SECTION II.

Of the different Methods *of calculating* Compound Quantities.

CHAP. I.

Of the Addition *of* Compound Quantities.

256. When two or more expressions, consisting of several terms, are to be added together, the operation is frequently represented merely by signs, placing each expression between two parentheses, and connecting it with the rest by means of the sign +. Thus, for example, if it be required, to add the expressions $a + b + c$ and $d + e + f$, we represent the sum by

$$(a + b + c) + (d + e + f).$$

257. It is evident that this is not to perform addition, but only to represent it. We see, however, at the same time, that in order to perform it actually, we have only to leave out the parentheses; for as the number $d + e + f$ is to be added to $a + b + c$, we know that this is done by joining to it first $+d$, then $+e$, and then $+f$; which therefore gives the sum $a + b + c + d + e + f$; and the same method is to be observed, if any of the terms are affected by the sign −; as they must be connected in the same way, by means of their proper sign.

258. To make this more evident, we shall consider an example in pure numbers, proposing to add the expression $15 - 6$ to $12 - 8$. Here, if we begin by adding 15, we shall have $12 - 8 + 15$; but this is adding too much, since we had only to add $15 - 6$, and it is evident that 6 is the number which we have added too much; let us therefore take this 6 away by writing it with the negative sign, and we shall have the true sum,

$$12 - 8 + 15 - 6;$$

which shews that the sums are found by writing all the terms, each with its proper sign.

259. If it were required therefore to add the expression $d - e - f$ to $a - b + c$, we should express the sum thus,

$$a - b + c + d - e - f;$$

remarking, however, that it is of no consequence in what order we write these terms; for their places may be changed at pleasure, provided their signs be preserved; so that this sum might have been written thus,

$$c - e + a - f + d - b.$$

260. It is evident, therefore, that addition is attended with no difficulty, whatever be the form of the terms to be added: thus, if it were necessary to add together the expressions $2a^3 + 6\sqrt{b} - 4\,log.\,c$ and $5\sqrt[5]{a} - 7c$, we should write them

$$2a^3 + 6\sqrt{b} - 4\,log.\,c + 5\sqrt[5]{a} - 7c,$$

either in this or in any other order of the terms; for if the signs are not changed, the sum will always be the same.

261. But it frequently happens that the sums represented in this manner may be considerably abridged, as is the case when two or more terms destroy each other; for example, if we find in the same sum the terms $+a - a$, or $3a - 4a + a$: or when two or more terms may be reduced to one, &c. Thus, in the following examples:

$3a + 2a = 5a,$	$7b - 3b = +4b$
$-6c + 10c = +4c;$	$4d - 2d = 2d$
$5a - 8a = -3a,$	$-7b + b = -6b$
$-3c - 4c = -7c,$	$-3d - 5d = -8d$
$2a - 5a + a = -2a,$	$-3b - 5b + 2b = -6b.$

Whenever two or more terms, therefore, are entirely the same with regard to letters, their sum may be abridged; but those cases must not be confounded with such as these, $2a^2 + 3a$, or $2b^3 - b^4$, which admit of no abridgment.

262. Let us consider now some other examples of reduction, as the following, which will lead us immediately to an important truth. Suppose it were required to add together the expressions $a + b$ and $a - b$; our rule gives $a + b + a - b$; now $a + a = 2a$, and $b - b = 0$; the sum therefore is $2a$: consequently, if we add together the sum of two numbers $(a + b)$ and their difference $(a - b)$, we obtain the double of the greater of those two numbers.

This will be better understood perhaps from the following examples:

$3a - 2b - c$	$a^3 - 2a^2b + 2ab^2$
$5b - 6c + a$	$- a^2b + 2ab^2 - b^3$
$4a + 3b - 7c$	$a^3 - 3a^2b + 4ab^2 - b^3$

$4a^2-3b+2c$	$a^4+2ab+b^3$
$3a^2+2b-12c$	$-a^4-2a^2b+3b^3$
$7a^2-b+10c$	$-2a^2b+2ab+4b^3$

CHAP. II.

Of the Subtraction *of* Compound Quantities.

263. If we wish merely to represent subtraction, we enclose each expression within two parentheses, joining, by the sign —, the expression which is to be subtracted, to that from which we have to subtract it.

When we subtract, for example, the expression $d-e+f$ from the expression $a-b+c$, we write the remainder thus:

$$(a-b+c)-(d-e+f);$$

and this method of representing it sufficiently shews which of the two expressions is to be subtracted from the other.

264. But if we wish to perform the actual subtraction, we must observe, first, that when we subtract a positive quantity $+b$ from another quantity a, we obtain $a-b$: and secondly, when we subtract a negative quantity $-b$ from a, we obtain $a+b$; because to free a person from a debt is the same as to give him something.

265. Suppose now it were required to subtract the expression $b-d$ from $a-c$. We first take away b, which gives $a-c-b$: but this is taking away too much by the quantity d, since we had to subtract only $b-d$; we must therefore restore the value of d, and then shall have

$$a-c-b+d;$$

whence it is evident that the terms of the expression to be subtracted must change their signs, and then be joined, with those contrary signs, to the terms of the other expression.

266. Subtraction is therefore easily performed by this rule, since we have only to write the expression from which we are to subtract, joining the other to it without any change beside that of the signs. Thus, in the first example, where it was required to subtract the expression $d-e+f$ from $a-b+c$, we obtain $a-b+c-d+e-f$.

An example in numbers will render this still more clear;

for if we subtract $6-2+4$ from $9-3+2$, we evidently obtain

$$9-3+2-6+2-4=0;$$

for $9-3+2=8$; also, $6-2+4=8$; and $8-8=0$.

267. Subtraction being therefore subject to no difficulty, we have only to remark, that if there are found in the remainder two or more terms, which are entirely similar with regard to the letters, that remainder may be reduced to an abridged form, by the same rules which we have given in addition.

268. Suppose we have to subtract $a-b$ from $a+b$; that is, to take the difference of two numbers from their sum: we shall then have $(a+b)-(a-b)$; but $a-a=0$, and $b+b=2b$; the remainder sought is therefore $2b$; that is to say, the double of the less of the two quantities.

269. The following examples will supply the place of further illustrations:

a^2+ab+b^2	$3a-4b+5c$	$a^3+3a^2b+3ab^2+b^3$	$\sqrt{a}+2\sqrt{b}$
$-a^2+ab+b^2$	$2b+4c-6a$	$a^3-3a^2b+3ab^2-b^3$	$\sqrt{a}-3\sqrt{b}$
$2a^2$.	$9a-6b+c$.	$6a^2b+2b^3$.	$5\sqrt{b}$.

CHAP. III.

Of the Multiplication *of* Compound Quantities.

270. When it is only required to represent multiplication, we put each of the expressions, that are to be multiplied together, within two parentheses, and join them to each other, sometimes without any sign, and sometimes placing the sign × between them. Thus, for example, to represent the product of the two expressions $a-b+c$ and $d-e+f$, we write

$$(a-b+c)\times(d-e+f)$$

or barely, $(a-b+c)\ (d-e+f)$

which method of expressing products is much used, because it immediately exhibits the factors of which they are composed.

271. But in order to shew how multiplication is actually performed, we may remark, in the first place, that to multiply, for example, a quantity, such as $a-b+c$, by 2,

each term of it is separately multiplied by that number; so that the product is

$$2a - 2b + 2c.$$

And the same thing takes place with regard to all other numbers; for if d were the number by which it was required to multiply the same expression, we should obtain

$$ad - bd + cd.$$

272. In the last article, we have supposed d to be a positive number; but if the multiplier were a negative number, as $-e$, the rule formerly given must be applied; namely, that unlike signs multiplied together produce $-$, and like signs $+$. Thus we should have

$$-ae + be - ce.$$

273. Now, in order to shew how a quantity, A, is to be multiplied by a compound quantity, $d - e$; let us first consider an example in numbers, supposing that A is to be multiplied by $7 - 3$. Here it is evident, that we are required to take the quadruple of A: for if we first take A seven times, it will then be necessary to subtract 3A from that product.

In general, therefore, if it be required to multiply A by $d - e$, we multiply the quantity A first by d, and then by e, and subtract this last product from the first: whence results $dA - eA$.

If we now suppose $A = a - b$, and that this is the quantity to be multiplied by $d - e$; we shall have

$$dA = ad - bd$$
$$eA = ae - be$$

whence $dA - eA = ad - bd - ae + be$ is the product required.

274. Since therefore we know accurately the product $(a - b) \times (d - e)$, we shall now exhibit the same example of multiplication under the following form:

$$\begin{array}{l} a - b \\ d - e \\ \hline ad - bd - ae + be. \end{array}$$

Which shews, that we must multiply each term of the upper expression by each term of the lower, and that, with regard to the signs, we must strictly observe the rule before given; a rule which this circumstance would completely confirm, if it admitted of the least doubt.

275. It will be easy, therefore, according to this method, to calculate the following example, which is, to multiply $a + b$ by $a - b$;

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline \text{Product} \quad a^2-b^2. \end{array}$$

276. Now, we may substitute for a and b any numbers whatever; so that the above example will furnish the following theorem; viz. The sum of two numbers, multiplied by their difference, is equal to the difference of the squares of those numbers: which theorem may be expressed thus:

$$(a + b) \times (a - b) = a^2 - b^2.$$

And from this another theorem may be derived; namely, The difference of two square numbers is always a product, and divisible both by the sum and by the difference of the roots of those two squares; consequently, the difference of two squares can never be a prime number *.

277. Let us now calculate some other examples:

$$\begin{array}{l} 2a-3 \\ a+2 \\ \hline 2a^2-3a \\ \quad 4a-6 \\ \hline 2a^2+ a-6 \end{array} \qquad \begin{array}{l} 4a^2-6a+9 \\ 2a+3 \\ \hline 8a^3-12a^2+18a \\ \quad 12a^2-18a+27 \\ \hline 8a^3+27 \end{array}$$

$$\begin{array}{l} 3a^2-2ab \\ 2a-4b \\ \hline 6a^3-4a^2b \\ \quad -12a^2b+8ab^2 \\ \hline 6a^3-16a^2b+8ab^2 \end{array} \qquad \begin{array}{l} a^2+ab^3 \\ a^4-a^3b^3 \\ \hline a^6+a^5b^3 \\ \quad -a^5b^3-a^4b^6 \\ \hline a^6-a^4b^6 \end{array}$$

* This theorem is general, except when the difference of the two numbers is only 1, and their sum is a prime; then it is evident that the difference of the two squares will also be a prime: thus, $6^2 - 5^2 = 11$, $7^2 - 6^2 = 13$, $9^2 - 8^2 = 17$, &c.

$$\begin{array}{l}
a^2+2ab+2b^2 \\
a^2-2ab+2b^2 \\
\hline
a^4+2a^3b+2a^2b^2 \\
\quad -2a^3b-4a^2b^2-4ab^3 \\
\qquad\qquad 2a^2b^2+4ab^3+4b^4 \\
\hline
\qquad a^4+b^4 \\
\hline
\end{array}$$

$$\begin{array}{l}
2a^2-3ab-4b^2 \\
3a^2-2ab+b^2 \\
\hline
6a^4-9a^3b-12a^2b^2 \\
\quad -4a^3b+6a^2b^2+8ab^3 \\
\qquad\qquad 2a^2b^2-3ab^3-4b^4 \\
\hline
6a^4-13a^3b-4a^2b^2+5ab^3-4b^4 \\
\hline
\end{array}$$

$$\begin{array}{l}
a^2+b^2+c^2-ab-ac-bc \\
a+b+c \\
\hline
a^3+ab^2+ac^2-a^2b-a^2c-abc \\
\quad a^2b+b^3+bc^2-ab^2-abc-b^2c \\
\qquad\quad a^2c+b^2c+c^3-abc-ac^2-bc^2 \\
\hline
a^3-3abc+b^3+c^3 \\
\hline
\end{array}$$

278. When we have more than two quantities to multiply together, it will easily be understood that, after having multiplied two of them together, we must then multiply that product by one of those which remain, and so on: but it is indifferent what order is observed in those multiplications.

Let it be proposed, for example, to find the value, or product, of the four following factors, *viz.*

I.	II.	III.	IV.
$(a+b)$	(a^2+ab+b^2)	$(a-b)$	(a^2-ab+b^2).

1st. The product of the factors I. and II.

$$\begin{array}{l}
a^2+ab+b^2 \\
a+b \\
\hline
a^3+a^2b+ab^2 \\
\quad +a^2b+ab^2+b^3 \\
\hline
a^3+2a^2b+2ab^2+b^3 \\
\hline
\end{array}$$

2d. The product of the factors III. and IV.

$$\begin{array}{l}
a^2-ab+b^2 \\
a-b \\
\hline
a^3-a^2b+ab^2 \\
\quad -a^2b+ab^2-b^3 \\
\hline
a^3-2a^2b+2ab^2-b^3 \\
\hline
\end{array}$$

It remains now to multiply the first product I. II. by this second product III. IV.

$$\begin{array}{l}
a^3+2a^2b+2ab^2+b^3 \\
a^3-2a^2b+2ab^2-b^3 \\
\hline
a^6+2a^5b+2a^4b^2+a^3b^3 \\
\quad -2a^5b-4a^4b^2-4a^3b^3-2a^2b^4 \\
\qquad\qquad 2a^4b^2+4a^3b^3+4a^2b^4+2ab^5 \\
\qquad\qquad\quad -a^3b^3-2a^2b^4-2ab^5-b^6 \\
\hline
\qquad\qquad a^6-b^6 \\
\hline
\end{array}$$

which is the product required.

279. Now let us resume the same example, but change the order of it, first multiplying the factors I. and III. and then II. and IV. together.

$$\begin{array}{l}
a+b \\
a-b \\
\hline
a^2+ab \\
\quad -ab-b^2 \\
\hline
a^2-b^2 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
a^2+ab+b^2 \\
a^2-ab+b^2 \\
\hline
a^4+a^3b+a^2b^2 \\
\quad -a^3b-a^2b^2-ab^3 \\
\qquad\qquad a^2b^2+ab^3+b^4 \\
\hline
a^4+a^2b^2+b^4 \\
\hline
\end{array}$$

Then multiplying the two products I. III. and II. IV.

$$\begin{array}{l}
a^4+a^2b^2+b^4 \\
a^2-b^2 \\
\hline
a^6+a^4b^2+a^2b^4 \\
\quad -a^4b^2-a^2b^4-b^6 \\
\hline
a^6-b^6 \\
\hline
\end{array}$$

which is the product required.

280. We may perform this calculation in a manner still more concise, by first multiplying the I[st]. factor by the IV[th]. and then the II[d]. by the III[d].

$$\begin{array}{l}
a^2-ab+b^2 \\
a+b \\
\hline
a^3-a^2b+ab^2 \\
\quad a^2b-ab^2+b^3 \\
\hline
\quad a^3+b^3 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
a^2+ab+b^2 \\
a-b \\
\hline
a^3+a^2b+ab^2 \\
\quad -a^2b-ab^2-b^3 \\
\hline
a^3-b^3 \\
\hline
\end{array}$$

It remains to multiply the product I. IV. by that of II. and III.

$$\begin{array}{l} a^3+b^3 \\ a^3-b^3 \\ \hline a^6+a^3b^3 \\ \quad -a^3b^3-b^6 \\ \hline a^6-b^6 \\ \hline \end{array}$$

the same result as before.

281. It will be proper to illustrate this example by a numerical application. For this purpose, let us make $a=3$ and $b=2$, we shall then have $a+b=5$, and $a-b=1$; farther, $a^2=9$, $ab=6$, and $b^2=4$: therefore $a^2+ab+b^2=19$, and $a^2-ab+b^2=7$: so that the product required is that of $5\times 19\times 1\times 7$, which is 665.

Now, $a^6=729$, and $b^6=64$; consequently, the product required is $a^6-b^6=665$, as we have already seen.

CHAP. IV.

Of the Division *of* Compound Quantities.

282. When we wish simply to represent division, we make use of the usual mark of fractions; which is, to write the denominator under the numerator, separating them by a line; or to enclose each quantity between parentheses, placing two points between the divisor and dividend, and a line between them. Thus, if it were required, for example, to divide $a+b$ by $c+d$, we should represent the quotient thus; $\frac{a+b}{c+d}$, according to the former method; and thus,

$$(a+b)\div(c+d)$$

according to the latter, where each expression is read $a+b$ *divided by* $c+d$.

283. When it is required to divide a compound quantity by a simple one, we divide each term separately, as in the following examples:

$$(6a-8b+4c)\div 2=3a-4b+2c$$
$$(a^2-2ab)\div a=a-2b$$
$$(a^3-2a^2b+3ab^2)\div a=a^2-2ab+3b^2$$

$$(4a^2-6a^2c+8abc)\div 2a=2a-3ac+4bc$$
$$(9a^2bc-12ab^2c+15abc^2)\div 3abc=3a-4b+5c.$$

284. If it should happen that a term of the dividend is not divisible by the divisor, the quotient is represented by a fraction, as in the division of $a+b$ by a, which gives $1+\frac{b}{a}$. Likewise, $(a^2-ab+b^2)\div a^2=1-\frac{b}{a}+\frac{b^2}{a^2}$.

In the same manner, if we divide $2a+b$ by 2, we obtain $a+\frac{b}{2}$: and here it may be remarked, that we may write $\frac{1}{2}b$, instead of $\frac{b}{2}$, because $\frac{1}{2}$ times b is equal to $\frac{b}{2}$; and, in the same manner, $\frac{b}{3}$ is the same as $\frac{1}{3}b$, and $\frac{2b}{3}$ the same as $\frac{2}{3}b$, &c.

285. But when the divisor is itself a compound quantity, division becomes more difficult. This frequently occurs where we least expect it; and when it cannot be performed, we must content ourselves with representing the quotient by a fraction, in the manner that we have already described. At present, we will begin by considering some cases in which actual division takes place.

286. Suppose, for example, it were required to divide $ac-bc$ by $a-b$, the quotient must here be such as, when multiplied by the divisor $a-b$, will produce the dividend $ac-bc$. Now, it is evident, that this quotient must include c, since without it we could not obtain ac; in order therefore to try whether c is the whole quotient, we have only to multiply it by the divisor, and see if that multiplication produces the whole dividend, or only a part of it. In the present case, if we multiply $a-b$ by c, we have $ac-bc$, which is exactly the dividend; so that c is the whole quotient. It is no less evident, that

$$(a^2+ab)\div(a+b)=a;$$
$$(3a^2-2ab)\div(3a-2b)=a;$$
$$(6a^2-9ab)\div(2a-3b)=3a, \text{ \&c.}$$

287. We cannot fail, in this way, to find a part of the quotient; if, therefore, what we have found, when multiplied by the divisor, does not exhaust the dividend, we have only to divide the remainder again by the divisor, in order to obtain a second part of the quotient; and to continue the same method, until we have found the whole.

Let us, as an example, divide $a^2+3ab+2b^2$ by $a+b$.

It is evident, in the first place, that the quotient will include the term a, since otherwise we should not obtain a^2. Now, from the multiplication of the divisor $a+b$ by a, arises a^2+ab; which quantity being subtracted from the dividend, leaves the remainder, $2ab+2b^2$; and this remainder must also be divided by $a+b$, where it is evident that the quotient of this division must contain the term $2b$. Now, $2b$, multiplied by $a+b$, produces $2ab+2b^2$; consequently, $a+2b$ is the quotient required; which multiplied by the divisor $a+b$, ought to produce the dividend $a^2+3ab+2b^2$. See the operation.

$$a+b)a^2+3ab+2b^2(a+2b$$
$$a^2+ab$$
$$2ab+2b^2$$
$$2ab+2b^2$$
$$0.$$

288. This operation will be considerably facilitated by choosing one of the terms of the divisor, which contains the highest power, to be written first, and then, in arranging the terms of the dividend, begin with the highest powers of that first term of the divisor, continuing it according to the powers of that letter. This term in the preceding example was a. The following examples will render the process more perspicuous.

$$a-b)a^3-3a^2b+3ab^2-b^3(a^2-2ab+b^2$$
$$a^3-a^2b$$
$$-2a^2b+3ab^2$$
$$-2a^2b+2ab^2$$
$$ab^2-b^3$$
$$ab^2-b^3$$
$$0.$$

$$a+b)a^2-b^2(a-b$$
$$a^2+ab$$
$$-ab-b^2$$
$$-ab-b^2$$
$$0.$$

$$3a-2b)18a^2-8b^2(6a+4b$$
$$18a^2-12ab$$
$$12ab-8b^2$$
$$12ab-8b^2$$
$$0.$$

$$a+b)a^3+b^3(a^2-ab+b^2$$
$$a^3+a^2b$$
$$-a^2b+b^3$$
$$-a^2b-ab^2$$
$$ab^2+b^3$$
$$ab^2+b^3$$
$$0.$$

$$2a-b)8a^3-b^3(4a^2+2ab+b^2$$
$$8a^3-4a^2b$$
$$4a^2b-b^3$$
$$4a^2b-2ab^2$$
$$2ab^2-b^3$$
$$2ab^2-b^3$$
$$0.$$

$$a^2-2ab+b^2)a^4-4a^3b+6a^2b^2-4ab^3+b^4(a^2-2ab+b^2$$
$$a^4-2a^3b+a^2b^2$$
$$-2a^3b+5a^2b^2-4ab^3$$
$$-2a^3b+4a^2b^2-2ab^3$$
$$a^2b^2-2ab^3+b^4$$
$$a^2b^2-2ab^3+b^4$$
$$0.$$

$$a^2-2ab+4b^2)a^4+4a^2b^2+16b^4(a^2+2ab+4b^2$$
$$a^4-2a^3b+4a^2b^2$$
$$2a^3b+16b^4$$
$$2a^3b-4a^2b^2+8ab^3$$
$$16b^4$$
$$4a^2b^2$$
$$4a^2b^2-8ab^3+16b^4$$
$$4a^2b^2-8ab^3+16b^4$$
$$0.$$

$$a^2-2ab+2b^2)a^4+4b^4(a^2+2ab+2b^2$$
$$a^4-2a^3b+2a^2b^2$$
$$2a^3b-2a^2b^2+4b^4$$
$$2a^3b-4a^2b^2+4ab^3$$
$$2a^2b^2-4ab^3+4b^4$$
$$2a^2b^2-4ab^3+4b^4$$
$$0.$$

$$1-2x+x^2)1-5x+10x^2-10x^3+5x^4-x^5(1-3x+3x^2-x^3$$
$$1-2x+x^2$$
$$-3x+9x^2-10x^3$$
$$-3x+6x^2-3x^3$$
$$3x^2-7x^3+5x^4$$
$$3x^2-6x^3+3x^4$$
$$-x^3+2x^4-x^5$$
$$-x^3+2x^4-x^5$$
$$0.$$

CHAP. V.

Of the Resolution *of* Fractions *into* Infinite Series*.

289. When the dividend is not divisible by the divisor,

the quotient is expressed, as we have already observed, by a fraction: thus, if we have to divide 1 by $1-a$, we obtain the fraction $\frac{1}{1-a}$. This, however, does not prevent us from attempting the division according to the rules that have been given, nor from continuing it as far as we please; and we shall not fail thus to find the true quotient, though under different forms.

290. To prove this, let us actually divide the dividend 1 by the divisor $1-a$, thus:

$$1-a)1 \quad * \quad (1+\frac{a}{1-a}$$
$$1-a$$
remainder a

$$\text{or, } 1-a)1 \quad * \quad (1+a+\frac{a^2}{1-a}$$
$$1-a$$
$$a$$
$$a-a^2$$
remainder a^2

To find a greater number of forms, we have only to continue dividing the remainder a^2 by $1-a$;

$$1-a)a^2 \quad * \quad (a^2+\frac{a^3}{1-a}$$
$$a^2-a^3$$
$$a^3$$

* The Theory of Series is one of the most important in all the mathematics. The series considered in this chapter were discovered by Mercator, about the middle of the last century; and soon after, Newton discovered those which derived from the extraction of roots, and which are treated of in Chapter XII. of this section. This theory has gradually received improvements from several other distinguished mathematicians. The works of James Bernoulli, and the second part of the "Differential Calculus" of Euler, are the books in which the fullest information is to be obtained on these subjects. There is likewise in the Memoirs of Berlin for 1768, a new method by M. de la Grange for resolving, by means of infinite series, all literal equations of any dimensions whatever. F. T.

$$\text{then, } 1-a)a^3 \quad * \; (a^3 + \frac{a^4}{1-a}$$
$$\frac{a^3-a^4}{a^4}$$

$$\text{and again, } 1-a)a^4 \quad * \; (a^4 + \frac{a^5}{1-a}$$
$$\frac{a^4-a^5}{a^5, \&c.}$$

291. This shews that the fraction $\frac{1}{1-a}$ may be exhibited under all the following forms:

I. $1 + \frac{a}{1-a}$. II. $1 + a + \frac{a^2}{1-a}$;

III. $1 + a + a^2 + \frac{a^3}{1-a}$. IV. $1+a+a^2+a^3 + \frac{a^4}{1-a}$;

V. $1 + a + a^2 + a^3 + a^4 + \frac{a^5}{1-a}$, &c.

Now, by considering the first of these expressions, which is $1 + \frac{a}{1-a}$, and remembering that 1 is the same as $\frac{1-a}{1-a}$, we have

$$1 + \frac{a}{1-a} = \frac{1-a}{1-a} + \frac{a}{1-a} = \frac{1-a+a}{1-a} = \frac{1}{1-a}.$$

If we follow the same process, with regard to the second expression, $1 + a + \frac{a^2}{1-a}$, that is to say, if we reduce the integral part $1 + a$ to the same denominator, $1 - a$, we shall have $\frac{1-a^2}{1-a}$, to which if we add $+ \frac{a^2}{1-a}$, we shall have $\frac{1-a^2+a^2}{1-a}$, that is to say, $\frac{1}{1-a}$.

In the third expression, $1 + a + a^2 + \frac{a^3}{1-a}$, the integers reduced to the denominator $1 - a$ make $\frac{1-a^3}{1-a}$; and if we add to that the fraction $\frac{a^3}{1-a}$, we have $\frac{1}{1-a}$, as before; therefore all these expressions are equal in value to $\frac{1}{1-a}$, the proposed fraction.

292. This being the case, we may continue the series as far as we please, without being under the necessity of performing any more calculations; and thus we shall have

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 + a^7 + \frac{a^8}{1-a};$$

or we might continue this farther, and still go on without end; for which reason it may be said that the proposed fraction has been resolved into an infinite series, which is, $1+a+a^2+a^3+a^4+a^5+a^6+a^7+a^8+a^9+a^{10}+a^{11}+a^{12}$, &c. to infinity: and there are sufficient grounds to maintain, that the value of this infinite series is the same as that of the fraction $\frac{1}{1-a}$.

293. What we have said may at first appear strange; but the consideration of some particular cases will make it easily understood. Let us suppose, in the first place, $a=1$; our series will become $1+1+1+1+1+1+1$, &c; and the fraction $\frac{1}{1-a}$, to which it must be equal, becomes $\frac{1}{0}$. Now, we have before remarked, that $\frac{1}{0}$ is a number infinitely great; which is therefore here confirmed in a satisfactory manner. See Art. 83 and 84.

Again, if we suppose $a = 2$, our series becomes $1+2+4+8+16+32+64$, &c. to infinity, and its value must be the same as $\frac{1}{1-2}$, that is to say $\frac{1}{-1} = -1$; which at first sight will appear absurd. But it must be remarked, that if we wish to stop at any term of the above series, we cannot do so without annexing to it the fraction which remains. Suppose, for example, we were to stop at 64, after having written $1+2+4+8+16+32+64$, we must add the fraction $\frac{128}{1-2}$, or $\frac{128}{-1}$, or -128; we shall therefore have $127-128$, that is in fact -1.

Were we to continue the series without intermission, the fraction would be no longer considered; but, in that case, the series would still go on.

294. These are the considerations which are necessary, when we assume for a numbers greater than unity; but if we suppose a less than 1, the whole becomes more intelligible: for example, let $a = \frac{1}{2}$; and we shall then have $\frac{1}{1-a} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$, which will be equal to the following series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128}$, &c. to in-

finity. Now, if we take only two terms of this series, we shall have $1 + \frac{1}{2}$, and it wants $\frac{1}{2}$ of being equal to $\frac{1}{1-a} = 2$. If we take three terms, it wants $\frac{1}{4}$; for the sum is $1\frac{3}{4}$. If we take four terms, we have $1\frac{7}{8}$, and the deficiency is only $\frac{1}{8}$. Therefore, the more terms we take, the less the difference becomes; and, consequently, if we continue the series to infinity, there will be no difference at all between its sum and the value of the fraction $\frac{1}{1-a}$, or 2.

295. Let $a = \frac{1}{3}$; and our fraction $\frac{1}{1-a}$ will then be $= \frac{1}{1-\frac{1}{3}} = \frac{3}{2} = 1\frac{1}{2}$, which, reduced to an infinite series, becomes $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$, &c. which is consequently equal to $\frac{1}{1-a}$.

Here, if we take two terms, we have $1\frac{1}{3}$, and there wants $\frac{1}{6}$. If we take three terms, we have $1\frac{4}{9}$, and there will still be wanting $\frac{1}{18}$. If we take four terms, we shall have $1\frac{13}{27}$, and the difference will be $\frac{1}{54}$; since, therefore, the error always becomes three times less, it must evidently vanish at last.

296. Suppose $a = \frac{2}{3}$; we shall have $\frac{1}{1-a} = \frac{1}{1-\frac{2}{3}} = 3$, $= 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243}$, &c. to infinity; and here, by taking first $1\frac{2}{3}$, the error is $1\frac{1}{3}$; taking three terms, which make $2\frac{1}{9}$, the error is $\frac{8}{9}$; taking four terms, we have $2\frac{11}{27}$, and the error is $\frac{16}{27}$.

297. If $a = \frac{1}{4}$, the fraction is $\frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = 1\frac{1}{3}$; and the series becomes $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}$, &c. The first two terms are equal to $1\frac{1}{4}$, which gives $\frac{1}{12}$ for the error; and taking one term more, we have $1\frac{5}{16}$, that is to say, only an error of $\frac{1}{48}$.

298. In the same manner we may resolve the fraction $\frac{1}{1+a}$, into an infinite series by actually dividing the numerator 1 by the denominator $1 + a$, as follows*.

* After a certain number of terms have been obtained, the law by which the following terms are formed will be evident; so that the series may be carried to any length without the trouble of continual division, as is shewn in this example.

$$
\begin{array}{l}
1+a)\ 1\ (1-a+a^2-a^3+a^4 \\
\quad 1+a \\
\hline
\quad -a \\
\quad -a-a^2 \\
\hline
\qquad a^2 \\
\qquad a^2+a^3 \\
\hline
\qquad\quad -a^3 \\
\qquad\quad -a^3-a^4 \\
\hline
\qquad\qquad a^4 \\
\qquad\qquad a^4+a^5 \\
\hline
\qquad\qquad\quad -a^5, \text{ \&c.}
\end{array}
$$

Whence it follows, that the fraction $\frac{1}{1+a}$ is equal to the series, $1 - a + a^2 - a^3 + a^4 - a^5 + a^6 - a^7$, &c.

299. If we make $a = 1$, we have this remarkable comparison:

$\frac{1}{1+a} = \frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1$, &c. to infinity; which appears rather contradictory; for if we stop at -1, the series gives 0; and if we finish at $+1$, it gives 1; but this is precisely what solves the difficulty; for since we must go on to infinity, without stopping either at -1 or at $+1$, it is evident, that the sum can neither be 0 nor 1, but that this result must lie between these two, and therefore be $\frac{1}{2}$*.

300. Let us now make $a = \frac{1}{2}$, and our fraction will be $\frac{1}{1+\frac{1}{2}} = \frac{2}{3}$, which must therefore express the value of the series $1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}$, &c. to infinity; here if we take only the two leading terms of this series, we have $\frac{1}{2}$, which is too small by $\frac{1}{6}$; if we take three terms, we have $\frac{3}{4}$, which is too much by $\frac{1}{12}$; if we take four terms, we have $\frac{5}{8}$, which is too small by $\frac{1}{24}$, &c.

* It may be observed, that no infinite series is in reality equal to the fraction from which it is derived, unless the remainder be considered, which, in the present case, is alternately $+\frac{1}{2}$ and $-\frac{1}{2}$; that is, $+\frac{1}{2}$ when the series is 0, and $-\frac{1}{2}$ when the series is 1, which still gives the same value for the whole expression. Vid. Art. 293.

301. Suppose again $a = \frac{1}{3}$, our fraction will then be $= \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$, which must be equal to this series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \frac{1}{729}$, &c. continued to infinity. Now, by considering only two terms, we have $\frac{2}{3}$, which is too small by $\frac{1}{12}$; three terms make $\frac{7}{9}$, which is too much by $\frac{1}{36}$; four terms give $\frac{20}{27}$, which is too small by $\frac{1}{108}$, and so on.

302. The fraction $\frac{1}{1+a}$ may also be resolved into an infinite series another way; namely, by dividing 1 by $a + 1$, as follows:

$$a + 1)\ 1\ \ (\frac{1}{a} - \frac{1}{a^2} + \frac{1}{a^3}, \text{ \&c.}$$

$$1 + \frac{1}{a}$$

$$-\frac{1}{a}$$

$$-\frac{1}{a} - \frac{1}{a^2}$$

$$\frac{1}{a^2}$$

$$\frac{1}{a^2} + \frac{1}{a^3}$$

$$-\frac{1}{a^3}, \text{ \&c.*}$$

Consequently, our fraction $\frac{1}{a+1}$, is equal to the infinite series $\frac{1}{a} - \frac{1}{a^2} + \frac{1}{a^3} - \frac{1}{a^4} + \frac{1}{a^5} - \frac{1}{a^6}$, &c. Let us make $a = 1$, and we shall have the series $1 - 1 + 1 - 1 + 1 - 1$, &c. $= \frac{1}{2}$, as before: and if we suppose $a = 2$, we shall have the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64}$, &c. $= \frac{1}{3}$.

* It is unnecessary to carry the actual division any farther, as the series may be continued to any length, from the law observable in the terms already obtained; for the signs are alternately *plus* and *minus*, and any subsequent term may be obtained by multiplying that immediately preceding it by $\frac{1}{a}$.

303. In the same manner, by resolving the general fraction $\frac{c}{a+b}$ into an infinite series, we shall have,

$$a + b)\ c\ \ (\frac{c}{a} - \frac{bc}{a^2} + \frac{b^2c}{a^3} - \frac{b^3c}{a^4}\ *$$

$$c + \frac{bc}{a}$$

$$-\frac{bc}{a}$$

$$-\frac{bc}{a} - \frac{b^2c}{a^2}$$

$$\frac{b^2c}{a^2}$$

$$\frac{b^2c}{a^2} + \frac{b^3c}{a^3}$$

$$-\frac{b^3c}{a^3}$$

Whence it appears, that we may compare $\frac{c}{a+b}$ with the series $\frac{c}{a} - \frac{bc}{a^2} + \frac{b^2c}{a^3} - \frac{b^3c}{a^4}$, &c. to infinity.

Let $a = 2$, $b = 4$, $c = 3$, and we shall have

$$\frac{c}{a+b} = \frac{3}{2+4} = \frac{3}{6} = \frac{1}{2} = \frac{3}{2} - 3 + 6 - 12, \text{ \&c.}$$

If $a = 10$, $b = 1$, and $c = 11$, we shall have

$$\frac{c}{a+b} = \frac{11}{10+1} = 1 = \frac{11}{10} - \frac{11}{100} + \frac{11}{1000} - \frac{11}{10000}, \text{ \&c.}$$

Here if we consider only one term of the series, we have $\frac{11}{10}$, which is too much by $\frac{1}{10}$; if we take two terms, we have $\frac{99}{100}$, which is too small by $\frac{1}{100}$; if we take three terms, we have $\frac{1001}{1000}$, which is too much by $\frac{1}{1000}$, &c.

304. When there are more than two terms in the divisor, we may also continue the division to infinity in the same

* Here again the law of continuation is manifest; the signs being alternately + and −, and each succeeding term is formed by multiplying the foregoing one by $\frac{b}{a}$.

manner. Thus, if the fraction $\frac{1}{1-a+a^2}$ were proposed, the infinite series, to which it is equal, will be found as follows:

$$1-a+a^2)\ 1\ *\ *(1+a,\ \&c.$$
$$1-a+a^2$$
$$a-a^2$$
$$a-a^2+a^3$$
$$-a^3$$
$$-a^3+a^4-a^5$$
$$-a^4+a^5$$
$$-a^4+a^5-a^6$$
$$a^6$$
$$a^6-a^7+a^8$$
$$a^7-a^8$$
$$a^7-a^8+a^9$$
$$-a^9$$

We have therefore the equation $\frac{1}{1-a+a^2} = 1 + a - a^3 - a^4 + a^6 + a^7$, &c.; where, if we make $a = 1$, we have $1 = 1 + 1 - 1 - 1 + 1 + 1 - 1 - 1$, &c. which series contains twice the series found above $1 - 1 + 1 - 1 + 1$, &c. Now, as we have found this to be $\frac{1}{2}$, it is not extraordinary that we should find $\frac{2}{2}$, or 1, for the value of that which we have just determined.

By making $a = \frac{1}{2}$, we shall have the equation $\frac{1}{\frac{3}{4}} = \frac{4}{3} = 1 + \frac{1}{2} - \frac{1}{8} - \frac{1}{16} + \frac{1}{64} + \frac{1}{128} - \frac{1}{512}$, &c.

If $a = \frac{1}{3}$, we shall have the equation $\frac{1}{\frac{7}{9}} = \frac{9}{7} = 1 + \frac{1}{3} - \frac{1}{27} - \frac{1}{81} + \frac{1}{729}$, &c. and if we take the four leading terms of this series, we have $\frac{104}{81}$, which is only $\frac{1}{567}$ less than $\frac{9}{7}$.

Suppose again $a = \frac{2}{3}$, we shall have $\frac{1}{\frac{7}{9}} = \frac{9}{7} = 1 + \frac{2}{3} - \frac{8}{27} - \frac{16}{81} + \frac{64}{729}$, &c. This series is therefore equal to the preceding one; and, by subtracting one from the other, we obtain $\frac{1}{3} - \frac{7}{27} - \frac{15}{81} + \frac{63}{729}$, &c. which is necessarily $=0$.

305. The method, which we have here explained, serves to resolve, generally, all fractions into infinite series; which is often found to be of the greatest utility. It is also remarkable, that an infinite series, though it never ceases, may have a determinate value. It should likewise be observed, that, from this branch of mathematics, inventions of the utmost importance have been derived; on which account the subject deserves to be studied with the greatest attention.

QUESTIONS FOR PRACTICE.

1. Resolve $\frac{ax}{a-x}$ into an infinite series.

Ans. $x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3}$, &c.

2. Resolve $\frac{b}{a+x}$ into an infinite series.

Ans. $\frac{b}{a} \times (1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} +$, &c.)

3. Resolve $\frac{a^2}{x+b}$ into an infinite series.

Ans. $\frac{a^2}{x} \times (1 - \frac{b}{x} + \frac{b^2}{x^2} - \frac{b^3}{x^3} +$, &c.)

4. Resolve $\frac{1+x}{1-x}$ into an infinite series.

Ans. $1 + 2x + 2x^2 + 2x^3 + 2x^4$, &c.

5. Resolve $\frac{a^2}{(a+x)^2}$ into an infinite series.

Ans. $1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3}$, &c.

CHAP. VI.

Of the Squares *of* Compound Quantities.

306. When it is required to find the square of a compound quantity, we have only to multiply it by itself, and the product will be the square required.

For example, the square of $a + b$ is found in the following manner:

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad\;\; ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

307. So that when the root consists of two terms added together, as $a+b$, the square comprehends, 1st, the squares of each term, namely, a^2 and b^2; and 2dly, twice the product of the two terms, namely, $2ab$: so that the sum $a^2+2ab+b^2$ is the square of $a+b$. Let, for example, $a=10$, and $b=3$; that is to say, let it be required to find the square of $10+3$, or 13, and we shall have $100+60+9$, or 169.

308. We may easily find, by means of this formula, the squares of numbers, however great, if we divide them into two parts. Thus, for example, the square of 57, if we consider that this number is the same as $50+7$, will be found $=2500+700+49=3249$.

309. Hence it is evident, that the square of $a+1$ will be a^2+2a+1: and since the square of a is a^2, we find the square of $a+1$ by adding to that square $2a+1$; and it must be observed, that this $2a+1$ is the sum of the two roots a and $a+1$.

Thus, as the square of 10 is 100, that of 11 will be $100+21$: the square of 57 being 3249, that of 58 is $3249+115=3364$; the square of $59=3364+117=3481$; the square of $60=3481+119=3600$, &c.

310. The square of a compound quantity, as $a+b$, is represented in this manner $(a+b)^2$. We have therefore $(a+b)^2=a^2+2ab+b^2$, whence we deduce the following equations:

$(a+1)^2=a^2+2a+1$; $\quad (a+2)^2=a^2+4a+4$;
$(a+3)^2=a^2+6a+9$; $\quad (a+4)^2=a^2+8a+16$; &c.

311. If the root be $a-b$, the square of it is $a^2-2ab+b^2$, which contains also the squares of the two terms, but in such a manner, that we must take from their sum twice the product of those two terms. Let, for example, $a=10$, and $b=-1$, then the square of 9 will be found equal to $100-20+1=81$.

312. Since we have the equation $(a-b)^2=a^2-2ab+b^2$, we shall have $(a-1)^2=a^2-2a+1$. The square of $a-1$ is found, therefore, by subtracting from a^2 the sum of the two roots a and $a-1$, namely, $2a-1$. Thus, for example, if $a=50$, we have $a^2=2500$, and $2a-1=99$; therefore $49^2=2500-99=2401$.

313. What we have said here may be also confirmed and illustrated by fractions; for if we take as the root $\frac{3}{5}+\frac{2}{5}=1$, the square will be, $\frac{9}{25}+\frac{4}{25}+\frac{12}{25}=\frac{25}{25}=1$.

Farther, the square of $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ will be $\frac{1}{4}-\frac{1}{3}+\frac{1}{9}=\frac{1}{36}$.

314. When the root consists of a greater number of terms, the method of determining the square is the same. Let us find, for example, the square of $a+b+c$:

$$\begin{array}{l} a+b+c \\ a+b+c \\ \hline a^2+ab+ac \\ \quad\;\; ab+b^2+bc \\ \quad\quad\quad\; ac+bc+c^2 \\ \hline a^2+2ab+2ac+b^2+2bc+c^2 \end{array}$$

We see that it contains, first, the square of each term of the root, and beside that, the double products of those terms multiplied two by two.

315. To illustrate this by an example, let us divide the number 256 into three parts, $200+50+6$; its square will then be composed of the following parts:

$$\begin{array}{rl} 200^2 = & 40000 \\ 50^2 = & 2500 \\ 6^2 = & 36 \\ 2\,(50\times 200) = & 20000 \\ 2\,(\;6\times 200) = & 2400 \\ 2\,(\;6\times\;\, 50) = & 600 \\ \hline & 65536 = 256\times 256, \text{ or } 256^2. \end{array}$$

316. When some terms of the root are negative, the square is still found by the same rule; only we must be careful what signs we prefix to the double products. Thus, $(a-b-c)^2=a^2+b^2+c^2-2ab-2ac+2bc$; and if we represent the number 256 by $300-40-4$, we shall have,

Positive Parts.		Negative Parts.	
$300^2 =$	90000	$2(40\times300) =$	24000
$40^2 =$	1600	$2(\ 4\times300) =$	2400
$2(40\times4) =$	320		-26400
$4^2 =$	16		
	91936		
	-26400		

65536, the square of 256 as before.

CHAP. VII.

Of the Extraction *of* Roots *applied to* Compound Quantities.

317. In order to give a certain rule for this operation, we must consider attentively the square of the root $a+b$, which is $a^2+2ab+b^2$, in order that we may reciprocally find the root of a given square.

318. We must consider therefore, first, that as the square, $a^2+2ab+b^2$, is composed of several terms, it is certain that the root also will comprise more than one term; and that if we write the terms of the square in such a manner, that the powers of one of the letters, as a, may go on continually diminishing, the first term will be the square of the first term of the root; and since, in the present case, the first term of the square is a^2, it is certain that the first term of the root is a.

319. Having therefore found the first term of the root, that is to say, a, we must consider the rest of the square, namely, $2ab+b^2$, to see if we can derive from it the second part of the root, which is b. Now, this remainder, $2ab+b^2$, may be represented by the product, $(2a+b)b$; wherefore the remainder having two factors, $(2a+b)$, and b, it is evident that we shall find the latter, b, which is the second part of the root, by dividing the remainder, $2ab+b^2$, by $2a+b$.

320. So that the quotient, arising from the division of the above remainder by $2a+b$, is the second term of the root required; and in this division we observe, that $2a$ is the double of the first term a, which is already determined: so that although the second term is yet unknown, and it is necessary, for the present, to leave its place empty, we may nevertheless attempt the division, since in it we attend only

to the first term $2a$; but as soon as the quotient is found, which in the present case is b, we must put it in the vacant place, and thus render the division complete.

321. The calculation, therefore, by which we find the root of the square $a^2+2ab+b^2$, may be represented thus:

$$\begin{array}{r|l} & a^2+2ab+b^2\ (a+b \\ & a^2 \\ \hline 2a+b) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline & 0. \end{array}$$

322. We may, also, in the same manner, find the square root of other compound quantities, provided they are squares, as will appear from the following examples:

$$\begin{array}{r|l} & a^2+6ab+9b^2\ (a+3b \\ & a^2 \\ \hline 2a+3b) & 6ab+9b^2 \\ & 6ab+9b^2 \\ \hline & 0. \end{array}$$

$$\begin{array}{r|l} & 4a^2-4ab+b^2\ (2a-b \\ & 4a^2 \\ \hline 4a-b) & -4ab+b^2 \\ & -4ab+b^2 \\ \hline & 0. \end{array}$$

$$\begin{array}{r|l} & 9p^2+24pq+16q^2\ (3p+4q \\ & 9p^2 \\ \hline 6p+4q) & 24pq+16q^2 \\ & 24pq+16q^2 \\ \hline & 0. \end{array}$$

$$\begin{array}{r|l} & 25x^2-60x+36\ (5x-6 \\ & 25x^2 \\ \hline 10x-6) & -60x+36 \\ & -60x+36 \\ \hline & 0. \end{array}$$

323. When there is a remainder after the division, it is a proof that the root is composed of more than two terms. We must in that case consider the two terms already found as forming the first part, and endeavour to derive the other from the remainder, in the same manner as we found the second term of the root from the first. The following examples will render this operation more clear.

$$\begin{array}{rl}
 & a^2+2ab-2ac-2bc+b^2+c^2\ (a+b-c \\
 & a^2 \\ \hline
2a+b) & 2ab-2ac-2bc+b^2+c^2 \\
 & 2ab \qquad\qquad +b^2 \\ \hline
2a+2b-c) & -2ac-2bc+c^2 \\
 & -2ac-2bc+c^2 \\ \hline
 & 0.
\end{array}$$

$$\begin{array}{rl}
 & a^4+2a^3+3a^2+2a+1\ (a^2+a+1 \\
 & a^4 \\ \hline
2a^2+a) & 2a^3+3a^2 \\
 & 2a^3+\ a^2 \\ \hline
2a^2+2a+1) & 2a^2+2a+1 \\
 & 2a^2+2a+1 \\ \hline
 & 0.
\end{array}$$

$$\begin{array}{rl}
 & a^4-4a^3b+8ab^3+4b^4\ (a^2-2ab-2b^2 \\
 & a^4 \\ \hline
2a^2-2ab) & -4a^3b+8ab^3+4b^4 \\
 & -4a^3b+4a^2b^2 \\ \hline
2a^2-4ab-2b^2) & -4a^2b^2+8ab^3+4b^4 \\
 & -4a^2b^2+8ab^3+4b^4 \\ \hline
 & 0.
\end{array}$$

$$\begin{array}{rl}
 & a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\ (a^3-3a^2b+3ab^2-b^3 \\
 & a^6 \\ \hline
2a^3-3a^2b) & -6a^5b+15a^4b^2 \\
 & -6a^5b+\ 9a^4b^2 \\ \hline
2a^3-6a^2b+3ab^2) & 6a^4b^2-20a^3b^3+15a^2b^4 \\
 & 6a^4b^2-18a^3b^3+\ 9a^2b^4 \\ \hline
2a^3-6a^2b+6ab^2-b^3) & -2a^3b^3+6a^2b^4-6ab^5+b^6 \\
 & -2a^3b^3+6a^2b^4-6ab^5+b^6 \\ \hline
 & 0.
\end{array}$$

324. We easily deduce from the rule which we have explained, the method which is taught in books of arithmetic for the extraction of the square root, as will appear from the following examples in numbers:

$$\begin{array}{rl}
 & 529\ (23 \\
 & 4 \\ \hline
43) & 129 \\
 & 129 \\ \hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & \dot{2}30\dot{4}\ (48 \\
 & 16 \\ \hline
88) & 704 \\
 & 704 \\ \hline
 & 0.
\end{array}$$

$$\begin{array}{rl}
 & 4\dot{0}9\dot{6}\ (64 \\
 & 36 \\ \hline
124) & 496 \\
 & 496 \\ \hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & 9\dot{6}0\dot{4}\ (98 \\
 & 81 \\ \hline
188) & 1504 \\
 & 1504 \\ \hline
 & 0.
\end{array}$$

$$\begin{array}{rl}
 & \dot{1}5\dot{6}2\dot{5}\ (125 \\
 & 1 \\ \hline
22) & 56 \\
 & 44 \\ \hline
245) & 1225 \\
 & 1225 \\ \hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & 99\dot{8}00\dot{1}\ (999 \\
 & 81 \\ \hline
189) & 1880 \\
 & 1701 \\ \hline
1989) & 17901 \\
 & 17901 \\ \hline
 & 0.
\end{array}$$

325. But when there is a remainder after all the figures have been used, it is a proof that the number proposed is

not a square; and, consequently, that its root cannot be assigned. In such cases, the radical sign, which we before employed, is made use of. This is written before the quantity, and the quantity itself is placed between parentheses, or under a line: thus, the square root of $a^2 + b^2$ is represented by $\sqrt{(a^2 + b^2)}$, or by $\sqrt{\overline{a^2 + b^2}}$; and $\sqrt{(1 - x^2)}$, or $\sqrt{\overline{1 - x^2}}$, expresses the square root of $1 - x^2$. Instead of this radical sign, we may use the fractional exponent $\frac{1}{2}$, and represent the square root of $a^2 + b^2$, for instance, by $(a^2 + b^2)^{\frac{1}{2}}$, or by $\overline{a^2 + b^2}|^{\frac{1}{2}}$

CHAP. VIII.

Of the Calculation *of* Irrational Quantities.

326. When it is required to add together two or more irrational quantities, this is to be done, according to the method before laid down, by writing all the terms in succession, each with its proper sign: and, with regard to abbreviations, we must remark that, instead of $\sqrt{a} + \sqrt{a}$, for example, we may write $2\sqrt{a}$; and that $\sqrt{a} - \sqrt{a} = 0$, because these two terms destroy one another. Thus, the quantities $3 + \sqrt{2}$ and $1 + \sqrt{2}$, added together, make $4 + 2\sqrt{2}$, or $4 + \sqrt{8}$; the sum of $5 + \sqrt{3}$ and $4 - \sqrt{3}$, is 9; and that of $2\sqrt{3} + 3\sqrt{2}$ and $\sqrt{3} - \sqrt{2}$, is $3\sqrt{3} + 2\sqrt{2}$.

327. Subtraction also is very easy, since we have only to add the proposed numbers, after having changed their signs; as will be readily seen in the following example, by subtracting the lower line from the upper.

$$\begin{array}{l} 4-\sqrt{2}+2\sqrt{3}-3\sqrt{5}+4\sqrt{6} \\ 1+2\sqrt{2}-2\sqrt{3}-5\sqrt{5}+6\sqrt{6} \\ \hline 3-3\sqrt{2}+4\sqrt{3}+2\sqrt{5}-2\sqrt{6}. \end{array}$$

328. In multiplication, we must recollect that $\sqrt{a}$ multiplied by $\sqrt{a}$ produces a; and that if the numbers which follow the sign $\sqrt{}$ are different, as a and b, we have $\sqrt{ab}$ for the product of $\sqrt{a}$ multiplied by $\sqrt{b}$. After this, it will be easy to calculate the following examples:

$$\begin{array}{l} 1+\sqrt{2} \\ 1+\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad\sqrt{2}+2 \\ \hline 1+2\sqrt{2}+2=3+2\sqrt{2}. \end{array} \qquad \begin{array}{l} 4+2\sqrt{2} \\ 2-\sqrt{2} \\ \hline 8+4\sqrt{2} \\ \quad -4\sqrt{2}-4 \\ \hline 8-4=4. \end{array}$$

329. What we have said applies also to imaginary quantities; we shall only observe farther, that $\sqrt{-a}$ multiplied by $\sqrt{-a}$ produces $-a$. If it were required to find the cube of $-1+\sqrt{-3}$, we should take the square of that number, and then multiply that square by the same number; as in the following operation:

$$\begin{array}{l} -1+\sqrt{-3} \\ -1+\sqrt{-3} \\ \hline 1-\sqrt{-3} \\ \quad -\sqrt{-3}-3 \\ \hline 1-2\sqrt{-3}-3=-2-2\sqrt{-3} \\ \qquad\qquad\qquad -1+\sqrt{-3} \\ \hline \qquad\qquad\qquad 2+2\sqrt{-3} \\ \qquad\qquad\qquad\quad -2\sqrt{-3}+6 \\ \hline \qquad\qquad\qquad 2+6=8. \end{array}$$

330. In the division of surds, we have only to express the proposed quantities in the form of a fraction; which may be then changed into another expression having a rational denominator; for if the denominator be $a+\sqrt{b}$, for example, and we multiply both this and the numerator by $a-\sqrt{b}$, the new denominator will be $a^2 - b$, in which there is no radical sign. Let it be proposed, for example, to divide $3 + 2\sqrt{2}$ by $1+\sqrt{2}$: we shall first have $\frac{3+2\sqrt{2}}{1+\sqrt{2}}$; then multiplying the two terms of the fraction by $1-\sqrt{2}$, we shall have for the numerator:

$$\begin{array}{l} 3+2\sqrt{2} \\ 1-\sqrt{2} \\ \hline 3+2\sqrt{2} \\ \quad -3\sqrt{2}-4 \\ \hline 3-\sqrt{2}-4=-\sqrt{2}-1; \end{array}$$

and for the denominator:

$$\begin{array}{l} 1+\sqrt{2} \\ 1-\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad -\sqrt{2}-2 \\ \hline 1-2=-1. \end{array}$$

Our new fraction therefore is $\frac{-\sqrt{2}-1}{-1}$; and if we again multiply the two terms by -1, we shall have for the numerator $\sqrt{2}+1$, and for the denominator $+1$. Now, it is easy to shew that $\sqrt{2}+1$ is equal to the proposed fraction $\frac{3+2\sqrt{2}}{1+\sqrt{2}}$; for $\sqrt{2}+1$ being multiplied by the divisor $1+\sqrt{2}$, thus,

$$\begin{array}{l} 1+\sqrt{2} \\ 1+\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad \sqrt{2}+2 \\ \hline \end{array}$$

we have $1+2\sqrt{2}+2=3+2\sqrt{2}$.

Another example. Let $8-5\sqrt{2}$ be divided by $3-2\sqrt{2}$. This, in the first instance, is $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$; and multiplying the two terms of this fraction by $3+2\sqrt{2}$, we have for the numerator,

$$\begin{array}{l} 8-5\sqrt{2} \\ 3+2\sqrt{2} \\ \hline 24-15\sqrt{2} \\ \quad 16\sqrt{2}-20 \\ \hline 24+\sqrt{2}-20=4+\sqrt{2}; \end{array}$$

and for the denominator,

$$\begin{array}{l} 3-2\sqrt{2} \\ 3+2\sqrt{2} \\ \hline 9-6\sqrt{2} \\ \quad 6\sqrt{2}-8 \\ \hline 9-8=1. \end{array}$$

Consequently, the quotient will be $4+\sqrt{2}$. The truth of this may be proved, as before, by multiplication; thus,

$$\begin{array}{l} 4+\sqrt{2} \\ 3-2\sqrt{2} \\ \hline 12+3\sqrt{2} \\ \quad -8\sqrt{2}-4 \\ \hline 12-5\sqrt{2}-4=8-5\sqrt{2}. \end{array}$$

331. In the same manner, we may transform irrational fractions into others, that have rational denominators. If we have, for example, the fraction $\frac{1}{5-2\sqrt{6}}$, and multiply its numerator and denominator by $5+2\sqrt{6}$, we transform it into this, $\frac{5+2\sqrt{6}}{1}=5+2\sqrt{6}$; in like manner, the fraction $\frac{2}{-1+\sqrt{-3}}$ assumes this form, $\frac{2+2\sqrt{-3}}{-4}=\frac{1+\sqrt{-3}}{-2}$; also $\frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}-\sqrt{5}}=\frac{11+2\sqrt{30}}{1}=11+2\sqrt{30}$.

332. When the denominator contains several terms, we may, in the same manner, make the radical signs in it vanish one by one. Thus, if the fraction $\frac{1}{\sqrt{10}-\sqrt{2}-\sqrt{3}}$ be proposed, we first multiply these two terms by $\sqrt{10}+\sqrt{2}+\sqrt{3}$, and obtain the fraction $\frac{\sqrt{10}+\sqrt{2}+\sqrt{3}}{5-2\sqrt{6}}$; then multiplying its numerator and denominator by $5+2\sqrt{6}$, we have $5\sqrt{10}+11\sqrt{2}+9\sqrt{3}+2\sqrt{60}$.

CHAP. IX.

Of Cubes, *and of the* Extraction *of* Cube Roots.

333. To find the cube of $a+b$, we have only to multiply its square, $a^2+2ab+b^2$, again by $a+b$, thus;

$$\begin{array}{l} a^2+2ab+b^2 \\ a+b \\ \hline a^3+2a^2b+ab^2 \\ \quad a^2b+2ab^2+b^3 \\ \hline \end{array}$$

and the cube will be $a^3+3a^2b+3ab^2+b^3$.

We see therefore that it contains the cubes of the two parts of the root, and, beside that, $3a^2b + 3ab^2$; which quantity is equal to $(3ab) \times (a + b)$; that is, the triple product of the two parts, a and b, multiplied by their sum.

334. So that whenever a root is composed of two terms, it is easy to find its cube by this rule: for example, the number $5 = 3 + 2$; its cube is therefore $27 + 8 + (18 \times 5) = 125$.

And if $7 + 3 = 10$ be the root; then the cube will be $343 + 27 + (63 \times 10) = 1000$.

To find the cube of 36, let us suppose the root $36 = 30 + 6$, and we have for the cube required, $27000 + 216 + (540 \times 36) = 46656$.

335. But if, on the other hand, the cube be given, namely, $a^3 + 3a^2b + 3ab^2 + b^3$, and it be required to find its root, we must premise the following remarks:

First, when the cube is arranged according to the powers of one letter, we easily know by the leading term a^3, the first term a of the root, since the cube of it is a^3; if, therefore, we subtract that cube from the cube proposed, we obtain the remainder, $3a^2b + 3ab^2 + b^3$, which must furnish the second term of the root.

336. But as we already know, from Art. 333, that the second term is $+b$, we have principally to discover how it may be derived from the above remainder. Now, that remainder may be expressed by two factors, thus, $(3a^2 + 3ab + b^2) \times (b)$; if, therefore, we divide by $3a^2 + 3ab + b^2$, we obtain the second part of the root $+b$, which is required.

337. But as this second term is supposed to be unknown, the divisor also is unknown; nevertheless we have the first term of that divisor, which is sufficient: for it is $3a^2$, that is, thrice the square of the first term already found; and by means of this, it is not difficult to find also the other part, b, and then to complete the divisor before we perform the division; for this purpose, it will be necessary to join to $3a^2$ thrice the product of the two terms, or $3ab$, and b^2, or the square of the second term of the root.

338. Let us apply what we have said to two examples of other given cubes.

$$\begin{array}{r|l} & a^3+12a^2+48a+64\ (a+4 \\ & a^3 \\ \hline 3a^2+12a+16) & 12a^2+48a+64 \\ & 12a^2+48a+64 \\ \hline & 0. \end{array}$$

$$\begin{array}{r|l} & a^6-6a^5+15a^4-20a^3+15a^2-6a+1\ (a^2-2a+1 \\ & a^6 \\ \hline 3a^4-6a^3+4a^2) & -6a^5+15a^4-20a^3 \\ & -6a^5+12a^4-8a^3 \\ \hline 3a^4-12a^3+12a^2+3a^2-6a+1) & 3a^4-12a^3+15a^2-6a+1 \\ & 3a^4-12a^3+15a^2-6a+1 \\ \hline & 0. \end{array}$$

339. The analysis which we have given is the foundation of the common rule for the extraction of the cube root in numbers. See the following example of the operation in the number 2197:

$$\begin{array}{r|l} & 2197\ (10 + 3 = 13 \\ & 1000 \\ \hline 300 & 1197 \\ 90 & \\ 9 & \\ \hline 399 & 1197 \\ \hline & 0. \end{array}$$

Let us also extract the cube root of 34965783:

$$\begin{array}{r|l} & 34965783\ (300 + 20 + 7, \text{ or } 327 \\ & 27000000 \\ \hline 270000 & 7965783 \\ 18000 & \\ 400 & \\ \hline 288400 & 5768000 \\ \hline 307200 & 2197783 \\ 6720 & \\ 49 & \\ \hline 313969 & 2197783 \\ \hline & 0. \end{array}$$

CHAP. X.

Of the higher Powers *of* Compound Quantities.

340. After squares and cubes, we must consider higher powers, or powers of a greater number of degrees; which are generally represented by exponents in the manner which we before explained: we have only to remember, when the root is compound, to enclose it in a parenthesis: thus, $(a+b)^5$ means that $a+b$ is raised to the fifth power, and $(a-b)^6$ represents the sixth power of $a-b$, and so on. We shall in this chapter explain the nature of these powers.

341. Let $a+b$ be the root, or the first power, and the higher powers will be found, by multiplication, in the following manner:

$$
\begin{array}{l}
(a+b)^1 = a+b \\
\qquad\quad a+b \\
\hline
\qquad\quad a^2+ab \\
\qquad\qquad\quad ab+b^2 \\
\hline
(a+b)^2 = a^2+2ab+b^2 \\
\qquad\quad a+b \\
\hline
\qquad\quad a^3+2a^2b+ab^2 \\
\qquad\qquad\quad a^2b+2ab^2+b^3 \\
\hline
(a+b)^3 = a^3+3a^2b+3ab^2+b^3 \\
\qquad\quad a+b \\
\hline
\qquad\quad a^4+3a^3b+3a^2b^2+ab^3 \\
\qquad\qquad\quad a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
\qquad\quad a+b \\
\hline
\qquad\quad a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
\qquad\qquad\quad a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\
\hline
\qquad\quad a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{array}
$$

$$
\begin{array}{l}
(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\
\qquad\quad a+b \\
\hline
\qquad\quad a^6+5a^5b+10a^4b^2+10a^3b^3+5a^2b^4+ab^5 \\
\qquad\qquad\quad a^5b+5a^4b^2+10a^3b^3+10a^2b^4+5ab^5+b^6 \\
\hline
(a+b)^6 = a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6, \text{ \&c.}
\end{array}
$$

342. The powers of the root $a-b$ are found in the same manner; and we shall immediately perceive that they do not differ from the preceding, excepting that the 2d, 4th, 6th, &c. terms are affected by the sign *minus*.

$$
\begin{array}{l}
(a-b)^1 = a-b \\
\qquad\quad a-b \\
\hline
\qquad\quad a^2-ab \\
\qquad\qquad\quad -ab+b^2 \\
\hline
(a-b)^2 = a^2-2ab+b^2 \\
\qquad\quad a-b \\
\hline
\qquad\quad a^3-2a^2b+ab^2 \\
\qquad\qquad\quad -a^2b+2ab^2-b^3 \\
\hline
(a-b)^3 = a^3-3a^2b+3ab^2-b^3 \\
\qquad\quad a-b \\
\hline
\qquad\quad a^4-3a^3b+3a^2b^2-ab^3 \\
\qquad\qquad\quad -a^3b+3a^2b^2-3ab^3+b^4 \\
\hline
(a-b)^4 = a^4-4a^3b+6a^2b^2-4ab^3+b^4 \\
\qquad\quad a-b \\
\hline
\qquad\quad a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4 \\
\qquad\qquad\quad -a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5 \\
\hline
(a-b)^5 = a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5 \\
\qquad\quad a-b \\
\hline
\qquad\quad a^6-5a^5b+10a^4b^2-10a^3b^3+5a^2b^4-ab^5 \\
\qquad\qquad\quad -a^5b+5a^4b^2-10a^3b^3+10a^2b^4-5ab^5+b^6 \\
\hline
(a-b)^6 = a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6, \text{ \&c.}
\end{array}
$$

Here we see that all the odd powers of b have the sign $-$, while the even powers retain the sign $+$. The reason of this is evident; for since $-b$ is a term of the root, the powers of that letter will ascend in the following series, $-b$, $+b^2$, $-b^3$, $+b^4$, $-b^5$, $+b^6$, &c. which clearly shews that the even powers must be affected by the sign $+$, and the odd ones by the contrary sign $-$.

343. An important question occurs in this place; namely, how we may find, without being obliged to perform the same calculation, all the powers either of $a+b$, or $a-b$.

We must remark, in the first place, that if we can assign all the powers of $a+b$, those of $a-b$ are also found; since we have only to change the signs of the even terms, that is to say, of the second, the fourth, the sixth, &c. The business then is to establish a rule, by which any power of $a+b$, however high, may be determined without the necessity of calculating all the preceding powers.

344. Now, if from the powers which we have already determined we take away the numbers that precede each term, which are called the *coefficients*, we observe in all the terms a singular order: first, we see the first term a of the root raised to the power which is required; in the following terms, the powers of a diminish continually by unity, and the powers of b increase in the same proportion; so that the sum of the exponents of a and of b is always the same, and always equal to the exponent of the power required; and, lastly, we find the term b by itself raised to the same power. If therefore the tenth power of $a+b$ were required, we are certain that the terms, without their coefficients, would succeed each other in the following order; a^{10}, a^9b, a^8b^2, a^7b^3, a^6b^4, a^5b^5, a^4b^6, a^3b^7, a^2b^8, ab^9, b^{10}.

345. It remains therefore to shew how we are to determine the coefficients, which belong to those terms, or the numbers by which they are to be multiplied. Now, with respect to the first term, its coefficient is always unity; and, as to the second, its coefficient is constantly the exponent of the power. With regard to the other terms, it is not so easy to observe any order in their coefficients; but, if we continue those coefficients, we shall not fail to discover the law by which they are formed; as will appear from the following Table.

Powers.	Coefficients.
1st	1, 1
2d	1, 2, 1
3d	1, 3, 3, 1
4th	1, 4, 6, 4, 1
5th	1, 5, 10, 10, 5, 1
6th	1, 6, 15, 20, 15, 6, 1
7th	1, 7, 21, 35, 35, 21, 7, 1
8th	1, 8, 28, 56, 70, 56, 28, 8, 1
9th	1, 9, 36, 84, 126, 126, 84, 36, 9, 1
10th	1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1, &c.

We see then that the tenth power of $a+b$ will be $a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$.

346. Now, with regard to the coefficients, it must be observed, that for each power their sum must be equal to the number 2 raised to the same power; for let $a=1$ and $b=1$, then each term, without the coefficients, will be 1; consequently the value of the power will be simply the sum of the coefficients. This sum, in the preceding example, is 1024, and accordingly $(1+1)^{10} = 2^{10} = 1024$. It is the same with respect to all other powers; thus, we have for the

1st $1+1=2=2^1$,
2d $1+2+1=4=2^2$,
3d $1+3+3+1=8=2^3$,
4th $1+4+6+4+1=16=2^4$,
5th $1+5+10+10+5+1=32=2^5$,
6th $1+6+15+20+15+6+1=64=2^6$,
7th $1+7+21+35+35+21+7+1=128=2^7$, &c.

347. Another necessary remark, with regard to the coefficients, is, that they increase from the beginning to the middle, and then decrease in the same order. In the even powers, the greatest coefficient is exactly in the middle; but in the odd powers, two coefficients, equal and greater than the others, are found in the middle, belonging to the mean terms.

The order of the coefficients likewise deserves particular attention; for it is in this order that we discover the means of determining them for any power whatever, without calculating all the preceding powers. We shall here explain this method, reserving the demonstration however for the next chapter.

348. In order to find the coefficients of any power proposed, the seventh for example, let us write the following fractions one after the other:

$$\tfrac{7}{1},\ \tfrac{6}{2},\ \tfrac{5}{3},\ \tfrac{4}{4},\ \tfrac{3}{5},\ \tfrac{2}{6},\ \tfrac{1}{7}.$$

In this arrangement, we perceive that the numerators begin by the exponent of the power required, and that they diminish successively by unity; while the denominators follow in the natural order of the numbers, 1, 2, 3, 4, &c. Now, the first coefficient being always 1, the first fraction gives the second coefficient; the product of the first two fractions, multiplied together, represents the third coefficient; the product of the three first fractions represents the fourth coefficient, and so on. Thus, the

1st coefficient is	1	$= 1$
2d - - - -	$\frac{7}{1}$	$= 7$
3d - - - -	$\frac{7.6}{1.2}$	$= 21$
4th - - - -	$\frac{7.6.5}{1.2.3}$	$= 35$
5th - - - -	$\frac{7.6.5.4}{1.2.3.4}$	$= 35$
6th - - - -	$\frac{7.6.5.4.3}{1.2.3.4.5}$	$= 21$
7th - - - -	$\frac{7.6.5.4.3.2}{1.2.3.4.5.6}$	$= 7$
8th - - - -	$\frac{7.6.5.4.3.2.1}{1.2.3.4.5.6.7}$	$= 1$

349. So that we have, for the second power, the fractions $\frac{2}{1}, \frac{1}{2}$; whence the first coefficient is 1, the second $\frac{2}{1} = 2$, and the third $2 \times \frac{1}{2} = 1$.

The third power furnishes the fractions $\frac{3}{1}, \frac{2}{2}, \frac{1}{3}$; wherefore the

1st coefficient $= 1$; 2d $= \frac{3}{1} = 3$;
3d $= 3 \cdot \frac{2}{2} = 3$; and 4th $= \frac{3}{1} \cdot \frac{2}{2} \cdot \frac{1}{3} = 1$.

We have, for the fourth power, the fractions $\frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}$, consequently, the

1st coefficient $= 1$;
2d $\frac{4}{1} = 4$; 3d $\frac{4}{1} \cdot \frac{3}{2} = 6$;
4th $\frac{4}{1} \cdot \frac{3}{2} \cdot \frac{2}{3} = 4$; and 5th $\frac{4}{1} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = 1$.

350. This rule evidently renders it unnecessary to find the coefficients of the preceding powers, as it enables us to discover immediately the coefficients which belong to any one proposed. Thus, for the tenth power, we write the fractions $\frac{10}{1}, \frac{9}{2}, \frac{8}{3}, \frac{7}{4}, \frac{6}{5}, \frac{5}{6}, \frac{4}{7}, \frac{3}{8}, \frac{2}{9}, \frac{1}{10}$, by means of which we find the

1st coefficient $= 1$;
2d $= \frac{10}{1} = 10$; 7th $= 252 \cdot \frac{5}{6} = 210$;
3d $= 10 \cdot \frac{9}{2} = 45$; 8th $= 210 \cdot \frac{4}{7} = 120$;
4th $= 45 \cdot \frac{8}{3} = 120$; 9th $= 120 \cdot \frac{3}{8} = 45$;
5th $= 120 \cdot \frac{7}{4} = 210$; 10th $= 45 \cdot \frac{2}{9} = 10$;
6th $= 210 \cdot \frac{6}{5} = 252$; and 11th $= 10 \cdot \frac{1}{10} = 1$.

351. We may also write these fractions as they are, without computing their value; and in this manner it is easy to express any power of $a + b$. Thus, $(a+b)^{100} = a^{100} + \frac{100}{1} a^{99}b + \frac{100 \cdot 99}{1 \cdot 2} a^{98}b^2 + \frac{100 \cdot 99 \cdot 98}{1 \cdot 2 \cdot 3} a^{97}b^3$ read × $+ \frac{100 \cdot 99 \cdot 98 \cdot 97}{1 \cdot 2 \cdot 3 \cdot 4} a^{96}b^4 +$, &c.* Whence the law of the succeeding terms may be easily deduced.

CHAP. XI.

Of the Transposition *of the* Letters, *on which the demonstration of the preceding* Rule *is founded.*

352. If we trace back the origin of the coefficients which we have been considering, we shall find, that each term is presented, as many times as it is possible to transpose the letters, of which that term is composed; or, to express the same thing differently, the coefficient of each term is equal to the number of transpositions which the letters composing that term admit of. In the second power, for example, the term ab is taken twice, that is to say, its coefficient is 2; and in fact we may change the order of the letters which compose that term twice, since we may write ab and ba.

* Or, which is a more general mode of expression,

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n \cdot (n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n \cdot (n-1) \cdot (n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$$
$$a^{n-4}b^4 \text{ \&c.} - - \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) - - - - 1}{1 \cdot 2 \cdot 3 \cdot \quad 4 - - - - - n}.$$

This elegant theorem for the involution of a compound quantity of two terms, evidently includes all powers whatever; and we shall afterwards shew how the same may be applied to the extraction of roots.

The term aa, on the contrary, is found only once, and here the order of the letters can undergo no change, or transposition. In the third power of $a + b$, the term aab may be written in three different ways; thus, aab, aba, baa; the coefficient therefore is 3. In the fourth power, the term a^3b or $aaab$ admits of four different arrangements, $aaab$, $aaba$, $abaa$, $baaa$; and consequently the coefficient is 4. The term $aabb$ admits of six transpositions, $aabb$, $abba$, $baba$, $abab$, $bbaa$, $baab$, and its coefficient is 6. It is the same in all other cases.

353. In fact, if we consider that the fourth power, for example, of any root consisting of more than two terms, as $(a + b + c + d)^4$, is found by the multiplication of the four factors, $(a + b + c + d)$ $(a + b + c + d)$ $(a + b + c + d)$ $(a + b + c + d)$, we readily see, that each letter of the first factor must be multiplied by each letter of the second, then by each letter of the third, and, lastly, by each letter of the fourth. So that every term is not only composed of four letters, but it also presents itself, or enters into the sum, as many times as those letters can be differently arranged with respect to each other; and hence arises its coefficient.

354. It is therefore of great importance to know, in how many different ways a given number of letters may be arranged; but, in this inquiry, we must particularly consider, whether the letters in question are the same, or different: for when they are the same, there can be no transposition of them; and for this reason the simple powers, as a^2, a^3, a^4, &c. have all unity for their coefficients.

355. Let us first suppose all the letters different; and, beginning with the simplest case of two letters, or ab, we immediately discover that two transpositions may take place, namely, ab and ba.

If we have three letters, abc, to consider, we observe that each of the three may take the first place, while the two others will admit of two transpositions; thus, if a be the first letter, we have two arrangements abc, acb; if b be in the first place, we have the arrangements bac, bca; lastly, if c occupy the first place, we have also two arrangements, namely, cab, cba; consequently the whole number of arrangements is $3 \times 2 = 6$.

If there be four letters $abcd$, each may occupy the first place; and in every case the three others may form six different arrangements, as we have just seen; therefore the whole number of transpositions is $4 \times 6 = 24 = 4 \times 3 \times 2 \times 1$.

If we have five letters, $abcde$, each of the five may be the

first, and the four others will admit of twenty-four transpositions; so that the whole number of transpositions will be $5 \times 24 = 120 = 5 \times 4 \times 3 \times 2 \times 1$.

356. Consequently, however great the number of letters may be, it is evident, provided they are all different, that we may easily determine the number of transpositions, and that we may for this purpose make use of the following Table:

Number of Letters.	Number of Transpositions.
1	$1 = 1$.
2	$2 . 1 = 2$.
3	$3 . 2 . 1 = 6$.
4	$4 . 3 . 2 . 1 = 24$.
5	$5 . 4 . 3 . 2 . 1 = 120$.
6	$6 . 5 . 4 . 3 . 2 . 1 = 720$.
7	$7 . 6 . 5 . 4 . 3 . 2 . 1 = 5040$.
8	$8 . 7 . 6 . 5 . 4 . 3 . 2 . 1 = 40320$.
9	$9 . 8 . 7 . 6 . 5 . 4 . 3 . 2 . 1 = 362880$.
10	$10 . 9 . 8 . 7 . 6 . 5 . 4 . 3 . 2 . 1 = 3628800$.

357. But, as we have intimated, the numbers in this Table can be made use of only when all the letters are different; for if two or more of them are alike, the number of transpositions becomes much less; and if all the letters are the same, we have only one arrangement: we shall therefore now shew how the numbers in the Table are to be diminished, according to the number of letters that are alike.

358. When two letters are given, and those letters are the same, the two arrangements are reduced to one, and consequently the number, which we have found above, is reduced to the half; that is to say, it must be divided by 2. If we have three letters alike, the six transpositions are reduced to one; whence it follows that the numbers in the Table must be divided by $6 = 3 . 2 . 1$; and, for the same reason, if four letters are alike, we must divide the numbers found by 24, or $4 . 3 . 2 . 1$, &c.

It is easy therefore to find how many transpositions the letters $aaabbc$, for example, may undergo. They are in number 6, and consequently, if they were all different, they would admit of $6 . 5 . 4 . 3 . 2 . 1$ transpositions; but since a is found thrice in those letters, we must divide that number of transpositions by $3 . 2 . 1$; and since b occurs twice, we must again divide it by $2 . 1$: the number of transpositions required will therefore be $\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 5 \cdot 4 \cdot 3 = 60$.

359. We may now readily determine the coefficients of all the terms of any power; as for example of the seventh power $(a + b)^7$.

The first term is a^7, which occurs only once; and as all the other terms have each seven letters, it follows that the number of transpositions for each term would be $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, if all the letters were different; but since in the second term, a^6b, we find six letters alike, we must divide the above product by $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, whence it follows that the coefficient is $\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7}{1}$.

In the third term, a^5b^2, we find the same letter a five times, and the same letter b twice; we must therefore divide that number first by $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, and then by $2 \cdot 1$; whence results the coefficient $\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = \frac{7 \cdot 6}{1 \cdot 2}$.

The fourth term a^4b^3 contains the letter a four times, and the letter b thrice; consequently, the whole number of the transpositions of the seven letters, must be divided, in the first place, by $4 \cdot 3 \cdot 2 \cdot 1$, and, secondly, by $3 \cdot 2 \cdot 1$, and the coefficient becomes $= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3}$.

In the same manner, we find $\frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4}$ for the coefficient of the fifth term, and so of the rest; by which the rule before given is demonstrated *.

360. These considerations carry us farther, and shew us

* From the *Theory of Combinations*, also, are frequently deduced the rules that have just been considered for determining the coefficients of terms of the power of a binomial; and this is perhaps attended with some advantage, as the whole is then reduced to a single formula.

In order to perceive the difference between *permutations* and *combinations*, it may be observed, that in the former we inquire in how many different ways the letters, which compose a certain formula, may change places; whereas, in combinations it is only necessary to know how many times these letters may be taken or multiplied together, one by one, two by two, three by three, &c.

also how to find all the powers of roots composed of more than two terms*. We shall apply them to the third power of $a + b + c$; the terms of which must be formed by all the possible combinations of three letters, each term having for its coefficient the number of its transpositions, as shewn, Art. 352.

Here, without performing the multiplication, the third power of $(a + b + c)$ will be, $a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2 + 3bc^2 + c^3$.

Suppose $a = 1$, $b = 1$, $c = 1$, the cube of $1 + 1 + 1$, or of 3, will be $1 + 3 + 3 + 3 + 6 + 3 + 1 + 3 + 3 + 1 = 27$;

Let us take the formula abc; here we know that the letters which compose it admit of six permutations, namely abc, acb, bac, bca, cab, cba; but as for combinations, it is evident that by taking these three letters, one by one, we have three combinations, namely a, b, and c; if two by two, we have three combinations, ab, ac, and bc; lastly, if we take them three by three, we have only the single combination abc.

Now, in the same manner as we prove that n different things admit of $1 \times 2 \times 3 \times 4 \cdots n$ different permutations, and that if r of these n things are equal, the number of permutations is $\frac{1 \times 2 \times 3 \times 4 \cdots n}{1 \times 2 \times 3 \times \cdots r}$; so likewise we prove that n things may be taken r by r, $\frac{n \times (n-1) \times (n-2) \cdots (n-r+1)}{1 \times 2 \times 3 \cdots r}$ number of times; or that we may take r of these n things in so many different ways. Hence, if we call n the exponent of the power to which we wish to raise the binomial $a + b$, and r the exponent of the letter b in any term, the coefficient of that term is always expressed by the formula $\frac{n \times (n-1) \times (n-2) \cdots (n-r+1)}{1 \times 2 \times 3 \cdots r}$. Thus, in the example, article 359, where $n = 7$, we have a^5b^2 for the third term, the exponent $r = 2$, and consequently the coefficient $= \frac{7 \times 6}{1 \times 2}$; for the fourth term we have $r = 3$, and the coefficient $= \frac{7 \times 6 \times 5}{1 \times 2 \times 3}$, and so on; which are evidently the same results as the permutations.

For complete and extensive treatises on the theory of combinations, we are indebted to *Frenicle*, *De Montmort*, *James Bernoulli*, &c. The two last have investigated this theory, with a view to its great utility in the calculation of probabilities. F. T.

* Roots, or quantities, composed of more than two terms, are called *polynomials*, in order to distinguish them from *binomials*, or quantities composed of two terms. F. T.

which result is accurate, and confirms the rule. But if we had supposed $a = 1$, $b = 1$, and $c = -1$, we should have found for the cube of $1 + 1 - 1$, that is of 1,

$1 + 3 - 3 + 3 - 6 + 3 + 1 - 3 + 3 - 1 = 1$, which is a still farther confirmation of the rule.

CHAP. XII.

Of the Expression *of* Irrational Powers *by* Infinite Series.

361. As we have shewn the method of finding any power of the root $a + b$, however great the exponent may be, we are able to express, generally, the power of $a + b$, whose exponent is undetermined; for it is evident that if we represent that exponent by n, we shall have by the rule already given (Art. 348 and the following):

$$(a + b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n}{1} \cdot \frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}a^{n-4}b^4 + \&c.$$

362. If the same power of the root $a - b$ were required, we need only change the signs of the second, fourth, sixth, &c. terms, and should have

$$(a - b)^n = a^n - \frac{n}{1}a^{n-1}b + \frac{n}{1} \cdot \frac{n-1}{2}a^{n-2}b^2 - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}a^{n-3}b^3 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}a^{n-4}b^4 - \&c.$$

363. These formulas are remarkably useful, since they serve also to express all kinds of radicals; for we have shewn that all irrational quantities may assume the form of powers whose exponents are fractional, and that $\sqrt[2]{a} = a^{\frac{1}{2}}$, $\sqrt[3]{a} = a^{\frac{1}{3}}$, and $\sqrt[4]{a} = a^{\frac{1}{4}}$, &c.: we have, therefore,

$$\sqrt[2]{(a + b)} = (a + b)^{\frac{1}{2}};\ \sqrt[3]{(a + b)} = (a + b)^{\frac{1}{3}};$$
$$\text{and } \sqrt[4]{(a + b)} = (a + b)^{\frac{1}{4}}, \&c.$$

Consequently, if we wish to find the square root of $a + b$, we have only to substitute for the exponent n the fraction $\frac{1}{2}$, in the general formula, Art. 361, and we shall have first, for the coefficients,

$\frac{n}{1} = \frac{1}{2}$; $\frac{n-1}{2} = -\frac{1}{4}$; $\frac{n-2}{3} = -\frac{3}{6}$; $\frac{n-3}{4} = -\frac{5}{8}$; $\frac{n-4}{5} = -\frac{7}{10}$; $\frac{n-5}{6} = -\frac{9}{12}$. Then, $a^n = a^{\frac{1}{2}} = \sqrt{a}$ and $a^{n-1} = \frac{1}{\sqrt{a}}$, $a^{n-2} = \frac{1}{a\sqrt{a}}$, $a^{n-3} = \frac{1}{a^2\sqrt{a}}$, &c. or we might express those powers of a in the following manner: $a^n = \sqrt{a}$; $a^{n-1} = \frac{\sqrt{a}}{a}$; $a^{n-2} = \frac{a^n}{a^2} = \frac{\sqrt{a}}{a^2}$; $a^{n-3} = \frac{a^n}{a^3} = \frac{\sqrt{a}}{a^3}$; $a^{n-4} = \frac{a^n}{a^4} = \frac{\sqrt{a}}{a^4}$, &c.

364. This being laid down, the square root of $a + b$ may be expressed in the following manner:

$$\sqrt{(a + b)} = \sqrt{a} + \tfrac{1}{2}b\frac{\sqrt{a}}{a} - \tfrac{1}{2} \cdot \tfrac{1}{4}b^2\frac{\sqrt{a}}{aa} + \tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \tfrac{3}{6}b^3\frac{\sqrt{a}}{a^3} - \tfrac{1}{2} \cdot \tfrac{1}{4} \cdot \tfrac{3}{6} \cdot \tfrac{5}{8}b^4\frac{\sqrt{a}}{a^4}, \&c.$$

365. If a therefore be a square number, we may assign the value of $\sqrt{a}$, and, consequently, the square root of $a + b$ may be expressed by an infinite series, without any radical sign.

Let, for example, $a = c^2$, we shall have $\sqrt{a} = c$; then

$$\sqrt{(c^2 + b)} = c + \tfrac{1}{2} \cdot \frac{b}{c} - \tfrac{1}{8} \cdot \frac{b^2}{c^3} + \tfrac{1}{16} \cdot \frac{b^3}{c^5} - \tfrac{5}{128} \cdot \frac{b^4}{c^7}, \&c.$$

We see, therefore, that there is no number, whose square root we may not extract in this manner; since every number may be resolved into two parts, one of which is a square represented by c^2. If, for example, the square root of 6 be required, we make $6 = 4 + 2$, consequently, $c^2 = 4$, $c = 2$, $b = 2$, whence results

$$\sqrt{6} = 2 + \tfrac{1}{2} - \tfrac{1}{16} + \tfrac{1}{64} - \tfrac{5}{1024}, \&c.$$

If we take only the two leading terms of this series, we shall have $2\frac{1}{2} = \frac{5}{2}$, the square of which, $\frac{25}{4}$, is $\frac{1}{4}$ greater than 6; but if we consider three terms, we have $2\frac{7}{16} = \frac{39}{16}$, the square of which, $\frac{1521}{256}$, is still $\frac{15}{256}$ too small.

366. Since, in this example, $\frac{5}{2}$ approaches very nearly to the true value of $\sqrt{6}$, we shall take for 6 the equivalent quantity $\frac{25}{4} - \frac{1}{4}$; thus $c^2 = \frac{25}{4}$; $c = \frac{5}{2}$; $b = \frac{1}{4}$; and calculating only the two leading terms, we find $\sqrt{6} = \frac{5}{2} + \frac{1}{2} \cdot \frac{-\frac{1}{4}}{\frac{5}{2}} = \frac{5}{2} - \frac{1}{2} \cdot \frac{\frac{1}{4}}{\frac{5}{2}} = \frac{5}{2} - \frac{1}{20} = \frac{49}{20}$; the square of which

fraction being $\frac{2401}{400}$, it exceeds the square of $\sqrt{6}$ only by $\frac{1}{400}$.

Now, making $6 = \frac{2401}{400} - \frac{1}{400}$, so that $c = \frac{49}{20}$ and $b = -\frac{1}{400}$; and still taking only the two leading terms, we have $\sqrt{6} = \frac{49}{20} + \frac{1}{2} \cdot \frac{-\frac{1}{400}}{\frac{49}{20}} = \frac{49}{20} - \frac{1}{2} \cdot \frac{\frac{1}{400}}{\frac{49}{20}} = \frac{49}{20} - \frac{1}{1960} = \frac{4801}{1960}$, the square of which is $\frac{23049601}{3841600}$; and 6, when reduced to the same denominator, is $= \frac{23049600}{3841600}$; the error therefore is only $\frac{1}{3841600}$.

367. In the same manner, we may express the cube root of $a+b$ by an infinite series; for since $\sqrt[3]{(a+b)} = (a+b)^{\frac{1}{3}}$, we shall have in the general formula, $n = \frac{1}{3}$, and for the coefficients, $\frac{n}{1} = \frac{1}{3}$; $\frac{n-1}{2} = -\frac{1}{3}$; $\frac{n-2}{3} = -\frac{5}{9}$; $\frac{n-3}{4} = -\frac{2}{3}$; $\frac{n-4}{5} = -\frac{11}{15}$, &c. and, with regard to the powers of a, we shall have $a^n = \sqrt[3]{a}$; $a^{n-1} = \frac{\sqrt[3]{a}}{a}$; $a^{n-2} = \frac{\sqrt[3]{a}}{a^2}$; $a^{n-3} = \frac{\sqrt[3]{a}}{a^3}$, &c. then

$$\sqrt[3]{(a+b)} = \sqrt[3]{a} + \tfrac{1}{3} \cdot b\frac{\sqrt[3]{a}}{a} - \tfrac{1}{9} \cdot b^2\frac{\sqrt[3]{a}}{a^2} + \tfrac{5}{81} \cdot b^3\frac{\sqrt[3]{a}}{a^3} - \tfrac{10}{243} \cdot b^4\frac{\sqrt[3]{a}}{a^4}, \text{\&c.}$$

368. If a therefore be a cube, or $a = c^3$, we have $\sqrt[3]{a} = c$, and the radical signs will vanish; for we shall have

$$\sqrt[3]{(c^3+b)} = c + \tfrac{1}{3} \cdot \frac{b}{c^2} - \tfrac{1}{9} \cdot \frac{b^2}{c^5} + \tfrac{5}{81} \cdot \frac{b^3}{c^8} - \tfrac{10}{243} \cdot \frac{b^4}{c^{11}} +, \text{\&c.}$$

369. We have therefore arrived at a formula, which will enable us to find, *by approximation*, the cube root of any number; since every number may be resolved into two parts, as $c^3 + b$, the first of which is a cube.

If we wish, for example, to determine the cube root of 2, we represent 2 by $1 + 1$, so that $c = 1$ and $b = 1$; consequently, $\sqrt[3]{2} = 1 + \frac{1}{3} - \frac{1}{9} + \frac{5}{81}$, &c. The two leading terms of this series make $1\frac{1}{3} = \frac{4}{3}$, the cube of which $\frac{64}{27}$ is too great by $\frac{10}{27}$: let us therefore make $2 = \frac{64}{27} - \frac{10}{27}$, we have $c = \frac{4}{3}$ and $b = -\frac{10}{27}$, and consequently $\sqrt[3]{2} = \frac{4}{3} + \frac{1}{3} \cdot \frac{-\frac{10}{27}}{\frac{16}{9}}$: these two terms give $\frac{4}{3} - \frac{5}{72} = \frac{91}{72}$, the cube of which is $\frac{753571}{373248}$: but, $2 = \frac{746496}{373248}$, so that the error is $\frac{7075}{373248}$; and in this way we might still approximate, the faster in proportion as we take a greater number of terms *.

* In the Philosophical Transactions for 1694, Dr. Halley has given a very elegant and general method for extracting roots of

CHAP. XIII.

Of the Resolution *of* Negative Powers.

370. We have already shewn, that $\frac{1}{a}$ may be expressed by a^{-1}; we may therefore express $\frac{1}{a+b}$ also by $(a+b)^{-1}$; so that the fraction $\frac{1}{a+b}$ may be considered as a power of $a+b$, namely, that power whose exponent is -1; from which it follows, that the series already found as the value of $(a+b)^n$ extends also to this case.

371. Since, therefore $\frac{1}{a+b}$ is the same as $(a+b)^{-1}$, let us suppose, in the general formula, [Art. 361.] $n = -1$; and we shall first have, for the coefficients, $\frac{n}{1} = -1$; $\frac{n-1}{2} = -1$; $\frac{n-2}{3} = -1$; $\frac{n-3}{4} = -1$, &c. And, for the powers of a, we have $a^n = a^{-1} = \frac{1}{a}$; $a^{n-1} = a^{-2} = \frac{1}{a^2}$; $a^{n-2} = \frac{1}{a^3}$; $a^{n-3} = \frac{1}{a^4}$ &c.: so that $(a+b)^{-1} = \frac{1}{a+b}$ $= \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} - \frac{b^5}{a^6}$, &c. which is the same series that we found before by division.

372. Farther, $\frac{1}{(a+b)^2}$ being the same with $(a+b)^{-2}$, let

any degree whatever by approximation; where he demonstrates this general formula,

$$\sqrt[m]{(a^m \pm b)} = \frac{m-2}{m-1}a + \sqrt{\left(\frac{a^2}{(m-1)^2} \pm \frac{2b}{(m^2-m)a^{m-1}}\right)}.$$

Those who have not an opportunity of consulting the Philosophical Transactions, will find the formation and the use of this formula explained in the new edition of Leçons Elementaires de Mathematiques by M. D'Abbé de la Caille, published by M. L'Abbé Marie. F. T. See also Dr. Hutton's Math. Dictionary.

us reduce this quantity also to an infinite series. For this purpose, we must suppose $n = -2$, and we shall first have, for the coefficients, $\frac{n}{1} = -\frac{2}{1}$; $\frac{n-1}{2} = -\frac{3}{2}$; $\frac{n-2}{3} = -\frac{4}{3}$; $\frac{n-3}{4} = -\frac{5}{4}$, &c.; and, for the powers of a, we obtain $a^n = \frac{1}{a^2}$; $a^{n-1} = \frac{1}{a^3}$; $a^{n-2} = \frac{1}{a^4}$; $a^{n-3} = \frac{1}{a^5}$, &c. We have therefore $(a+b)^{-2} = \frac{1}{(a+b)^2} = \frac{1}{a^2} - \frac{2.b}{1.a^3} + \frac{2.3.b^2}{1.2.a^4} - \frac{2.3.4.b^3}{1.2.3.a^5} + \frac{2.3.4.5.b^4}{1.2.3.4.a^5}$, &c. Now, $\frac{2}{1} = 2$; $\frac{2.3}{1.2} = 3$; $\frac{2.3.4}{1.2.3} = 4$; $\frac{2.3.4.5}{1.2.3.4} = 5$, &c. and consequently, $\frac{1}{(a+b)^2} = \frac{1}{a^2} - 2\frac{b}{a^3} + 3\frac{b^2}{a^4} - 4\frac{b^3}{a^5} + 5\frac{b^4}{a^6} - 6\frac{b^5}{a^7} + 7\frac{b^6}{a^8}$, &c.

373. Let us proceed, and suppose $n = -3$, and we shall have a series expressing the value of $\frac{1}{(a+b)^3}$, or of $(a+b)^{-3}$. Here the coefficients will be $\frac{n}{1} = -\frac{3}{1}$; $\frac{n-1}{2} = -\frac{4}{2}$; $\frac{n-2}{3} = -\frac{5}{3}$, &c. and the powers of a become, $a^n = \frac{1}{a^3}$; $a^{n-1} = \frac{1}{a^4}$; $a^{n-2} = \frac{1}{a^5}$, &c. which gives $\frac{1}{(a+b)^3} = \frac{1}{a^3} - \frac{3.b}{1.a^4} + \frac{3.4.b^2}{1.2.a^5} - \frac{3.4.5.b^3}{1.2.3.a^6} + \frac{3.4.5.6.b^4}{1.2.3.4.a^7} = \frac{1}{a^3} - 3\frac{b}{a^4} + 6\frac{b^2}{a^5} - 10\frac{b^3}{a^6} + 15\frac{b^4}{a^7} - 21\frac{b^5}{a^8} + 28\frac{b^6}{a^9}$, &c.

If now we make $n = -4$; we shall have for the coefficients $\frac{n}{1} = -\frac{4}{1}$; $\frac{n-1}{2} = -\frac{5}{2}$; $\frac{n-2}{3} = -\frac{6}{3}$; $\frac{n-3}{4} = -\frac{7}{4}$, &c. And for the powers, $a^n = \frac{1}{a^4}$; $a^{n-1} = \frac{1}{a^5}$; $a^{n-2} = \frac{1}{a^6}$; $a^{n-3} = \frac{1}{a^7}$; $a^{n-4} = \frac{1}{a^8}$, whence we obtain,

$$\frac{1}{(a+b)^4} = \frac{1}{a^4} - \frac{4b}{1a^5} + \frac{4.5.b^2}{1.2.a^6} - \frac{4.5.6.b^3}{1.2.3.a^7}, \text{ \&c.} = \frac{1}{a^4} - 4\frac{b}{a^5} + 10\frac{b^2}{a^6} - 20\frac{b^3}{a^7} + 35\frac{b^4}{a^8} - 56\frac{b^5}{a^9} +, \text{ \&c.}$$

374. The different cases that have been considered

enable us to conclude with certainty, that we shall have, generally, for any negative power of $a + b$;

$$\frac{1}{(a+b)^m} = \frac{1}{a^m} - \frac{m.b}{a^{m+1}} + \frac{m.(m-1).b^2}{2.a^{m+2}} - \frac{m.(m-1).(m-2).b^3}{2.3.a^{m+3}},$$

&c. And, by means of this formula, we may transform all such fractions into infinite series, substituting fractions also, or fractional exponents, for m, in order to express irrational quantities.

375. The following considerations will illustrate this subject still farther; for we have seen that,

$$\frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} - \frac{b^5}{a^6} +, \text{ \&c.}$$

If, therefore, we multiply this series by $a + b$, the product ought to be $= 1$; and this is found to be true, as will be seen by performing the multiplication:

$$\frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} + \frac{b^4}{a^5} - \frac{b^5}{a^6} +, \text{ \&c.}$$
$$a + b$$

$$1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{b^3}{a^3} + \frac{b^4}{a^4} - \frac{b^5}{a^5} +, \text{ \&c.}$$
$$+ \frac{b}{a} - \frac{b^2}{a^2} + \frac{b^3}{a^3} - \frac{b^4}{a^4} + \frac{b^5}{a^5} -, \text{ \&c.}$$

where all the terms but the first cancel each other.

376. We have also found, that

$$\frac{1}{(a+b)^2} = \frac{1}{a^2} - \frac{2b}{a^3} + \frac{3b^2}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6} - \frac{6b^5}{a^7}, \text{ \&c.}$$

And if we multiply this series by $(a+b)^2$, the product ought also to be equal to 1. Now, $(a+b)^2 = a^2 + 2ab + b^2$, and

$$\frac{1}{a^2} - \frac{2b}{a^3} + \frac{3b^2}{a^4} - \frac{4b^3}{a^5} + \frac{5b^4}{a^6} - \frac{6b^5}{a^7} + a, \text{ \&c.}$$
$$a^2 + 2ab + b^2$$

$$1 - \frac{2b}{a} + \frac{3b^2}{a^2} - \frac{4b^3}{a^3} + \frac{5b^4}{a^4} - \frac{6b^5}{a^5} +, \text{ \&c.}$$
$$+ \frac{2b}{a} - \frac{4b^2}{a^2} + \frac{6b^3}{a^3} - \frac{8b^4}{a^4} + \frac{10b^5}{a^5} -, \text{ \&c.}$$
$$+ \frac{b^2}{a^2} - \frac{2b^3}{a^3} + \frac{3b^4}{a^4} - \frac{4b^5}{a^5} +, \text{ \&c.}$$

which gives 1 for the product, as the nature of the thing required.

377. If we multiply the series which we found for the value of $\frac{1}{(a+b)^2}$, by $a+b$ only, the product ought to answer to the fraction $\frac{1}{a+b}$, or be equal to the series already found, namely, $\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}$, &c. and this the actual multiplication will confirm.

$$\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}, \text{ \&c.}$$
$$a+b$$
$$\frac{1}{a}-\frac{2b}{a^2}+\frac{3b^2}{a^3}-\frac{4b^3}{a^4}+\frac{5b^4}{a^5}, \text{ \&c.}$$
$$+\frac{b}{a^2}-\frac{2b^2}{a^3}+\frac{3b^3}{a^4}-\frac{4b^4}{a^5}, \text{ \&c.}$$
$$\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}, \text{ \&c. as required.}$$

SECTION III.

Of Ratios *and* Proportions.

CHAP. I.

Of Arithmetical Ratio, *or of the* Difference *between two* Numbers.

378. Two quantities are either equal to one another, or they are not. In the latter case, where one is greater than the other, we may consider their inequality under two different points of view: we may ask, *how much* one of the quantities is greater than the other? Or we may ask, *how many times* the one is greater than the other? The

results which constitute the answers to these two questions are both called *relations*, or *ratios*. We usually call the former an *arithmetical ratio*, and the latter a *geometrical ratio*, without however these denominations having any connexion with the subject itself. The adoption of these expressions has been entirely arbitrary.

379. It is evident, that the quantities of which we speak must be of one and the same kind; otherwise we could not determine any thing with regard to their equality, or inequality: for it would be absurd to ask if two pounds and three ells are equal quantities. So that in what follows, quantities of the same kind only are to be considered; and as they may always be expressed by numbers, it is of numbers only that we shall treat, as was mentioned at the beginning.

380. When of two given numbers, therefore, it is required how much the one is greater than the other, the answer to this question determines the arithmetical ratio of the two numbers; but since this answer consists in giving the difference of the two numbers, it follows, that an arithmetical ratio is nothing but the *difference* between two numbers; and as this appears to be a better expression, we shall reserve the words *ratio* and *relation* to express geometrical ratios.

381. As the difference between two numbers is found by subtracting the less from the greater, nothing can be easier than resolving the question how much one is greater than the other: so that when the numbers are equal, the difference being nothing, if it be required how much one of the numbers is greater than the other, we answer, by nothing; for example, 6 being equal to 2×3, the difference between 6 and 2×3 is 0.

382. But when the two numbers are not equal, as 5 and 3, and it is required how much 5 is greater than 3, the answer is, 2; which is obtained by subtracting 3 from 5. Likewise 15 is greater than 5 by 10; and 20 exceeds 8 by 12.

383. We have therefore three things to consider on this subject; 1st. the greater of the two numbers; 2d. the less; and 3d. the difference: and these three quantities are so connected together, that any two of the three being given, we may always determine the third.

Let the greater number be a, the less b, and the difference d; then d will be found by subtracting b from a, so that $d = a - b$; whence we see how to find d, when a and b are given.