x - 2) \times (x^2 + 2x + 4) = 0.$$

710. Now, the question is, to know what number we are to substitute instead of x, in order that $x^3 = 8$, or that $x^3 - 8 = 0$; and it is evident that this condition is answered, by supposing the product which we have just now found equal to 0: but this happens, not only when the first

factor $x - 2 = 0$, which gives us $x = 2$, but also when the second factor

$x^2 + 2x + 4 = 0$. Let us, therefore, make $x^2 + 2x + 4 = 0$; then we shall have $x^2 = -2x - 4$, and thence $x = -1 \pm \sqrt{-3}$.

711. So that beside the case, in which $x = 2$, which corresponds to the equation $x^3 = 8$, we have two other values of x, the cubes of which are also 8; and these are,

$x = -1 + \sqrt{-3}$, and $x = -1 - \sqrt{-3}$, as will be evident, by actually cubing these expressions;

$$\begin{array}{lcl}
-1+\sqrt{-3} & & -1-\sqrt{-3} \\
-1+\sqrt{-3} & & -1-\sqrt{-3} \\
\hline
1-\sqrt{-3} & & 1+\sqrt{-3} \\
\quad -\sqrt{-3}-3 & & \quad +\sqrt{-3}-3 \\
\hline
-2-2\sqrt{-3} & \text{square} & -2+2\sqrt{-3} \\
-1+\sqrt{-3} & & -1-\sqrt{-3} \\
\hline
2+2\sqrt{-3} & & 2-2\sqrt{-3} \\
\quad -2\sqrt{-3}+6 & & \quad +2\sqrt{-3}+6 \\
\hline
8 & \text{cube.} & 8.
\end{array}$$

It is true, that these values are imaginary, or impossible; but yet they deserve attention.

712. What we have said applies in general to every cubic equation, such as $x^3 = a$; namely, that beside the value $x = \sqrt[3]{a}$, we shall always find two other values. To abridge the calculation, let us suppose $\sqrt[3]{a} = c$, so that $a = c^3$, our equation will then assume this form, $x^3 - c^3 = 0$, which will be divisible by $x - c$, as the actual division shews:

$$\begin{array}{l}
x - c)\ x^3 - c^3\ (x^2 + cx + c^2 \\
\qquad\ \ x^3 - cx^2 \\
\hline
\qquad\qquad cx^2 - c^3 \\
\qquad\qquad cx^2 - c^2x \\
\hline
\qquad\qquad\quad c^2x - c^3 \\
\qquad\qquad\quad c^2x - c^3 \\
\hline
\qquad\qquad\qquad 0.
\end{array}$$

Consequently, the equation in question may be represented by the product $(x - c) \times (x^2 + cx + c^2) = 0$, which is in fact $= 0$, not only when $x - c = 0$, or $x = c$, but also

when $x^2 + cx + c^2 = 0$. Now, this expression contains two other values of x; for it gives

$x^2 = -cx - c^2$, and $x = -\frac{c}{2} \pm \sqrt{(\frac{c^2}{4} - c^2)}$, or . . . . .

$x = \frac{-c \pm \sqrt{-3c^2}}{2}$; that is to say, $x = \frac{-c \pm c\sqrt{-3}}{2}$

$= \frac{-1 \pm \sqrt{-3}}{2} \times c$.

713. Now, as c was substituted for $\sqrt[3]{a}$, we conclude, that every equation of the third degree, of the form $x^3 = a$, furnishes three values of x expressed in the following manner:

1. $x = \sqrt[3]{a}$,

2. $x = \frac{-1 + \sqrt{-3}}{2} \times \sqrt[3]{a}$,

3. $x = \frac{-1 - \sqrt{-3}}{2} \times \sqrt[3]{a}$.

This shews, that every cube root has three different values; but that one only is real, or possible, the two others being impossible. This is the more remarkable, since every square root has two values, and since we shall afterwards see, that a biquadratic root has four different values, that a fifth root has five values, and so on.

In ordinary calculations, indeed, we employ only the first of those values, because the other two are imaginary; as we shall shew by some examples.

714. *Question* 1. To find a number, whosse quare, multiplied by its fourth part, may produce 432.

Let x be that number; the product of x^2 multiplied by $\frac{1}{4}x$ must be equal to the number 432, that is to say, $\frac{1}{4}x^3 = 432$, and $x^3 = 1728$: whence, by extracting the cube root, we have $x = 12$.

The number sought therefore is 12; for its square 144, multiplied by its fourth part, or by 3, gives 432.

715. *Question* 2. Required a number such, that if we divide its fourth power by its half, and add $14\frac{1}{4}$ to the product, the sum may be 100.

Calling that number x, its fourth power will be x^4; dividing by the half, or $\frac{1}{2}x$, we have $2x^3$; and adding to that $14\frac{1}{4}$, the sum must be 100. We have therefore $2x^3 + 14\frac{1}{4} = 100$; subtracting $14\frac{1}{4}$, there remains $2x^3 = \frac{343}{4}$; dividing by 2, gives $x^3 = \frac{343}{8}$, and extracting the cube root, we find $x = \frac{7}{2}$.

716. *Question* 3. Some officers being quartered in a

country, each commands three times as many horsemen, and twenty times as many foot-soldiers, as there are officers. Also a horseman's monthly pay amounts to as many florins as there are officers, and each foot-soldier receives half that pay; the whole monthly expense is 13000 florins. Required the number of officers.

If x be the number required, each officer will have under him $3x$ horsemen and $20x$ foot-soldiers. So that the whole number of horsemen is $3x^2$, and that of foot-soldiers is $20x^2$.

Now, each horseman receiving x florins per month, and each foot-soldier receiving $\frac{1}{2}x$ florins, the pay of the horsemen, each month, amounts to $3x^3$, and that of the foot-soldiers to $10x^3$; consequently, they all together receive $13x^3$ florins, and this sum must be equal to 13000 florins: we have therefore $13x^3 = 13000$, or $x^3 = 1000$, and $x = 10$, the number of officers required.

717. *Question* 4. Several merchants enter into partnership, and each contributes a hundred times as many sequins as there are partners; they send a factor to Venice, to manage their capital, who gains, for every hundred sequins, twice as many sequins as there are partners, and he returns with 2662 sequins profit. Required the number of partners.

If this number be supposed $= x$, each of the partners will have furnished $100x$ sequins, and the whole capital must have been $100x^2$; now, the profit being $2x$ for 100, the capital must have produced $2x^3$; so that $2x^3 = 2662$, or $x^3 = 1331$; this gives $x = 11$, which is the number of partners.

718. *Question* 5. A country girl exchanges cheeses for hens, at the rate of two cheeses for three hens; which hens lay each $\frac{1}{2}$ as many eggs as there are cheeses. Farther, the girl sells at market nine eggs for as many sous as each hen had laid eggs, receiving in all 72 sous: how many cheeses did she exchange?

Let the number of cheeses $= x$, then the number of hens, which the girl received in exchange, will be $\frac{3}{2}x$, and each hen laying $\frac{1}{2}x$ eggs, the number of eggs will be $= \frac{3}{4}x^2$. Now, as nine eggs sell for $\frac{1}{2}x$ sous, the money which $\frac{3}{4}x^2$ eggs produce is $\frac{1}{24}x^3$, and $\frac{1}{24}x^3 = 72$. Consequently, $x^3 = 24 \times 72 = 8 \times 3 \times 8 \times 9 = 8 \times 8 \times 27$, whence $x = 12$; that is to say, the girl exchanged twelve cheeses for eighteen hens.

CHAP. XI.

Of the Resolution *of* Complete Equations *of the* Third Degree.

719. An equation of the third degree is called *complete*, when, beside the cube of the unknown quantity, it contains that unknown quantity itself, and its square: so that the general formula for these equations, bringing all the terms to one side, is

$$ax^3 \pm bx^2 \pm cx \pm d = 0.$$

And the purpose of this chapter is to shew how we are to derive from such equations the values of x, which are also called the roots of the equation. We suppose, in the first place, that every such an equation has three roots; since it has been seen, in the last chapter, that this is true even with regard to pure equations of the same degree.

720. We shall first consider the equation $x^3 - 6x^2 + 11x - 6 = 0$; and, since an equation of the second degree may be considered as the product of two factors, we may also represent an equation of the third degree by the product of three factors, which are in the present instance,

$$(x - 1) \times (x - 2) \times (x - 3) = 0;$$

since, by actually multiplying them, we obtain the given equation; for $(x - 1) \times (x - 2)$ gives $x^2 - 3x + 2$, and multiplying this by $x - 3$, we obtain $x^3 - 6x^2 + 11x - 6$, which are the given quantities, and which must be $= 0$. Now, this happens, when the product $(x - 1) \times (x - 2) \times (x - 3) = 0$; and, as it is sufficient for this purpose, that one of the factors become $= 0$, three different cases may give this result, namely, when $x - 1 = 0$, or $x = 1$; secondly, when $x - 2 = 0$, or $x = 2$; and thirdly, when $x - 3 = 0$, or $x = 3$.

We see immediately also, that if we substituted for x, any number whatever beside one of the above three, none of the three factors would become equal to 0; and, consequently, the product would no longer be 0: which proves that our equation can have no other root than these three.

721. If it were possible, in every other case, to assign the three factors of such an equation in the same manner,

we should immediately have its three roots. Let us, therefore, consider, in a more general manner, these three factors, $x-p$, $x-q$, $x-r$. Now, if we seek their product, the first, multiplied by the second, gives $x^2-(p+q)x+pq$, and this product, multiplied by $x-r$, makes

$$x^3-(p+q+r)x^2+(pq+pr+qr)x-pqr.$$

Here, if this formula must become $=0$, it may happen in three cases: the first is that, in which $x-p=0$, or $x=p$; the second is, when $x-q=0$, or $x=q$; the third is, when $x-r=0$, or $x=r$.

722. Let us now represent the quantity found, by the equation $x^3-ax^2+bx-c=0$. It is evident, in order that its three roots may be $x=p$, $x=q$, $x=r$, that we must have,

1. $a=p+q+r$,
2. $b=pq+pr+qr$, and
3. $c=pqr$.

We perceive, from this, that the second term of the equation contains the sum of the three roots; that the third term contains the sum of the products of the roots taken two by two; and lastly, that the fourth term consists of the product of all the three roots multiplied together.

From this last property we may deduce an important truth, which is, that an equation of the third degree can have no other rational roots than the divisors of the last term; for, since that term is the product of the three roots, it must be divisible by each of them: so that when we wish to find a root by trial, we immediately see what numbers we are to use *.

For example, let us consider the equation, $x^3=x+6$, or $x^3-x-6=0$. Now, as this equation can have no other rational roots than numbers which are factors of the last term 6, we have only 1, 2, 3, 6, to try with, and the result of these trials will be as follows:

If $x=1$, we have $1-1-6=-6$.
If $x=2$, we have $8-2-6=0$.
If $x=3$, we have $27-3-6=18$.
If $x=6$, we have $216-6-6=204$.

Hence we see, that $x=2$ is one of the roots of the given equation; and, knowing this, it is easy to find the other two;

* We shall find in the sequel, that this is a general property of equations of any dimension; and as this trial requires us to know all the divisors of the last term of the equation, we may for this purpose have recourse to the Table, Art. 66.

for $x=2$ being one of the roots, $x-2$ is a factor of the equation, and we have only to seek the other factor by means of division as follows:

$$\begin{array}{l} x-2)\; x^3-x-6\; (x^2+2x+3 \\ \quad\quad x^3-2x^2 \\ \hline \quad\quad\quad 2x^2-x-6 \\ \quad\quad\quad 2x^2-4x \\ \hline \quad\quad\quad\quad 3x-6 \\ \quad\quad\quad\quad 3x-6 \\ \hline \quad\quad\quad\quad\quad 0. \end{array}$$

Since, therefore, the formula is represented by the product $(x-2)\times(x^2+2x+3)$, it will become $=0$, not only when $x-2=0$, but also when $x^2+2x+3=0$: and, this last factor gives $x^2+2x=-3$; consequently,

$$x=-1\pm\sqrt{-2};$$

and these are the other two roots of our equation, which are evidently impossible, or imaginary.

723. The method which we have explained, is applicable only when the first term x^3 is multiplied by 1, and the other terms of the equation have integer coefficients; therefore, when this is not the case, we must begin by a preparation, which consists in transforming the equation into another form having the condition required; after which, we make the trial that has been already mentioned.

Let there be given, for example, the equation

$$x^3-3x^2+\tfrac{11}{4}x-\tfrac{3}{4}=0;$$

as it contains fourth parts, let us make $x=\frac{y}{2}$, which will give

$$\frac{y^3}{8}-\frac{3y^2}{4}+\frac{11y}{8}-\tfrac{3}{4}=0,$$

and, multiplying by 8, we shall obtain the equation

$$y^3-6y^2+11y-6=0,$$

the roots of which are, as we have already seen, $y=1$, $y=2$, $y=3$; whence it follows, that in the given equation, we have $x=\frac{1}{2}$, $x=1$, $x=\frac{3}{2}$.

724. Let there be an equation, where the coefficient of the first term is a whole number but not 1, and whose last term is 1; for example,

$$6x^3-11x^2+6x-1=0.$$

Here, if we divide by 6, we shall have $x^3 - \frac{11}{6}x^2 + x - \frac{1}{6} = 0$; which equation we may clear of fractions, by the method just explained.

First, by making $x = \frac{y}{6}$, we shall have

$$\frac{y^3}{216} - \frac{11y^2}{216} + \frac{y}{6} - \frac{1}{6} = 0;$$

and multiplying by 216, the equation will become $y^3 - 11y^2 + 36y - 36 = 0$. But as it would be tedious to make trial of all the divisors of the number 36, and as the last term of the original equation is 1, it is better to suppose, in this equation, $x = \frac{1}{z}$; for we shall then have $\frac{6}{z^3} - \frac{11}{z^2} + \frac{6}{z} - 1 = 0$, which, multiplied by z^3, gives $6 - 11z + 6z^2 - z^3 = 0$, and transposing all the terms, $z^3 - 6z^2 + 11z - 6 = 0$; where the roots are $z = 1$, $z = 2$, $z = 3$; whence it follows that in our equation $x = 1$, $x = \frac{1}{2}$, $x = \frac{1}{3}$.

725. It has been observed in the preceding articles, that in order to have all the roots in positive numbers, the signs *plus* and *minus* must succeed each other alternately; by means of which the equation takes this form, $x^3 - ax^2 + bx - c = 0$, the signs changing as many times as there are positive roots. If all the three roots had been negative, and we had multiplied together the three factors $x + p$, $x + q$, $x + r$, all the terms would have had the sign *plus*, and the form of the equation would have been $x^3 + ax^2 + bx + c = 0$, in which the same signs follow each other *three* times; that is, the number of negative roots.

We may conclude, therefore, that as often as the signs change, the equation has positive roots; and that as often as the same signs follow each other, the equation has negative roots. This remark is very important, because it teaches us whether the divisors of the last term are to be taken affirmatively or negatively, when we wish to make the trial which has been mentioned.

726. In order to illustrate what has been said by an example, let us consider the equation $x^3 + x^2 - 34x + 56 = 0$, in which the signs are changed twice, and in which the same sign returns but once. Here we conclude that the equation has two positive roots, and one negative root; and as these roots must be divisors of the last term 56, they must be included in the numbers $\pm 1, 2, 4, 7, 8, 14, 28, 56$.

Let us, therefore, make $x = 2$, and we shall have $8 + 4 - 68 + 56 = 0$; whence we conclude that $x = 2$ is a positive root, and that therefore $x - 2$ is a divisor of the equation, by means of which we easily find the two other roots; for, actually dividing by $x - 2$, we have

$$\begin{array}{l} x - 2)\ x^3 + x^2 - 34x + 56\ (x^2 + 3x - 28 \\ \quad\quad x^3 - 2x^2 \\ \hline \quad\quad\quad 3x^2 - 34x \\ \quad\quad\quad 3x^2 - 6x \\ \hline \quad\quad\quad\quad -28x + 56 \\ \quad\quad\quad\quad -28x + 56 \\ \hline \quad\quad\quad\quad\quad 0. \end{array}$$

And making the quotient $x^2 + 3x - 28 = 0$, we find the two other roots; which will be $x = -\frac{3}{2} \pm \sqrt{(\frac{9}{4} + 28)} = -\frac{3}{2} \pm \frac{11}{2}$; that is, $x = 4$; or $x = -7$; and taking into account the root found before, namely, $x = 2$, we clearly perceive that the equation has two positive, and one negative root. We shall give some examples to render this still more evident.

727. *Question* 1. There are two numbers, whose difference is 12, and whose product multiplied by their sum makes 14560. What are those numbers?

Let x be the less of the two numbers, then the greater will be $x + 12$, and their product will be $x^2 + 12x$, which multiplied by the sum $2x + 12$, gives

$$2x^3 + 36x^2 + 144x = 14560;$$

and dividing by 2, we have

$$x^3 + 18x^2 + 72x = 7280.$$

Now, the last term 7280 is too great for us to make trial of all its divisors; but as it is divisible by 8, we shall make $x = 2y$, because the new equation, $8y^3 + 72y^2 + 144y = 7280$, after the substitution, being divided by 8, will become $y^3 + 9y^2 + 18y = 910$; to solve which, we need only try the divisors 1, 2, 5, 7, 10, 13, &c. of the number 910: where it is evident, that the three first, 1, 2, 5, are too small; beginning therefore with supposing $y = 7$, we immediately find that number to be one of the roots; for the substitution gives $343 + 441 + 126 = 910$. It follows, therefore, that $x = 14$; and the two other roots will be found by dividing $y^3 + 9y^2 + 18y - 910$ by $y - 7$, thus:

$$y - 7\,)\,y^3 + 9y^2 + 18y - 910\,(\,y^2 + 16y + 130$$
$$y^3 - 7y^2$$
$$16y^2 + 18y$$
$$16y^2 - 112y$$
$$130y - 910$$
$$130y - 910$$
$$0.$$

Supposing now this quotient $y^2 + 16y + 130 = 0$, we shall have $y^2 + 16y = -130$, and thence $y = -8 \pm \sqrt{-66}$; a proof that the other two roots are impossible.

The two numbers sought are therefore 14, and (14 + 12) = 26; the product of which, 364, multiplied by their sum, 40, gives 14560.

728. *Question 2.* To find two numbers whose difference is 18, and such, that their sum multiplied by the difference of their cubes, may produce 275184.

Let x be the less of the two numbers, then $x + 18$ will be the greater; the cube of the first will be x^3, and the cube of the second

$$x^3 + 54x^2 + 972x + 5832;$$

the difference of the cubes

$$54x^2 + 972x + 5832 = 54\,(x^2 + 18x + 108),$$

which multiplied by the sum $2x + 18$, or $2\,(x + 9)$, gives the product

$$108\,(x^3 + 27x^2 + 270x + 972) = 275184.$$

And, dividing by 108, we have

$$x^3 + 27x^2 + 270x + 972 = 2548, \text{ or}$$
$$x^3 + 27x^2 + 270x = 1576.$$

Now, the divisors of 1576 are 1, 2, 4, 8, &c. the two first of which are too small; but if we try $x = 4$, that number is found to satisfy the terms of the equation.

It remains, therefore, to divide by $x - 4$, in order to find the two other roots; which division gives the quotient $x^2 + 31x + 394$; making therefore

$$x^2 + 31x = -394, \text{ we shall find}$$
$$x = -\tfrac{31}{2} \sqrt{\pm(\tfrac{961}{4} - \tfrac{1576}{4})};$$

that is, two imaginary roots.

Hence the numbers sought are 4, and (4 + 18) = 22.

729. *Question 3.* Required two numbers whose difference is 720, and such, that if the less be multiplied by the square root of the greater, the product may be 20736.

If the less be represented by x, the greater will evidently be $x + 720$; and, by the question,

$$x\sqrt{(x + 720)} = 20736 = 8\,.\,8\,.\,4\,.\,81.$$

Squaring both sides, we have

$$x^2\,(x + 720) = x^3 + 720x^2 = 8^2\,.\,8^2\,.\,4^2\,.\,81^2.$$

Let us now make $x = 8y$; this supposition gives

$$8^3y^3 + 720\,.\,8^2y^2 = 8^2\,.\,8^2\,.\,4^2\,.\,81^2;$$

and dividing by 8^3, we have $y^3 + 90y^2 = 8\,.\,4^2\,.\,81^2$. Farther, let us suppose $y = 2z$, and we shall have $8z^3 + 4\,.\,90z^2 = 8\,.\,4^2\,.\,81^2$; or, dividing by 8,

$$z^3 + 45z^2 = 4^2\,.\,81^2.$$

Again, make $z = 9u$, in order to have, in this last equation, $9^3u^3 + 45\,.\,9^2u^2 = 4^2\,.\,9^4$, because dividing now by 9^3, the equation becomes $u^3 + 5u^2 = 4^2\,.\,9$, or $u^2\,(u + 5) = 16\,.\,9 = 144$; where it is obvious, that $u = 4$; for in this case $u^2 = 16$, and $u + 5 = 9$: since, therefore, $u = 4$, we have $z = 36$, $y = 72$, and $x = 576$, which is the less of the two numbers sought; so that the greater is 1296, and the square root of this last, or 36, multiplied by the other number 576, gives 20736.

730. *Remark.* This question admits of a simpler solution; for since the square root of the greater number, multiplied by the less, must give a product equal to a given number, the greater of the two numbers must be a square. If, therefore, from this consideration, we suppose it to be x^2, the other number will be $x^2 - 720$, which being multiplied by the square root of the greater, or by x, we have $x^3 - 720x = 20736 = 64\,.\,27\,.\,12$.

If we make $x = 4y$, we shall have

$$64y^3 - 720\,.\,4y = 64\,.\,27\,.\,12, \text{ or}$$
$$y^3 - 45y = 27\,.\,12.$$

Supposing, farther, $y = 3z$, we find $27z^3 - 135z = 27\,.\,12$; or, dividing by 27, $z^3 - 5z = 12$, or $z^3 - 5z - 12 = 0$. The divisors of 12 are 1, 2, 3, 4, 6, 12: the first two are too small; but the supposition of $z = 3$ gives exactly $27 - 15 - 12 = 0$. Consequently, $z = 3$, $y = 9$, and $x = 36$; whence we conclude, that the greater of the two numbers sought, or x^2, = 1296, and that the less, or $x^2 - 720$, = 576, as before.

731. *Question 4.* There are two numbers, whose difference is 12; the product of this difference by the sum of their cubes is 102144; what are the numbers?

Calling the less of the two numbers x, the greater will be $x + 12$: also the cube of the first is x^3, and of the second

$x^3 + 36x^2 + 432x + 1728$; the product also of the sum of these cubes by the difference 12, is

$$12\,(2x^3 + 36x^2 + 432x + 1728) = 102144;$$

and, dividing successively by 12 and by 2, we have

$$x^3 + 18x^2 + 216x + 864 = 4256, \text{ or}$$
$$x^3 + 18x^2 + 216x = 3392 = 8 \,.\, 8 \,.\, 53.$$

If now we substitute $x = 2y$, and divide by 8, we shall have $y^3 + 9y^2 + 54y = 8 \,.\, 53 = 424$.

Now, the divisors of 424 are 1, 2, 4, 8, 53, &c. 1 and 2 are evidently too small; but if we make $y = 4$, we find $64 + 144 + 216 = 424$. So that $y = 4$, and $x = 8$; whence we conclude that the two numbers sought are 8 and $(8 + 12) = 20$.

732. *Question 5.* Several persons form a partnership, and establish a certain capital, to which each contributes ten times as many pounds as there are persons in company: they gain 6 *plus* the number of partners per cent; and the whole profit is 392 pounds: required how many partners there are?

Let x be the number required; then each partner will have furnished $10x$ pounds, and conjointly $10x^2$ pounds; and since they gain $x + 6$ per cent, they will have gained with the whole capital, $\frac{x^3+6x^2}{10}$, which is to be made equal to 392.

We have, therefore, $x^3 + 6x^2 = 3920$, consequently, making $x = 2y$, and dividing by 8, we have

$$y^3 + 3y^2 = 490.$$

Now, the divisors of 490 are 1, 2, 5, 7, 10, &c. the first three of which are too small; but if we suppose $y = 7$, we have $343 + 147 = 490$; so that $y = 7$, and $x = 14$.

There are therefore fourteen partners, and each of them put 140 pounds into the common stock.

733. *Question 6.* A company of merchants have a common stock of 8240 pounds; and each contributes to it forty times as many pounds as there are partners; with which they gain as much per cent as there are partners: now, on dividing the profit, it is found, after each has received ten times as many pounds as there are persons in the company, that there still remains 224*l.* Required the number of merchants?

If x be made to represent the number, each will have contributed $40x$ to the stock; consequently, all together will have contributed $40x^2$, which makes the stock

$= 40x^2 + 8240$. Now, with this sum they gain x per cent; so that the whole gain is

$$\frac{40x^3}{100} + \frac{8240x}{100} = \tfrac{4}{10}x^3 + \tfrac{824}{10}x = \tfrac{2}{5}x^3 + \tfrac{412}{5}x.$$

From which sum each receives $10x$, and consequently they all together receive $10x^2$, leaving a remainder of 224; the profit must therefore have been $10x^2 + 224$, and we have the equation

$$\frac{2x^3}{5} + \frac{412x}{5} = 10x^2 + 224.$$

Multiplying by 5 and dividing by 2, we have $x^3 + 206x = 25x^2 + 560$, or $x^3 - 25x^2 + 206x - 560 = 0$: the first, however, will be more convenient for trial. Here the divisors of the last term are 1, 2, 4, 5, 7, 8, 10, 14, 16, &c. and they must be taken positively; because in the second form of the equation the signs vary three times, which shews that all the three roots are positive.

Here, if we first try $x = 1$, and $x = 2$, it is evident that the first side will become less than the second. We shall therefore make trial of other divisors.

When $x = 4$, we have $64 + 824 = 400 + 560$, which does not satisfy the terms of the equation.

If $x = 5$, we have $125 + 1030 = 625 + 560$, which likewise does not succeed.

But if $x = 7$, we have $343 + 1442 = 1225 + 560$, which answers to the equation; so that $x = 7$ is a root of it. Let us now seek for the other two, by dividing the second form of our equation by $x - 7$.

$$x - 7)\; x^3 - 25x^2 + 260x - 560 \;(\, x^2 - 18x + 80$$
$$x^3 - 7x^2$$
$$\overline{-18x^2 + 206x}$$
$$-18x^2 + 126x$$
$$\overline{80x - 560}$$
$$80x - 560$$
$$\overline{0.}$$

Now, making this quotient equal to nothing, we have $x^2 - 18x + 80 = 0$, or $x^2 - 18x = -80$; which gives $x = 9 \pm 1$, so that the two other roots are $x = 8$; or $x = 10$.

This question therefore admits of three answers. According to the first, the number of merchants is 7; according to the second, it is 8; and, according to the third, it is 10. The following statement shews, that all these will answer the conditions of the question:

Number of merchants	7	8	10
Each contributes $40x$	280	320	400
In all they contribute $40x^2$	1960	2560	4000
The original stock was	8240	8240	8240
The whole stock is $40x^2 + 8240$	10200	10800	12240
With this capital they gain as much per cent as there are partners	714	864	1224
Each takes from it	70	80	100
So that they all together take $10x^2$	490	640	1000
Therefore there remains	224	224	224

CHAP. XII.

Of the Rule *of* Cardan, *or of* Scipio Ferreo.

734. When we have removed fractions from an equation of the third degree, according to the manner which has been explained, and none of the divisors of the last term are found to be a root of the equation, it is a certain proof, not only that the equation has no root in integer numbers; but also that a fractional root cannot exist; which may be proved as follows.

Let there be given the equation $x^3 - ax^2 + bx - c = 0$, in which, a, b, c, express integer numbers. If we suppose, for example, $x = \frac{3}{2}$, we shall have $\frac{27}{8} - \frac{9}{4}a + \frac{3}{2}b - c = 0$. Now, the first term only has 8 for the denominator; the others being either integer numbers, or numbers divided only by 4 or by 2, and therefore cannot make 0 with the first term. The same thing happens with every other fraction.

735. As in those fractions the roots of the equation are neither integer numbers, nor fractions, they are irrational, and, as it often happens, imaginary. The manner, therefore, of expressing them, and of determining the radical signs which affect them, forms a very important point, and deserves to be carefully explained. This method, called *Cardan's Rule*, is ascribed to *Cardan*, or more properly to *Scipio Ferreo*, both of whom lived some centuries since*.

736. In order to understand this rule, we must first attentively consider the nature of a cube, whose root is a binomial.

Let $a + b$ be that root; then the cube of it will be $a^3 + 3a^2b + 3ab^2 + b^3$, and we see that it is composed of the cubes of the two terms of the binomial, and beside that, of the two middle terms, $3a^2b + 3ab^2$, which have the common factor $3ab$, multiplying the other factor, $a + b$; that is to say, the two terms contain thrice the product of the two terms of the binomial, multiplied by the sum of those terms.

737. Let us now suppose $x = a + b$; taking the cube of each side, we have $x^3 = a^3 + b^3 + 3ab(a + b)$: and, since $a + b = x$, we shall have the equation, $x^3 = a^3 + b^3 + 3abx$, or $x^3 = 3abx + a^3 + b^3$, one of the roots of which we know to be $x = a + b$. Whenever, therefore, such an equation occurs, we may assign one of its roots.

For example, let $a = 2$ and $b = 3$; we shall then have the equation $x^3 = 18x + 35$, which we know with certainty to have $x = 5$ for one of its roots.

738. Farther, let us now suppose $a^3 = p$, and $b^3 = q$; we shall then have $a = \sqrt[3]{p}$ and $b = \sqrt[3]{q}$, consequently, $ab = \sqrt[3]{pq}$; therefore, whenever we meet with an equation, of the form $x^3 = 3x\sqrt[3]{pq} + p + q$, we know that one of the roots is $\sqrt[3]{p} + \sqrt[3]{q}$.

Now, we can determine p and q, in such a manner, that both $3\sqrt[3]{pq}$ and $p + q$ may be quantities equal to determinate numbers; so that we can always resolve an equation of the third degree, of the kind which we speak of.

739. Let, in general, the equation $x^3 = fx + g$ be proposed. Here, it will be necessary to compare f with $3\sqrt[3]{pq}$, and g with $p + q$; that is, we must determine p and q in

* This rule when first discovered by Scipio Ferreo was only for particular forms of cubics, but it was afterwards generalised by Tartalea and Cardan. See Montucla's Hist. Math.; also Dr. Hutton's Dictionary, article Algebra; and Professor Bonnycastle's Introduction to his Treatise on Algebra, Vol. 1. p. xii—xv.

such a manner, that $3\sqrt[3]{pq}$ may become equal to f, and $p + q = g$; for we then know that one of the roots of our equation will be $x = \sqrt[3]{p} + \sqrt[3]{q}$.

740. We have therefore to resolve these two equations,

$$3\sqrt[3]{pq} = f,$$
$$p + q = g.$$

The first gives $\sqrt[3]{pq} = \frac{f}{3}$; or $pq = \frac{f^3}{27} = \frac{1}{27}f^3$, and $4pq = \frac{4}{27}f^3$. The second equation, being squared, gives $p^2 + 2pq + q^2 = g^2$; if we subtract from it $4pq = \frac{4}{27}f^3$, we have $q^2 - 2pq + q^2 = g^2 - \frac{4}{27}f^3$, and taking the square root of both sides, we have

$$p - q = \sqrt{(g^2 - \tfrac{4}{27}f^3)}.$$

Now, since $p + q = g$, we have, by adding $p + q$ to one side of the equation, and its equal, g, to the other, $2p = g + \sqrt{(g^2 - \frac{4}{27}f^3)}$; and, by subtracting $p - q$ from $p + q$, we have $2q = g - \sqrt{(g^2 - \frac{4}{27}f^3)}$; consequently,

$$p = \frac{g + \sqrt{(g^2 - \frac{4}{27}f^3)}}{2} \text{ and } q = \frac{g - \sqrt{(g^2 - \frac{4}{27}f^3)}}{2}.$$

741. In a cubic equation, therefore, of the form $x^3 = fx + g$, whatever be the numbers f and g, we have always for one of the roots

$$x = \sqrt[3]{\left(\frac{g + \sqrt{g^2 - \frac{4}{27}f^3}}{2}\right)} + \sqrt[3]{\left(\frac{g - \sqrt{g^2 - \frac{4}{27}f^3}}{2}\right)};$$

that is, an irrational quantity, containing not only the sign of the square root, but also the sign of the cube root; and this is the formula which is called *the Rule of Cardan*.

742. Let us apply it to some examples, in order that its use may be better understood.

Let $x^3 = 6x + 9$. First, we shall have $f = 6$, and $g = 9$; so that $g^2 = 81$, $f^3 = 216$, $\frac{4}{27}f^3 = 32$; then $g^2 - \frac{4}{27}f^3 = 49$, and $\sqrt{(g^2 - \frac{4}{27}f^3)} = 7$. Therefore, one of the roots of the given equation is

$$x = \sqrt[3]{\left(\frac{9+7}{2}\right)} + \sqrt[3]{\left(\frac{9-7}{2}\right)} = \sqrt[3]{\tfrac{16}{2}} + \sqrt[3]{\tfrac{2}{2}} = \sqrt[3]{8} + \sqrt[3]{1} = 2 + 1 = 3.$$

743. Let there be proposed the equation $x^3 = 3x + 2$. Here, we shall have $f = 3$ and $g = 2$; and consequently, $g^2 = 4$, $f^3 = 27$, and $\frac{4}{27}f^3 = 4$; which gives $\sqrt{(g^2 - \frac{4}{27}f^3)} = 0$; whence it follows, that one of the roots is $x = \sqrt[3]{\left(\frac{2+0}{2}\right)} + \sqrt[3]{\left(\frac{2-0}{2}\right)} = 1 + 1 = 2$.

744. It often happens, however, that, though such an equation has a rational root, that root cannot be found by the rule which we are now considering.

Let there be given the equation $x^3 = 6x + 40$, in which $x = 4$ is one of the roots. We have here $f = 6$ and $g = 40$; farther, $g^2 = 1600$, and $\frac{4}{27}f^3 = 32$; so that $g^2 - \frac{4}{27}f^3 = 1568$, and $\sqrt{(g^2 - \frac{4}{27}f^3)} = \sqrt{1568} = \ldots\ldots \sqrt{(4 \cdot 4 \cdot 49 \cdot 2)} = 28\sqrt{2}$; consequently one of the roots will be

$$x = \sqrt[3]{\left(\frac{40 + 28\sqrt{2}}{2}\right)} + \sqrt[3]{\left(\frac{40 - 28\sqrt{2}}{2}\right)} \text{ or}$$

$$x = \sqrt[3]{(20 + 14\sqrt{2})} + \sqrt[3]{(20 - 14\sqrt{2})};$$

which quantity is really $= 4$, although, upon inspection, we should not suppose it. In fact, the cube of $2 + \sqrt{2}$ being $20 + 14\sqrt{2}$, we have, reciprocally, the cube root of $20 + 14\sqrt{2}$ equal to $2 + \sqrt{2}$; in the same manner, $\sqrt[3]{(20 - 14\sqrt{2})} = 2 - \sqrt{2}$; wherefore our root $x = 2 + \sqrt{2} + 2 - \sqrt{2} = 4$*.

745. To this rule it might be objected, that it does not extend to all equations of the third degree, because the square of x does not occur in it; that is to say, the second term of the equation is wanting. But we may remark, that every complete equation may be transformed into another, in which the second term is wanting, which will therefore enable us to apply the rule.

To prove this, let us take the complete equation $x^3 - 6x^2 + 11x - 6 = 0$: where, if we take the third of the coefficient 6 of the second term, and make $x - 2 = y$, we shall have

$$x = y + 2, \quad x^2 = y^2 + 4y + 4, \text{ and}$$
$$x^3 = y^3 + 6y^2 + 12y + 8;$$

$$\begin{aligned}
\text{Consequently, } x^3 &= y^3 + 6y^2 + 12y + 8 \\
-6x^2 &= \quad\; - 6y^2 - 24y - 24 \\
11x &= \qquad\qquad\; 11y + 22 \\
-6 &= \qquad\qquad\qquad\; - 6
\end{aligned}$$

or, $x^3 - 6x^2 + 11x - 6 = y^3 - y$.

We have, therefore, the equation $y^3 - y = 0$, the resolu-

* We have no general rules for extracting the cube root of these binomials, as we have for the square root; those that have been given by various authors, all lead to a mixt equation of the third degree similar to the one proposed. However, when the extraction of the cube root is possible, the sum of the two radicals which represent the root of the equation, always becomes rational; so that we may find it immediately by the method explained, Art. 722. F. T.

tion of which it is evident, since we immediately perceive that it is the product of the factors

$$y(y^2-1) = y\,(y+1) \times (y-1) = 0.$$

If we now make each of these factors $= 0$, we have

$$1\begin{cases} y \pm 0, \\ x = 2, \end{cases} \qquad 2\begin{cases} y = -1, \\ x = \ \ 1, \end{cases} \qquad 3\begin{cases} y = 1, \\ x = 3, \end{cases}$$

that is to say, the three roots which we have already found.

746. Let there now be given the general equation of the third degree, $x^3 + ax^2 + bx + c = 0$, of which it is required to destroy the second term.

For this purpose, we must add to x the third of the coefficient of the second term, preserving the same sign, and then write for this sum a new letter, as for example y, so that we shall have $x + \frac{1}{3}a = y$, and $x = y - \frac{1}{3}a$; whence results the following calculation:

$$x = y - \tfrac{1}{3}a,\ x^2 = y^2 - \tfrac{2}{3}ay + \tfrac{1}{9}a^2,$$
$$\text{and } x^3 = y^3 - ay^2 + \tfrac{1}{3}a^2y - \tfrac{1}{27}a^3;$$

Consequently,

$$\begin{aligned} x^3 &= y^3 - ay^2 + \tfrac{1}{3}a^2y - \tfrac{1}{27}a^3 \\ ax^2 &= \qquad\ \ ay^2 - \tfrac{2}{3}a^2y + \tfrac{1}{9}a^3 \\ bx &= \qquad\qquad\qquad by - \tfrac{1}{3}ab \\ c &= \qquad\qquad\qquad\qquad\quad c \end{aligned}$$

or, $y^3 - (\frac{1}{3}a - b)\,y + \frac{2}{27}a^3 - \frac{1}{3}ab + c = 0$, an equation in which the second term is wanting.

747. We are enabled, by means of this transformation, to find the roots of all equations of the third degree, as the following example will shew.

Let it be proposed to resolve the equation

$$x^3 - 6x^2 + 13x - 12 = 0.$$

Here it is first necessary to destroy the second term; for which purpose, let us make $x - 2 = y$, and then we shall have $x = y + 2$, $x^2 = y^2 + 4y + 4$, and $x^3 = y^3 + 6y^2 + 12y + 8$; therefore,

$$\begin{aligned} x^3 &= y^3 + 6y^2 + 12y + 8 \\ -6x^2 &= \quad\ \ - 6y^2 - 24y - 24 \\ 13x &= \qquad\qquad\ \ 13y + 26 \\ -12 &= \qquad\qquad\qquad\ \ - 12 \end{aligned}$$

which gives $y^3 + y - 2 = 0$; or $y^3 = -y + 2$.

And if we compare this equation with the formula, (Art. 741) $x^3 = fx + g$, we have $f = -1$, and $g = 2$; wherefore, $g^2 = 4$, and $\frac{4}{27}f^3 = -\frac{4}{27}$; also, $g^2 - \frac{4}{27}f^3 = 4 + \frac{4}{27} = \frac{112}{27}$, and $\sqrt{(g^2 - \frac{4}{27}f^3)} = \sqrt{\frac{112}{27}} = \frac{4\sqrt{21}}{9}$;

consequently,

$$y = \left(\frac{\frac{4\sqrt{21}}{9} + 2}{2}\right)^{\frac{1}{3}} + \left(\frac{2 - \frac{4\sqrt{21}}{9}}{2}\right)^{\frac{1}{3}}, \text{ or}$$

$$y = \sqrt[3]{(1 + \frac{2\sqrt{21}}{9})} + \sqrt[3]{(1 - \frac{2\sqrt{21}}{9})}, \text{ or}$$

$$y = \sqrt[3]{(9 + \frac{2\sqrt{21}}{9})} + \sqrt[3]{(9 - \frac{2\sqrt{21}}{9})}$$

$$y = \sqrt[3]{(\frac{27+6\sqrt{21}}{27})} + \sqrt[3]{(\frac{27-6\sqrt{21}}{27})} \text{ or}$$

$$y = \tfrac{1}{3}\sqrt[3]{(27 + 6\sqrt{21})}\ (+\tfrac{1}{3}\sqrt[3]{(27 - 6\sqrt{21})};$$

and it remains to substitute this value in $x = y + 2$.

748. In the solution of this example, we have been brought to a quantity doubly irrational; but we must not immediately conclude that the root is irrational: because the binomials $27 \pm 6\sqrt{21}$ might happen to be real cubes; and this is the case here; for the cube of $\frac{3+\sqrt{21}}{2}$ being $\frac{216+48\sqrt{21}}{8} = 27 + 6\sqrt{21}$, it follows that the cube root of $27 + 6\sqrt{21}$ is $\frac{3+\sqrt{21}}{2}$, and that the cube root of $27 - 6\sqrt{21}$ is $\frac{3-\sqrt{21}}{2}$. Hence the value which we found for y becomes

$$y = \tfrac{1}{3}(\frac{3+\sqrt{21}}{2}) + \tfrac{1}{3}(\frac{3-\sqrt{21}}{2}) = \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

Now, since $y = 1$, we have $x = 3$ for one of the roots of the equation proposed, and the other two will be found by dividing the equation by $x - 3$.

$$\begin{array}{r|l} x - 3\ )\ x^3 - 6x^2 + 13x - 12 & (x^2 - 3x + 4 \\ x^3 - 3x^2 \qquad\qquad\quad & \\ \hline -3x^2 + 13x \qquad & \\ -3x^2 + \ \ 9x \qquad & \\ \hline 4x - 12 & \\ 4x - 12 & \\ \hline 0. & \end{array}$$

Also making the quotient $x^2 - 3x + 4 = 0$, we have $x^2 = 3x - 4$; and

$$x = \frac{3}{2} \pm \sqrt{(\tfrac{9}{4} - \tfrac{16}{4})} = \tfrac{3}{2} \pm \sqrt{-\tfrac{7}{4}} = \frac{3 \pm \sqrt{-7}}{2};$$

which are the other two roots, but they are imaginary.

749. It was, however, by chance, as we have remarked, that we were able, in the preceding example, to extract the cube root of the binomials that we obtained, which is the case only when the equation has a rational root; consequently, the rules of the preceding chapter are more easily employed for finding that root. But when there is no rational root, it is, on the other hand, impossible to express the root which we obtain in any other way, than according to the rule of Cardan; so that it is then impossible to apply reductions. For example, in the equation $x^3 = 6x + 4$, we have $f = 6$ and $g = 4$; so that $x = \sqrt[3]{(2 + 2\sqrt{-1})} + \sqrt[3]{(2 - 2\sqrt{-1})}$, which cannot be otherwise expressed*.

* In this example, we have $\frac{4}{27}f^3$ less than g^2, which is the well-known *irreducible case*; a case which is so much the more remarkable, as the three roots are then always real. We cannot here make use of Cardan's formula, except by applying the methods of approximation, such as transforming it into an infinite series. In the work spoken of in the Note, Art. 40, Lambert has given particular Tables, by which we may easily find the numerical values of the roots of cubic equations, in the irreducible, as well as the other cases. For this purpose we may also employ the ordinary Tables of sines. See the Spherical Astronomy of Mauduit, printed at Paris in 1765.

In the present work of EULER, we are not to look for all that might have been said on the direct and approximate resolutions of equations. He had too many curious and important objects, to dwell long upon this; but by consulting *l'Histoire des Mathematiques, l'Algebre de M. Clairaut, le Cours de Mathematiques de M. Bezout*, and the latter volumes of the Academical Memoirs of Paris and Berlin, the reader will obtain all that is known at present concerning the resolution of equations. F. T.

For a clear and explicit investigation of this method, the reader is also referred to Bonnycastle's Trigonometry; from which the following formulæ for the solution of the different cases of cubic equations are extracted.

1. $x^3 + px - q = 0$.

Put $\frac{q}{2}(\frac{3}{p})^{\frac{3}{2}} = \tan. z$, and $\sqrt[3]{(\tan. (45° - \frac{1}{2}z))} = \tan. u$;

Then $x = 2\sqrt{\frac{p}{3}} \times \cot. 2u$. Or, putting

Log. $\frac{q}{2} + 10 - \frac{3}{2}\log. \frac{p}{2} = \log. \tan. z$, and

QUESTIONS FOR PRACTICE.

1. Given $y^3 + 30y = 117$, to determine y. *Ans.* $y = 3$.
2. Given $y^3 - 36y = 91$, to find the value of y. *Ans.* $y = 7$.

$\frac{1}{3}(\log. \tan. (45° - \frac{1}{2}z) + 20) = \log. \tan. u$,

Then $\log. x = \frac{1}{3}\log. \frac{4p}{3} + \log. \cot. 2u - 10$.

2. $x^3 + px + q = 0$.

Put $\frac{q}{2}(\frac{3}{p})^{\frac{3}{2}} = \tan. z$, and $\sqrt[3]{(\tan. (45° - \frac{1}{2}z))} = \tan. u$,

Then $x = -2\sqrt{\frac{p}{3}} \times \cot. 2u$. Or, putting

Log. $\frac{q}{2} + 10 - \frac{3}{2}\log. \frac{p}{3} = \log. \tan. z$, and

$\frac{1}{3}(\log. \tan. (45° - \frac{1}{2}z) + 20) = \log. \tan. u$,

Then $\log. x = 10 - \frac{1}{3}\log. \frac{4p}{3} - \log. \cot. 2u$.

3. $x^3 - px - q = 0$.

This form has 2 cases, according as $\frac{2}{q}(\frac{p}{3})^{\frac{3}{2}}$ is less, or greater than 1.

In the 1st case, put $\frac{2}{q}(\frac{p}{3})^{\frac{3}{2}} = \cos. z$.

And $\sqrt[3]{(\tan. 45° - \frac{1}{2}z)} = \tan. u$;

Then $x = 2\sqrt{\frac{p}{3}} \times \text{cosec}. 2u$. Or, putting

$10 + \frac{3}{2}\log. \frac{p}{3} - \log. \frac{q}{2} = \log. \cos. z$, and

$\frac{1}{3}(\log. \tan. (45° - \frac{1}{2}z) + 20) = \log. \tan. u$;

Then $\log. x = 10 + \log. \frac{4p}{3} - \log. \sin. 2u$.

In the 2d case, put $\frac{q}{2}(\frac{3}{p})^{\frac{3}{2}} = \cos. z$, and x will have the 3 following values:

$$x = +2\sqrt{\frac{p}{3}} \times \cos. \frac{z}{3}$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos. (60° - \frac{z}{3})$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos. (60° + \frac{z}{3}) \text{ or,}$$

3. Given $y^3 + 24y = 250$, to find the value of y. *Ans.* $y = 5{\cdot}05$.

4. Given $y^6 - 3y^4 - 2y^2 - 8 = 0$, to find y. *Ans.* $y = 2$.

$$\text{Log.}\ x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.\frac{z}{3} - 10,$$

$$\text{Log.}\ x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.\left(60° - \frac{z}{3}\right) + 10,$$

$$\text{Log.}\ x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.\left(60° + \frac{z}{3}\right) + 10,$$

Taking the value of x, answering to log. x, positively in the first equation, and negatively in the two latter.

4. $x^3 - px + q = 0$.

This form, like the former, has also two cases, according as $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}}$ is less, or greater than 1.

In the 1st case, put $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}} = \cos.\ z$,

And $\sqrt[3]{(\tan.(45° - \frac{1}{2}z))} = \tan.\ u$, as before;

Then $x = -2\sqrt{\frac{p}{3}}\ \text{cosec.}\ 2u$. Or, putting

$$10 + \tfrac{3}{2}\log.\frac{p}{3} - \log.\frac{q}{2} = \log.\cos.\ z, \text{ and}$$

$$\tfrac{1}{3}\{\log.(\tan.45° - \tfrac{1}{2}z) + 20\} = \log.\tan.\ u;$$

$$\text{Then, } -\log.\ x = 10 + \log\frac{4p}{3} - \log.\sin.\ 2u.$$

In the 2d case, put $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = \cos.\ z$, and x will have the 3 following values:

$$x = -2\sqrt{\frac{p}{3}} \times \cos.\frac{z}{3}$$

$$x = +2\sqrt{\frac{p}{3}} \times \cos.\left(60° - \frac{z}{3}\right)$$

$$x = +2\sqrt{\frac{p}{3}} \times \cos.\left(60° + \frac{z}{3}\right). \text{ Or,}$$

$$\text{Log.}\ x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.\frac{z}{3} - 10,$$

$$\text{Log.}\ x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.\left(60° - \frac{x}{3}\right) - 10,$$

$$\text{Log.}\ x = \tfrac{1}{2}\log.\frac{4p}{3} + \log.\cos.\left(60° + \frac{z}{3}\right) - 10,$$

5. Given $y^3 + 3y^2 + 9y = 13$, to determine y. *Ans.* $y = 1$.

6. Given $x^3 - 6x = -9$, to find the value of x. *Ans.* $x = -3$.

7. Given $x^3 - 6x^2 + 10x = 8$, to find x. *Ans.* $x = 4$.

8. Given $p^3 - \frac{193}{3}p = \frac{1150}{27}$, to find p. *Ans.* $p = 8\frac{1}{3}$.

9. Given $x^3 - \frac{13}{3}x = \frac{70}{27}$, to find x. *Ans.* $x = 2\frac{1}{3}$.

10. Given $a^3 - 91a = -330$, to find a. *Ans.* $a = 5$.

11. Given $y^3 - 19y = 30$, what is the value of y? *Ans.* $y = 5$.

Taking the value of x, answering to log. x, negatively in the first equation, and positively in the two latter.

As an example of this mode of solution, in what is usually called the *Irreducible Case of Cubic Equations*, Let $x^3 - 3x = 1$, to find its 3 roots.

Here $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = \frac{1}{2}(\frac{3}{3})^{\frac{3}{2}} = \frac{1}{2} = .5 = \cos.60° = z$, hence

$$x = 2\sqrt{\frac{p}{3}} \times \cos.\frac{z}{3} = 2\cos.20° = 1.8793852$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos.\left(60° - \frac{z}{3}\right) = -2\cos.40° = -1.5320888$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos.\left(60° + \frac{z}{3}\right) = -2\cos.80° = -0.3472964.$$

Also, let $x^3 - 3x = -1$, to find its three roots.

Here, as before, $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}} = .5 = \cos.60° = z$, hence

$$x = -2\sqrt{\frac{p}{3}} \times \cos.\frac{z}{3} = -2\cos.20° = -1.8793852$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos.\left(60° - \frac{z}{3}\right) = 2\cos.40° = 1.5320888$$

$$x = -2\sqrt{\frac{p}{3}} \times \cos.\left(60° + \frac{z}{3}\right) = 2\cos.80° = 0.3472964.$$

Where the roots are the negatives of those of the first case. For the mode of investigating these kinds of formulæ, see, in addition to the references already given, Cagnoli, Traité de Trigon. and Article Irreducible Case, in the Supplement to Dr. Hutton's Mathematical Dictionary.

CHAP. XIII.

Of the Resolution *of* Equations *of the* Fourth Degree.

750. When the highest power of the quantity x rises to the fourth degree, we have *equations of the fourth degree*, the general form of which is

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

We shall, in the first place, consider *pure* equations of the fourth degree, the expression for which is simply $x^4 = f$; the root of which is immediately found by extracting the biquadrate root of both sides, since we obtain $x = \sqrt[4]{f}$.

751. As x^4 is the square of x^2, the calculation is greatly facilitated by beginning with the extraction of the square root; for we shall then have $x^2 = \sqrt{f}$; and, taking the square root again, we have $x = \sqrt[4]{f}$; so that $\sqrt[4]{f}$ is nothing but the square root of the square root of f.

For example, if we had the equation $x^4 = 2401$, we should immediately have $x^2 = 49$, and then $x = 7$.

752. It is true this is only one root; and since there are always three roots in an equation of the third degree, so also there are four roots in an equation of the fourth degree: but the method which we have explained will actually give those four roots. For, in the above example, we have not only $x^2 = 49$, but also $x^2 = -49$; now, the first value gives the two roots $x = 7$ and $x = -7$, and the second value gives $x = \sqrt{-49}, = 7\sqrt{-1}$, and $x = -\sqrt{-49} = -7\sqrt{-1}$; which are the four biquadrate roots of 2401. The same also is true with respect to other numbers.

753. Next to these pure equations, we shall consider others, in which the second and fourth terms are wanting, and which have the form $x^4 + fx^2 + g = 0$. These may be resolved by the rule for equations of the second degree; for if we make $x^2 = y$, we have $y^2 + fy + g = 0$, or $y^2 = -fy - g$, whence we deduce

$$y = -\tfrac{1}{2}f \pm \sqrt{(\tfrac{1}{4}f^2 - g)} = \left(\frac{-f \pm \sqrt{(f^2 - 4g)}}{2}\right).$$

Now, $x^2 = y$; so that $x = \pm\sqrt{\left(\frac{-f \pm \sqrt{(f^2 - 4g)}}{2}\right)}$, in which the double signs $\pm$ indicate all the four roots.

754. But whenever the equation contains all the terms, it may be considered as the product of four factors. In fact, if we multiply these four factors together, $(x - p) \times (x - q) \times (x - r) \times (x - s)$, we get the product $x^4 - (p + q + r + s)\,x^3 + (pq + pr + ps + qr + qs + rs)\,x^2 - (pqr + pqs + prs + qrs)\,x + pqrs$; and this quantity cannot be equal to 0, except when one of these four factors is $= 0$. Now, that may happen in four ways;

1. when $x = p$; 2. when $x = q$;
3. when $x = r$; 4. when $x = s$;

and consequently these are the four roots of the equation.

755. If we consider this formula with attention, we observe, in the second term, the sum of the four roots multiplied by $-x^3$; in the third term, the sum of all the possible products of two roots, multiplied by x^2; in the fourth term, the sum of the products of the roots combined three by three, multiplied by $-x$; lastly, in the fifth term, the product of all the four roots multiplied together.

756. As the last term contains the product of all the roots, it is evident that such an equation of the fourth degree can have no rational root, which is not a divisor of the last term. This principle, therefore, furnishes an easy method of determining all the rational roots, when there are any; since we have only to substitute successively for x all the divisors of the last term, till we find one which satisfies the terms of the equation: for having found such a root, for example, $x = p$, we have only to divide the equation by $x - p$, after having brought all the terms to one side, and then suppose the quotient $= 0$. We thus obtain an equation of the third degree, which may be resolved by the rules already given.

757. Now, for this purpose, it is absolutely necessary that all the terms should consist of integers, and that the first should have only unity for the coefficient; whenever, therefore, any terms contain fractions, we must begin by destroying those fractions, and this may always be done by substituting, instead of x, the quantity y, divided by a number which contains all the denominators of those fractions.

For example, if we have the equation

$$x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{3}x^2 - \tfrac{3}{4}x + \tfrac{1}{18} = 0,$$

as we find here fractions which have for denominators 2, 3, and multiples of these numbers, let us suppose $x = \frac{y}{6}$, and we shall then have

$$\frac{y^4}{6^4} - \frac{\frac{1}{2}y^3}{6^3} + \frac{\frac{1}{3}y^2}{6^2} - \frac{\frac{3}{4}y}{6} + \tfrac{1}{18} = 0,$$

an equation, which, multiplied by 6^4, becomes

$$y^4 - 3y^3 + 12y^2 - 162y + 72 = 0.$$

If we now wish to know whether this equation has rational roots, we must write, instead of y, the divisors of 72 successively, in order to see in what cases the formula would really be reduced to 0.

758. But as the roots may as well be positive as negative, we must make two trials with each divisor; one, supposing that divisor positive, the other, considering it as negative. However, the following rule will frequently enable us to dispense with this*. Whenever the signs + and − succeed each other regularly, the equation has as many positive roots as there are changes in the signs; and as many times as the same sign recurs without the other intervening, so many negative roots belong to the equation. Now, our example contains four changes of the signs, and no succession; so that all the roots are positive: and we have no need to take any of the divisors of the last term negatively.

759. Let there be given the equation

$$x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$$

We see here two changes of signs, and also two successions; whence we conclude, with certainty, that this equation contains two positive, and as many negative roots, which must all be divisors of the number 12. Now, its divisors being 1, 2, 3, 4, 6, 12, let us first try $x = +1$, which actually produces 0; therefore one of the roots is $x = 1$.

If we next make $x = -1$, we find $+1 - 2 - 7 + 8 + 12 = 21 - 9 = 12$: so that $x = -1$ is not one of the roots of the equation. Let us now make $x = 2$, and we again find the quantity $= 0$; consequently, another of the roots is $x = 2$; but $x = -2$, on the contrary, is found not to be a root. If we suppose $x = 3$, we have $81 + 54 - 63 - 24 + 12 = 60$, so that the supposition does not answer; but $x = -3$, giving $81 - 54 - 63 + 24 + 12 = 0$, this is evidently one of the roots sought. Lastly, when we try $x = -4$, we likewise see the equation reduced to nothing; so that all the four roots are rational, and have the following values: $x = 1$, $x = 2$, $x = -3$, and $x = -4$; and, according to the rule given above, two of these roots are positive, and the two others are negative.

* This rule is general for equations of all dimensions, provided there are no imaginary roots. The French ascribe it to Descartes, the English to Harriot; but the general demonstration of it was first given by M. l'Abbé de Gua. See the Memoires de l'Academie des Sciences de Paris, for 1741. F. T.

760. But as no root could be determined by this method, when the roots are all irrational, it was necessary to devise other expedients for expressing the roots whenever this case occurs; and two different methods have been discovered for finding such roots, whatever be the nature of the equation of the fourth degree.

But before we explain those general methods, it will be proper to give the solution of some particular cases, which may frequently be applied with great advantage.

761. When the equation is such, that the coefficients of the terms succeed in the same manner, both in the direct and in the inverse order of the terms, as happens in the following equation*;

$$x^4 + mx^3 + nx^2 + mx + 1 = 0;$$

or in this other equation, which is more general:

$$x^4 + max^3 + na^2x^2 + ma^3x + a^4 = 0;$$

we may always consider such a formula as the product of two factors, which are of the second degree, and are easily resolved. In fact, if we represent this last equation by the product

$$(x^2 + pax + a^2) \times (x^2 + qax + a^2) = 0,$$

in which it is required to determine p and q in such a manner, that the above equation may be obtained, we shall find, by performing the multiplication,

$$x^4 + (p + q)ax^3 + (pq + 2)a^2x^2 + (p + q)a^3x + a^4 = 0;$$

and, in order that this equation may be the same as the former, we must have,

1. $p + q = m$,
2. $pq + 2 = n$,

and, consequently, $pq = n - 2$.

* These equations may be called *reciprocal*, for they are not at all changed by substituting $\frac{1}{x}$ for x. From this property it follows, that if a, for instance, be one of the roots, $\frac{1}{a}$ will be one likewise; for which reason such equations may be reduced to others of a dimension one half less. De Moivre has given, in his Miscellanea Analytica, page 71, general formulæ for the reduction of such equations, whatever be their dimension. F. T.

See also Wood's Algebra, the Complément des Elemens d'Algebra, by Lacroix, and Waring's Medit. Algeb. chap. 3.

Now, squaring the first of those equations, we have $p^2 + 2pq + q^2 = m^2$; and if from this we subtract the second, taken four times, or $4pq = 4n - 8$, there remains $p^2 - 2pq + q^2 = m^2 - 4n + 8$; and taking the square root, we find $p - q = \sqrt{(m^2 - 4n + 8)}$; also, $p + q = m$; we shall therefore have, by addition, $2p = m + \sqrt{(m^2 - 4n + 8)}$, or $p = \frac{m + \sqrt{(m^2 - 4n + 8)}}{2}$; and, by subtraction, $2q = m - \sqrt{(m^2 - 4n + 8)}$, or $q = \frac{m - \sqrt{(m^2 - 4n + 8)}}{2}$. Having therefore found p and q, we have only to suppose each factor $= 0$, in order to determine the value of x. The first gives $x^2 + pax + a^2 = 0$, or $x^2 = -pax - a^2$, whence we obtain $x = -\frac{pa}{2} \pm \sqrt{(\frac{p^2a^2}{4} - a^2)}$,

$$\text{or } x = -\frac{pa}{2} \pm \tfrac{1}{2}a\sqrt{(p^2 - 4)}.$$

The second factor gives $x = -\frac{qa}{2} \pm \frac{1}{2}a\sqrt{(q^2 - 4)}$;

and these are the four roots of the given equation.

762. To render this more clear, let there be given the equation $x^4 - 4x^3 - 3x^2 - 4x + 1 = 0$. We have here $a = 1, m = -4, n = -3$; consequently, $m^2 - 4n + 8 = 36$, and the square root of this quantity is $= 6$; therefore $p = \frac{-4+6}{2} = 1$, and $q = \frac{-4-6}{2} = -5$; whence result the four roots,

1st and 2d $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3} = \frac{-1 \pm \sqrt{(-3)}}{2}$; and

3d and 4th $x = \frac{5}{2} \pm \frac{1}{2}\sqrt{21} = \frac{5 \pm \sqrt{21}}{2}$; that is, the

four roots of the given equation are:

1. $x = \frac{-1+\sqrt{-3}}{2}$, 2. $x = \frac{-1-\sqrt{-3}}{2}$,

3. $x = \frac{5+\sqrt{21}}{2}$, 4. $x = \frac{5-\sqrt{21}}{2}$.

The first two of these roots are imaginary, or impossible; but the last two are possible; since we may express $\sqrt{21}$ to any degree of exactness, by means of decimal fractions. In fact, 21 being the same with 21·00000000, we have only to extract the square root, which gives $\sqrt{21} = 4{\cdot}5825$.

Since, therefore, $\sqrt{21} = 4{\cdot}5825$, the third root is very nearly $x = 4{\cdot}7912$, and the fourth, $x = 0{\cdot}2087$. It would have been easy to have determined these roots with still more precision: for we observe that the fourth root is very nearly $\frac{2}{10}$, or $\frac{1}{5}$, which value will answer the equation with sufficient exactness. In fact, if we make $x = \frac{1}{5}$, we find $\frac{1}{625} - \frac{4}{125} - \frac{3}{25} - \frac{4}{5} + 1 = \frac{31}{625}$. We ought however to have obtained 0, but the difference is evidently not great.

763. The second case in which such a resolution takes place, is the same as the first with regard to the coefficients, but differs from it in the signs, for we shall suppose that the second and the fourth terms have different signs; such, for example, as the equation

$$x^4 + max^3 + na^2x^2 - ma^3x + a^4 = 0,$$

which may be represented by the product,

$$(x^2 + pax - a^2) \times (x^2 + qax - a^2) = 0.$$

For the actual multiplication of these factors gives

$$x^4 + (p + q)ax^3 + (pq - 2)a^2x^2 - (p + q)a^3x + a^4,$$

a quantity equal to that which was given, if we suppose, in the first place, $p + q = m$, and in the second place, $pq - 2 = n$, or $pq = n + 2$; because in this manner the fourth terms become equal of themselves. If now we square the first equation, as before, (Art. 761.) we shall have $p^2 + 2pq + q^2 = m^2$; and if from this we subtract the second, taken four times, or $4pq = 4n + 8$, there will remain $p^2 - 2pq + q^2 = m^2 - 4n - 8$; the square root of which is $p - q = \sqrt{(m^2 - 4n - 8)}$, and thence, by adding $p + q = m$, we obtain

$p = \frac{m + \sqrt{(m^2 - 4n - 8)}}{2}$; and, by subtracting $p + q$, ...

$q = \frac{m - \sqrt{(m^2 - 4n - 8)}}{2}$. Having therefore found p and q, we shall obtain from the first factor (as in Art. 761.) the two roots $x = -\frac{1}{2}pa \pm \frac{1}{2}a\sqrt{(p^2 + 4)}$, and from the second factor the two roots $x = -\frac{1}{2}qa \pm \frac{1}{2}a\sqrt{(q^2 + 4)}$, that is, we have the four roots of the equation proposed.

764. Let there be given the equation

$$x^4 - 3 \times 2x^3 + 3 \times 8x + 16 = 0.$$

Here we have $a = 2$, $m = -3$, and $n = 0$; so that $\sqrt{(m^2 - 4n - 8)} = 1, = p - q$; and, consequently,

$$p = \frac{-3+1}{2} = -1, \text{ and } q = \frac{-3-1}{2} = -2.$$

Therefore the first two roots are $x = 1 \pm \sqrt{5}$, and the last two are $x = 2 \pm \sqrt{8}$; so that the four roots sought will be,

$$1.\ x = 1 + \sqrt{5}, \qquad 2.\ x = 1 - \sqrt{5},$$
$$3.\ x = 2 + \sqrt{8}, \qquad 4.\ x = 2 - \sqrt{8}.$$

Consequently, the four factors of our equation will be $(x - 1 - \sqrt{5}) \times (x - 1 + \sqrt{5}) \times (x - 2 - \sqrt{8}) \times (x - 2 + \sqrt{8})$, and their actual multiplication produces the given equation; for the first two being multiplied together, give $x^2 - 2x - 4$, and the other two give $x^2 - 4x - 4$: now, these products multiplied together, make $x^4 - 6x^3 + 24x + 16$, which is the same equation that was proposed.

CHAP. XIV.

Of the Rule *of* Bombelli *for reducing the* Resolution *of* Equations *of the* Fourth Degree *to that of* Equations *of the* Third Degree.

765. We have already shewn how equations of the third degree are resolved by the rule of Cardan; so that the principal object, with regard to equations of the fourth degree, is to reduce them to equations of the third degree. For it is impossible to resolve, generally, equations of the fourth degree, without the aid of those of the third; since, when we have determined one of the roots, the others always depend on an equation of the third degree. And hence we may conclude, that the resolution of equations of higher dimensions presupposes the resolution of all equations of lower degrees.

766. It is now some centuries since Bombelli, an Italian, gave a rule for this purpose, which we shall explain in this chapter *.

Let there be given the general equation of the fourth degree, $x^4 + ax^3 + bx^2 + cx + d = 0$, in which the letters a, b, c, d, represent any possible numbers; and let us suppose that this equation is the same as $(x^2 + \frac{1}{2}ax + p)^2 - (qx + r)^2 = 0$, in which it is required to determine the letters p, q, and r, in order that we may obtain the equation proposed. By squaring, and ordering this new equation, we shall have

$$\begin{aligned} x^4 + ax^3 + \tfrac{1}{4}a^2x^2 &+ apx + p^2 \\ 2px^2 &- 2qrx - r^2 \\ - q^2x^2. & \end{aligned}$$

Now, the first two terms are already the same here as in the given equation; the third term requires us to make $\frac{1}{4}a^2 + 2p - q^2 = b$, which gives $q^2 = \frac{1}{4}a^2 + 2p - b$; the fourth term shews that we must make $ap - 2qr = c$, or $2qr = ap - c$; and, lastly, we have for the last term $p^2 - r^2 = d$, or $r^2 = p^2 - d$. We have therefore three equations which will give the values of p, q, and r.

767. The easiest method of deriving those values from them is the following: if we take the first equation four times, we shall have $4q^2 = a^2 + 8p - 4b$; which equation, multiplied by the last, $r^2 = p^2 - d$, gives

$$4q^2r^2 = 8p^3 + (a^2 - 4b)p^2 - 8dp - d)a^2 - 4b).$$

Farther, if we square the second equation, we have $4q^2r^2 = a^2p^2 - 2acp + c^2$. So that we have two values of $4q^2r^2$, which, being made equal, will furnish the equation

$$8p^3 + (a^2 - 4b)p^2 - 8dp - d(a^2 - 4b) = a^2p^2 - 2acp + c^2,$$

or, bringing all the terms to one side, and arranging,

$$8p^3 - 4bp^2 + (2ac - 8d)p - a^2d + 4bd - c^2 = 0,$$

an equation of the third degree, which will always give the value of p by the rules already explained.

768. Having therefore determined three values of p by the given quantities a, b, c, d, when it was required to find only one of those values, we shall also have the values of the two other letters q and r; for the first equation will give $q = \sqrt{(\frac{1}{4}a^2 + 2p - b)}$, and the second gives $r = \frac{ap - c}{2q}$.

Now, these three values being determined for each given case, the four roots of the proposed equation may be found in the following manner:

This equation having been reduced to the form $(x^2 + \frac{1}{2}ax + p)^2 - (qx + r)^2 = 0$, we shall have

$$(x^2 + \tfrac{1}{2}ax + p)^2 = (qx + r)^2,$$

and, extracting the root, $x^2 + \frac{1}{2}ax + p = qx + r$, or $x^2 + \frac{1}{2}ax + p = -qx - r$. The first equation gives $x^2 = (q - \frac{1}{2}a)x - p + r$, from which we may find two roots; and the second equation, to which we may give the form $x^2 = -(q + \frac{1}{2}a)x - p - r$, will furnish the two other roots.

* This rule rather belongs to Louis Ferrari. It is improperly called the Rule of Bombelli, in the same manner as the rule discovered by Scipio Ferreo has been ascribed to Cardan. F. T.

769. Let us illustrate this rule by an example, and suppose that the equation

$$x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$$

was given. If we compare it with our general formula (at the end of Art. 767.), we have $a = -10$, $b = 35$, $c = -50$, $d = 24$; and, consequently, the equation which must give the value of p is

$$8p^3 - 140p^2 + 808p - 1540 = 0, \text{ or}$$
$$2p^3 - 35p^2 + 202p - 385 = 0.$$

The divisors of the last term are 1, 5, 7, 11, &c.; the first of which does not answer; but making $p = 5$, we get $250 - 875 + 1010 - 385 = 0$, so that $p = 5$; and if we farther suppose $p = 7$, we get $686 - 1715 + 1414 - 385 = 0$, a proof that $p = 7$ is the second root. It remains now to find the third root; let us therefore divide the equation by 2, in order to have $p^3 - \frac{35}{2}p^2 + 101p - \frac{385}{2} = 0$, and let us consider that the coefficient of the second term, or $\frac{35}{2}$, being the sum of all the three roots, and the first two making together 12, the third must necessarily be $\frac{11}{2}$.

We consequently know the three roots required. But it may be observed that one would have been sufficient, because each gives the same four roots for our equation of the fourth degree.

770. To prove this, let $p = 5$; we shall then have, by the formula, $\sqrt{(\frac{1}{4}a^2 + 2p - b)}$, $q = \sqrt{(25 + 10 - 35)} = 0$, and $r = \dfrac{-50+50}{0} = \frac{0}{0}$. Now, nothing being determined by this, let us take the third equation,

$$r^2 = p^2 - d = 25 - 24 = 1,$$

so that $r = 1$; our two equations of the second degree will then be:

$$1.\ x^2 = 5x - 4, \qquad 2.\ x^2 = 5x - 6.$$

The first gives the two roots

$$x = \tfrac{5}{2} \pm \sqrt{\tfrac{9}{4}}, \text{ or } x = \frac{5 \pm 3}{2},$$

that is to say, $x = 4$ and $x = 1$.

The second equation gives

$$x = \tfrac{5}{2} \pm \sqrt{\tfrac{1}{4}} = \frac{5 \pm 1}{2},$$

that is to say, $x = 3$, and $x = 2$.

But suppose now $p = 7$, we shall have

$$q = \sqrt{(25 + 14 - 35)} = 2, \text{ and } r = \frac{-70+50}{4} = -5,$$

whence result the two equations of the second degree,

$$1.\ x^2 = 7x - 12, \qquad 2.\ x^2 = 3x - 2;$$

the first gives $x = \frac{7}{2} \pm \sqrt{\frac{1}{4}}$, or $x = \dfrac{7 \pm 1}{2}$,

so that $x = 4$, and $x = 3$; the second furnishes the root

$$x = \tfrac{3}{2} \pm \sqrt{\tfrac{1}{4}} = \frac{3 \pm 1}{2},$$

and, consequently, $x = 2$, and $x = 1$; therefore by this second supposition the same four roots are found as by the first.

Lastly, the same roots are found, by the third value of p, $= \frac{11}{2}$: for, in this case, we have

$$q = \sqrt{(25 + 11 - 35)} = 1, \text{ and } r = \frac{-55+50}{2} = -\tfrac{5}{2};$$

so that the two equations of the second degree become,

$$1.\ x^2 = 6x - \tfrac{7}{2}, \qquad 2.\ x^2 = 4x - 3.$$

Whence we obtain from the first, $x = 3 \pm \sqrt{1}$, that is to say, $x = 4$, and $x = 2$; and from the second, $x = 2 \pm \sqrt{1}$, that is to say, $x = 3$, and $x = 1$, which are the same roots that we originally obtained.

771. Let there now be proposed the equation

$$x^4 - 16x - 12 = 0,$$

in which $a = 0$, $b = 0$, $c = -16$, $d = -12$; and our equation of the third degree will be

$$8p^3 + 96p - 256 = 0, \text{ or } p^3 + 12p - 32 = 0,$$

and we may make this equation still more simple, by writing $p = 2t$; for we have then

$$8t^3 + 24t - 32 = 0, \text{ or } t^3 + 3t - 4 = 0.$$

The divisors of the last term are 1, 2, 4; whence one of the roots is found to be $t = 1$; therefore $p = 2$, $q = \sqrt{4} = 2$, and $r = \frac{16}{4} = 4$. Consequently, the two equations of the second degree are

$$x^2 = 2x + 2, \text{ and } x^2 = -2x - 6;$$

which give the roots

$$x = 1 \pm \sqrt{3}, \text{ and } x = -1 \pm \sqrt{-5}.$$

772. We shall endeavour to render this resolution still more familiar, by a repetition of it in the following example. Suppose there were given the equation

$$x^4 - 6x^3 + 12x^2 - 12x + 4 = 0,$$

which must be contained in the formula

$$(x^2 - 3x + p)^2 - (qx + r)^2 = 0,$$

in the former part of which we have put $-3x$, because -3 is half the coefficient -6, of the given equation. This formula being expanded, gives

$$x^4 - 6x^3 + (2p + 9 - q^2)x^2 - (6p + 2qr)x + p^2 - r^2 = 0;$$

which, compared with our equation, there will result from that comparison the following equations:

1. $2p + 9 - q^2 = 12$,
2. $6p + 2qr = 12$,
3. $p^2 - r^2 = 4$.

The first gives $q^2 = 2p - 3$;
the second, $2qr = 12 - 6p$, or $qr = 6 - 3p$;
the third, $r^2 = p^2 - 4$.

Multiplying r^2 by q^2, and $p^2 - 4$ by $2p - 3$, we have

$$q^2r^2 = 2p^3 - 3p^2 - 8p + 12;$$

and if we square the value of qr, we have

$$q^2r^2 = 36 - 36p + 9p^2;$$

so that we have the equation

$$2p^3 - 3p^2 - 8p + 12 = 9p^2 - 36p + 36, \text{ or}$$
$$2p^3 - 12p^2 + 28p - 24 = 0, \text{ or}$$
$$p^3 - 6p^2 + 14p - 12 = 0,$$

one of the roots of which is $p = 2$; and it follows that $q^2 = 1$, $q = 1$, and $qr - r = 0$. Therefore our equation will be $(x^2 - 3x + 2)^2 = x^2$, and its square root will be $x^2 - 3x + 2 = \pm x$. If we take the upper sign, we have $x^2 = 4x - 2$; and taking the lower sign, we obtain $x^2 = 2x - 2$, whence we derive the four roots $x = 2 \pm \sqrt{2}$, and $x = 1 \pm \sqrt{-1}$.

CHAP. XV.

Of a new Method *of resolving* Equations *of the* Fourth Degree.

773. The rule of Bombelli, as we have seen, resolves equations of the fourth degree by means of an equation of the third degree; but since the invention of that Rule, another method has been discovered of performing the same resolution: and, as it is altogether different from the first, it deserves to be separately explained*.

774. We suppose that the root of an equation of the fourth degree has the form, $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$, in which the letters p, q, r, express the roots of an equation of the third degree, $z^3 - fz^2 + gz - h = 0$; so that $p + q + r = f$; $pq + pr + qr = g$; and $pqr = h$. This being laid down, we square the assumed formula, $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$, and we obtain

$$x^2 = p + q + r + 2\sqrt{pq} + 2\sqrt{pr} + 2\sqrt{qr};$$

and, since $p + q + r = f$, we have

$$x^2 - f = 2\sqrt{pq} + 2\sqrt{pr} + 2\sqrt{qr};$$

we again take the squares, and find

$$x^4 - 2fx^2 + f^2 = 4pq + 4pr + 4qr + 8\sqrt{p^2qr} + 8\sqrt{pq^2r} + 8\sqrt{pqr^2}.$$

Now, $4pq + 4pr + 4qr = 4g$; so that the equation becomes $x^4 - 2fx^2 + f^2 - 4g = 8\sqrt{pqr} \times (\sqrt{p} + \sqrt{q} + \sqrt{r})$; but $\sqrt{p} + \sqrt{q} + \sqrt{r} = x$, and $pqr = h$, or $\sqrt{pqr} = \sqrt{h}$; wherefore we arrive at this equation of the fourth degree, $x^4 - 2fx^2 - 8x\sqrt{h} + f^2 - 4g = 0$, one of the roots of which is $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$; and in which p, q, and r, are the roots of an equation of the third degree,

$$z^3 - fz^2 + gz - h = 0.$$

775. The equation of the fourth degree, at which we have arrived, may be considered as general, although the second term x^3y is wanting; for we shall afterwards shew, that every complete equation may be transformed into another, from which the second term has been taken away.

Let there be proposed the equation $x^4 - ax^2 - bx - c = 0$, in order to determine its root. This we must first compare with the formula, in order to obtain the values of f, g, and h; and we shall have,

1. $2f = a$, and, consequently, $f = \frac{a}{2}$;

2. $8\sqrt{h} = b$, so that $h = \frac{b^2}{64}$;

3. $f^2 - 4g = -c$, or $\frac{a^2}{4} - 4g + c = 0$,

or $\frac{1}{4}a^2 + c = 4g$; consequently, $g = \frac{1}{16}a^2 + \frac{1}{4}c$.

* This method was the invention of Euler himself. He has explained it in the sixteenth volume of the Ancient Commentaries of Petersburg. F. T.

776. Since, therefore, the equation

$$x^4 - ax^2 - bx - c = 0,$$

gives the values of the letters f, g, and h, so that

$$f = \tfrac{1}{2}a,\ g = \tfrac{1}{16}a^2 + \tfrac{1}{4}c, \text{ and } h = \tfrac{1}{64}b^2, \text{ or } \sqrt{h} = \tfrac{1}{8}b,$$

we form from these values the equation of the third degree $z^3 - fz^2 + gz - h = 0$, in order to obtain its roots by the known rule. And if we suppose those roots, 1. $z = p$, 2. $z = q$, 3. $z = r$, one of the roots of our equation of the fourth degree must be, by the supposition, Art. 774,

$$x = \sqrt{p} + \sqrt{q} + \sqrt{r}.$$

777. This method appears at first to furnish only one root of the given equation; but if we consider that every sign $\sqrt{}$ may be taken negatively, as well as positively, we shall immediately perceive that this formula contains all the four roots.

Farther, if we chose to admit all the possible changes of the signs, we should have eight different values of x, and yet four only can exist. But it is to be observed, that the product of those three terms, or $\sqrt{pqr}$, must be equal to $\sqrt{h} = \frac{1}{8}b$, and that if $\frac{1}{8}b$ be positive, the product of the terms $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, must likewise be positive; so that all the variations that can be admitted are reduced to the four following:

1. $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$,
2. $x = \sqrt{p} - \sqrt{q} - \sqrt{r}$,
3. $x = -\sqrt{p} + \sqrt{q} - \sqrt{r}$.
4. $x = -\sqrt{p} - \sqrt{q} + \sqrt{r}$.

In the same manner, when $\frac{1}{8}b$ is negative, we have only the four following values of x:

1. $x = \sqrt{p} + \sqrt{q} - \sqrt{r}$,
2. $x = \sqrt{p} - \sqrt{q} + \sqrt{r}$,
3. $x = -\sqrt{p} + \sqrt{q} + \sqrt{r}$,
4. $x = -\sqrt{p} - \sqrt{q} - \sqrt{r}$.

This circumstance enables us to determine the four roots in all cases; as may be seen in the following example.

778. Let there be proposed the equation of the fourth degree, $x^4 - 25x^2 + 60x - 36 = 0$, in which the second term is wanting. Now, if we compare this with the general formula, we have $a = 25$, $b = -60$, and $c = 36$; and after that,

$$f = \tfrac{25}{2},\ g = \tfrac{625}{16} + 9 = \tfrac{769}{16}, \text{ and } h = \tfrac{225}{4};$$

by which means our equation of the third degree becomes,

$$z^3 - \tfrac{25}{2}z^2 + \tfrac{769}{16}z - \tfrac{225}{4} = 0.$$

First, to remove the fractions, let us make $z = \frac{u}{4}$; and we shall have $\frac{u^3}{64} - \frac{25u^2}{32} + \frac{769u}{64} - \frac{225}{4} = 0$, and multiplying by the greatest denominator, we obtain

$$u^3 - 50u^2 + 769u - 3600 = 0.$$

We have now to determine the three roots of this equation; which are all three found to be positive; one of them being $u = 9$: then dividing the equation by $u - 9$, we find the new equation $u^2 - 41u + 400 = 0$, or $u^2 = 41u - 400$, which gives

$$u = \tfrac{41}{2} \pm \sqrt{(\tfrac{1681}{4} - \tfrac{1600}{4})} = \frac{41 \pm 9}{2};$$

so that the three roots are $u = 9$, $u = 16$, and $u = 25$.

Consequently, as $z = \frac{u}{4}$ the roots are

$$1.\ z = \tfrac{9}{4},\ 2.\ z = 4,\ 3.\ z = \tfrac{25}{4}.$$

These, therefore, are the values of the letters p, q, and r; that is to say, $p = \frac{9}{4}$, $q = 4$, and $r = \frac{25}{4}$. Now, if we consider that $\sqrt{pqr} = \sqrt{h} = -\frac{15}{2}$, and that therefore this value $= \frac{1}{8}b$ is negative, we must, agreeably to what has been said with regard to the signs of the roots $\sqrt{p}$, $\sqrt{q}$, and $\sqrt{r}$, take all those three roots negatively, or take only one of them negatively; and consequently, as $\sqrt{p} = \frac{3}{2}$, $\sqrt{q} = 2$, and $\sqrt{r} = \frac{5}{2}$, the four roots of the given equation are found to be:

1. $x = \frac{3}{2} + 2 - \frac{5}{2} = 1$,
2. $x = \frac{3}{2} - 2 + \frac{5}{2} = 2$,
3. $x = -\frac{3}{2} + 2 + \frac{5}{2} = 3$,
4. $x = -\frac{3}{2} - 2 - \frac{5}{2} = -6$.

From these roots are formed the four factors,

$$(x - 1) \times (x - 2) \times (x - 3) \times (x + 6) = 0.$$

The first two, multiplied together, give $x^2 - 3x + 2$; the product of the last two is $x^2 + 3x - 18$; again multiplying these two products together, we obtain exactly the equation proposed.

779. It remains now to shew how an equation of the fourth degree, in which the second term is found, may be transformed into another, in which that term is wanting: for which we shall give the following rule *.

* An investigation of this rule may be seen in Maclaurin's Algebra, Part II. chap. 3.

Let there be proposed the general equation $y^4 + ay^3 + by^2 + cy + d = 0$. If we add to y the fourth part of the coefficient of the second term, or $\frac{1}{4}a$, and write, instead of the sum, a new letter x, so that $y + \frac{1}{4}a = x$, and consequently $y = x - \frac{1}{4}a$: we shall have

$y^2 = x^2 - \frac{1}{2}ax + \frac{1}{16}a^2$, $y^3 = x^3 - \frac{3}{4}ax^2 + \frac{3}{16}a^2x - \frac{1}{64}a^3$,

and, lastly, as follows:

$$\begin{array}{rl} y^4 = & x^4 - ax^3 + \frac{3}{8}a^2x^2 - \frac{1}{16}a^3x + \frac{1}{256}a^4 \\ ay^3 = & \quad ax^3 - \frac{3}{4}a^2x^2 + \frac{3}{16}a^3x - \frac{1}{64}a^4 \\ by^2 = & \quad\quad bx^2 - \frac{1}{2}abx + \frac{1}{16}a^2b \\ cy = & \quad\quad\quad cx - \frac{1}{4}ac \\ d = & \quad\quad\quad\quad d \end{array}$$

And hence, by addition,

$$x^4 + 0 \left.\begin{array}{l} - \frac{3}{8}a^2x^2 + \frac{1}{8}a^3x - \frac{3}{256}a^4 \\ \quad + bx^2 - \frac{1}{2}abx + \frac{1}{16}a^2b \\ \quad\quad + cx - \frac{1}{4}ac \\ \quad\quad\quad + d \end{array}\right\} = 0.$$

We have now an equation from which the second term is taken away, and to which nothing prevents us from applying the rule before given for determining its four roots. After the values of x are found, those of y will easily be determined, since $y = x - \frac{1}{4}a$.

780. This is the greatest length to which we have yet arrived in the resolution of algebraic equations. All the pains that have been taken in order to resolve equations of the fifth degree, and those of higher dimensions, in the same manner, or, at least, to reduce them to inferior degrees, have been unsuccessful: so that we cannot give any general rules for finding the roots of equations, which exceed the fourth degree.

The only success that has attended these attempts has been the resolution of some particular cases; the chief of which is that, in which a rational root takes place; for this is easily found by the method of divisors, because we know that such a root must be always a factor of the last term. The operation, in other respects, is the same as that we have explained for equations of the third and fourth degree.

781. It will be necessary, however, to apply the rule of Bombelli to an equation which has no rational roots.

Let there be given the equation $y^4 - 8y^3 + 14y^2 + 4y - 8 = 0$. Here we must begin with destroying the second term, by adding the fourth of its coefficient to y, supposing $y - 2 = x$, and substituting in the equation, instead of y, its new value $x + 2$, instead of y^2, its value $x^2 + 4x + 4$; and doing the same with regard to y^3 and y^4, we shall have,

$$\begin{array}{rl} y^4 = & x^4 + 8x^3 + 24x^2 + 32x + 16 \\ -8y^3 = & \quad - 8x^3 - 48x^2 - 96x - 64 \\ 14y^2 = & \quad\quad 14x^2 + 56x + 56 \\ 4y = & \quad\quad\quad 4x + 8 \\ -8 = & \quad\quad\quad\quad - 8 \\ \hline & x^4 + 0 \; - 10x^2 - 4x + 8 = 0. \end{array}$$

This equation being compared with our general formula, gives $a = 10$, $b = 4$, $c = -8$; whence we conclude, that $f = 5$, $g = \frac{17}{4}$, $h = \frac{1}{4}$, and $\sqrt{h} = \frac{1}{2}$; that the product $\sqrt{pqr}$ will be positive; and that it is from the equation of the third degree, $z^3 - 5z^2 + \frac{17}{4}z - \frac{1}{4} = 0$, that we are to seek for the three roots p, q, r. (Art. 774.)

782. Let us first remove the fractions from this equation, by making $z = \frac{u}{2}$, and we shall thus have, after multiplying by 8, the equation $u^3 - 10u^2 + 17u - 2 = 0$, in which all the roots are positive. Now, the divisors of the last term are 1 and 2; if we try $u = 1$, we find $1 - 10 + 17 - 2 = 6$; so that the equation is not reduced to nothing: but trying $u = 2$, we find $8 - 40 + 34 - 2 = 0$, which answers to the equation, and shews that $u = 2$ is one of the roots. The two others will be found by dividing by $u - 2$, as usual; then the quotient $u^2 - 8u + 1 = 0$ will give $u^2 = 8u - 1$, and $u = 4 \pm \sqrt{15}$. And since $z = \frac{1}{2}u$, the three roots of the equation of the third degree are,

$$1,\; z = p = 1,$$

$$2,\; z = q = \frac{4 + \sqrt{15}}{2},$$

$$3,\; z = r = \frac{4 - \sqrt{15}}{2}.$$

783. Having therefore determined p, q, r, we have also their square roots; namely, $\sqrt{p} = 1$,

$$\sqrt{q} = \frac{\sqrt{(8 + 2\sqrt{15})}}{2}\,^{*}, \text{ and } \sqrt{r} = \frac{\sqrt{(8 - 2\sqrt{15})}}{2}.$$

* This expression for the square root of q is obtained by multiplying the numerator and denominator of $\frac{4+\sqrt{15}}{2}$ by 2, and extracting the root of the latter, in order to remove the surd: Thus, $\frac{4+\sqrt{15}}{2} \times 2 = \frac{8+2\sqrt{15}}{4}$; and $\frac{\sqrt{(8+2\sqrt{15})}}{\sqrt{4}} = \frac{\sqrt{(8+2\sqrt{15}}}{2}$.

But we have already seen, (Art. 675, and 676), that the square root of $a \pm \sqrt{b}$, when $\sqrt{(a^2 - b)} = c$, is expressed by $\sqrt{(a \pm \sqrt{b})} = \sqrt{(\frac{a+c}{2})} \pm \sqrt{(\frac{a-c}{2})}$: so that, as in the present case, $a = 8$, and $\sqrt{b} = 2\sqrt{15}$; and consequently, $b = 60$, and $c = \sqrt{(a^2 - b)} = 2$, we have $\sqrt{(8 + 2\sqrt{15})} = \sqrt{5} + \sqrt{3}$, and $\sqrt{(8 - 2\sqrt{15})} \ldots\ldots = \sqrt{5} - \sqrt{3}$. Hence, we have $\sqrt{p} = 1$, $\sqrt{q} = \frac{\sqrt{5} + \sqrt{3}}{2}$, and $\sqrt{r} = \frac{\sqrt{5} - \sqrt{3}}{2}$; wherefore, since we also know that the product of these quantities is positive, the four values of x will be:

1. $x = \sqrt{p} + \sqrt{q} + \sqrt{r} = 1 + \frac{\sqrt{5} + \sqrt{3} + \sqrt{5} - \sqrt{3}}{2} \ldots\ldots\ldots$
$= 1 + \sqrt{5}$,

2. $x = \sqrt{p} - \sqrt{q} - \sqrt{r} = 1 + \frac{-\sqrt{5} - \sqrt{3} - \sqrt{5} + \sqrt{3}}{2} \ldots\ldots$
$= 1 + \sqrt{5}$,

3. $x = -\sqrt{p} + \sqrt{q} - \sqrt{r} = -1 + \frac{\sqrt{5} + \sqrt{3} - \sqrt{5} + \sqrt{3}}{2} \ldots\ldots$
$= -1 + \sqrt{3}$,

4. $x = -1\sqrt{p} - \sqrt{q} + \sqrt{r} = -1 + \frac{-\sqrt{5} - \sqrt{3} + \sqrt{5} - \sqrt{3}}{2}$.
$= -1 - \sqrt{3}$.

Lastly, as we have $y = x + 2$, the four roots of the given equation are:

1. $y = 3 + \sqrt{5}$, 2. $y = 3 - \sqrt{5}$,
3. $y = 1 + \sqrt{3}$, 4. $y = 1 - \sqrt{3}$.

QUESTIONS FOR PRACTICE.

1. Given $z^4 - 4z^3 - 8z + 32 = 0$, to find the values of z. *Ans.* $4, 2, -1 + \sqrt{-3}, -1 - \sqrt{-3}$.

2. Given $y^4 - 4y^3 - 3y^2 - 4y + 1 = 0$, to find the values of y. *Ans.* $\frac{-1 \pm \sqrt{-3}}{2}$, and $\frac{5 \pm \sqrt{21}}{2}$.

3. Given $x^4 - 3x^2 - 4x = 3$, to find the values of x. *Ans.* $\frac{1 \pm \sqrt{13}}{2}$, and $\frac{-1 \pm \sqrt{-3}}{2}$.

CHAP. XVI.

Of the Resolution *of* Equations *by* Approximation.

784. When the roots of an equation are not rational, whether they may be expressed by radical quantities, or even if we have not that resource, as is the case with equations which exceed the fourth degree, we must be satisfied with determining their values by approximation; that is to say, by methods which are continually bringing us nearer to the true value, till at last the error being very small, it may be neglected. Different methods of this kind have been proposed, the chief of which we shall explain.

785. The first method which we shall mention, supposes that we have already determined, with tolerable exactness, the value of one root *; that we know, for example, that such a value exceeds 4, and that it is less than 5. In this case, if we suppose this value $= 4 + p$, we are certain that p expresses a fraction. Now, as p is a fraction, and consequently less than unity, the square of p, its cube, and, in general, all the higher powers of p, will be much less with respect to unity; and, for this reason, since we require only an approximation, they may be neglected in the calculation. When we have, therefore, nearly determined the fraction p, we shall know more exactly the root $4 + p$; from that we proceed to determine a new value still more exact, and continue the same process till we come as near the truth as we desire.

786. We shall illustrate this method first by an easy example, requiring by approximation the root of the equation $x^2 = 20$.

Here we perceive, that x is greater than 4 and less than 5; making, therefore, $x = 4 + p$, we shall have $x^2 = 16 + 8p + p^2 = 20$; but as p^2 must be very small, we shall neglect it, in order that we may have only the equation $16 +$

* This is the method given by Sir Is. Newton at the beginning of his Method of Fluxions. When investigated, it is found subject to different imperfections; for which reason we may with advantage substitute the method given by M. de la Grange, in the Memoirs of Berlin for 1767 and 1768. F. T.

This method has since been published by De la Grange, in a separate Treatise, where the subject is discussed in the usual masterly style of this author.

$8p = 20$, or $8p = 4$. This gives $p = \frac{1}{2}$, and $x = 4\frac{1}{2}$, which already approaches nearer the true root. If, therefore, we now suppose $x = 4\frac{1}{2} + p'$; we are sure that p' expresses a fraction much smaller than before, and that we may neglect p'^2 with greater propriety. We have, therefore, $x^2 = 20\frac{1}{4} + 9p' = 20$, or $9p' = -\frac{1}{4}$; and consequently, $p' = -\frac{1}{36}$; therefore $x = 4\frac{1}{2} - \frac{1}{36} = 4\frac{17}{36}$.

And if we wished to approximate still nearer to the true value, we must make $x = 4\frac{17}{36} + p''$, and should thus have $x^2 = 20\frac{1}{1296} + 8\frac{34}{36}p'' = 20$; so that $8\frac{34}{36}p'' = -\frac{1}{1296}$, $322p'' = -\frac{36}{1296} = -\frac{1}{36}$, and

$$p = -\frac{1}{36 \times 322} = -\frac{1}{11592}:$$

therefore $x = 4\frac{17}{36} - \frac{1}{11592} = 4\frac{4473}{11592}$, a value which is so near the truth, that we may consider the error as of no importance.

787. Now, in order to generalise what we have here laid down, let us suppose the given equation to be $x^2 = a$, and that we previously know x to be greater than n, but less than $n + 1$. If we now make $x = n + p$, p must be a fraction, and p^2 may be neglected as a very small quantity, so that we shall have $x^2 = n^2 + 2np = a$; or $2np = a - n^2$, and $p = \frac{a-n^2}{2n}$; consequently, $x = n + \frac{a-n^2}{2n} = \frac{n^2+a}{2n}$.

Now, if n approximated towards the true value, this new value $\frac{n^2+a}{2n}$ will approximate much nearer; and, by substituting it for n, we shall find the result much nearer the truth; that is, we shall obtain a new value, which may again be substituted, in order to approach still nearer; and the same operation may be continued as long as we please.

For example, let $x^2 = 2$; that is to say, let the square root of 2 be required; and as we already know a value sufficiently near, which is expressed by n, we shall have a still nearer value of the root expressed by $\frac{n^2+2}{2n}$. Let therefore,

1. $n = 1$, and we shall have $x = \frac{3}{2}$,
2. $n = \frac{3}{2}$, and we shall have $x = \frac{17}{12}$,
3. $n = \frac{17}{12}$, and we shall have $x = \frac{577}{408}$.

This last value approaches so near $\sqrt{2}$, that its square $\frac{332929}{166464}$ differs from the number 2 only by the small quantity $\frac{1}{166464}$, by which it exceeds it.

788. We may proceed in the same manner, when it is

required to find by approximation cube roots, biquadrate roots, &c.

Let there be given the equation of the third degree, $x^3 = a$; or let it be proposed to find the value of $\sqrt[3]{a}$.

Knowing that it is nearly n, we shall suppose $x = n + p$; neglecting p^2 and p^3, we shall have $x^3 = n^3 + 3n^2p = a$; so that $3n^2p = a - n^3$, and $p = \frac{a - n^3}{3n^2}$; whence $x = \frac{2n^3 + a}{3n^2}$. If, therefore, n is nearly $= \sqrt[3]{a}$, the quantity which we have now found will be much nearer it. But for still greater exactness, we may again substitute this new value for n, and so on.

For example, let $x^3 = 2$; and let it be required to determine $\sqrt[3]{2}$. Here, if n is nearly the value of the number sought, the formula $\frac{2n^3+2}{3n^2}$ will express that number still more nearly; let us therefore make

1. $n = 1$, and we shall have $x = \frac{4}{3}$,
2. $n = \frac{4}{3}$, and we shall have $x = \frac{91}{72}$,
3. $n = \frac{91}{72}$, and we shall have $x = \frac{1621130896}{1286342294}$.

789. This method of approximation may be employed, with the same success, in finding the roots of all equations.

To shew this, suppose we have the general equation of the third degree, $x^3 + ax^2 + bx + c = 0$, in which n is very nearly the value of one of the roots. Let us make $x = n - p$; and, since p will be a fraction, neglecting the powers of this letter, which are higher than the first degree, we shall have $x^2 = n^2 - 2np$, and $x^3 = n^3 - 3n^2p$; whence we have the equation $n^3 - 3n^2p + an^2 - 2anp + bn - bp + c = 0$, or $n^3 + an^2 + bn + c = 3n^2p + 2anp + bp = (3n^2 + 2an + b)p$; so that $p = \frac{n^3 + an^2b + n - c}{3n^2 + 2an + b}$, and

$$x = n - \left(\frac{n^3+an^2+bn+c}{3n^2+2an+b}\right) = \frac{2n^3 + an^2 - c}{3n^2 + 2an + b}.$$

This value, which is more exact than the first, being substituted for n, will furnish a new value still more accurate.

790. In order to apply this operation to an example, let $x^3 + 2x^2 + 3x - 50 = 0$, in which $a = 2$, $b = 3$, and $c = -50$. If n is supposed to be nearly the value of one of the roots, $x = \frac{2n^3+2n^2+50}{3n^2+4n+3}$, will be a value still nearer the truth.

Now, the assumed value of $x = 3$ not being far from the true one, we shall suppose $n = 3$, which gives us $x = \frac{62}{21}$; and if we were to substitute this new value instead of n, we should find another still more exact.

791. We shall give only the following example, for equations of higher dimensions than the third.

Let $x^5 = 6x + 10$, or $x^5 - 6x - 10 = 0$, where we readily perceive that 1 is too small, and that 2 is too great. Now, if $x = n$ is a value not far from the true one, and we make $x = n + p$, we shall have $x^5 = n^5 + 5n^4p$; and, consequently,

$$n^5 + 5n^4p = 6n + 6p + 10; \text{ or}$$
$$p(5n^4 - 6) = 6n + 10 - n^5.$$

Wherefore $p = \frac{6n+10-n^5}{5n^4-6}$, and $x = \frac{4n^5+10}{5n^4-6}$. If we suppose $n = 1$, we shall have $x = \frac{14}{-1} = -14$; this value is altogether inapplicable, a circumstance which arises from the approximated value of n having been taken by much too small. We shall therefore make $n = 2$, and shall thus obtain $x = \frac{138}{74} = \frac{69}{37}$, a value which is much nearer the truth. And if we were now to substitute for n, the fraction $\frac{69}{37}$, we should obtain a still more exact value of the root x.

792. Such is the most usual method of finding the roots of an equation by approximation, and it applies successfully to all cases.

We shall however explain another method *, which deserves attention, on account of the facility of the calculation. The foundation of this method consists in determining for each equation a series of numbers, as a, b, c, &c. such, that each term of the series, divided by the preceding one, may express the value of the root with so much the more exactness, according as this series of numbers is carried to a greater length.

Suppose we have already got the terms p, q, r, s, t, &c.

* The theory of approximation here given, is founded on the theory of what are called *recurring* series, invented by M. de Moivre. This method was given by Daniel Bernoulli, in vol. iii. of the Ancient Commentaries of Petersburg. But Euler has here presented it in rather a different point of view. Those who wish to investigate these matters, may consult chapters 13 and 17 of vol. i. of our author's Introd. in Anal. Infin.; an excellent work, in which several subjects treated of in this first part, beside others equally connected with pure mathematics, are profoundly analysed and clearly explained. F. T.

$\frac{q}{p}$ must express the root x with tolerable exactness; that is to say, we have $\frac{q}{p} = x$ nearly. We shall have also $\frac{r}{q} = x$*, and the multiplication of the two values will give $\frac{r}{p} = x^2$. Farther as $\frac{s}{r} = x$, we shall also have $\frac{s}{p} = x^3$; then, since $\frac{t}{s} = x$, we shall have $\frac{t}{p} = x^4$, and so on.

793. For the better explanation of this method, we shall begin with an equation of the second degree, $x^2 = x + 1$, and shall suppose that in the above series we have found the terms p, q, r, s, t, &c. Now, as $\frac{q}{p} = x$, and $\frac{r}{p} = x^2$, we shall have the equation $\frac{r}{p} = \frac{q}{p} + 1$, or $q + p = r$. And as we find, in the same manner, that $s = r + q$, and $t = s + r$; we conclude that each term of our series is the sum of the two preceding terms; so that having the first two terms, we can easily continue the series to any length. With regard to the first two terms, they may be taken at pleasure: if we therefore suppose them to be 0, 1, our series will be 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, &c. and such, that if we divide any term by that which immediately precedes it, we shall have a value of x so much nearer the true one, according as we have chosen a term more distant. The error, indeed, is very great at first, but it diminishes as we advance. The series of those values of x, in the order in which they are always approximating towards the true one, is as follows:

$$x = \tfrac{1}{0}, \tfrac{1}{1}, \tfrac{2}{1}, \tfrac{3}{2}, \tfrac{5}{3}, \tfrac{8}{5}, \tfrac{13}{8}, \tfrac{21}{13}, \tfrac{34}{21}, \tfrac{55}{34}, \tfrac{89}{55}, \tfrac{144}{89}, \text{ \&c.}$$

If, for example, we make $x = \frac{21}{13}$, we have $\frac{441}{169} = \frac{21}{13} + 1 = \frac{442}{169}$, in which the error is only $\frac{1}{169}$. Any of the succeeding terms will render it still less.

794. Let us also consider the equation $x^2 = 2x + 1$; and since, in all cases, $x = \frac{q}{p}$, and $x^2 = \frac{r}{p}$, we shall have

* It must only be understood here that $\frac{r}{q}$ is nearly equal to x.

$\frac{r}{p} = \frac{2q}{p} + 1$, or $r = 2q + p$; whence we infer that the double of each term, added to the preceding term, will give the succeeding one. If, therefore, we begin again with 0, 1, we shall have the series,

$$0, 1, 2, 5, 12, 29, 70, 169, 408, \&c.$$

Whence it follows, that the value of x will be expressed still more accurately by the following fractions:

$$x = \tfrac{1}{0}, \tfrac{2}{1}, \tfrac{5}{2}, \tfrac{12}{5}, \tfrac{29}{12}, \tfrac{70}{29}, \tfrac{169}{70}, \tfrac{408}{169}, \&c.$$

which, consequently, will always approximate nearer and nearer the true value of $x = 1 + \sqrt{2}$; so that if we take unity from these fractions, the value of $\sqrt{2}$ will be expressed more and more exactly by the succeeding fractions:

$$\tfrac{1}{0}, \tfrac{1}{1}, \tfrac{3}{2}, \tfrac{7}{5}, \tfrac{17}{12}, \tfrac{41}{29}, \tfrac{99}{70}, \tfrac{239}{169}, \&c.$$

For example, $\frac{99}{70}$ has for its square $\frac{9801}{4900}$, which differs only by $\frac{1}{4900}$ from the number 2.

795. This method is no less applicable to equations, which have a greater number of dimensions. If, for example, we have the equation of the third degree $x^3 = x^2 + 2x + 1$, we must make $x = \frac{q}{p}$, $x^2 = \frac{r}{p}$, and $x^3 = \frac{s}{p}$; we shall then have $s = r + 2q + p$; which shews how, by means of the three terms p, q, and r, we are to determine the succeeding one, s; and, as the beginning is always arbitrary, we may form the following series:

$$0, 0, 1, 1, 3, 6, 13, 28, 60, 129, \&c.$$

from which result the following fractions for the approximate values of x:

$$x = \tfrac{0}{0}, \tfrac{1}{0}, \tfrac{1}{1}, \tfrac{3}{1}, \tfrac{6}{3}, \tfrac{13}{6}, \tfrac{28}{13}, \tfrac{60}{28}, \tfrac{129}{60}, \&c.$$

The first of these values would be very far from the truth; but if we substitute in the equation $\frac{60}{28}$, or $\frac{15}{7}$, instead of x, we obtain

$$\tfrac{3375}{343} = \tfrac{225}{49} + \tfrac{30}{7} + 1 = \tfrac{3388}{343},$$

in which the error is only $\frac{13}{343}$.

796. It must be observed, however, that all equations are not of such a nature as to admit the application of this method; and, particularly, when the second term is wanting, it cannot be made use of. For example, let $x^2 = 2$; if we wished to make $x = \frac{q}{p}$, and $x^2 = \frac{r}{p}$, we should have

$\frac{r}{p} = 2$, or $r = 2p$, that is to say, $r = 0q + 2p$, whence would result the series

$$1, 1, 2, 2, 4, 4, 8, 8, 16, 16, 32, 32, \&c.$$

from which we can draw no conclusion, because each term, divided by the preceding, gives always $x = 1$, or $x = 2$. But we may obviate this inconvenience, by making $x = y - 1$; for by these means we have $y^2 - 2y + 1 = 2$; and if we now make $y = \frac{q}{p}$, and $y^2 = \frac{r}{p}$, we shall obtain the same approximation that has been already given.

797. It would be the same with the equation $x^3 = 2$. This method would not furnish such a series of numbers as would express the value of $\sqrt[3]{2}$. But we have only to suppose $x = y - 1$, in order to have the equation $y^3 - 3y^2 + 3y - 1 = 2$, or $y^3 = 3y^2 - 3y + 3$; and then making $y = \frac{q}{p}$, $y^2 = \frac{r}{p}$, and $y^3 = \frac{s}{p}$, we have $s = 3r - 3q + 3p$, by means of which we see how three given terms determine the succeeding one.

Assuming then any three terms for the first, for example 0, 0, 1, we have the following series:

$$0, 0, 1, 3, 6, 12, 27, 63, 144, 324, \&c.$$

The last two terms of this series give $y = \frac{324}{144}$ and $x = \frac{5}{4}$. This fraction approaches sufficiently near the cube root of 2; for the cube of $\frac{5}{4}$ is $\frac{125}{64}$, and $2 = \frac{128}{64}$.

798. We must farther observe, with regard to this method, that when the equation has a rational root, and the beginning of the period is chosen such, that this root may result from it, each term of the series, divided by the preceding term, will give the root with equal accuracy.

To shew this, let there be given the equation $x^2 = x + 2$, one of the roots of which is $x = 2$; as we have here, for the series, the formula $r = q + 2p$, if we take 1, 2, for the first two terms, we have the series 1, 2, 4, 8, 16, 32, 64, &c. a geometrical progression, whose exponent $= 2$. The same property is proved by the equation of the third degree $x^3 = x^2 + 3x + 9$, which has $x = 3$ for one of the roots. If we suppose the first terms to be 1, 3, 9, we shall find, by the formula, $s = r + 3q + 9p$, and the series 1, 3, 9, 27, 81, 243, &c. which is likewise a geometrical progression.

799. But if the beginning of the series exceed the root, we shall not approximate towards that root at all; for when the equation has more than one root, the series gives by approximation only the greatest: and we do not find one of the less roots, unless the first terms have been properly chosen for that purpose. This will be illustrated by the following example.

Let there be given the equation $x^2 = 4x - 3$, whose two roots are $x = 1$, and $x = 3$. The formula for the series is $r = 4q - 3p$, and if we take 1, 1, for the first two terms of the series, which consequently expresses the least root, we have for the whole series, 1, 1, 1, 1, 1, 1, 1, 1, &c. but assuming for the leading terms the numbers 1, 3, which contain the greatest root, we have the series, 1, 3, 9, 27, 81, 243, 729, &c. in which all the terms express precisely the root 3. Lastly, if we assume any other beginning, provided it be such that the least term is not comprised in it, the series will continually approximate towards the greatest root 3; which may be seen by the following series:

Beginning,

0,	1,	4,	13,	40,	121,	364,	&c.		
1,	2,	5,	14,	41,	122,	365,	&c.		
2,	3,	6,	15,	42,	123,	366,	1095,	&c.	
2,	1,	−2,	−11,	−38,	−118,	−362,	−1091,	−3278,	&c.

in which the quotients of the division of the last terms by the preceding always approximate towards the greater root 3, and never towards the less.

800. We may even apply this method to equations which go on to infinity. The following will furnish an example:

$$x^{\infty} = x^{\infty-1} + x^{\infty-2} + x^{\infty-3} + x^{\infty-4} +, \text{ \&c.}$$

The series for this equation must be such, that each term may be equal to the sum of all the preceding; that is, we must have

1, 1, 2, 4, 8, 16, 32, 64, 128, &c.

whence we see that the greater root of the given equation is exactly $x = 2$; and this may be shewn in the following manner. If we divide the equation by x^{∞}, we shall have

$$1 = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} +, \text{ \&c.}$$

a geometrical progression, whose sum is found $= \frac{1}{x-1}$; so that $1 = \frac{1}{x-1}$; multiplying therefore by $x - 1$, we have $x - 1 = 1$, and $x = 2$.

801. Beside these methods of determining the roots of an equation by approximation, some others have been invented, but they are all either too tedious, or not sufficiently general*. The method which deserves the preference to all others, is

* This remark does not apply to the method of finding the roots of equations of all degrees, and however affected, by The Rule of Double Position. In order, therefore, that this chapter might be more complete, we shall explain this method as briefly as possible.

Substitute in the given equation two numbers, as near the true root as possible, and observe the separate results. Then, as the difference of these results is to the difference of the two numbers; so is the difference between the true result, and either of the former, to the respective correction of each. This being added to the number, when too small, or substracted from it, when too great, will give the true root nearly.

The number thus found, with any other that may be supposed to approach still nearer to the true root, may be assumed for another operation, which may be repeated, till the root shall be determined to any degree of exactness that may be required.

Example. Given $x^3 + x^2 + x = 100$.

Having ascertained by a few trials, or by inspecting a Table of roots and powers, that x is more than 4, and less than 5, let us substitute these two numbers in the given equation, and calculate the results.

By the first supposition $\begin{cases} x = 4 \\ x^2 = 16 \\ x^3 = 64 \end{cases}$ By the second supposition $\begin{cases} x = 5 \\ x^2 = 25 \\ x^3 = 125 \end{cases}$

	84	. . . Results . . .	155	
	155	5	100	true result.
	84	4	84	
Differences	71	1	16	
Then,	As 71 :	1 ::	16 : ·2253 +	

Therefore 4 + ·2253, or 4·2253 approximates nearly to the true root.

If now 4·2 and 4·3 were taken as the assumed numbers, and substituted in the given equation, we should obtain the value of $x = 4·264$ very nearly.

that which we explained first; for it applies successfully to all kinds of equations: whereas the other often requires the equation to be prepared in a certain manner, without which it cannot be employed; and of this we have seen a proof in different examples.

QUESTIONS FOR PRACTICE.

1. Given $x^3 + 2x^2 - 23x - 70 = 0$, to find x. *Ans.* $x = 5{\cdot}13450$.

2. Given $x^3 - 15x^2 + 63x - 50 = 0$, to find x. *Ans.* $x = 1{\cdot}028039$.

3. Given $x^4 - 3x^2 - 75x = 10000$, to find x. *Ans.* $x = 10{\cdot}2615$.

4. Given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$, to find x. *Ans.* $x = 8{\cdot}4144$.

5. Let $120x^3 + 3657x^2 - 38059x = 8007115$, to find x. *Ans.* $x = 34{\cdot}6532$.

END OF PART I.

ELEMENTS OF ALGEBRA.

PART II.

Containing the Analysis *of* Indeterminate Quantities.

CHAP. I.

Of the Resolution *of* Equations *of the* First Degree, *which contain more than one unknown* Quantity.

ARTICLE I.

IT has been shewn, in the First Part, how one unknown quantity is determined by a single equation, and how we may determine two unknown quantities by means of two equations, three unknown quantities by three equations, and so on; so that there must always be as many equations as there are unknown quantities to determine, at least when the question itself is determinate.

When a question, therefore, does not furnish as many equations as there are unknown quantities to be determined, some of these must remain undetermined, and depend on our will; for which reason, such questions are said to be *indeterminate*; forming the subject of a particular branch of algebra, which is called *Indeterminate Analysis.*

2. As in those cases we may assume any numbers for one, or more unknown quantities, they also admit of several solutions: but, on the other hand, as there is usually annexed the condition, that the numbers sought are to be integer and positive, or at least rational, the number of all the possible solutions of those questions is greatly limited: so that often there are very few of them possible; at other