which gives 1 for the product, as the nature of the thing required.

377. If we multiply the series which we found for the value of $\frac{1}{(a+b)^2}$, by $a+b$ only, the product ought to answer to the fraction $\frac{1}{a+b}$, or be equal to the series already found, namely, $\frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5}$, &c. and this the actual multiplication will confirm.

$$\frac{1}{a^2} - \frac{2b}{a^3} + \frac{3b^2}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6}, \text{ \&c.}$$
$$a+b$$

$$\frac{1}{a} - \frac{2b}{a^2} + \frac{3b^2}{a^3} - \frac{4b^3}{a^4} + \frac{5b^4}{a^5}, \text{ \&c.}$$
$$+\frac{b}{a^2} - \frac{2b^2}{a^3} + \frac{3b^3}{a^4} - \frac{4b^4}{a^5}, \text{ \&c.}$$

$$\frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5}, \text{ \&c. as required.}$$

SECTION III.

Of Ratios *and* Proportions.

CHAP. I.

Of Arithmetical Ratio, *or of the* Difference *between two* Numbers.

378. Two quantities are either equal to one another, or they are not. In the latter case, where one is greater than the other, we may consider their inequality under two different points of view: we may ask, *how much* one of the quantities is greater than the other? Or we may ask, *how many times* the one is greater than the other? The results which constitute the answers to these two questions are both called *relations*, or *ratios*. We usually call the former an *arithmetical ratio*, and the latter a *geometrical ratio*, without however these denominations having any connexion with the subject itself. The adoption of these expressions has been entirely arbitrary.

379. It is evident, that the quantities of which we speak must be of one and the same kind; otherwise we could not determine any thing with regard to their equality, or inequality: for it would be absurd to ask if two pounds and three ells are equal quantities. So that in what follows, quantities of the same kind only are to be considered; and as they may always be expressed by numbers, it is of numbers only that we shall treat, as was mentioned at the beginning.

380. When of two given numbers, therefore, it is required how much the one is greater than the other, the answer to this question determines the arithmetical ratio of the two numbers; but since this answer consists in giving the difference of the two numbers, it follows, that an arithmetical ratio is nothing but the *difference* between two numbers; and as this appears to be a better expression, we shall reserve the words *ratio* and *relation* to express geometrical ratios.

381. As the difference between two numbers is found by subtracting the less from the greater, nothing can be easier than resolving the question how much one is greater than the other: so that when the numbers are equal, the difference being nothing, if it be required how much one of the numbers is greater than the other, we answer, by nothing; for example, 6 being equal to 2×3, the difference between 6 and 2×3 is 0.

382. But when the two numbers are not equal, as 5 and 3, and it is required how much 5 is greater than 3, the answer is, 2; which is obtained by subtracting 3 from 5. Likewise 15 is greater than 5 by 10; and 20 exceeds 8 by 12.

383. We have therefore three things to consider on this subject; 1st. the greater of the two numbers; 2d. the less; and 3d. the difference: and these three quantities are so connected together, that any two of the three being given, we may always determine the third.

Let the greater number be a, the less b, and the difference d; then d will be found by subtracting b from a, so that $d = a - b$; whence we see how to find d, when a and b are given.

384. But if the difference and the less of the two numbers, that is, if d and b were given, we might determine the greater number by adding together the difference and the less number, which gives $a = b + d$; for if we take from $b + d$ the less number b, there remains d, which is the known difference: suppose, for example, the less number is 12, and the difference 8, then the greater number will be 20.

385. Lastly, if beside the difference d, the greater number a be given, the other number b is found by subtracting the difference from the greater number, which gives $b = a - d$; for if the number $a - d$ be taken from the greater number a, there remains d, which is the given difference.

386. The connexion, therefore, among the numbers, a, b, d, is of such a nature as to give the three following results: 1st. $d = a - b$; 2d. $a = b + d$; 3d. $b = a - d$; and if one of these three comparisons be just, the others must necessarily be so also: therefore, generally, if $z = x + y$, it necessarily follows, that $y = z - x$, and $x = z - y$.

387. With regard to these arithmetical ratios we must remark, that if we add to the two numbers a and b, any number c, assumed at pleasure, or subtract it from them, the difference remains the same; that is, if d is the difference between a and b, that number d will also be the difference between $a + c$ and $b + c$, and between $a - c$ and $b - c$. Thus, for example, the difference between the numbers 20 and 12 being 8, that difference will remain the same, whatever number we add to, or subtract from, the numbers 20 and 12.

388. The proof of this is evident: for if $a - b = d$, we have also $(a + c) - (b + c) = d$; and likewise $(a - c) - (b - c) = d$.

389. And if we double the two numbers a and b, the difference will also become double; thus, when $a - b = d$, we shall have $2a - 2b = 2d$; and, generally, $na - nb = nd$, whatever value we give to n.

CHAP. II.

Of Arithmetical Proportion.

390. When two arithmetical ratios, or relations, are equal,
this equality is called an *arithmetical proportion.*
Thus, when $a - b = d$ and $p - q = d$, so that the dif-
ference is the same between the numbers p and q as between
the numbers a and b, we say that these four numbers form
an arithmetical proportion; which we write thus, $a - b =$
$p - q$, expressing clearly by this, that the difference between
a and b is equal to the difference between p and q.

391. An arithmetical proportion consists therefore of four
terms, which must be such, that if we subtract the second
from the first, the remainder is the same as when we sub-
tract the fourth from the third; thus, the four numbers 12,
7, 9, 4, form an arithmetical proportion, because $12 -$
$7 = 9 - 4$.

392. When we have an arithmetical proportion, as $a - b$
$= p - q$, we may make the second and third terms change
places, writing $a - p = b - q$: and this equality will be
no less true; for, since $a - b = p - q$, add b to both sides,
and we have $a = b + p - q$: then subtract p from both
sides, and we have $a - p = b - q$.

In the same manner, as $12 - 7 = 9 - 4$, so also $12 -$
$9 = 7 - 4$*.

393. We may in every arithmetical proportion put the
second term also in the place of the first, if we make the
same transposition of the third and fourth; that is, if $a -$
$b = p - q$, we have also $b - a = q - p$; for $b - a$ is
the negative of $a - b$, and $q - p$ is also the negative of
$p - q$; and thus, since $12 - 7 = 9 - 4$, we have also,
$7 - 12 = 4 - 9$.

394. But the most interesting property of every arithmetical proportion is this, that the sum of the second and third term is always equal to the sum of the first and fourth. This property, which we must particularly consider, is expressed also by saying that the sum of the *means* is equal to the sum of the *extremes*. Thus, since $12 - 7 = 9 - 4$, we have $7 + 9 = 12 + 4$; the sum being in both cases 16.

* To indicate that those numbers form such a proportion, some authors write them thus: 12 . 7 : : 9 . 4.

395. In order to demonstrate this principal property, let $a - b = p - q$; then if we add to both $b + q$, we have $a + q = b + p$; that is, the sum of the first and fourth terms is equal to the sum of the second and third: and inversely, of four numbers, a, b, p, q, are such, that the sum of the second and third is equal to the sum of the first and fourth; that is, if $b + p = a + q$, we conclude, without a possibility of mistake, that those numbers are in arithmetical proportion, and that $a - b = p - q$; for, since $a + q = b + p$, if we subtract from both sides $b + q$, we obtain $a - b = p - q$.

Thus, the numbers 18, 13, 15, 10, being such, that the sum of the means ($13 + 15 = 28$) is equal to the sum of the extremes ($18 + 10 = 28$), it is certain that they also form an arithmetical proportion; and, consequently, that $18 - 13 = 15 - 10$.

396. It is easy, by means of this property, to resolve the following question. The first three terms of an arithmetical proportion being given, to find the fourth? Let a, b, p, be the first three terms, and let us express the fourth by q, which it is required to determine, then $a + q = b + p$; by subtracting a from both sides, we obtain $q = b + p - a$.

Thus, the fourth term is found by adding together the second and third, and subtracting the first from that sum. Suppose, for example, that 19, 28, 13, are the three first given terms, the sum of the second and third is 41; and taking from it the first, which is 19, there remains 22 for the fourth term sought, and the arithmetical proportion will be represented by $19 - 28 = 13 - 22$, or by $28 - 19 = 22 - 13$, or, lastly, by $28 - 22 = 19 - 13$.

397. When in arithmetical proportion the second term is equal to the third, we have only three numbers; the property of which is this, that the first, *minus* the second, is equal to the second, *minus* the third; or that the difference between the first and second number is equal to the difference between the second and third: the three numbers 19, 15, 11, are of this kind, since $19 - 15 = 15 - 11$.

398. Three such numbers are said to form a continued arithmetical proportion, which is sometimes written thus, 19 : 15 : 11. Such proportions are also called *arithmetical progressions*, particularly if a greater number of terms follow each other according to the same law.

An arithmetical progression may be either *increasing*, or *decreasing*. The former distinction is applied when the terms go on increasing; that is to say, when the second exceeds the first, and the third exceeds the second by the same quantity; as in the numbers 4, 7, 10; and the decreasing progression is that in which the terms go on always diminishing by the same quantity, such as the numbers 9, 5, 1.

399. Let us suppose the numbers a, b, c, to be in arithmetical progression; then $a - b = b - c$, whence it follows, from the equality between the sum of the extremes and that of the means, that $2b = a + c$; and if we subtract a from both, we have $2b - a = c$.

400. So that when the first two terms a, b, of an arithmetical progression are given, the third is found by taking the first from twice the second. Let 1 and 3 be the first two terms of an arithmetical progression, the third will be $2 \times 3 - 1 = 5$; and these three numbers 1, 3, 5, give the proportion

$$1 - 3 = 3 - 5.$$

401. By following the same method, we may pursue the arithmetical progression as far as we please; we have only to find the fourth term by means of the second and third, in the same manner as we determined the third by means of the first and second, and so on. Let a be the first term, and b the second, the third will be $2b - a$, the fourth $4b - 2a - b = 3b - 2a$, the fifth $6b - 4a - 2b + a = 4b - 3a$, the sixth $8b - 6a - 3b + 2a = 5b - 4a$, the seventh $10b - 8a - 4b + 3a = 6b - 5a$, &c.

CHAP. III.

Of Arithmetical Progressions.

402. We have already remarked, that a series of numbers composed of any number of terms, which always increase, or decrease, by the same quantity, is called an *arithmetical progression*.

Thus, the natural numbers written in their order, as 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. form an arithmetical progression, because they constantly increase by unity; and the series 25, 22, 19, 16, 13, 10, 7, 4, 1, &c. is also such a progression, since the numbers constantly decrease by 3.

403. The number, or quantity, by which the terms of an arithmetical progression become greater or less, is called the

difference; so that when the first term and the difference are given, we may continue the arithmetical progression to any length.

For example, let the first term be 2, and the difference 3, and we shall have the following increasing progression: 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, &c. in which each term is found by adding the difference to the preceding one.

404. It is usual to write the natural numbers, 1, 2, 3, 4, 5, &c. above the terms of such an arithmetical progression, in order that we may immediately perceive the rank which any term holds in the progression, which numbers, when written above the terms, are called *indices;* thus, the above example will be written as follows:

Indices.	1	2	3	4	5	6	7	8	9	10	
Arith. Prog.	2,	5,	8,	11,	14,	17,	20,	23,	26,	29,	&c.

where we see that 29 is the tenth term.

405. Let a be the first term, and d the difference, the arithmetical progression will go on in the following order:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ a, & a \pm d, & a \pm 2d, & a \pm 3d, & a \pm 4d, & a \pm 5d, & a \pm 6d, & \text{\&c.}\end{matrix}$$

according as the series is increasing, or decreasing, whence it appears that any term of the progression might be easily found, without the necessity of finding all the preceding ones, by means only of the first term a and the difference d; thus, for example, the tenth term will be $a \pm 9d$, the hundredth term $a \pm 99d$, and, generally, the nth term will be $a \pm (n-1)d$.

406. When we stop at any point of the progression, it is of importance to attend to the first and the last term, since the index of the last term will represent the number of terms. If, therefore, the first term be a, the difference d, and the number of terms n, we shall have for the last term $a \pm (n-1)d$, according as the series is increasing or decreasing, which is consequently found by multiplying the difference by the number of terms *minus* one, and adding, or subtracting, that product from the first term. Suppose, for example, in an ascending arithmetical progression of a hundred terms, the first term is 4, and the difference 3; then the last term will be $99 \times 3 + 4 = 301$.

407. When we know the first term a, and the last z, with the number of terms n, we can find the difference d; for, since the last term $z = a \pm (n-1)d$, if we subtract a from both sides, we obtain $z - a = (n-1)d$. So that by taking the difference between the first and last term, we have the product of the difference multiplied by the number of terms *minus* 1; we have therefore only to divide $z - a$ by $n - 1$

in order to obtain the required value of the difference d, which will be $\frac{z-a}{n-1}$. This result furnishes the following rule: Subtract the first term from the last, divide the remainder by the number of terms *minus* 1, and the quotient will be the common difference: by means of which we may write the whole progression.

408. Suppose, for example, that we have an increasing arithmetical progression of nine terms, whose first is 2, and last 26, and that it is required to find the difference. We must subtract the first term 2 from the last 26, and divide the remainder, which is 24, by $9 - 1$, that is, by 8; the quotient 3 will be equal to the difference required, and the whole progression will be:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2, & 5, & 8, & 11, & 14, & 17, & 20, & 23, & 26. \end{matrix}$$

To give another example, let us suppose that the first term is 1, the last 2, the number of terms 10, and that the arithmetical progression, answering to these suppositions, is required; we shall immediately have for the difference $\frac{2-1}{10-1} = \frac{1}{9}$, and thence conclude that the progression is:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1, & 1\frac{1}{9}, & 1\frac{2}{9}, & 1\frac{3}{9}, & 1\frac{4}{9}, & 1\frac{5}{9}, & 1\frac{6}{9}, & 1\frac{7}{9}, & 1\frac{8}{9}, & 2. \end{matrix}$$

Another example. Let the first term be $2\frac{1}{3}$, the last term $12\frac{1}{2}$, and the number of terms 7; the difference will be $\frac{12\frac{1}{2} - 2\frac{1}{3}}{7-1} = \frac{10\frac{1}{6}}{6} = \frac{61}{36} = 1\frac{25}{36}$, and consequently the progression:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2\frac{1}{3}, & 4\frac{1}{36}, & 5\frac{13}{18}, & 7\frac{5}{12}, & 9\frac{1}{9}, & 10\frac{29}{36}, & 12\frac{1}{2}. \end{matrix}$$

409. If now the first term a, the last term z, and the difference d, are given, we may from them find the number of terms n; for since $z - a = (n-1)d$, by dividing both sides by d, we have $\frac{z-a}{d} = n - 1$; also n being greater by 1 than $n - 1$, we have $n = \frac{z-a}{d} + 1$; consequently the number of terms is found by dividing the difference between the first and the last term, or $z - a$, by the difference of the progression, and adding unity to the quotient.

For example, let the first term be 4, the last 100, and the difference 12, the number of terms will be $\frac{100-4}{12} + 1 = 9$;

and these nine terms will be,

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4, & 16, & 28, & 40, & 52, & 64, & 76, & 88, & 100. \end{matrix}$$

If the first term be 2, the last 6, and the difference $1\frac{1}{3}$, the number of terms will be $\frac{4}{1\frac{1}{3}} + 1 = 4$; and these four terms will be,

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2, & 3\frac{1}{3}, & 4\frac{2}{3}, & 6. \end{matrix}$$

Again, let the first term be $3\frac{1}{3}$, the last $7\frac{2}{3}$, and the difference $1\frac{4}{9}$, the number of terms will be $\frac{7\frac{2}{3} - 3\frac{1}{3}}{1\frac{4}{9}} + 1 = 4$; which are,

$$3\tfrac{1}{3},\ 4\tfrac{7}{9},\ 6\tfrac{2}{9},\ 7\tfrac{2}{3}.$$

410. It must be observed, however, that as the number of terms is necessarily an integer, if we had not obtained such a number for n, in the examples of the preceding article, the questions would have been absurd.

Whenever we do not obtain an integer number for the value of $\frac{z-a}{d}$, it will be impossible to resolve the question; and consequently, in order that questions of this kind may be possible, $z - a$ must be divisible by d.

411. From what has been said, it may be concluded, that we have always four quantities, or things, to consider in an arithmetical progression:

1st. The first term, a; 2d. The last term, z;

3d. The difference, d; and 4th. The number of terms, n.

The relations of these quantities to each other are such, that if we know three of them, we are able to determine the fourth; for,

1. If a, d, and n, are known, we have $z = a \pm (n-1)d$.
2. If z, d, and n, are known, we have

$$a = z - (n-1)d.$$

3. If a, z, and n, are known, we have $d = \frac{z-a}{n-1}$.
4. If a, z, and d, are known, we have $n = \frac{z-a}{d} + 1$.

CHAP. IV.

Of the Summation *of* Arithmetical Progressions.

412. It is often necessary also to find the sum of an arithmetical progression. This might be done by adding all the terms together; but as the addition would be very tedious, when the progression consisted of a great number of terms, a rule has been devised, by which the sum may be more readily obtained.

413. We shall first consider a particular given progression, such that the first term is 2, the difference 3, the last term 29, and the number of terms 10;

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2, & 5, & 8, & 11, & 14, & 17, & 20, & 23, & 26, & 29. \end{matrix}$$

In this progression we see that the sum of the first and last term is 31; the sum of the second and the last but one 31; the sum of the third and the last but two 31, and so on: hence we conclude, that the sum of any two terms equally distant, the one from the first, and the other from the last term, is always equal to the sum of the first and the last term.

414. The reason of this may be easily traced; for if we suppose the first to be a, the last z, and the difference d, the sum of the first and the last term is $a + z$; and the second term being $a + d$, and the last but one $z - d$, the sum of these two terms is also $a + z$. Farther, the third time being $a + 2d$, and the last but two $z - 2d$, it is evident that these two terms also, when added together, make $a + z$; and the demonstration may be easily extended to any other two terms equally distant from the first and last.

415. To determine, therefore, the sum of the progression proposed, let us write the same progression term by term, inverted, and add the corresponding terms together, as follows:

$$\begin{array}{r} 2+\ 5+\ 8+11+14+17+20+23+26+29 \\ 29+26+23+20+17+14+11+\ 8+\ 5+\ 2 \\ \hline 31+31+31+31+31+31+31+31+31+31 \end{array}$$

This series of equal terms is evidently equal to twice the sum of the given progression: now, the number of those equal terms is 10, as in the progression, and their sum consequently is equal to $10 \times 31 = 310$. Hence, as this sum is twice the sum of the arithmetical progression, the sum required must be 155.

416. If we proceed in the same manner with respect to any arithmetical progression, the first term of which is a, the last z, and the number of terms n; writing under the given progression the same progression inverted, and adding term to term, we shall have a series of n terms, each of which will be expressed by $a + z$; therefore the sum of this series will be $n(a + z)$, which is twice the sum of the proposed arithmetical progression; the latter, therefore, will be represented by $\frac{n(a+z)}{2}$.

417. This result furnishes an easy method of finding the sum of any arithmetical progression; and may be reduced to the following rule:

Multiply the sum of the first and the last term by the number of terms, and half the product will be the sum of the whole progression. Or, which amounts to the same, multiply the sum of the first and the last term by half the number of terms. Or, multiply half the sum of the first and the last term by the whole number of terms.

418. It will be necessary to illustrate this rule by some examples.

First, let it be required to find the sum of the progression of the natural numbers, 1, 2, 3, &c. to 100. This will be, by the first rule, $\frac{100 \times 101}{2} = \frac{10100}{2} = 5050$.

If it were required to tell how many strokes a clock strikes in twelve hours; we must add together the numbers 1, 2, 3, as far as 12; now this sum is found immediately to be $\frac{12 \times 13}{2} = 6 \times 13 = 78$. If we wished to know the sum of the same progression continued to 1000, we should find it to be 500500; and the sum of this progression, continued to 10000, would be 50005000.

419. Suppose a person buys a horse, on condition that for the first nail he shall pay 5 pence, for the second 8 pence, for the third 11 pence, and so on, always increasing 3 pence more for each nail, the whole number of which is 32; required the purchase of the horse?

In this question it is required to find the sum of an arithmetical progression, the first term of which is 5, the difference 3, and the number of terms 32; we must there-

fore begin by determining the last term; which is found by the rule, in Articles 406 and 411, to be $5 + (31 \times 3) = 98$; after which the sum required is easily found to be $\frac{103 \times 32}{2}$ $= 103 \times 16$; whence we conclude that the horse costs 1648 pence, or 6*l.* 17*s.* 4*d.*

420. Generally, let the first term be a, the difference d, and the number of terms n; and let it be required to find, by means of these data, the sum of the whole progression. As the last term must be $a \pm (n - 1)d$, the sum of the first and the last will be $2a \pm (n - 1)d$; and multiplying this sum by the number of terms n, we have $2na \pm n(n - 1)d$; the sum required therefore will be $na \pm \frac{n(n-1)d}{2}$.

Now, this formula, if applied to the preceding example, or to $a = 5$, $d = 3$, and $n = 32$, gives $5 \times 32 + \frac{32 \cdot 31 \cdot 3}{2} = 160 + 1488 = 1648$; the same sum that we obtained before.

421. If it be required to add together all the natural numbers from 1 to n, we have, for finding this sum, the first term 1, the last term n, and the number of terms n; therefore the sum required is $\frac{n^2+n}{2} = \frac{n(n+1)}{2}$. If we make $n = 1766$, the sum of all the numbers, from 1 to 1766, will be 883, or half the number of terms, multiplied by $1767 = 1560261$.

422. Let the progression of uneven numbers be proposed, 1, 3, 5, 7, &c. continued to n terms, and let the sum of it be required. Here the first term is 1, the difference 2, the number of terms n; the last term will therefore be $1 + (n - 1)2 = 2n - 1$, and consequently the sum required $= n^2$.

The whole therefore consists in multiplying the number of terms by itself; so that whatever number of terms of this progression we add together, the sum will be always a square, namely, the square of the number of terms; which we shall exemplify as follows:

Indices,	1	2	3	4	5	6	7	8	9	10, &c.
Progress.	1,	3,	5,	7,	9,	11,	13,	15,	17,	19, &c.
Sum.	1,	4,	9,	16,	25,	36,	49,	64,	81,	100, &c.

423. Let the first term be 1, the difference 3, and the number of terms n; we shall have the progression 1, 4, 7, 10, &c. the last term of which will be $1+(n-1)3=3n-2$;

wherefore the sum of the first and the last term is $3n-1$, and consequently the sum of this progression is equal to $\frac{n(3n-1)}{2}=\frac{3n^2-n}{2}$; and if we suppose $n=20$, the sum will be $10\times59=590$.

424. Again, let the first term be 1, the difference d, and the number of terms n; then the last term will be $1+(n-1)d$; to which adding the first, we have $2+(n-1)d$, and multiplying by the number of terms, we have $2n+n(n-1)d$; whence we deduce the sum of the progression $n+\frac{n(n-1)d}{2}$.

And by making d successively equal to 1, 2, 3, 4, &c., we obtain the following particular values, as shewn in the subjoined Table.

If $d=1$, the sum is $n+\frac{n(n-1)}{2}=\frac{n^2+n}{2}$

$d=2$, - - - $n+\frac{2n(n-1)}{2}=n^2$

$d=3$, - - - $n+\frac{3n(n-1)}{2}=\frac{3n^2-n}{2}$

$d=4$, - - - $n+\frac{4n(n-1)}{2}=2n^2-n$

$d=5$, - - - $n+\frac{5n(n-1)}{2}=\frac{5n^2-3n}{2}$

$d=6$, - - - $n+\frac{6n(n-1)}{2}=3n^2-2n$

$d=7$, - - - $n+\frac{7n(n-1)}{2}=\frac{7n^2-5n}{2}$

$d=8$, - - - $n+\frac{8n(n-1)}{2}=4n^2-3n$

$d=9$, - - - $n+\frac{9n(n-1)}{2}=\frac{9n^2-7n}{2}$

$d=10$, - - - $n+\frac{10n(n-1)}{2}=5n^2-4n$

QUESTIONS FOR PRACTICE.

1. Required the sum of an increasing arithmetical progression, having 3 for its first term, 2 for the common difference, and the number of terms 20. *Ans.* 440.

2. Required the sum of a decreasing arithmetical progression, having 10 for its first term, $\frac{1}{3}$ for the common difference, and the number of terms 21. *Ans.* 140.

3. Required the number of all the strokes of a clock in twelve hours, that is, a complete revolution of the index. *Ans.* 78.

4. The clocks of Italy go on to 24 hours; how many strokes do they strike in a complete revolution of the index? *Ans.* 300.

5. One hundred stones being placed on the ground, in a straight line, at the distance of a yard from each other, how far will a person travel who shall bring them one by one to a basket, which is placed one yard from the first stone? *Ans.* 5 *miles* and 1300 *yards*.

CHAP. V.

Of Figurate*, *or* Polygonal Numbers.

425. The summation of arithmetical progressions, which begin by 1, and the difference of which is 1, 2, 3, or any

* The French translator has justly observed, in his note at the conclusion of this chapter, that algebraists make a distinction between figurate and polygonal numbers; but as he has not entered far upon this subject, the following illustration may not be unacceptable.

It will be immediately perceived in the following Table, that each series is derived immediately from the foregoing one, being the sum of all its terms from the beginning to that place; and hence also the law of continuation, and the general term of each series, will be readily discovered.

Natural	1, 2, 3, 4, 5	- -	n general term
Triangular	1, 3, 6, 10, 15	- -	$\frac{n.(n+1)}{2}$
Pyramidal	1, 4, 10, 20, 35	- -	$\frac{n.(n+1).(n+2)}{2.3}$
Triangular-pyramidal	1, 5, 15, 35, 70	- -	$\frac{n.(n+1).(n+2).(n+3)}{2.3.4}$

And, in general, the figurate number of any order m will be expressed by the formula,

$$\frac{n.(n+1).(n+2).(n+3) - - (n+m-1)}{1.2\ .\ 3\ .\ 4\ - -\ m}.$$

Now, one of the principal properties of these numbers, and other integer, leads to the theory of *polygonal numbers*, which are formed by adding together the terms of any such progression.

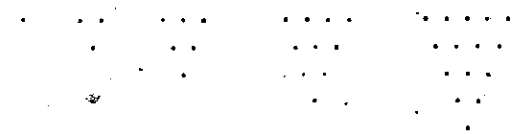
426. Suppose the difference to be 1; then, since the first term is 1 also, we shall have the arithmetical progression, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c. and if in this progression we take the sum of one, of two, of three, &c. terms, the following series of numbers will arise:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, &c.

for $1=1$, $1+2=3$, $1+2+3=6$, $1+2+3+4=10$, &c.

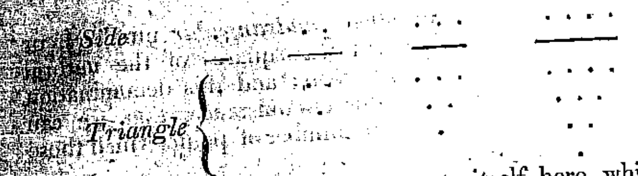
Which numbers are called *triangular*, or *trigonal* numbers, because we may always arrange as many points in the form of a triangle as they contain units, thus:

1 3 6 10 15



427. In all these triangles, we see how many points each side contains In the first triangle there is only one point; in the second there are two; in the third there are three; in the fourth there are four, &c.: so that the triangular numbers, or the number of points, which is simply called the *triangle*, are arranged according to the number of points which the side contains, which number is called the *side;* that is, the third triangular number, or the third triangle, is that whose side has three points; the fourth, that whose side has four; and so on; which may be represented thus:

which Fermat considered as very interesting, (*see his notes on Diophantus, page* 16), is this: that if from the nth term of any series the $(n-1)$ term of the same series be subtracted, the remainder will be the nth term of the preceding series. Thus, in the third series above given, the nth term is $\frac{n.(n+1).(n+2)}{2.3}$; consequently, the $(n-1)$ term, by substituting $(n-1)$ instead of n, is $\frac{(n-1).n.(n+1)}{2.3}$; and if the latter be subtracted from the former, the remainder is $\frac{n.(n-1)}{2}$, which is the nth term of the preceding order of numbers. The same law will be observed between two consecutive terms of any one of these sums.



428. A question therefore presents itself here, which is, how to determine the triangle when the side is given? and, after what has been said, this may be easily resolved.

For if the side be n, the triangle will be $1+2+3+4+\cdots n$. Now, the sum of this progression is $\frac{n^2+n}{2}$; consequently the value of the triangle is $\frac{n^2+n}{2}$ *.

Thus, if $\left\{\begin{array}{l} n=1, \\ n=2, \\ n=3, \\ n=4, \end{array}\right\}$ the triangle is $\left\{\begin{array}{l} 1, \\ 3, \\ 6, \\ 10, \end{array}\right.$

and so on: and when $n=100$, the triangle will be 5050.

429. This formula $\frac{n^2+n}{2}$ is called the general formula of triangular numbers; because by it we find the triangular number, or the triangle, which answers to any side indicated by n.

This may be transformed into $\frac{n(n+1)}{2}$; which serves also to facilitate the calculation; since one of the two numbers n, or $n+1$, must always be an even number, and consequently divisible by 2.

So, if $n=12$, the triangle is $\frac{12\times13}{2}=6\times13=78$; and if $n=15$, the triangle is $\frac{15\times16}{2}=15\times8=120$, &c.

430. Let us now suppose the difference to be 2, and we shall have the following arithmetical progression:

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, &c.

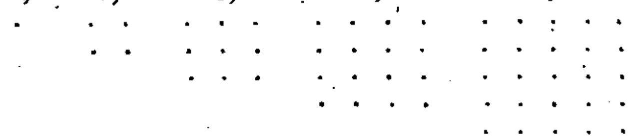
the sums of which, taking successively one, two, three, four terms, &c. form the following series:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, &c.

* M. de Joncourt published at the Hague, in 1762, a Table of trigonal numbers answering to all the natural numbers from 1 to 20000; which Tables are found useful in facilitating a great number of arithmetical operations, as the author shews in a very long introduction. F. T.

the terms of which are called *quadrangular* numbers, or *squares*; since they represent the squares of the natural numbers, as we have already seen; and this denomination is the more suitable from this circumstance, that we can always form a square with the number of points which those terms indicate, thus:

1, 4, 9, 16, 25,



431. We see here, that the side of any square contains precisely the number of points which the square root indicates. Thus, for example, the side of the square 16 consists of 4 points; that of the square 25 consists of 5 points; and, in general, if the side be n, that is, if the number of the terms of the progression, 1, 3, 5, 7, &c. which we have taken, be expressed by n, the square, or the quadrangular number, will be equal to the sum of those terms; that is to n^2, as we have already seen, Article 422; but it is unnecessary to extend our consideration of square numbers any farther, having already treated of them at length.

432. If now we call the difference 3, and take the sums in the same manner as before, we obtain numbers which are called *pentagons*, or *pentagonal* numbers, though they cannot be so well represented by points*.

* It is not, however, that we are unable to represent, by points, polygons of any number of sides; but the rule which I am going to explain for this purpose, seems to have escaped all the writers on algebra whom I have consulted.

I begin with drawing a small polygon that has the number of sides required; this number remains constant for one and the same series of polygonal numbers, and it is equal to 2 *plus* the difference of the arithmetical progression from which the series is produced. I then choose one of its angles, in order to draw from the angular point all the diagonals of this polygon, which, with the two sides containing the angle that has been taken, are to be indefinitely produced; after that, I take these two sides, and the diagonals of the first polygon on the indefinite lines, each as often as I choose; and draw, from the corresponding points marked by the compass, lines parallel to the sides of the first polygon; and divide them into as many equal parts, or by as many points as there are actually in the diagonals and the two sides produced. This rule is general, from the triangle up to the polygon of an infinite number of sides: and the division of these figures into triangles might furnish matter for many curious considerations, and for elegant transformations of the general formulæ, by which the polygonal numbers are expressed in this chapter; but it is unnecessary to dwell on them at present. F. T.

Indices,	1	2	3	4	5	6	7	8	9	&c.
Arith. Prog.	1,	4,	7,	10,	13,	16,	19,	22,	25,	&c.
Pentagon,	1,	5,	12,	22,	35,	51,	70,	92,	117,	&c.

the indices shewing the side of each pentagon.

433. It follows from this, that if we make the side n, the pentagonal number will be $\frac{3n^2-n}{2} = \frac{n(3n-1)}{2}$.

Let, for example, $n = 7$, the pentagon will be 70; and if the pentagon, whose side is 100, be required, we make $n =$ 100, and obtain 14950 for the number sought.

434. If we suppose the difference to be 4, we arrive at *hexagonal* numbers, as we see by the following progressions:

Indices,	1	2	3	4	5	6	7	8	9	&c.
Arith. Prog.	1,	5,	9,	13,	17,	21,	25,	29,	33,	&c.
Hexagon,	1,	6,	15,	28,	45,	66,	91,	120,	153,	&c.

where the indices still shew the side of each hexagon.

435. So that when the side is n, the hexagonal number is $2n^2 - n = n(2n-1)$; and we have farther to remark, that all the hexagonal numbers are also triangular; since, if we take of these last the first, the third, the fifth, &c. we have precisely the series of hexagons.

436. In the same manner, we may find the numbers which are heptagonal, octagonal, &c. It will be sufficient therefore to exhibit the following Table of formulæ for all numbers that are comprehended under the general name of *polygonal* numbers.

Supposing the side to be represented by n, we have for the

Triangle $\frac{n^2+n}{2} = \frac{n(n+1)}{2}$.

Square - $\frac{2n^2+0n}{2} = n^2$.

Vgon - $\frac{3n^2-n}{2} = \frac{n(3n-1)}{2}$.

VIgon - $\frac{4n^2-2n}{2} = 2n^2 - n = n(2n-1)$.

VIIgon - $\frac{5n^2-3n}{2} = \frac{n(5n-3)}{2}$.

VIIIgon	$\frac{6n^2-4n}{2} = 3n^2 - 2n = n(3n-2)$.
IXgon -	$\frac{7n^2-5n}{2} = \frac{n(7n-5)}{2}$.
Xgon -	$\frac{8n^2-6n}{2} = 4n^2 - 3n = n(4n-3)$.
XIgon -	$\frac{9n^2-7n}{2} = \frac{n(9n-7)}{2}$.
XIIgon -	$\frac{10n^2-8n}{2} = 5n^2 - 4n = n(5n-4)$.
XXgon -	$\frac{18n^2-16n}{2} = 9n^2 - 8n = n(9n-8)$.
XXVgon -	$\frac{23n^2-21n}{2} = \frac{n(23n-21)}{2}$.
$m$gon -	$\frac{(m-2)n^2-(m-4)n}{2}$ *.

437. So that the side being n, the m-gonal number will be represented by $\frac{(m-2)n^2-(m-4)n}{2}$; whence we may deduce all the possible polygonal numbers which have the side n. Thus, for example, if the bigonal numbers were required, we should have $m = 2$, and consequently the number sought $= n$; that is to say, the bigonal numbers are the natural numbers, 1, 2, 3, &c.*

If we make $m = 3$, we have $\frac{n^2+n}{2}$ for the triangular number required.

If we make $m = 4$, we have the square number n^2, &c.

438. To illustrate this rule by examples, suppose that the XXV-gonal number, whose side is 36, were required; we look first in the Table for the XXV-gonal number, whose side is n, and it is found to be $\frac{23n^2-21n}{2}$. Then making $n = 36$, we find 14526 for the number sought.

439. *Question.* A person bought a house, and he is asked how much he paid for it. He answers, that the 365th gonal number of 12 is the number of crowns which it cost him.

In order to find this number, we make $m = 365$, and $n = 12$; and substituting these values in the general formula, we find for the price of the house 23970 crowns *.

* The general expression for the m-gonal number is easily derived from the summation of an arithmetical progression, whose first term is 1, common difference d, and number of terms n; as in the following series; viz. $1 + (1+d) + (1+2d) +$, &c. $(1+(n-1).d)$, the sum of which is expressed by $\frac{(2+(n-1).d)n}{2}$; but in all cases $d = m-2$, therefore substituting this value for d, the expression becomes $\frac{2n+(n^2-n).(m-2)}{2} = \frac{(m-2)n^2-(m-4)n}{2}$ as in the formula.

* This chapter is entitled "Of Figurate or Polygonal Numbers." It is not however without foundation that some algebraists make a distinction between *figurate* numbers and *polygonal* numbers. For the numbers commonly called *figurate* are all derived from a single arithmetical progression, and each series of numbers is formed from it by adding together the terms of the series which goes before. On the other hand, every series of *polygonal* numbers is produced from a different arithmetical progression. Hence, in strictness, we cannot speak of a single series of figurate numbers, as being at the same time a series of polygonal numbers. This will be made more evident by the following Tables.

TABLE OF FIGURATE NUMBERS.

Constant numbers - -	1.	1.	1.	1.	1.	1.	&c.
Natural - - - - -	1.	2.	3.	4.	5.	6.	&c.
Triangular - - - -	1.	3.	6.	10.	15.	21.	&c.
Pyramidal - - - -	1.	4.	10.	20.	35.	56.	&c.
Triangular-pyramidal -	1.	5.	15.	35.	70.	126.	&c.

TABLE OF POLYGONAL NUMBERS.

Diff. of the progr.	Numbers						
1	triangular	1.	3.	6.	10.	15.	&c.
2	square - -	1.	4.	9.	16.	25.	&c.
3	pentagon -	1.	5.	12.	22.	35.	&c.
4	hexagon -	1.	6.	15.	28.	45.	&c.

Powers likewise form particular series of numbers. The first two are to be found among the figurate numbers, and the third among the polygonal; which will appear by successively substituting for a the numbers 1, 2, 3, &c.

TABLE OF POWERS.

a^0 - - - - -	1.	1.	1.	1.	1.	&c.
a^1 - - - - -	1.	2.	3.	4.	5.	&c.
a^2 - - - - -	1.	4.	9.	16.	25.	&c.
a^3 - - - - -	1.	8.	27.	64.	125.	&c.
a^4 - - - - -	1.	16.	81.	256.	625.	&c.

CHAP. VI.

Of Geometrical Ratio.

440. The *Geometrical ratio* of two numbers is found by resolving the question, *How many times* is one of those numbers greater than the other? This is done by dividing one by the other; and the quotient will express the ratio required.

441. We have here three things to consider; 1st, the first of the two given numbers, which is called the *antecedent*; 2dly, the other number, which is called the *consequent*; 3dly, the ratio of the two numbers, or the quotient arising from the division of the antecedent by the consequent. For example, if the relation of the numbers 18 and 12 be required, 18 is the antecedent, 12 is the consequent, and the ratio will be $\frac{18}{12} = 1\frac{1}{2}$; whence we see that the antecedent contains the consequent once and a half.

442. It is usual to represent geometrical relation by two points, placed one above the other, between the antecedent and the consequent. Thus, $a : b$ means the geometrical relation of these two numbers, or the ratio of a to b.

We have already remarked that this sign is employed to represent division*, and for this reason we make use of it here; because, in order to know the ratio, we must divide a by b; the relation expressed by this sign being read simply, a is to b.

443. Relation therefore is expressed by a fraction, whose numerator is the antecedent, and whose denominator is the consequent; but perspicuity requires that this fraction should be always reduced to its lowest terms: which is done, as we have already shewn, by dividing both the numerator and denominator by their greatest common divisor. Thus, the fraction $\frac{18}{12}$ becomes $\frac{3}{2}$, by dividing both terms by 6.

The algebraists of the sixteenth and seventeenth centuries paid great attention to these different kinds of numbers and their mutual connexion, and they discovered in them a variety of curious properties; but as their utility is not great, they are now seldom introduced into the systems of mathematics. F. T.

* It will be observed that we have made use of the symbol ÷ for division, as is now usually done in books on this subject.

444. So that relations only differ according as their ratios are different; and there are as many different kinds of geometrical relations as we can conceive different ratios.

The first kind is undoubtedly that in which the ratio becomes unity. This case happens when the two numbers are equal, as in $3 : 3 :: 4 : 4 :: a : a$; the ratio is here 1, and for this reason we call it the relation of equality.

Next follow those relations in which the ratio is another whole number. Thus, $4 : 2$ the ratio is 2, and is called *double* ratio; $12 : 4$ the ratio is 3, and is called *triple* ratio: $24 : 6$ the ratio is 4, and is called *quadruple* ratio, &c.

We may next consider those relations whose ratios are expressed by fractions; such as $12 : 9$, where the ratio is $\frac{4}{3}$, or $1\frac{1}{3}$, and $18 : 27$, where the ratio is $\frac{2}{3}$, &c. We may also distinguish those relations in which the consequent contains exactly twice, thrice, &c. the antecedent: such are the relations $6 : 12$, $5 : 15$, &c. the ratio of which some call *subduple*, *subtriple*, &c. ratios.

Farther, we call that ratio *rational* which is an expressible number; the antecedent and consequent being integers, such as $11 : 7$, $8 : 15$, &c. and we call that an *irrational* or *surd* ratio, which can neither be exactly expressed by integers, nor by fractions, such as $\sqrt{5} : 8$, or $4 : \sqrt{3}$.

445. Let a be the antecedent, b the consequent, and d the ratio, we know already that a and b being given, we find $d = \frac{a}{b}$: if the consequent b were given with the ratio, we should find the antecedent $a = bd$, because bd divided by b gives d: and lastly, when the antecedent a is given, and the ratio d, we find the consequent $b = \frac{a}{d}$; for, dividing the antecedent a by the consequent $\frac{a}{d}$, we obtain the quotient d, that is to say, the ratio.

446. Every relation $a : b$ remains the same, if we multiply or divide the antecedent and consequent by the same number, because the ratio is the same: thus, for example, let d be the ratio of $a : b$, we have $d = \frac{a}{b}$; now the ratio of the relation $na : nb$ is also $\frac{na}{nb} = d$, and that of the relation $\frac{a}{n} : \frac{b}{n}$ is likewise $\frac{na}{nb} = d$.

447. When a ratio has been reduced to its lowest terms,

it is easy to perceive and enunciate the relation. For example, when the ratio $\frac{a}{b}$ has been reduced to the fraction $\frac{p}{q}$, we say $a : b = p : q$, or $a : b :: p : q$, which is read, a is to b as p is to q. Thus, the ratio of 6 : 3 being $\frac{2}{1}$, or 2, we say 6 : 3 :: 2 : 1. We have likewise 18 : 12 :: 3 : 2, and 24 : 18 :: 4 : 3, and 30 : 45 :: 2 : 3, &c. But if the ratio cannot be abridged, the relation will not become more evident; for we do not simplify it by saying 9 : 7 :: 9 . 7.

448. On the other hand, we may sometimes change the relation of two very great numbers into one that shall be more simple and evident, by reducing both to their lowest terms. Thus, for example, we can say, 28844 : 14422 :: 2 : 1; or, 10566 : 7044 :: 3 : 2; or, 57600 : 25200 :: 16 : 7.

449. In order, therefore, to express any relation in the clearest manner, it is necessary to reduce it to the smallest possible numbers; which is easily done, by dividing the two terms of it by their greatest common divisor. Thus, to reduce the relation 57600 : 25200 to that of 16 : 7, we have only to perform the single operation of dividing the numbers 57600 and 25200 by 3600, which is their greatest common divisor.

450. It is important, therefore, to know how to find the greatest common divisor of two given numbers; but this requires a Rule, which we shall explain in the following chapter.

CHAP. VII.

Of the Greatest Common Divisor *of two given* Numbers.

451. There are some numbers which have no other common divisor than unity; and when the numerator and denominator of a fraction are of this nature, it cannot be reduced to a more convenient form*. The two numbers 48 and 35, for example, have no common divisor, though each has its own divisors; for which reason, we cannot express the relation 48 : 35 more simply, because the division of two numbers by 1 does not diminish them.

* In this case, the two numbers are said to be prime to each other. See Art. 66.

452. But when the two numbers have a common divisor, it is found, and even the greatest which they have, by the following Rule:

Divide the greater of the two numbers by the less; next, divide the preceding divisor by the remainder; what remains in this second division will afterwards become a divisor for a third division, in which the remainder of the preceding divisor will be the dividend. We must continue this operation till we arrive at a division that leaves no remainder; and this last divisor will be the greatest common divisor of the two given numbers.

Thus, for the two numbers 576 and 252.

```
252) 576 (2
     504
     ---
      72) 252 (3
          216
          ---
           36) 72 (2
               72
               --
                0.
```

So that, in this instance, the greatest common divisor is 36.

453. It will be proper to illustrate this rule by some other examples; and, for this purpose, let the greatest common divisor of the numbers 504 and 312 be required.

```
312) 504 (1
     312
     ---
     192) 312 (1
          192
          ---
          120) 192 (1
               120
               ---
                72) 120 (1
                     72
                     --
                     48) 72 (1
                         48
                         --
                         24) 48 (2
                             48
                             --
                              0.
```

CHAP. VIII.

Of Geometrical Proportions.

461. Two geometrical relations are equal when their ratios are equal; and this equality of two relations is called a *geometrical proportion*. Thus, for example, we write $a : b = c : d$, or $a : b :: c : d$, to indicate that the relation $a : b$ is equal to the relation $c : d$; but this is more simply expressed by saying a is to b as c to d. The following is such a proportion, $8 : 4 :: 12 : 6$; for the ratio of the relation $8 : 4$ is $\frac{2}{1}$, or 2, and this is also the ratio of the relation $12 : 6$.

462. So that $a : b :: c : d$ being a geometrical proportion, the ratio must be the same on both sides, consequently $\frac{a}{b} = \frac{c}{d}$; and, reciprocally, if the fractions $\frac{a}{b} = \frac{c}{d}$, we have $a : b :: c : d$.

463. A geometrical proportion consists therefore of four terms, such, that the first divided by the second gives the same quotient as the third divided by the fourth; and hence we deduce an important property, common to all geometrical proportions, which is, that the product of the first and the last term is always equal to the product of the second and third; or, more simply, that the product of the extremes is equal to the product of the means.

464. In order to demonstrate this property, let us take the geometrical proportion $a : b :: c : d$, so that $\frac{a}{b} = \frac{c}{d}$. Now, if we multiply both these fractions by b, we obtain $a = \frac{bc}{d}$, and multiplying both sides farther by d, we have $ad = bc$; but ad is the product of the extreme terms, and bc is that of the means, which two products are found to be equal.

465. Reciprocally, if the four numbers a, b, c, d, are such, that the product of the two extremes, a and d, is equal to the product of the two means, b and c, we are certain that they form a geometrical proportion: for, since $ad = bc$, we

have only to divide both sides by bd, which gives us $\frac{ad}{bd} = \frac{bc}{bd}$, or $\frac{a}{b} = \frac{c}{d}$, and consequently $a : b :: c : d$.

466. The four terms of a geometrical proportion, as $a : b :: c : d$, may be transposed in different ways, without destroying the proportion; for the rule being always, that the product of the extremes is equal to the product of the means, or $ad = bc$, we may say,

1st. $b : a :: d : c$; 2dly. $a : c :: b : d$;
3dly. $d : b :: c : a$; 4thly. $d : c :: b : a$.

467. Beside these four geometrical proportions, we may deduce some others from the same proportion, $a : b :: c : d$; for we may say, $a + b : a :: c + d : c$, or the first term, *plus* the second, is to the first, as the third, *plus* the fourth, is to the third; that is, $a + b : a :: c + d : c$.

We may farther say, the first, *minus* the second, is to the first, as the third, *minus* the fourth, is to the third, or $a - b : a :: c - d : c$. For, if we take the product of the extremes and the means, we have $ac - bc = ac - ad$, which evidently leads to the equality $ad = bc$.

And, in the same manner, we may demonstrate that $a + b : b :: c + d : d$; and that $a - b : b :: c - d : d$.

468. All the proportions which we have deduced from $a : b :: c : d$ may be represented generally as follows:

$$ma + nb : pa + qb :: mc + nd : pc + qd.$$

For the product of the extreme terms is $mpac + npbc + mqad + nqbd$; which, since $ad = bc$, becomes $mpac + npbc + mqbc + nqbd$; also the product of the mean terms is $mpac + mqbc + npad + nqbd$; or, since $ad = bc$, it is $mpac + mqbc + npbc + nqbd$; so that the two products are equal.

469. It is evident, therefore, that a geometrical proportion being given, for example, $6 : 3 :: 10 : 5$, an infinite number of others may be deduced from it. We shall, however, give only a few:

$3 : 6 :: 5 : 10$; $6 : 10 :: 3 : 5$; $9 : 6 :: 15 : 10$;
$3 : 3 :: 5 : 5$; $9 : 15 :: 3 : 5$; $9 : 3 :: 15 : 5$.

470. Since in every geometrical proportion the product of the extremes is equal to the product of the means, we may, when the three first terms are known, find the fourth from them. Thus, let the three first terms be $24 : 15 :: 40$ to the fourth term: here, as the product of the means is 600, the fourth term multiplied by the first, that is by 24, must

also make 600; consequently, by dividing 600 by 24 the quotient 25 will be the fourth term required, and the whole proportion will be 24 : 15 :: 40 : 25. In general, therefore, if the three first terms are $a : b :: c$; we put d for the unknown fourth letter; and since $ad = bc$, we divide both sides by a, and have $d = \frac{bc}{a}$; so that the fourth term is $\frac{bc}{a}$, which is found by multiplying the second term by the third, and dividing that product by the first.

471. This is the foundation of the celebrated *Rule of Three* in Arithmetic; for in that rule we suppose three numbers given, and seek a fourth, in geometrical proportion with those three; so that the first may be to the second, as the third is to the fourth.

472. But here it will be necessary to pay attention to some particular circumstances. First, if in two proportions the first and the third terms are the same, as in $a : b :: c : d$, and $a : f :: c : g$, then the two second and the two fourth terms will also be in geometrical proportion, so that $b : d :: f : g$; for the first proportion being transformed into this, $a : c :: b : d$, and the second into this, $a : c :: f : g$, it follows that the relations $b : d$ and $f : g$ are equal, since each of them is equal to the relation $a : c$. Thus, for example, if 5 : 100 :: 2 : 40, and 5 : 15 :: 2 : 6, we must have 100 : 40 :: 15 : 6.

473. But if the two proportions are such, that the mean terms are the same in both, I say that the first terms will be in an inverse proportion to the fourth terms: that is, if $a : b :: c : d$, and $f : b :: c : g$, it follows that $a : f :: g : d$. Let the proportions be, for example, 24 : 8 :: 9 : 3, and 6 : 8 :: 9 : 12, we have 24 : 6 :: 12 : 3; the reason is evident; for the first proportion gives $ad = bc$; and the second gives $fg = bc$; therefore $ad = fg$, and $a : f :: g : d$, or $a : g :: f : d$.

474. Two proportions being given, we may always produce a new one by separately multiplying the first term of the one by the first term of the other, the second by the second, and so on with respect to the other terms. Thus, the proportions $a : b :: c : d$, and $e : f :: g : h$ will furnish this, $ae : bf :: cg : dh$; for the first giving $ad = bc$, and the second giving $eh = fg$, we have also $adeh = bcfg$; but now $adeh$ is the product of the extremes, and $bcfg$ is the product of the means in the new proportion: so that the two products being equal, the proportion is true.

475. Let the two proportions be 6 : 4 :: 15 : 10 and 9 : 12 :: 15 : 20; their combination will give the proportion

6 × 9 : 4 × 12 :: 15 × 15 : 10 × 20,
or 54 : 48 :: 225 : 200,
or 9 : 8 :: 9 : 8.

476. We shall observe, lastly, that if two products are equal, $ad = bc$, we may reciprocally convert this equality into a geometrical proportion; for we shall always have one of the factors of the first product in the same proportion to one of the factors of the second product, as the other factor of the second product is to the other factor of the first product; that is, in the present case, $a : c :: b : d$, or $a : b :: c : d$. Let 3 × 8 = 4 × 6, and we may form from it this proportion, 8 : 4 :: 6 : 3, or this, 3 : 4 :: 6 : 8. Likewise, if 3 × 5 = 1 × 15, we shall have 3 : 15 :: 1 : 5, or 5 : 1 :: 15 : 3, or 3 : 1 :: 15 : 5.

CHAP. IX.

Observations on the Rules *of* Proportion *and their* Utility.

477. This theory is so useful in the common occurrences of life, that scarcely any person can do without it. There is always a proportion between prices and commodities; and when different kinds of money are the subject of exchange, the whole consists in determining their mutual relations. The examples furnished by these reflections will be very proper for illustrating the principles of proportion, and shewing their utility by the application of them.

478. If we wished to know, for example, the relation between two kinds of money; suppose an old *louis d'or* and a *ducat*: we must first know the value of those pieces when compared with others of the same kind. Thus, an old louis being, at Berlin, worth 5 rixdollars and 8 drachms, and a ducat being worth 3 rixdollars, we may reduce these two values to one denomination; either to rixdollars, which gives the proportion 1L : 1D :: $5\frac{1}{3}$R : 3R, or :: 16 : 9; or to drachms, in which case we have 1L : 1D :: 128 : 72 :: 16 : 9; which proportions evidently give the true relation of the old louis to the ducat; for the equality of the products of the extremes and the means gives, in both cases, 9 louis

= 16 ducats; and, by means of this comparison, we may change any sum of old louis into ducats, and vice-versa. Thus, suppose it were required to find how many ducats there are in 1000 old louis, we have this proportion:

Lou. Lou. Duc. Duc.
As 9 : 1000 :: 16 : $1777\frac{7}{9}$, the number sought.

If, on the contrary, it were required to find how many old louis d'or there are in 1000 ducats, we have the following proportion:

Duc. Duc. Lou.
As 16 : 1000 :: 9 : $562\frac{1}{2}$ louis. *Ans.*

479. At Petersburgh the value of the ducat varies, and depends on the course of exchange; which course determines the value of the ruble in stivers, or Dutch halfpence, 105 of which make a ducat. So that when the exchange is at 45 stivers per ruble, we have this proportion:

As 45 : 105 :: 3 : 7;

and hence this equality, 7 rubles = 3 ducats.

Hence again we shall find the value of a ducat in rubles; for

Du. Du. Ru.
As 3 : 1 :: 7 : $2\frac{1}{3}$ rubles;

that is, 1 ducat is equal to $2\frac{1}{3}$ rubles.

But if the exchange were at 50 stivers, the proportion would be,

As 50 : 105 :: 10 : 21;

which would give 21 rubles = 10 ducats; whence 1 ducat = $2\frac{1}{10}$ rubles. Lastly, when the exchange is at 44 stivers, we have

As 44 : 105 :: 1 : $2\frac{17}{44}$ rubles;

which is equal to 2 rubles, $38\frac{7}{11}$ copecks.

480. It follows also from this, that we may compare different kinds of money, which we have frequently occasion to do in bills of exchange.

Suppose, for example, that a person of Petersburgh has 1000 rubles to be paid to him at Berlin, and that he wishes to know the value of this sum in ducats at Berlin.

The exchange is at $47\frac{1}{2}$; that is to say, one ruble makes $47\frac{1}{2}$ stivers; and in Holland, 20 stivers make a florin; $2\frac{1}{2}$ Dutch florins make a Dutch dollar: also, the exchange of Holland with Berlin is at 142; that is to say, for 100 Dutch dollars, 142 dollars are paid at Berlin; and lastly, the ducat is worth 3 dollars at Berlin.

481. To resolve the question proposed, let us proceed

step by step. Beginning therefore with the stivers, since 1 ruble = $47\frac{1}{2}$ stivers, or 2 rubles = 95 stivers, we shall have

Ru. Ru. Stiv.
As 2 : 1000 :: 95 : 47500 stivers;

then again,

Stiv. Stiv. Flor.
As 20 : 47500 :: 1 : 2375 florins.

Also, since $2\frac{1}{2}$ florins = 1 Dutch dollar, or 5 florins = 2 Dutch dollars; we shall have

Flor. Flor. D.D.
As 5 : 2375 :: 2 : 950 Dutch dollars.

Then, taking the dollars of Berlin, according to the exchange, at 142, we shall have

D.D. D.D. Dollars.
As 100 : 950 :: 142 : 1349 dollars of Berlin.

And lastly,

Dol. Dol. Du.
As 3 : 1349 :: 1 : $449\frac{2}{3}$ ducats,

which is the number sought.

482. Now, in order to render these calculations still more complete, let us suppose that the Berlin banker refuses, under some pretext or other, to pay this sum, and to accept the bill of exchange without five per cent. discount; that is, paying only 100 instead of 105. In that case, we must make use of the following proportion:

As 105 : 100 :: $449\frac{2}{3}$: $428\frac{16}{63}$ ducats;

which is the answer under those conditions.

483. We have shewn that six operations are necessary in making use of the Rule of Three; but we can greatly abridge those calculations by a rule which is called the *Rule of Reduction*, or *Double Rule of Three*. To explain which, we shall first consider the two antecedents of each of the six preceding operations:

1st.	2 rubles	:	95 stivers.
2d.	20 stivers	:	1 Dutch florin.
3d.	5 Dutch flor.	:	2 Dutch dollars.
4th.	100 Dutch doll.	:	142 dollars.
5th.	3 dollars	:	1 ducat.
6th.	105 ducats	:	100 ducats.

If we now look over the preceding calculations, we shall observe, that we have always multiplied the given sum by the third terms, or second antecedents, and divided the products by the first: it is evident, therefore, that we shall arrive at the same results by multiplying at once the sum proposed by the product of all the third terms, and dividing by the product of all the first terms: or, which amounts to the same thing, that we have only to make the following proportion: As the product of all the first terms, is to the given number of rubles, so is the product of all the second terms, to the number of ducats payable at Berlin.

484. This calculation is abridged still more, when amongst the first terms some are found that have common divisors with the second or third terms; for, in this case, we destroy those terms, and substitute the quotient arising from the division by that common divisor. The preceding example will, in this manner, assume the following form.

As (2.20.5.100.3.105) : 1000 : : (95.2.142.100) :

$\frac{1000.95.2.142.100}{2.20.5.100.3.105}$; and after cancelling the common divisors in the numerator and denominator, this will become

$\frac{10.19.142}{3.21} = \frac{26980}{63} = 428\frac{16}{63}$ ducats, as before.

485. The method which must be observed in using the Rule of Reduction is this: we begin with the kind of money in question, and compare it with another which is to begin the next relation, in which we compare this second kind with a third, and so on. Each relation, therefore, begins with the same kind as the preceding relation ended with; and the operation is continued till we arrive at the kind of money which the answer requires; at the end of which we must reckon the fractional remainders.

486. Let us give some other examples, in order to facilitate the practice of this calculation.

If ducats gain at Hamburgh 1 per cent. on two dollars banco; that is to say, if 50 ducats are worth, not 100, but 101 dollars banco; and if the exchange between Hamburgh and Konigsberg is 119 drachms of Poland; that is, if 1 dollar banco is equal to 119 Polish drachms; how many Polish florins are equivalent to 1000 ducats?

It being understood that 30 Polish drachms make 1 Polish florin,

Here	1 : 1000	: :	2 dollars banco
	100	—	101 dollars banco
	1	—	119 Polish drachms
	30	—	1 Polish florin;

therefore,

(100.30) : 1000 : : (2.101.119) : $\frac{1000.2.101.119}{100.30}$ =

$\frac{2.101.119}{3} = 8012\frac{2}{3}$ Polish florins. *Ans.*

487. We will propose another example, which may still farther illustrate this method.

Ducats of Amsterdam are brought to Leipsic, having in the former city the value of 5 flor. 4 stivers current; that is to say, 1 ducat is worth 104 stivers, and 5 ducats are worth 26 Dutch florins. If, therefore, the *agio of the bank* at Amsterdam is 5 per cent.; that is, if 105 currency are equal to 100 banco; and if the exchange from Leipsic to Amsterdam, in bank money, is $133\frac{1}{4}$ per cent.; that is, if for 100 dollars we pay at Leipsic $133\frac{1}{4}$ dollars; and lastly, 2 Dutch dollars making 5 Dutch florins; it is required to determine how many dollars we must pay at Leipsic, according to these exchanges, for 1000 ducats?

By the rule,

	5 : 1000	: :	26 flor. Dutch curr.
	105	—	100 flor. Dutch banco
	400	—	533 doll. of Leipsick
	5	—	2 doll. banco;

therefore,

As (5.105.400.5) : 1000 : : (26.100.533.2) :

$\frac{1000.26.100.533.2}{5.105.400.5} = \frac{4.26.533}{21} = 2639\frac{13}{21}$ dollars, the number sought.

CHAP. X.

Of Compound Relations.

488. *Compound Relations* are obtained by multiplying the terms of two or more relations, the antecedents by the antecedents, and the consequents by the consequents; we then say, that the relation between those two products is *compounded* of the relations given.

Thus the relations $a : b$, $c : d$, $e : f$, give the compound relation $ace : bdf$*.

* Each of these three *ratios* is said to be one of the *roots* of the compound ratio.

489. A relation continuing always the same, when we divide both its terms by the same number, in order to abridge it, we may greatly facilitate the above composition by comparing the antecedents and the consequents, for the purpose of making such reductions as we performed in the last chapter.

For example, we find the compound relation of the following given relations thus:

Relations given.

12 : 25, 28 : 33, and 55 : 56.

Which becomes

$$(12.28.55) : (25.33.56) = 2 : 5$$

by cancelling the common divisors.

So that 2 : 5 is the compound relation required.

490. The same operation is to be performed, when it is required to calculate generally by letters; and the most remarkable case is that in which each antecedent is equal to the consequent of the preceding relation. If the given relations are

$a : b$
$b : c$
$c : d$
$d : e$
$e : a$

the compound relation is 1 : 1.

491. The utility of these principles will be perceived, when it is observed, that the relation between two square fields is compounded of the relations of the lengths and the breadths.

Let the two fields, for example, be A and B; A having 500 feet in length by 60 feet in breadth; the length of B being 360 feet, and its breadth 100 feet; the relation of the lengths will be 500 : 360, and that of the breadths 60 : 100. So that we have

$$(500.60) : (360.100) = 5 : 6$$

Wherefore the field A is to the field B, as 5 to 6.

492. Again, let the field A be 720 feet long, 88 feet broad; and let the field B be 660 feet long, and 90 feet broad; the relations will be compounded in the following manner:

Relation of the lengths 720 : 660
Relation of the breadths 88 : 90

and by cancelling, the

Relation of the fields A and B is 16 : 15.

493. Further, if it be required to compare two rooms with respect to the space, or contents, we observe, that that relation is compounded of three relations; namely, that of the lengths, breadths, and heights. Let there be, for example, a room A, whose length is 36 feet, breadth 16 feet, and height 14 feet, and a room B, whose length is 42 feet, breadth 24 feet, and height 10 feet; we shall have these three relations:

For the length 36 : 42
For the breadth 16 : 24
For the height 14 : 10

And cancelling the common measures, these become 4 : 5. So that the contents of the room A, is to the contents of the room B, as 4 to 5.

494. When the relations which we compound in this manner are equal, there result multiplicate relations. Namely, two equal relations give a *duplicate ratio*, or *ratio of the squares*; three equal relations produce the *triplicate ratio*, or *ratio of the cubes*; and so on. For example, the relations $a : b$ and $a : b$ give the compound relation $a^2 : b^2$; wherefore we say, that the squares are in the duplicate ratio of their roots. And the ratio $a : b$ multiplied twice, giving the ratio $a^3 : b^3$, we say that the cubes are in the triplicate ratio of their roots.

495. Geometry teaches, that two circular spaces are in the duplicate relation of their diameters; this means, that they are to each other as the squares of their diameters.

Let A be such a space, having its diameter 45 feet, and B another circular space, whose diameter is 30 feet; the first space will be to the second as 45 × 45 is to 30 × 30; or, compounding these two equal relations, as 9 : 4. Therefore the two areas are to each other as 9 to 4.

496. It is also demonstrated, that the solid contents of spheres are in the ratio of the cubes of their diameters: so that the diameter of a globe, A, being 1 foot, and the diameter of a globe, B, being 2 feet, the solid content of A will be to that of B, as $1^3 : 2^3$; or as 1 to 8. If, therefore, the spheres are formed of the same substance, the latter will weigh 8 times as much as the former.

497. It is evident that we may in this manner find the weight of cannon balls, their diameters, and the weight of one, being given. For example, let there be the ball A, whose diameter is 2 inches, and weight 5 pounds; and if the weight of another ball be required, whose diameter is 8 inches, we have this proportion,

$$2^3 : 8^3 :: 5 : 320 \text{ pounds},$$

which gives the weight of the ball B: and for another ball C, whose diameter is 15 inches, we should have,

$$2^3 : 15^3 :: 5 : 2109\tfrac{3}{8}\text{lb}.$$

498. When the ratio of two fractions, as $\frac{a}{b} : \frac{c}{d}$, is required, we may always express it in integer numbers; for we have only to multiply the two fractions by bd, in order to obtain the ratio $ad : bc$, which is equal to the other; and from hence results the proportion $\frac{a}{b} : \frac{c}{d} :: ad : bc$. If, therefore, ad and bc have common divisors, the ratio may be reduced to fewer terms. Thus $\frac{15}{24} : \frac{25}{36} :: (15.36) : (24.25) :: 9 : 10$.

499. If we wished to know the ratio of the fractions $\frac{1}{a}$ and $\frac{1}{b}$, it is evident that we should have $\frac{1}{a} : \frac{1}{b} :: b : a$; which is expressed by saying, that two fractions, which have unity for their numerator, are in the *reciprocal*, or *inverse* ratio of their denominators: and the same thing is said of two fractions which have any common numerator; for $\frac{c}{a} : \frac{c}{b} :: b : a$. But if two fractions have their denominators equal, as $\frac{a}{c} : \frac{b}{c}$, they are in the *direct ratio* of the numerators; namely, as $a : b$. Thus, $\frac{3}{8} : \frac{3}{16} :: \frac{6}{16} : \frac{3}{16}$, or $6 : 3 :: 2 : 1$, and $\frac{10}{7} : \frac{15}{7} :: 10 : 15$, or $2 : 3$.

500. It has been observed, in the free descent of bodies, that a body falls about 16 English feet in a second, that in two seconds of time it falls from the height of 64 feet, and in three seconds it falls 144 feet. Hence it is concluded, that the heights are to each other as the squares of the times; and, reciprocally, that the times are in the subduplicate ratio of the heights, or as the square roots of the heights *.

If, therefore, it be required to determine how long a stone will be in falling from the height of 2304 feet; we have $16 : 2304 :: 1 : 144$, the square of the time; and consequently the time required is 12 seconds.

501. If it be required to determine how far, or through what height, a stone will pass by descending for the space of an hour, or 3600 seconds; we must say,

$$\text{As } 1^2 : 3600^2 :: 16 : 207360000 \text{ feet},$$

the height required.

Which being reduced is found equal to 39272 miles; and consequently nearly five times greater than the diameter of the earth.

502. It is the same with regard to the price of precious stones, which are not sold in the proportion of their weight; every body knows that their prices follow a much greater ratio. The rule for diamonds is, that the price is in the duplicate ratio of the weight; that is to say, the ratio of the prices is equal to the square of the ratio of the weights. The weight of diamonds is expressed in carats, and a carat is equivalent to 4 grains; if, therefore, a diamond of one carat is worth 10 livres, a diamond of 100 carats will be worth as many times 10 livres as the square of 100 contains 1; so that we shall have, according to the Rule of Three,

$$\text{As } 1 : 10000 :: 10 : 100000 \text{ liv. } \textit{Ans.}$$

There is a diamond in Portugal which weighs 1680 carats; its price will be found, therefore, by making

$$1^2 : 1680^2 :: 10 : 28224000 \text{ livres}.$$

503. The posts, or mode of travelling, in France, furnish sufficient examples of compound ratios; because the price is regulated by the compound ratio of the number of horses, and the number of leagues, or posts. Thus, for example, if one horse cost 20 sous per post, it is required to find how much must be paid for 28 horses for $4\frac{1}{2}$ posts.

We write first the ratio of the horses - -	1 : 28
Under this ratio we put that of the stages -	2 : 9
And, compounding the two ratios, we have -	2 : 252

Or, by abridging the two terms, $1 : 126 :: 1$ liv. to 126 fr. or 42 crowns.

Again, If I pay a ducat for eight horses for 3 miles, how much must I pay for thirty horses for four miles? The calculation is as follows:

$$8 : 30$$
$$3 : 4$$

By compounding these two ratios, and abridging,

$$1 : 5 :: 1 \text{ duc.} : 5 \text{ ducats; the sum required.}$$

504. The same composition occurs when workmen are to be paid, since those payments generally follow the ratio

* The space, through which a heavy body descends, in the latitude of London, and in the first second of time, has been found by experiment to be $16\frac{1}{12}$ English feet; but in calculations where great accuracy is not required, the fraction may be omitted.

compounded of the number of workmen and that of the days which they have been employed.

If, for example, 25 sous per day be given to one mason, and it is required what must be paid to 24 masons who have worked for 50 days, we state the calculation thus:

1 : 24
1 : 50

1 : 1200 :: 25 : 30000 sous, or 1500 francs.

In these examples, five things being given, the rule which serves to resolve them is called, in books of arithmetic, The Rule of Five, or Double Rule of Three.

CHAP. XI.

Of Geometrical Progressions.

505. A series of numbers, which are always becoming a certain number of times greater, or less, is called a *geometrical progression*, because each term is constantly to the following one in the same geometrical ratio: and the number which expresses how many times each term is greater than the preceding, is called the *exponent*, or *ratio*. Thus, when the first term is 1 and the exponent, or ratio, is 2, the geometrical progression becomes,

Terms	1	2	3	4	5	6	7	8	9	&c.
Prog.	1,	2,	4,	8,	16,	32,	64,	128,	256,	&c.

The numbers 1, 2, 3, &c. always marking the place which each term holds in the progression.

506. If we suppose, in general, the first term to be a, and the ratio b, we have the following geometrical progression:

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\ \ldots\ n.$$
$$Prog.\ a,\ ab,\ ab^2,\ ab^3,\ ab^4,\ ab^5,\ ab^6,\ ab^7\ \ldots\ ab^{n-1}.$$

So that, when this progression consists of n terms, the last term is ab^{n-1}. We must, however, remark here, that if the ratio b be greater than unity, the terms increase continually; if $b = 1$, the terms are all equal; lastly, if b be less than 1, or a fraction, the terms continually decrease. Thus, when $a = 1$, and $b = \frac{1}{2}$, we have this geometrical progression:

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \tfrac{1}{32},\ \tfrac{1}{64},\ \tfrac{1}{128},\ \&c.$$

507. Here therefore we have to consider:

1. The first term, which we have called a.
2. The exponent, which we call b.
3. The number of terms, which we have expressed by n.
4. And the last term, which, we have already seen, is expressed by ab^{n-1}.

So that, when the first three of these are given, the last term is found by multiplying the $n-1$ power of b, or b^{n-1}, by the first term a.

If, therefore, the 50th term of the geometrical progression 1, 2, 4, 8, &c. were required, we should have $a = 1$, $b = 2$, and $n = 50$; consequently the 50th term would be 2^{49}; and as $2^9 = 512$, we shall have $2^{10} = 1024$; wherefore the square of 2^{10}, or 2^{20}, $= 1048576$, and the square of this number, which is 1099511627776, $= 2^{40}$. Multiplying therefore this value of 2^{40} by 2^9, or 512, we have $2^{49} = 562949953421312$ for the 50th term.

508. One of the principal questions which occurs on this subject, is to find the sum of all the terms of a geometrical progression; we shall therefore explain the method of doing this. Let there be given, first, the following progression, consisting of ten terms:

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256,\ 512,$$

the sum of which we shall represent by s, so that

$$s = 1+2+4+8+16+32+64+128+256+512;$$

doubling both sides, we shall have

$$2s = 2+4+8+16+32+64+128+256+512+1024;$$

and subtracting from this the progression represented by s, there remains $s = 1024 - 1 = 1023$; wherefore the sum required is 1023.

509. Suppose now, in the same progression, that the number of terms is undetermined, that is, let them be generally represented by n, so that the sum in question, or

$$s,\ = 1+2+2^2+2^3+2^4\ \ldots\ 2^{n-1}$$

If we multiply by 2, we have

$$2s = 2+2^2+2^3+2^4\ \ldots\ldots\ 2^n,$$

then subtracting from this equation the preceding one, we have $s = 2^n - 1$. It is evident, therefore, that the sum required is found, by multiplying the last term, 2^{n-1}, by the exponent 2, in order to have 2^n, and subtracting unity from that product,

510. This is made still more evident by the following

examples, in which we substitute successively for n, the numbers 1, 2, 3, 4, &c.

$1 = 1$; $1 + 2 = 3$; $1 + 2 + 4 = 7$; $1 + 2 + 4 + 8 = 15$; $1 + 2 + 4 + 8 + 16 = 31$; $1 + 2 + 4 + 8 + 16 + 32 = 63$, &c.

511. On this subject, the following question is generally proposed. A man offers to sell his horse on the following condition; that is, he demands 1 penny for the first nail, 2 for the second, 4 for the third, 8 for the fourth, and so on, doubling the price of each succeeding nail. It is required to find the price of the horse, the nails being 32 in number?

This question is evidently reduced to finding the sum of all the terms of the geometrical progression 1, 2, 4, 8, 16, &c. continued to the 32d term. Now, that last term is 2^{31}; and, as we have already found $2^{20} = 1048576$, and $2^{10} = 1024$, we shall have $2^{20} \times 2^{10} = 2^{30} = 1073741824$; and multiplying again by 2, the last term $2^{31} = 2147483648$; doubling therefore this number, and subtracting unity from the product, the sum required becomes 4294967295 pence; which being reduced, we have 17895697*l.* 1*s.* 3*d.* for the price of the horse.

512. Let the ratio now be 3, and let it be required to find the sum of the geometrical progression 1, 3, 9, 27, 81, 243, 729, consisting of 7 terms.

Calling the sum s as before, we have

$$s = 1 + 3 + 9 + 27 + 81 + 243 + 729.$$

And multiplying by 3,

$$3s = 3 + 9 + 27 + 81 + 243 + 729 + 2187.$$

Then subtracting the former series from the latter, we have $2s = 2187 - 1 = 2186$: so that the double of the sum is 2186, and consequently the sum required is 1093.

513. In the same progression, let the number of terms be n, and the sum s; so that

$$s = 1 + 3 + 3^2 + 3^3 + 3^4 + \ldots . 3^{n-1}.$$

If now we multiply by 3, we have

$$3s = 3 + 3^2 + 3^3 + 3^4 + \ldots . 3^n.$$

Then subtracting from this expression the value of s, as before, we shall have $2s = 3^n - 1$; therefore $s = \frac{3^n - 1}{2}$. So that the sum required is found by multiplying the last term by 3, subtracting 1 from the product, and dividing the remainder by 2; as will appear, also, from the following particular cases:

$$1 \quad \frac{(1 \times 3) - 1}{2} = 1$$

$$1 + 3 \quad \frac{(3 \times 3) - 1}{2} = 4$$

$$1 + 3 + 9 \quad \frac{(3 \times 9) - 1}{2} = 13$$

$$1 + 3 + 9 + 27 \quad \frac{(3 \times 27) - 1}{2} = 40$$

$$1 + 3 + 9 + 27 + 81 \quad \frac{(3 \times 81) - 1}{2} = 121.$$

514. Let us now suppose, generally, the first term to be a, the ratio b, the number of terms n, and their sum s, so that

$$s = a + ab + ab^2 + ab^3 + ab^4 + \ldots . ab^{n-1}.$$

If we multiply by b, we have

$$bs = ab + ab^2 + ab^3 + ab^4 + ab^5 + \ldots . ab^n,$$

and taking the difference between this and the above equation, there remains $(b - 1)s = ab^n - a$; whence we easily deduce the sum required $s = \frac{a.(b^n - 1)}{b - 1}$. Consequently, the sum of any geometrical progression is found, by multiplying the last term by the ratio, or exponent of the progression, and dividing the difference between this product and the first term, by the difference between 1 and the ratio.

515. Let there be a geometrical progression of seven terms, of which the first is 3; and let the ratio be 2: we shall then have $a = 3$, $b = 2$, and $n = 7$; therefore the last term is 3×2^6, or 3×64, $= 192$; and the whole progression will be

$$3, 6, 12, 24, 48, 96, 192.$$

Farther, if we multiply the last term 192 by the ratio 2, we have 384; subtracting the first term, there remains 381; and dividing this by $b - 1$, or by 1, we have 381 for the sum of the whole progression.

516. Again, let there be a geometrical progression of six terms, of which the first is 4; and let the ratio be $\frac{3}{2}$: then the progression is

$$4, 6, 9, \tfrac{27}{2}, \tfrac{81}{4}, \tfrac{243}{8}.$$

If we multiply the last term by the ratio, we shall have $\frac{729}{16}$; and subtracting the first term $= \frac{64}{16}$, the remainder is $\frac{665}{16}$; which, divided by $b - 1 = \frac{1}{2}$, gives $\frac{665}{8} = 83\frac{1}{8}$ for the sum of the series.

517. When the exponent is less than 1, and, consequently, when the terms of the progression continually diminish, the sum of such a decreasing progression, carried on to infinity, may be accurately expressed.

For example, let the first term be 1, the ratio $\frac{1}{2}$, and the sum s, so that:

$$s = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64}, \text{\&c.}$$

ad infinitum.

If we multiply by 2, we have

$$2s = 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} +, \text{\&c.}$$

ad infinitum: and, subtracting the preceding progression, there remains $s = 2$ for the sum of the proposed infinite progression.

518. If the first term be 1, the ratio $\frac{1}{3}$, and the sum s; so that

$$s = 1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} +, \text{\&c. ad infinitum:}$$

Then multiplying the whole by 3, we have

$$3s = 3 + 1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} +, \text{\&c. ad infinitum;}$$

and subtracting the value of s, there remains $2s = 3$; wherefore the sum $s = 1\frac{1}{2}$.

519. Let there be a progression whose sum is s, the first term 2, and the ratio $\frac{3}{4}$; so that

$$s = 2 + \tfrac{3}{2} + \tfrac{9}{8} + \tfrac{27}{32} + \tfrac{81}{128} +, \text{\&c. ad infinitum.}$$

Multiplying by $\frac{4}{3}$, we have

$$\tfrac{4}{3}s = \tfrac{8}{3} + 2 + \tfrac{3}{2} + \tfrac{9}{8} + \tfrac{27}{32} + \tfrac{81}{128} +, \text{\&c. ad infinitum;}$$

and subtracting from this progression s, there remains $\frac{1}{3}s = \frac{8}{3}$: wherefore the sum required is 8.

520. If we suppose, in general, the first term to be a, and the ratio of the progression to be $\dfrac{b}{c}$, so that this fraction may be less than 1, and consequently c greater than b; the sum of the progression, carried on ad infinitum, will be found thus:

$$\text{Make } s = a + \frac{ab}{c} + \frac{ab^2}{c^2} + \frac{ab^3}{c^3} + \frac{ab^4}{c^4} +, \text{\&c.}$$

Then multiplying by $\dfrac{b}{c}$, we shall have

$$\frac{b}{c}s = \frac{ab}{c} + \frac{ab^2}{c^2} + \frac{ab^3}{c^3} + \frac{ab^4}{c^4} +, \text{\&c. ad infinitum;}$$

and subtracting this equation from the preceding, there remains $(1 - \dfrac{b}{c})\, s = a$.

Consequently, $s = \dfrac{a}{1-\dfrac{b}{c}} = \dfrac{ac}{c-b}$, by multiplying both the numerator and denominator by c.

The sum of the infinite geometrical progression proposed is, therefore, found by dividing the first term a by 1 minus the ratio, or by multiplying the first term a by the denominator of the ratio, and dividing the product by the same denominator diminished by the numerator of the ratio*.

521. In the same manner we find the sums of progressions, the terms of which are alternately affected by the signs + and −. Suppose, for example,

$$s = a - \frac{ab}{c} + \frac{ab^2}{c^2} - \frac{ab^3}{c^3} + \frac{ab^4}{c^4} -, \text{\&c.}$$

Multiplying by $\dfrac{b}{c}$, we have,

$$\frac{b}{c}s = \frac{ab}{c} - \frac{ab^2}{c^2} + \frac{ab^3}{c^3} - \frac{ab^4}{c^4}, \text{\&c.}$$

And adding this equation to the preceding, we obtain $(1 + \dfrac{b}{c})s = a$: whence we deduce the sum required, $s = \dfrac{a}{1+\dfrac{b}{c}}$, or $s = \dfrac{ac}{c+b}$.

522. It is evident, therefore, that if the first term $a = \frac{3}{5}$, and the ratio be $\frac{2}{5}$, that is to say, $b = 2$, and $c = 5$, we shall find the sum of the progression $\frac{3}{5} + \frac{6}{25} + \frac{12}{125} + \frac{24}{625} +$, &c. $= 1$; since, by subtracting the ratio from 1, there remains $\frac{3}{5}$, and by dividing the first term by that remainder, the quotient is 1.

It is also evident, if the terms be alternately positive and negative, and the progression assume this form:

$$\tfrac{3}{5} - \tfrac{6}{25} + \tfrac{12}{125} - \tfrac{24}{625} +, \text{\&c.}$$

that the sum will be

$$\frac{a}{1+\dfrac{b}{c}} = \frac{\frac{3}{5}}{\frac{7}{5}} = \tfrac{3}{7}.$$

523. Again: let there be proposed the infinite progression,

* This particular case is included in the general Rule, Art. 514.

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} +, \&c.$$

The first term is here $\frac{3}{10}$, and the ratio is $\frac{1}{10}$; therefore subtracting this last from 1, there remains $\frac{9}{10}$, and, if we divide the first term by this fraction, we have $\frac{1}{3}$ for the sum of the given progression. So that taking only one term of the progression, namely, $\frac{3}{10}$, the error would be $\frac{1}{30}$.

And taking two terms, $\frac{3}{10} + \frac{3}{100} = \frac{33}{100}$, there would still be wanting $\frac{1}{300}$ to make the sum, which we have seen is $\frac{1}{3}$.

524. Let there now be given the infinite progression,

$$9 + \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} +, \&c.$$

The first term is 9, and the ratio is $\frac{1}{10}$. So that 1 minus the ratio is $\frac{9}{10}$; and $\dfrac{9}{\frac{9}{10}} = 10$, the sum required: which series is expressed by a decimal fraction, thus, 9·9999999, &c.

QUESTIONS FOR PRACTICE.

1. A servant agreed with a master to serve him eleven years without any other reward for his service than the produce of one grain of wheat for the first year; and that product to be sown the second year, and so on from year to year till the end of the time, allowing the increase to be only in a tenfold proportion. What was the sum of the whole produce?

Ans. 11111111110 grains.

N. B. It is farther required, to reduce this number of grains to the proper measures of capacity, and then by supposing an average price of wheat to compute the value of the corns in money.

2. A servant agreed with a gentleman to serve him twelve months, provided he would give him a farthing for his first month's service, a penny for the second, and 4*d.* for the third, &c. What did his wages amount to?

Ans. 5825*l.* 8*s.* 5¼*d.*

3. One *Sessa*, an *Indian*, having first invented the game of chess, shewed it to his prince, who was so delighted with it, that he promised him any reward he should ask; upon which Sessa requested that he might be allowed one grain of wheat for the first square on the chess board, two for the second, and so on, doubling continually, to 64, the whole number of squares. Now, supposing a pint to contain 7680 of those grains, and one quarter to be worth 1*l.* 7*s.* 6*d.*, it is required to compute the value of the whole sum of grains.

Ans. 64481488296*l.*

CHAP. XII.

Of Infinite Decimal Fractions.

525. We have already seen, in logarithmic calculations, that Decimal Fractions are employed instead of Vulgar Fractions: the same are also advantageously employed in other calculations. It will therefore be very necessary to shew how a vulgar fraction may be transformed into a decimal fraction; and, conversely, how we may express the value of a decimal, by a vulgar fraction.

526. Let it be required, in general, to change the fraction $\frac{a}{b}$, into a decimal. As this fraction expresses the quotient of the division of the numerator a by the denominator b, let us write, instead of a, the quantity a·0000000, whose value does not at all differ from that of a, since it contains neither tenth parts, hundredth parts, nor any other parts whatever. If we now divide the quantity by the number b, according to the common rules of division, observing to put the point in the proper place, which separates the decimal and the integers, we shall obtain the decimal sought. This is the whole of the operation, which we shall illustrate by some examples.

Let there be given first the fraction $\frac{1}{2}$, and the division in decimals will assume this form:

$$\frac{2)1{\cdot}0000000}{0{\cdot}5000000} = \frac{1}{2}.$$

Hence it appears, that $\frac{1}{2}$ is equal to 0·5000000 or to 0·5; which is sufficiently evident, since this decimal fraction represents $\frac{5}{10}$, which is equivalent to $\frac{1}{2}$.

527. Let now $\frac{1}{3}$ be the given fraction, and we shall have,

$$\frac{3)1{\cdot}0000000}{0{\cdot}3333333} = \frac{1}{3}.$$

This shews, that the decimal fraction, whose value is $\frac{1}{3}$, cannot, strictly, ever be discontinued, but that it goes on, ad infinitum, repeating always the number 3; which agrees with what has been already shewn, Art. 523; namely, that the fractions

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000}, \&c. \textit{ ad infinitum}, = \frac{1}{3}.$$

The decimal fraction which expresses the value of $\frac{2}{3}$, is also continued ad infinitum; for we have,

$$\frac{3)2{\cdot}0000000}{0{\cdot}6666666} = \frac{2}{3}.$$

Which is also evident from what we have just said, because $\frac{2}{3}$ is the double of $\frac{1}{3}$.

528. If $\frac{1}{4}$ be the fraction proposed, we have

$$\frac{4)1{\cdot}0000000}{0{\cdot}2500000} = \frac{1}{4}.$$

So that $\frac{1}{4}$ is equal to 0·2500000, or to 0·25: which is evidently true, since $\frac{2}{10}$, or $\frac{20}{100}$, $+ \frac{5}{100} = \frac{25}{100} = \frac{1}{4}$.

In like manner, we should have for the fraction $\frac{3}{4}$,

$$\frac{4)3{\cdot}0000000}{0{\cdot}7500000} = \frac{3}{4}.$$

So that $\frac{3}{4} = 0{\cdot}75$: and in fact

$$\frac{7}{10} + \frac{5}{100} = \frac{75}{100} = \frac{3}{4}.$$

The fraction $\frac{5}{4}$ is changed into a decimal fraction, by making

$$\frac{4)5{\cdot}0000000}{1{\cdot}2500000} = \frac{5}{4}.$$

Now, $1 + \frac{25}{100} = \frac{5}{4}$.

529. In the same manner, $\frac{1}{5}$ will be found equal to 0·2; $\frac{2}{5} = 0{\cdot}4$; $\frac{3}{5} = 0{\cdot}6$; $\frac{4}{5} = 0{\cdot}8$; $\frac{5}{5} = 1$; $\frac{6}{5} = 1{\cdot}2$, &c.

When the denominator is 6, we find $\frac{1}{6} = 0{\cdot}1666666$, &c. which is equal to $0{\cdot}666666 - 0{\cdot}5$: but $0{\cdot}666666 = \frac{2}{3}$, and $0{\cdot}5 = \frac{1}{2}$, wherefore $0{\cdot}1666666 = \frac{2}{3} - \frac{1}{2}$; or $\frac{4}{6} - \frac{3}{6} = \frac{1}{6}$.

We find, also, $\frac{2}{6} = 0{\cdot}333333$, &c. $= \frac{1}{3}$; but $\frac{3}{6}$ becomes $0{\cdot}5000000 = \frac{1}{2}$; also, $\frac{5}{6} = 0{\cdot}833333 = 0{\cdot}333333 + 0{\cdot}5$, that is to say, $\frac{1}{3} + \frac{1}{2}$; or $\frac{2}{6} + \frac{3}{6} = \frac{5}{6}$.

530. When the denominator is 7, the decimal fractions become more complicated. For example, we find $\frac{1}{7} = 0{\cdot}142857$; however it must be observed that these six figures are continually repeated. To be convinced, therefore, that this decimal fraction precisely expresses the value of $\frac{1}{7}$, we may transform it into a geometrical progression, whose first term is $\frac{142857}{1000000}$, the ratio being $\frac{1}{1000000}$; and consequently, the sum $= \dfrac{\frac{142857}{1000000}}{1 - \frac{1}{1000000}} = \frac{142857}{1000000} = \frac{1}{7}$.

531. We may prove, in a manner still more easy, that the decimal fraction, which we have found, is exactly equal to $\frac{1}{7}$; for, by substituting for its value the letter s, we have

$$\begin{aligned}
s &= 0{\cdot}142857142857142857, \text{ \&c.}\\
10s &= 1{\cdot}\ 42857142857142857, \text{ \&c.}\\
100s &= 14{\cdot}\ 2857142857142857, \text{ \&c.}\\
1000s &= 142{\cdot}\ 857142857142857, \text{ \&c.}\\
10000s &= 1428{\cdot}\ 57142857142857, \text{ \&c.}\\
100000s &= 14285{\cdot}\ 7142857142857, \text{ \&c.}\\
1000000s &= 142857{\cdot}\ 142857142857, \text{ \&c.}\\
\text{Subtract } s &= \quad\ 0{\cdot}\ 142857142857, \text{ \&c.}\\
\hline
999999s &= 142857.
\end{aligned}$$

And, dividing by 999999, we have $s = \frac{142857}{999999} = \frac{1}{7}$. Wherefore the decimal fraction, which was represented by s, is $= \frac{1}{7}$.

532. In the same manner, $\frac{2}{7}$ may be transformed into a decimal fraction, which will be 0·28571428, &c. and this enables us to find more easily the value of the decimal fraction which we have represented by s; because 0·28571428, &c. must be the double of it, and, consequently, $= 2s$. Now we have seen that

$$\begin{aligned}
100s &= 14{\cdot}28571428571, \text{ \&c.}\\
\text{So that subtracting}\quad 2s &= \ \ 0{\cdot}28571428571, \text{ \&c.}\\
\hline
\text{there remains } 98s &= 14\\
\text{wherefore}\quad s &= \tfrac{14}{98} = \tfrac{1}{7}.
\end{aligned}$$

We also find $\frac{3}{7} = 0{\cdot}42857142857$, &c. which, according to our supposition, must be equal to $3s$; and we have found that

$$\begin{aligned}
10s &= 1{\cdot}42857142857, \text{ \&c.}\\
\text{So that subtracting}\quad 3s &= 0{\cdot}42857142857, \text{ \&c.}\\
\hline
\text{we have}\quad 7s &= 1, \text{ wherefore } s = \tfrac{1}{7}.
\end{aligned}$$

533. When a proposed fraction, therefore, has the denominator 7, the decimal fraction is infinite, and 6 figures are continually repeated; the reason of which is easy to perceive, namely, that when we continue the division, a remainder must return, sooner or later, which we have had already. Now, in this division, 6 different numbers only can form the remainder, namely 1, 2, 3, 4, 5, 6; so that, at least, after the sixth division, the same figures must return; but when the denominator is such as to lead to a division without remainder, these cases do not happen.

534. Suppose now that 8 is the denominator of the fraction proposed: we shall find the following decimal fractions:

$\frac{1}{8} = 0{\cdot}125$; $\frac{2}{8} = 0{\cdot}25$; $\frac{3}{8} = 0{\cdot}375$; $\frac{4}{8} = 0{\cdot}5$;
$\frac{5}{8} = 0{\cdot}625$; $\frac{6}{8} = 0{\cdot}75$; $\frac{7}{8} = 0{\cdot}875$, &c.

535. If the denominator be 9, we have

$\frac{1}{9} = 0{\cdot}111$, &c. $\frac{2}{9} = 0{\cdot}222$, &c. $\frac{3}{9} = 0{\cdot}333$, &c.

And if the denominator be 10, we have $\frac{1}{10} = 0{\cdot}1$, $\frac{2}{10} = 0{\cdot}2$, $\frac{3}{10} = 0{\cdot}3$. This is evident from the nature of decimals, as also that $\frac{1}{100} = 0{\cdot}01$; $\frac{37}{100} = 0{\cdot}37$; $\frac{256}{1000} = 0{\cdot}256$; $\frac{24}{10000} = 0{\cdot}0024$, &c.

536. If 11 be the denominator of the given fraction, we shall have $\frac{1}{11} = 0{\cdot}0909090$, &c. Now, suppose it were required to find the value of this decimal fraction: let us call it s, and we shall have

$$s = 0{\cdot}090909,$$
$$10s = 00{\cdot}909090,$$
$$100s = 9{\cdot}09090.$$

If, therefore, we subtract from the last the value of s, we shall have $99s = 9$, and consequently $s = \frac{9}{99} = \frac{1}{11}$: thus, also,

$$\tfrac{2}{11} = 0{\cdot}181818, \text{ \&c.}$$
$$\tfrac{3}{11} = 0{\cdot}272727, \text{ \&c.}$$
$$\tfrac{6}{11} = 0{\cdot}545454, \text{ \&c.}$$

537. There are a great number of decimal fractions, therefore, in which one, two, or more figures constantly recur, and which continue thus to infinity. Such fractions are curious, and we shall shew how their values may be easily found *.

* These recurring decimals furnish many interesting researches; I had entered upon them, before I saw the present *Algebra*, and should perhaps have prosecuted my inquiry, had I not likewise found a Memoir in the *Philosophical Transactions* for 1769, entitled *The Theory of circulating Fractions*. I shall content myself with stating here the reasoning with which I began.

Let $\frac{n}{d}$ be any real fraction irreducible to lower terms. And suppose it were required to find how many decimal places we must reduce it to, before the same terms will return again. In order to determine this, I begin by supposing that $10n$ is greater than d; if that were not the case, and only $100n$ or $1000n > d$, it would be necessary to begin with trying to reduce $\frac{10n}{d}$ or $\frac{100n}{d}$, &c. to less terms, or to a fraction $\frac{n^1}{d^1}$.

This being established, I say that the same period can return only when the same remainder n returns in the continual division.

Let us first suppose, that a single figure is constantly repeated, and let us represent it by a, so that $s = 0{\cdot}aaaaaaa$. We have

$$10s = a{\cdot}aaaaaaa$$
and subtracting
$$s = 0{\cdot}aaaaaaa$$

we have $9s = a$; wherefore $s = \frac{a}{9}$.

538. When two figures are repeated, as ab, we have $s = 0{\cdot}ababab$. Therefore $100s = ab{\cdot}ababab$; and if we subtract s from it, there remains $99s = ab$; consequently, $s = \frac{ab}{99}$.

When three figures, as abc, are found repeated, we have $s = 0{\cdot}abcabcabc$; consequently, $1000s = abc{\cdot}abcabc$; and subtracting s from it, there remains $999s = abc$; wherefore $s = \frac{abc}{999}$, and so on.

Whenever, therefore, a decimal fraction of this kind oc-

Suppose that when this happens we have added s cyphers, and that q is the integral part of the quotient; then abstracting from the point, we shall have $\frac{n \times 10^s}{d} = q + \frac{n}{d}$; wherefore $q = \frac{n}{d} \times (10^s - 1)$. Now, as q must be an integer number, it is required to determine the least integer number for s, such that $\frac{n}{d} \times (10^s - 1$, or only that $\frac{10^s - 1}{d}$, may be an integer number.

This problem requires several cases to be distinguished: the first is that in which d is a divisor of 10, or of 100, or of 1000, &c. and it is evident that in this case there can be no circulating fraction. For the second case, we shall take that in which d is an odd number, and not a factor of any power of 10; in this case, the value of s may rise to $d - 1$, but frequently it is less. A third case is that in which d is even, and, consequently, without being a factor of any power of 10, has nevertheless a common divisor with one of those powers: this common divisor can only be a number of the form 2^c; so that if, $\frac{d}{2^c} = e$, I say, the period will be the same as for the fraction $\frac{n}{d}$, but will not commence before the figure represented by c. This case comes to the same therefore with the second case, on which it is evident the theory depends. F. T.

curs, it is easy to find its value. Let there be given, for example, 0·296296: its value will be $\frac{296}{999} = \frac{8}{27}$, by dividing both its terms by 37.

This fraction ought to give again the decimal fraction proposed; and we may easily be convinced that this is the real result, by dividing 8 by 9, and then that quotient by 3, because $27 = 3 \times 9$: thus, we have

$$
\begin{array}{r}
9)\ 8{\cdot}000000 \\ \hline
3)\ 0{\cdot}888888 \\ \hline
0{\cdot}296296, \&c.
\end{array}
$$

which is the decimal fraction that was proposed.

539. Suppose it were required to reduce the fraction $\frac{1}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10}$, to a decimal. The operation would be as follows:

$$
\begin{array}{r}
2)\ 1{\cdot}00000000000000 \\ \hline
3)\ 0{\cdot}50000000000000 \\ \hline
4)\ 0{\cdot}16666666666666 \\ \hline
5)\ 0{\cdot}04166666666666 \\ \hline
6)\ 0{\cdot}00833333333333 \\ \hline
7)\ 0{\cdot}00138888888888 \\ \hline
8)\ 0{\cdot}00019841269841 \\ \hline
9)\ 0{\cdot}00002480158730 \\ \hline
10)\ 0{\cdot}00000275573192 \\ \hline
0{\cdot}00000027557319
\end{array}
$$

CHAP. XIII.

Of the Calculation *of* Interest*.

540. We are accustomed to express the interest of any principal by *per cents*, signifying how much interest is annually paid for the sum of 100 pounds. And it is very usual to put out the principal sum at 5 *per cent*, that is, on such terms, that we receive 5 pounds interest for every 100 pounds principal. Nothing therefore is more easy than to calculate the interest for any sum; for we have only to say, according to the Rule of Three:

As 100 is to the principal proposed, so is the rate *per cent* to the interest required. Let the principal, for example, be 860*l*., its annual interest is found by this proportion;

$$\text{As } 100 : 5 :: 860 : 43.$$

Therefore 43*l*. is the annual interest.

541. We shall not dwell any longer on examples of Simple Interest, but pass on immediately to the calculation of *Compound Interest*; where the chief subject of inquiry is, to what sum does a given principal amount, after a certain number of years, the interest being annually added to the principal. In order to resolve this question, we begin with the consideration, that 100*l*. placed out at 5 per cent, becomes, at the end of a year, a principal of 105*l*.: therefore let the principal be a; its amount, at the end of the year, will be found, by saying; As 100 is to a, so is 105 to the answer; which gives

* The theory of the calculation of interest owes its first improvements to Leibneitz, who delivered the principal elements of it in the *Acta Eruditorum* of Leipsic for 1683. It was afterwards the subject of several detached dissertations written in a very interesting manner. It has been most indebted to those mathematicians who have cultivated political arithmetic; in which are combined, in a manner truly useful, the calculation of interest, and the calculation of probabilities, founded on the data furnished by the bills of mortality. We are still in want of a good elementary treatise of political arithmetic, though this extensive branch of science has been much attended to in England, France, and Holland. F. T.

$$\frac{105a}{100} = \frac{21a}{20} = \tfrac{21}{20} \times a, = a + \tfrac{1}{20} \times a.$$

542. So that, when we add to the original principal its twentieth part, we obtain the amount of the principal at the end of the first year: and adding to this its twentieth part, we know the amount of the given principal at the end of two years, and so on. It is easy, therefore, to compute the successive and annual increases of the principal, and to continue this calculation to any length.

543. Suppose, for example, that a principal, which is at present 1000*l.*, is put out at five per cent; that the interest is added every year to the principal; and that it were required to find its amount at any time. As this calculation must lead to fractions, we shall employ decimals, but without carrying them farther than the thousandth parts of a pound, since smaller parts do not at present enter into consideration.

The given principal of 1000*l.* will be worth

after 1 year - - -	1050*l.*
	52·5,
after 2 years - - -	1102·5
	55·125,
after 3 years - - -	1157·625
	57·881,
after 4 years - - -	1215·506
	60·775,
after 5 years - - -	1276·281, &c.

which sums are formed by always adding $\frac{1}{20}$ of the preceding principal.

544. We may continue the same method, for any number of years; but when this number is very great, the calculation becomes long and tedious; but it may always be abridged, in the following manner:

Let the present principal be a, and since a principal of 20*l.* amounts to 21*l.* at the end of a year, the principal a will amount to $\frac{21}{20} \cdot a$ at the end of a year: and the same principal will amount, the following year, to $\frac{21^2}{20^2} \cdot a = (\frac{21}{20})^2 \cdot a$. Also, this principal of two years will amount to $(\frac{21}{20})^3 \cdot a$, the year after: which will therefore be the principal of three years; and still increasing in the same manner, the given

principal will amount to $(\frac{21}{20})^4 \cdot a$ at the end of four years; to $(\frac{21}{20})^5 \cdot a$, at the end of five years; and after a century, it will amount to $(\frac{21}{20})^{100} \cdot a$; so that, in general, $(\frac{21}{20})^n \cdot a$ will be the amount of this principal, after n years; and this formula will serve to determine the amount of the principal, after any number of years.

545. The fraction $\frac{21}{20}$, which is used in this calculation, depends on the interest having been reckoned at 5 per cent., and on $\frac{21}{20}$ being equal to $\frac{105}{100}$. But if the interest were estimated at 6 per cent, the principal a would amount to $\frac{106}{100} \cdot a$, at the end of a year; to $(\frac{106}{100})^2 \cdot a$, at the end of two years; and to $\frac{106^n}{100} \cdot a$, at the end of n years.

If the interest is only at 4 per cent. the principal a will amount only to $(\frac{104}{100})^n \cdot a$, after n years.

546. When the principal a, as well as the number of years, is given, it is easy to resolve these formulæ by logarithms. For if the question be according to our first supposition, we shall take the logarithm of $(\frac{21}{20})^n \cdot a$, which is $= log. (\frac{21}{20})^n + log. a$; because the given formula is the product of $(\frac{21}{20})^n$ and a. Also, as $(\frac{21}{20})^n$ is a power, we shall have $log. (\frac{21}{20})^n = n\ log. \frac{21}{20}$: so that the logarithm of the amount required is $n\ log. \frac{21}{20} + log.\ a$; and farther, the logarithm of the fraction $\frac{21}{20} = log. 21 - log. 20$.

547. Let now the principal be 1000*l.* and let it be required to find how much this principal will amount to at the end of 100 years, reckoning the interest at 5 per cent.

Here we have $n = 100$; and, consequently, the logarithm of the amount required will be $100\ log. \frac{21}{20} + log. 1000$, which is calculated thus:

log. 21	= 1·3222193
subtracting *log.* 20	= 1·3010300
log. $\frac{21}{20}$	= 0·0211893
multiplying by	100
100 *log.* $\frac{21}{20}$	= 2·1189300
adding *log.* 1000	= 3·0000000
gives	5·1189300

which is the logarithm of the principal required.

We perceive, from the characteristic of this logarithm, that the principal required will be a number consisting of six figures, and it is found to be 131501*l.*

548. Again, suppose a principal of 3452*l.* were put out at 6 per cent, what would it amount to at the end of 64 years?

We have here $a = 3452$, and $n = 64$. Wherefore the logarithm of the amount sought is

$64\ log.\ \frac{53}{50} + log.\ 3452$, which is calculated thus:

$$\begin{aligned} log.\ 53 &= 1{\cdot}7242759 \\ \text{subtracting } log.\ 50 &= 1{\cdot}6989700 \\ \hline log.\ \tfrac{53}{50} &= 0{\cdot}0253059 \\ \text{multiplying by } & \quad 64 \\ \hline 64\ log.\ \tfrac{53}{50} &= 1{\cdot}6195776 \\ log.\ 3452 &= 3{\cdot}5380708 \\ \hline \text{which gives } & \quad 5{\cdot}1576484 \end{aligned}$$

And taking the number of this logarithm, we find the amount required equal to 143763*l*.

549. When the number of years is very great, as it is required to multiply this number by the logarithm of a fraction, a considerable error might arise from the logarithms in the Tables not being calculated beyond 7 figures of decimals; for which reason it will be necessary to employ logarithms carried to a greater number of figures, as in the following example.

A principal of 1*l*. being placed at 5 per cent., compound interest, for 500 years, it is required to find to what sum this principal will amount, at the end of that period.

We have here $a = 1$ and $n = 500$; consequently, the logarithm of the principal sought is equal to $500\ log.\ \frac{21}{20} + log.\ 1$, which produces this calculation:

$$\begin{aligned} log.\ 21 &= 1{\cdot}3222192947733919 \\ \text{subtracting } log.\ 20 &= 1{\cdot}3010299956663981 \\ \hline log.\ \tfrac{21}{20} &= 0{\cdot}0211892990699938 \\ \text{multiply by } & \quad 500 \\ \hline 500\ log.\ \tfrac{21}{20} &= 10{\cdot}5946495349969000, \end{aligned}$$

the logarithm of the amount required; which will be found equal to 39323200000*l*.

550. If we not only add the interest annually to the principal, but also increase it every year by a new sum b, the original principal, which we call a, would increase each year in the following manner:

after 1 year, $\frac{21}{20}a + b$,
after 2 years, $(\frac{21}{20})^2a + \frac{21}{20}b + b$,
after 3 years, $(\frac{21}{20})^3a + (\frac{21}{20})^2b + \frac{21}{20}b + b$,

after 4 years, $(\frac{21}{20})^4a + (\frac{21}{20})^3b + (\frac{21}{20})^2b + \frac{21}{20}b + b$,
after n years, $(\frac{21}{20})^na + (\frac{21}{20})^{n-1}b + (\frac{21}{20})^{n-2}b + \frac{21}{20}b$, &c.

This amount evidently consists of two parts, of which the first is $(\frac{21}{20})^na$; and the other, taken inversely, forms the series $b + \frac{21}{20}b + (\frac{21}{20})^2b + (\frac{21}{20})^3b + \ldots (\frac{21}{20})^{n-1}b$; which series is evidently a geometrical progression, the ratio of which is equal to $\frac{21}{20}$; and we shall therefore find its sum, by first multiplying the last term $(\frac{21}{20})^{n-1}b$ by the exponent $\frac{21}{20}$; which gives $(\frac{21}{20})^nb$. Then, subtracting the first term b, there remains $(\frac{21}{20})^nb - b$; and, lastly, dividing by the exponent *minus* 1, that is to say by $\frac{1}{20}$, we shall find the sum required to be $20(\frac{21}{20})^nb - 20b$; therefore the amount sought is, $(\frac{21}{20})^na + 20(\frac{21}{20})^nb - 20b = (\frac{21}{20})^n \times (a + 20b) - 20b$.

551. The resolution of this formula requires us to calculate separately, its first term $(\frac{21}{20})^n \times (a + 20b)$, which is $n\ log.\ \frac{21}{20} + log.\ (a + 20b)$; for the number which answers to this logarithm in the Tables, will be the first term; and if from this we subtract $20b$, we shall have the amount sought.

552. A person has a principal of 1000*l*. placed out at five per cent, compound interest, to which he adds annually 100*l*. beside the interest: what will be the amount of this principal at the end of twenty-five years?

We have here $a = 1000$; $b = 100$; $n = 25$; the operation is therefore as follows:

$$log.\ \tfrac{21}{20} = 0{\cdot}021189299$$

Multiplying by 25, we have

$$\begin{aligned} 25\ log.\ \tfrac{21}{20} &= 0{\cdot}5297324750 \\ log.\ (a + 20b) &= 3{\cdot}4771213135 \\ \hline \text{And the sum } &= 4{\cdot}0068537885. \end{aligned}$$

So that the first part, or the number which answers to this logarithm, is 10159·1, and if we subtract $20b = 2000$, we find that the principal in question, after twenty-five years, will amount to 8159·1*l*.

553. Since then this principal of 1000*l*. is always increasing, and after twenty-five years amounts to $8159\frac{1}{10}$*l*. we may require, in how many years it will amount to 1000000*l*.

Let n be the number of years required: and, since $a = 1000$, $b = 100$, the principal will be, at the end of n years, $(\frac{21}{20})^n \cdot (3000) - 2000$, which sum must make 1000000; from it therefore results this equation;

$$3000 \cdot (\tfrac{21}{20})^n - 2000 = 1000000;$$

And adding 2000 to both sides, we have

$$3000 \cdot (\tfrac{21}{20})^n = 1002000.$$

Then dividing both sides by 3000, we have $(\frac{21}{20})^n = 334$.

Taking the logarithms, $n\ log.\ \frac{21}{20} = log.\ 334$; and dividing by $log.\ \frac{21}{20}$, we obtain $n = \dfrac{log.\ 334}{log.\ \frac{21}{20}}$. Now, $log.\ 334 = 2{\cdot}5237465$, and $log.\ \frac{21}{20} = 0{\cdot}0211893$; therefore $n = \dfrac{2{\cdot}5237465}{0{\cdot}0211893}$; and, lastly, if we multiply the two terms of this fraction by 10000000, we shall have $n = \frac{25237465}{211893} = 119$ years, 1 month, 7 days; and this is the time, in which the principal of 1000*l.* will be increased to 1000000*l.*

554. But if we supposed that a person, instead of annually increasing his principal by a certain fixed sum, diminished it, by spending a certain sum every year, we should have the following gradations, as the values of that principal a, year after year, supposing it put out at 5 per cent, compound interest, and representing the sum which is annually taken from it by b:

after 1 year, it would be $\frac{21}{20}a - b$,
after 2 years, $(\frac{21}{20})^2a - \frac{21}{20}b - b$,
after 3 years, $(\frac{21}{20})^3a - (\frac{21}{20})^2b - \frac{21}{20}b - b$,
after n years, $(\frac{21}{20})^na - (\frac{21}{20})^{n-1}b - (\frac{21}{20})^{n-2}b \ldots - (\frac{21}{20})b - b$.

555. This principal consists of two parts, one of which is $(\frac{21}{20})^n \cdot a$, and the other, which must be subtracted from it, taking the terms inversely, forms the following geometrical progression:

$$b + (\tfrac{21}{20})b + (\tfrac{21}{20})^2b + (\tfrac{21}{20})^3b + \ldots (\tfrac{21}{20})^{n-1}b.$$

Now we have already found (Art. 550.) that the sum of this progression is $20(\frac{21}{20})^nb - 20b$; if, therefore, we subtract this quantity from $(\frac{21}{20})^n \cdot a$, we shall have for the principal required, after n years =

$$(\tfrac{21}{20})^n \cdot (a - 20b) + 20b.$$

556. We might have deduced this formula immediately from that of Art. 550. For, in the same manner as we annually added the sum b, in the former supposition; so, in the present, we subtract the same sum b every year. We have therefore only to put in the former formula, $-b$ every where, instead of $+b$. But it must here be particularly remarked, that if $20b$ is greater than a, the first part becomes negative, and, consequently, the principal will continually diminish. This will be easily perceived; for if we annually take away from the principal more than is added to it by the interest, it is evident that this principal must continually become less, and at last it will be absolutely reduced to nothing; as will appear from the following example:

557. A person puts out a principal of 100000*l.* at 5 per cent interest; but he spends annually 6000*l.*; which is more than the interest of his principal, the latter being only 5000*l.*; consequently, the principal will continually diminish; and it is required to determine, in what time it will be all spent.

Let us suppose the number of years to be n, and since $a = 100000$, and $b = 6000$, we know that after n years the amount of the principal will be $-20000\ (\frac{21}{20})^n + 120000$, or $120000 - 20000(\frac{21}{20})^n$, where the factor, -20000, is the result of $a - 20b$; or $100000 - 120000$.

So that the principal will become nothing, when $20000(\frac{21}{20})^n$ amounts to 120000; or when $20000(\frac{21}{20})^n = 120000$. Now, dividing both sides by 20000, we have $(\frac{21}{20})^n = 6$; and taking the logarithm, we have $n\ log.\ (\frac{21}{20}) = log.\ 6$; then dividing by $log.\ \frac{21}{20}$, we have $n = \dfrac{log.\ 6}{log.\ \frac{21}{20}}$, or $n = \dfrac{0{\cdot}7781513}{0{\cdot}0211893}$; and, consequently, $n = 36$ years, 8 months, 22 days; at the end of which time, no part of the principal will remain.

558. It will here be proper also to shew how, from the same principles, we may calculate interest for times shorter than whole years. For this purpose, we make use of the formula $(\frac{21}{20})^n \cdot a$ already found, which expresses the amount of a principal, at 5 per cent, compound interest, at the end of n years; for if the time be less than a year, the exponent n becomes a fraction, and the calculation is performed by logarithms as before. If, for example, the amount of a principal at the end of one day were required, we should make $n = \frac{1}{365}$; if after two days, $n = \frac{2}{365}$, and so on.

559. Suppose the amount of 100000*l.* for 8 days were required, the interest being at 5 per cent.

Here $a = 100000$, and $n = \frac{8}{365}$, consequently, the amount sought is $(\frac{21}{20})^{\frac{8}{365}} \times 100000$; the logarithm of which quantity is $log.\ (\frac{21}{20})^{\frac{8}{365}} + log.\ 100000 = \frac{8}{365}\ log.\ \frac{20}{21} + log.\ 100000$. Now, $log.\ \frac{21}{20} = 0{\cdot}0211893$, which, multiplied by $\frac{8}{365}$, gives $0{\cdot}0004644$, to which adding

$$log.\ 100000 = 5{\cdot}0000000$$

the sum is $5{\cdot}0004644$,

The natural number of this logarithm is found to be 100107. So that, subtracting the principal, 100000 from this amount, the interest, for eight days, is 107*l.*

560. To this subject belong also the calculation of the present value of a sum of money, which is payable only after a term of years. For as 20*l.*, in ready money, amounts to 21*l.* in a year; so, reciprocally, a sum of 21*l.*, which cannot be received till the end of one year, is really worth only 20*l.* If, therefore, we express, by a, a sum whose payment is due at the end of a year, the present value of this sum is $\frac{20}{21}a$; and therefore to find the present worth of a principal a, payable a year hence, we must multiply it by $\frac{20}{21}$; to find its value two years before the time of payment, we multiply it by $(\frac{20}{21})^2a$; and in general, its value, n years before the time of payment, will be expressed by $(\frac{20}{21})^na$.

561. Suppose, for example, a man has to receive for five successive years, an annual rent of 100*l.* and that he wishes to give it up for ready money, the interest being at 5 per cent; it is required to find how much he is to receive. Here the calculations may be made in the following manner:

For 100*l.* due

after 1 year, he receives	95·239
after 2 years - - -	90·704
after 3 years - - -	86·385
after 4 years - - -	82·272
after 5 years - - -	78·355
Sum of the 5 terms =	432·955

So that the possessor of the rent can claim, in ready money, only 432·955*l.*

562. If such a rent were to last a greater number of years, the calculation, in the manner we have performed it, would become very tedious; but in that case it may be facilitated as follows:

Let the annual rent be a, commencing at present and lasting n years, it will be actually worth

$$a + (\tfrac{20}{21})a + (\tfrac{20}{21})^2a + (\tfrac{20}{21})^3a + (\tfrac{20}{21})^4a \ldots . + (\tfrac{20}{21})^na.$$

Which is a geometrical progression, and the whole is reduced to finding its sum. We therefore multiply the last term by the exponent, the product of which is $(\frac{20}{21})^{n+1}a$; then, subtracting the first term, there remains $(\frac{20}{21})^{n+1}a - a$; and, lastly, dividing by the exponent *minus* 1, that is, by $-\frac{1}{21}$, or, which amounts to the same, multiplying by -21, we shall have the sum required,

$-21 \cdot (\frac{20}{21})^{n+1} \cdot a + 21a$, or, $21a - 21 \cdot (\frac{20}{21})^{n+1} \cdot a$; and the value of the second term, which it is required to subtract, is easily calculated by logarithms.

QUESTIONS FOR PRACTICE.

1. What will 375*l.* 10*s.* amount to in 9 years at 6 *per cent.* compound interest? *Ans.* 634*l.* 8*s.*

2. What is the interest of 1*l.* for one day, at the rate of 5 *per cent.*? *Ans.* 0·0001369863 parts of a pound.

3. What will 365*l.* amount to in 875 days, at the rate of 4 *per cent.*? *Ans.* 400*l.*

4. What will 256*l.* 10*s.* amount to in seven years, at the rate of 6 *per cent.* compound interest? *Ans.* 385*l.* 13*s.* 7½*d.*

5. What will 563*l.* amount to in 7 years and 99 days, at the rate of 6 *per cent.* compound interest? *Ans.* 860*l.*

6. What is the amount of 400*l.* at the end of 3½ years, at 6 *per cent.* compound interest? *Ans.* 490*l.* 11*s.* 7½*d.*

7. What will 320*l.* 10*s.* amount to in four years, at 5 *per cent.* compound interest? *Ans.* 389*l.* 11*s.* 4¼*d.*

8. What will 650*l.* amount to in 5 years, at 5 *per cent.* compound interest? *Ans.* 829*l.* 11*s.* 7½*d.*

9. What will 550*l.* 10*s.* amount to in 3 years and 6 months, at 6 *per cent.* compound interest? *Ans.* 675*l.* 6*s.* 5*d.*

10. What will 15*l.* 10*s.* amount to in 9 years, at 3½ *per cent.* compound interest? *Ans.* 21*l.* 2*s.* 4¼*d.*

11. What is the amount of 550*l.* at 4 *per cent.* in seven months? *Ans.* 562*l.* 16*s.* 8*d.*

12. What is the amount of 100*l.* at 7·37 *per cent.* in nine years and nine months? *Ans.* 200*l.*

13. If a principal x be put out at compound interest for x years, at x *per cent.* required the time in which it will gain x. *Ans.* 8·49824 years.

14. What sum, in ready money, is equivalent to 600*l.* due nine months hence, reckoning the interest at 5 *per cent.*? *Ans.* 578*l.* 6*s.* 3¾*d.*

15. What sum, in ready money, is equivalent to an annuity of 70*l.* to commence 6 years hence, and then to continue for 21 years at 5 *per cent*? *Ans.* 669*l.* 14*s.* 0¾*d.*

16. A man puts out a sum of money, at 6 *per cent.*, to continue 40 years; and then both principal and interest are to sink. What is that *per cent.* to continue for ever? *Ans.* 52 *per cent.*