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ELEMENTS

OF

ALGEBRA.

PART I.

Containing the Analysis *of* Determinate Quantities.

SECTION I.

Of the different Methods *of calculating* Simple Quantities.

CHAP. I.

Of Mathematics *in general.*

ARTICLE I.

WHATEVER is capable of increase or diminution, is called *magnitude*, or *quantity*.

A sum of money therefore is a quantity, since we may increase it or diminish it. It is the same with a weight, and other things of this nature.

2. From this definition, it is evident, that the different kinds of magnitude must be so various, as to render it difficult to enumerate them: and this is the origin of the different branches of the Mathematics, each being employed on a particular kind of magnitude. Mathematics, in general, is the *science of quantity*; or, the science which investigates the means of measuring quantity.

3. Now, we cannot measure or determine any quantity, except by considering some other quantity of the same kind as known, and pointing out their mutual relation. If it were proposed, for example, to determine the quantity of a sum of money, we should take some known piece of money,

as a louis, a crown, a ducat, or some other coin, and shew how many of these pieces are contained in the given sum. In the same manner, if it were proposed to determine the quantity of a weight, we should take a certain known weight; for example, a pound, an ounce, &c. and then shew how many times one of these weights is contained in that which we are endeavouring to ascertain. If we wished to measure any length or extension, we should make use of some known length, such as a foot.

4. So that the determination, or the measure of magnitude of all kinds, is reduced to this: fix at pleasure upon any one known magnitude of the same species with that which is to be determined, and consider it as the *measure* or *unit*; then, determine the proportion of the proposed magnitude to this known measure. This proportion is always expressed by numbers; so that a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit.

5. From this it appears, that all magnitudes may be expressed by numbers; and that the foundation of all the Mathematical Sciences must be laid in a complete treatise on the science of Numbers, and in an accurate examination of the different possible methods of calculation.

This fundamental part of mathematics is called Analysis, or Algebra *.

6. In Algebra then we consider only numbers, which represent quantities, without regarding the different kinds of quantity. These are the subjects of other branches of the mathematics.

7. Arithmetic treats of numbers in particular, and is the *science of numbers properly so called*; but this science extends only to certain methods of calculation which occur in common practice: Algebra, on the contrary, comprehends in general all the cases that can exist in the doctrine and calculation of numbers.

* Several mathematical writers make a distinction between *Analysis* and *Algebra*. By the term *Analysis*, they understand the method of determining those general rules, which assist the understanding in all mathematical investigations; and by *Algebra*, the instrument which this method employs for accomplishing that end. This is the definition given by M. Bezout in the preface to his Algebra. F. T.

CHAP. II.

Explanation of the Signs + Plus *and* − Minus.

8. When we have to add one given number to another, this is indicated by the sign +, which is placed before the second number, and is read *plus*. Thus $5 + 3$ signifies that we must add 3 to the number 5, in which case, every one knows that the result is 8; in the same manner $12 + 7$ make 19; $25 + 16$ make 41; the sum of $25 + 41$ is 66, &c.

9. We also make use of the same sign + *plus*, to connect several numbers together; for example, $7 + 5 + 9$ signifies that to the number 7 we must add 5, and also 9, which make 21. The reader will therefore understand what is meant by

$$8 + 5 + 13 + 11 + 1 + 3 + 10,$$

viz. the sum of all these numbers, which is 51.

10. All this is evident; and we have only to mention, that in Algebra, in order to generalise numbers, we represent them by letters, as a, b, c, d, &c. Thus, the expression $a + b$, signifies the sum of two numbers, which we express by a and b, and these numbers may be either very great, or very small. In the same manner, $f + m + b + x$, signifies the sum of the numbers represented by these four letters.

If we know therefore the numbers that are represented by letters, we shall at all times be able to find, by arithmetic, the sum or value of such expressions.

11. When it is required, on the contrary, to subtract one given number from another, this operation is denoted by the sign −, which signifies *minus*, and is placed before the number to be subtracted: thus, $8 - 5$ signifies that the number 5 is to be taken from the number 8; which being done, there remain 3. In like manner $12 - 7$ is the same as 5; and $20 - 14$ is the same as 6, &c.

12. Sometimes also we may have several numbers to subtract from a single one; as, for instance, $50 - 1 - 3 - 5 - 7 - 9$. This signifies, first, take 1 from 50, and there remain 49; take 3 from that remainder, and there will remain 46; take away 5, and 41 remain; take away 7, and 34 remain; lastly, from that take 9, and there remain 25: this last remainder is the value of the expression. But as the numbers 1, 3, 5, 7, 9, are all to be subtracted, it is the

same thing if we subtract their sum, which is 25, at once from 50, and the remainder will be 25 as before.

13. It is also easy to determine the value of similar expressions, in which both the signs + *plus* and — *minus* are found. For example;

$$12 - 3 - 5 + 2 - 1 \text{ is the same as } 5.$$

We have only to collect separately the sum of the numbers that have the sign + before them, and subtract from it the sum of those that have the sign —. Thus, the sum of 12 and 2 is 14; and that of 3, 5, and 1, is 9; hence 9 being taken from 14, there remain 5.

14. It will be perceived, from these examples, that the order in which we write the numbers is perfectly indifferent and arbitrary, provided the proper sign of each be preserved. We might with equal propriety have arranged the expression in the preceding article thus; $12 + 2 - 5 - 3 - 1$, or $2 - 1 - 3 - 5 + 12$, or $2 + 12 - 3 - 1 - 5$, or in still different orders; where it must be observed, that in the arrangement first proposed, the sign + is supposed to be placed before the number 12.

15. It will not be attended with any more difficulty if, in order to generalise these operations, we make use of letters instead of real numbers. It is evident, for example, that

$$a - b - c + d - e,$$

signifies that we have numbers expressed by a and d, and that from these numbers, or from their sum, we must subtract the numbers expressed by the letters b, c, e, which have before them the sign —.

16. Hence it is absolutely necessary to consider what sign is prefixed to each number: for in Algebra, simple quantities are numbers considered with regard to the signs which precede, or affect them. Farther, we call those *positive quantities*, before which the sign + is found; and those are called *negative quantities*, which are affected by the sign —.

17. The manner in which we generally calculate a person's property, is an apt illustration of what has just been said. For we denote what a man really possesses by positive numbers, using, or understanding the sign +; whereas his debts are represented by negative numbers, or by using the sign —. Thus, when it is said of any one that he has 100 crowns, but owes 50, this means that his real possession amounts to $100 - 50$; or, which is the same thing, $+ 100 - 50$, that is to say, 50.

18. Since negative numbers may be considered as debts, because positive numbers represent real possessions, we may say that negative numbers are less than nothing. Thus, when a man has nothing of his own, and owes 50 crowns, it is certain that he has 50 crowns less than nothing; for if any one were to make him a present of 50 crowns to pay his debts, he would still be only at the point nothing, though really richer than before.

19. In the same manner, therefore, as positive numbers are incontestably greater than nothing, negative numbers are less than nothing. Now, we obtain positive numbers by adding 1 to 0, that is to say, 1 to nothing; and by continuing always to increase thus from unity. This is the origin of the series of numbers called *natural numbers*; the following being the leading terms of this series:

$$0, +1, +2, +3, +4, +5, +6, +7, +8, +9, +10,$$

and so on to infinity.

But if, instead of continuing this series by successive additions, we continued it in the opposite direction, by perpetually subtracting unity, we should have the following series of negative numbers:

$$0, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10,$$

and so on to infinity.

20. All these numbers, whether positive or negative, have the known appellation of whole numbers, or *integers*, which consequently are either greater or less than nothing. We call them *integers*, to distinguish them from fractions, and from several other kinds of numbers, of which we shall hereafter speak. For instance, 50 being greater by an entire unit than 49, it is easy to comprehend that there may be, between 49 and 50, an infinity of intermediate numbers, all greater than 49, and yet all less than 50. We need only imagine two lines, one 50 feet, the other 49 feet long, and it is evident that an infinite number of lines may be drawn, all longer than 49 feet, and yet shorter than 50.

21. It is of the utmost importance through the whole of Algebra, that a precise idea should be formed of those negative quantities, about which we have been speaking. I shall, however, content myself with remarking here, that all such expressions as

$$+1 - 1, +2 - 2, +3 - 3, +4 - 4, \text{ \&c.}$$

are equal to 0, or nothing. And that

$$+2 - 5 \text{ is equal to } -3:$$

for if a person has 2 crowns, and owes 5, he has not only nothing, but still owes 3 crowns. In the same manner,

$7 - 12$ is equal to -5, and $25 - 40$ is equal to -15.

22. The same observations hold true, when, to make the expression more general, letters are used instead of numbers;

thus 0, or nothing, will always be the value of $+ a - a$; but if we wish to know the value of $+ a - b$, two cases are to be considered.

The first is when a is greater than b; b must then be subtracted from a, and the remainder (before which is placed, or understood to be placed, the sign $+$) shews the value sought.

The second case is that in which a is less than b: here a is to be subtracted from b, and the remainder being made negative, by placing before it the sign $-$, will be the value sought.

CHAP. III.

Of the Multiplication *of* Simple Quantities.

23. When there are two or more equal numbers to be added together, the expression of their sum may be abridged: for example,

$a + a$ is the same with $2 \times a$,
$a + a + a$ - - - - - $3 \times a$,
$a + a + a + a$ - - - $4 \times a$, and so on; where $\times$ is the sign of multiplication. In this manner we may form an idea of multiplication; and it is to be observed that,

$2 \times a$ signifies 2 times, or twice a
$3 \times a$ - - - - 3 times, or thrice a
$4 \times a$ - - - - 4 times a, &c.

24. If therefore a number expressed by a letter is to be multiplied by any other number, we simply put that number before the letter, thus;

a multiplied by 20 is expressed by $20a$, and
b multiplied by 30 is expressed by $30b$, &c.

It is evident, also, that c taken once, or $1c$, is the same as c.

25. Farther, it is extremely easy to multiply such products again by other numbers; for example:

2 times, or twice $3a$, makes $6a$
3 times, or thrice $4b$, makes $12b$
5 times $7x$ makes $35x$.

and these products may be still multiplied by other numbers at pleasure.

26. When the number by which we are to multiply is also represented by a letter, we place it immediately before the other letter; thus, in multiplying b by a, the product is

written ab; and pq will be the product of the multiplication of the number q by p. Also, if we multiply this pq again by a, we shall obtain apq.

27. It may be farther remarked here, that the order in which the letters are joined together is indifferent; thus ab is the same thing as ba; for b multiplied by a is the same as a multiplied by b. To understand this, we have only to substitute, for a and b, known numbers, as 3 and 4; and the truth will be self-evident; for 3 times 4 is the same as 4 times 3.

28. It will not be difficult to perceive, that when we substitute numbers for letters joined together, in the manner we have described, they cannot be written in the same way by putting them one after the other. For if we were to write 34 for 3 times 4, we should have 34 and not 12. When therefore it is required to multiply common numbers, we must separate them by the sign $\times$, or by a point: thus, 3×4, or 3.4, signifies 3 times 4; that is, 12. So, 1×2 is equal to 2; and $1 \times 2 \times 3$ makes 6. In like manner, $1 \times 2 \times 3 \times 4 \times 56$ makes 1344; and $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$ is equal to 3628800, &c.

29. In the same manner we may discover the value of an expression of this form, $5.7.8.abcd$. It shews that 5 must be multiplied by 7, and that this product is to be again multiplied by 8; that we are then to multiply this product of the three numbers by a, next by b, then by c, and lastly by d. It may be observed, also, that instead of $5.7.8$, we may write its value, 280; for we obtain this number when we multiply the product of 5 by 7, or 35, by 8.

30. The results which arise from the multiplication of two or more numbers are called *products*; and the numbers, or individual letters, are called *factors*.

31. Hitherto we have considered only positive numbers, and there can be no doubt, but that the products which we have seen arise are positive also: viz. $+ a$ by $+ b$ must necessarily give $+ ab$. But we must separately examine what the multiplication of $+a$ by $-b$, and of $-a$ by $-b$, will produce.

32. Let us begin by multiplying $-a$ by 3 or $+3$. Now, since $-a$ may be considered as a debt, it is evident that if we take that debt three times, it must thus become three times greater, and consequently the required product is $-3a$. So if we multiply $-a$ by $+b$, we shall obtain $-ba$, or, which is the same thing, $-ab$. Hence we conclude, that if a positive quantity be multiplied by a negative quantity, the product will be negative; and it may be laid down

as a rule, that + by + makes + or *plus*; and that, on the contrary, + by −, or − by +, gives −, or *minus*.

33. It remains to resolve the case in which − is multiplied by −; or, for example, $-a$ by $-b$. It is evident, at first sight, with regard to the letters, that the product will be ab; but it is doubtful whether the sign +, or the sign −, is to be placed before it; all we know is, that it must be one or the other of these signs. Now, I say that it cannot be the sign −: for $-a$ by $+b$ gives $-ab$, and $-a$ by $-b$ cannot produce the same result as $-a$ by $+b$; but must produce a contrary result, that is to say, $+ab$; consequently, we have the following rule: − multiplied by − produces +, that is, the same as + multiplied by + *.

* A further illustration of this rule is generally given by algebraists as follows:

First, we know that $+a$ multiplied by $+b$ gives the product $+ab$; and if $+a$ be multiplied by a quantity less than b, as $b-c$, the product must necessarily be less than ab; in short, from ab we must subtract the product of a, multiplied by c; hence $a \times (b-c)$ must be expressed by $ab-ac$; therefore it follows that $a \times -c$ gives the product $-ac$.

If now we consider the product arising from the multiplication of the two quantities $(a-b)$, and $(c-d)$, we know that it is less than that of $(a-b) \times c$, or of $ac-bc$; in short, from this product we must subtract that of $(a-b) \times d$; but the product $(a-b) \times (c-d)$ becomes $ac-bc-ad$, together with the product of $-b \times -d$ annexed; and the question is only what sign we must employ for this purpose, whether + or −. Now we have seen that from the product $ac-bc$ we must subtract the product of $(a-b) \times d$, that is, we must subtract a quantity less than ad; we have therefore subtracted already too much by the quantity bd; this product must therefore be added; that is, it must have the sign + prefixed; hence we see that $-b \times -d$ gives $+bd$ for a product; or − *minus* multiplied by − *minus* gives + *plus*. See Art. 273, 274.

Multiplication has been erroneously called a compendious method of performing addition: whereas it is the taking, or repeating of one given number as many times, as the number by which it is to be multiplied, contains units. Thus, 9×3 means that 9 is to be taken 3 times, or that the measure of multiplication is 3; again, $9 \times \frac{1}{2}$ means that 9 is to be taken half a time, or that the measure of multiplication is $\frac{1}{2}$. In multiplication there are two factors, which are sometimes called the multiplicand and the multiplier. These, it is evident, may reciprocally change places, and the product will be still the same: for $9 \times 3 = 3 \times 9$, and $9 \times \frac{1}{2} = \frac{1}{2} \times 9$. Hence it appears, that numbers may be diminished by multiplication, as well as increased, in any given ratio, which is wholly inconsistent with

34. The rules which we have explained are expressed more briefly as follows:

Like signs, multiplied together, give +; unlike or contrary signs give −. Thus, when it is required to multiply the following numbers, $+a$, $-b$, $-c$, $+d$; we have first $+a$ multiplied by $-b$, which makes $-ab$; this by $-c$, gives $+abc$; and this by $+d$, gives $+abcd$.

35. The difficulties with respect to the signs being removed, we have only to shew how to multiply numbers that are themselves products. If we were, for instance, to multiply the number ab by the number cd, the product would be $abcd$, and it is obtained by multiplying first ab by c, and then the result of that multiplication by d. Or, if we had to multiply 36 by 12; since 12 is equal to 3 times 4, we

the nature of Addition; for $9 \times \frac{1}{2} = 4\frac{1}{2}$, $9 \times \frac{1}{9} = 1$, $9 \times \frac{1}{100} = \frac{9}{100}$, &c. The same will be found true with respect to algebraic quantities, $a \times b = ab$, $-9 \times 3 = -27$, that is, 9 negative integers multiplied by 3, or taken 3 times, are equal to −27, because the measure of multiplication is 3. In the same manner, by inverting the factors, 3 positive integers multiplied by −9, or taken 9 times negatively, must give the same result. Therefore a positive quantity taken negatively, or a negative quantity taken positively, gives a negative product.

From these considerations, we may illustrate the present subject in a different way, and shew, that the product of two negative quantities must be positive. First, algebraic quantities may be considered as a series of numbers increasing in any ratio, on each side of nothing, to infinity. Let us assume a small part only of such a series for the present purpose, in which the ratio is unity; and let us multiply every term of it by −2, [illegible]

5, 4, 3, 2, 1, 0, −1, −2, −3, −4, −5,
−2, −2, −2, −2, −2, −2, −2, −2, −2, −2, −2,
−10, −8, −6, −4, −2, 0, +2, +4, +6, +8, +10.

Here, of course, we find the series inverted, and the ratio doubled. Farther, in order to illustrate the subject, we may consider the ratio of a series of fractions between 1 and 0, as indefinitely small, till the last term being multiplied by −2, the product would be equal to 0. If, after this, the multiplier having passed the middle term, 0, be multiplied into any negative term, however small, between 0 and −1, on the other side of the series, the product, it is evident, must be positive, otherwise the series could not go on. Hence it appears, that the taking of a negative quantity negatively destroys the very property of negation, and is the conversion of negative into positive numbers. So that if + × − = −, it necessarily follows that − × − must give a contrary product, that is, +. See Art. 176, 177.

should only multiply 36 first by 3, and then the product 108 by 4, in order to have the whole product of the multiplication of 12 by 36, which is consequently 432.

36. But if we wished to multiply $5ab$ by $3cd$, we might write $3cd \times 5ab$. However, as in the present instance the order of the numbers to be multiplied is indifferent, it will be better, as is also the custom, to place the common numbers before the letters, and to express the product thus: $5 \times 3abcd$, or $15abcd$; since 5 times 3 is 15.

So if we had to multiply $12pqr$ by $7xy$, we should obtain $12 \times 7pqrxy$, or $84pqrxy$.

CHAP. IV.

Of the Nature of whole Numbers, *or* Integers, *with respect to their* Factors.

37. We have observed that a product is generated by the multiplication of two or more numbers together, and that these numbers are called *factors*. Thus, the numbers a, b, c, d, are the factors of the product $abcd$.

38. If, therefore, we consider all whole numbers as products of two or more numbers multiplied together, we shall soon find that some of them cannot result from such a multiplication, and consequently have not any factors; while others may be the products of two or more numbers multiplied together, and may consequently have two or more factors. Thus 4 is produced by 2×2; 6 by 2×3; 8 by $2 \times 2 \times 2$; 27 by $3 \times 3 \times 3$; and 10 by 2×5, &c.

39. But on the other hand, the numbers 2, 3, 5, 7, 11, 13, 17, &c. cannot be represented in the same manner by factors, unless for that purpose we make use of unity, and represent 2, for instance, by 1×2. But the numbers which are multiplied by 1 remaining the same, it is not proper to reckon unity as a factor.

All numbers, therefore, such as 2, 3, 5, 7, 11, 13, 17, &c. which cannot be represented by factors, are called *simple*, or *prime numbers*; whereas others, as 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, &c. which may be represented by factors, are called *composite numbers*.

40. *Simple* or *prime numbers* deserve therefore particular attention, since they do not result from the multiplication of two or more numbers. It is also particularly worthy of observation, that if we write these numbers in succession as they follow each other, thus,

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, &c.*

we can trace no regular order; their increments being sometimes greater, sometimes less; and hitherto no one has been able to discover whether they follow any certain law or not.

41. All *composite* numbers, which may be represented by factors, result from the prime numbers above mentioned; that is to say, all their factors are prime numbers. For, if we find a factor which is not a prime number, it may always be decomposed and represented by two or more prime numbers. When we have represented, for instance, the number

* All the prime numbers from 1 to 100000 are to be found in the tables of divisors, which I shall speak of in a succeeding note. But particular tables of the prime numbers from 1 to 101000 have been published at Halle, by M. Kruger, in a German work entitled "Thoughts on Algebra;" M. Kruger had received them from a person called Peter Jaeger, who had calculated them. M. Lambert has continued these tables as far as 102000, and republished them in his supplements to the logarithmic and trigonometrical tables, printed at Berlin in 1770; a work which contains, likewise, several tables that are of great use in the different branches of mathematics, and explanations which it would be too long to enumerate here.

The Royal Parisian Academy of Sciences is in possession of tables of prime numbers, presented to it by P. Mercastel de l'Oratoire, and by M. du Tour; but they have not been published. They are spoken of in Vol. V. of the Foreign Memoirs, with a reference to a memoir, contained in that volume, by M. Rallier des Ourmes, Honorary Counsellor of the Presidial Court at Rennes, in which the author explains an easy method of finding prime numbers.

In the same volume we find another method by M. Rallier des Ourmes, which is entitled, "A new Method for Division, when the Dividend is a Multiple of the Divisor, and may therefore be divided without a Remainder; and for the Extraction of Roots when the Power is perfect." This method, more curious, indeed, than useful, is almost totally different from the common one: it is very easy, and has this singularity, that, provided we know as many figures on the right of the dividend, or the power, as there are to be in the quotient, or the root, we may pass over the figures which precede them, and thus obtain the quotient. M. Rallier des Ourmes was led to this new method by reflecting on the numbers terminating the numerical expressions of products or powers, a species of numbers which I have remarked also, on other occasions, it would be useful to consider. F. T.

30 by 5×6; it is evident that 6 not being a prime number, but being produced by 2×3, we might have represented 30 by $5 \times 2 \times 3$, or by $2 \times 3 \times 5$; that is to say, by factors which are all prime numbers.

42. If we now consider those composite numbers which may be resolved into prime factors, we shall observe a great difference among them; thus we shall find that some have only two factors, that others have three, and others a still greater number. We have already seen, for example, that

4 is the same as 2×2,	6 is the same as 2×3,
8 - - - $2\times2\times2$,	9 - - - 3×3,
10 - - - - 2×5,	12 - - - $2\times3\times2$,
14 - - - - 2×7,	15 - - - - 3×5,
16 - - $2\times2\times2\times2$,	and so on.

43. Hence, it is easy to find a method for analysing any number, or resolving it into its simple factors. Let there be proposed, for instance, the number 360; we shall represent it first by 2×180. Now 180 is equal to 2×90, and

$$\left.\begin{matrix}90\\45\\15\end{matrix}\right\}\text{ is the same as }\left\{\begin{matrix}2\times45,\\3\times15,\text{ and lastly}\\3\times5.\end{matrix}\right.$$

So that the number 360 may be represented by these simple factors, $2 \times 2 \times 2 \times 3 \times 3 \times 5$; since all these numbers multiplied together produce 360 *.

44. This shews, that prime numbers cannot be divided by other numbers; and, on the other hand, that the simple factors of compound numbers are found most conveniently, and with the greatest certainty, by seeking the simple, or prime numbers, by which those compound numbers are divisible. But for this *division* is necessary; we shall therefore explain the rules of that operation in the following chapter.

* There is a table at the end of a German book of arithmetic, published at Leipsic, by Poetius, in 1728, in which all the numbers from 1 to 10000 are represented in this manner by their simple factors. F. T.

CHAP. V.

Of the Division *of* Simple Quantities.

45. When a number is to be separated into two, three, or more equal parts, it is done by means of *division*, which enables us to determine the magnitude of one of those parts. When we wish, for example, to separate the number 12 into three equal parts, we find by division that each of those parts is equal to 4.

The following terms are made use of in this operation. The number which is to be decompounded, or divided, is called the *dividend*; the number of equal parts sought is called the *divisor*; the magnitude of one of those parts, determined by the division, is called the *quotient*: thus, in the above example,

12 is the dividend,
3 is the divisor, and
4 is the quotient.

46. It follows from this, that if we divide a number by 2, or into two equal parts, one of those parts, or the quotient, taken twice, makes exactly the number proposed; and, in the same manner, if we have a number to divide by 3, the quotient taken thrice must give the same number again. In general, the multiplication of the quotient by the divisor must always reproduce the dividend.

47. It is for this reason that division is said to be a rule, which teaches us to find a number or quotient, which, being multiplied by the divisor, will exactly produce the dividend. For example, if 35 is to be divided by 5, we seek for a number which, multiplied by 5, will produce 35. Now, this number is 7, since 5 times 7 is 35. The manner of expression employed in this reasoning, is; 5 in 35 goes 7 times; and 5 times 7 makes 35.

48. The dividend therefore may be considered as a product, of which one of the factors is the divisor, and the other the quotient. Thus, supposing we have 63 to divide by 7, we endeavour to find such a product, that, taking 7 for one of its factors, the other factor multiplied by this may exactly give 63. Now 7×9 is such a product, and consequently 9 is the quotient obtained when we divide 63 by 7.

49. In general, if we have to divide a number ab by a, it is evident that the quotient will be b; for a multiplied by b

gives the dividend ab. It is clear also, that if we had to divide ab by b, the quotient would be a. And in all examples of division that can be proposed, if we divide the dividend by the quotient, we shall again obtain the divisor; for as 24 divided by 4 gives 6, so 24 divided by 6 will give 4.

50. As the whole operation consists in representing the dividend by two factors, of which one may be equal to the divisor, and the other to the quotient, the following examples will be easily understood. I say first that the dividend abc, divided by a, gives bc; for a, multiplied by bc, produces abc: in the same manner abc, being divided by b, we shall have ac; and abc, divided by ac, gives b. It is also plain, that $12mn$, divided by $3m$, gives $4n$; for $3m$, multiplied by $4n$, makes $12mn$. But if this same number $12mn$ had been divided by 12, we should have obtained the quotient mn.

51. Since every number a may be expressed by $1a$, or a, it is evident that if we had to divide a, or $1a$, by 1, the quotient would be the same number a. And, on the contrary, if the same number a, or $1a$, is to be divided by a, the quotient will be 1.

52. It often happens that we cannot represent the dividend as the product of two factors, of which one is equal to the divisor; hence, in this case, the division cannot be performed in the manner we have described.

When we have, for example, 24 to divide by 7, it is at first sight obvious, that the number 7 is not a factor of 24; for the product of 7×3 is only 21, and consequently too small; and 7×4 makes 28, which is greater than 24. We discover, however, from this, that the quotient must be greater than 3, and less than 4. In order therefore to determine it exactly, we employ another species of numbers, which are called *fractions*, and which we shall consider in one of the following chapters.

53. Before we proceed to the use of fractions, it is usual to be satisfied with the whole number which approaches nearest to the true quotient, but at the same time paying attention to the *remainder* which is left; thus we say, 7 in 24 goes 3 times, and the remainder is 3, because 3 times 7 produces only 21, which is 3 less than 24. We may also consider the following examples in the same manner:

6)34(5, that is to say, the divisor is 6, the dividend 34, the quotient 5, and the remainder 4.
  30
  ——
   4

9)41(4, here the divisor is 9, the dividend 41, the quotient 4, and the remainder 5.
  36
  ——
   5

The following rule is to be observed in examples where there is a remainder.

54. Multiply the divisor by the quotient, and to the product add the remainder, and the result will be the dividend. This is the method of proving the division, and of discovering whether the calculation is right or not. Thus, in the first of the two last examples, if we multiply 6 by 5, and to the product 30 add the remainder 4, we obtain 34, or the dividend. And in the last example, if we multiply the divisor 9 by the quotient 4, and to the product 36 add the remainder 5, we obtain the dividend 41.

55. Lastly, it is necessary to remark here, with regard to the signs + *plus* and − *minus*, that if we divide $+ab$ by $+a$, the quotient will be $+b$, which is evident. But if we divide $+ab$ by $-a$, the quotient will be $-b$; because $-a \times -b$ gives $+ab$. If the dividend is $-ab$, and is to be divided by the divisor $+a$, the quotient will be $-b$; because it is $-b$ which, multiplied by $+a$, makes $-ab$. Lastly, if we have to divide the dividend $-ab$ by the divisor $-a$, the quotient will be $+b$; for the dividend $-ab$ is the product of $-a$ by $+b$.

56. With regard, therefore, to the signs + and −, division requires the same rules to be observed that we have seen take place in multiplication; viz.

+ by + makes +; + by − makes −;
− by + makes −; − by − makes +:

or, in few words, like signs give *plus*, and unlike signs give *minus*.

57. Thus, when we divide $18pq$ by $-3p$, the quotient is $-6q$. Farther;

$-30xy$ divided by $+6y$ gives $-5x$, and
$-54abc$ divided by $-9b$ gives $+6ac$;

for, in this last example, $-9b$ multiplied by $+6ac$ makes $-6 \times 9abc$, or $-54abc$. But enough has been said on the division of simple quantities; we shall therefore hasten to the explanation of fractions, after having added some further remarks on the nature of numbers, with respect to their divisors.

CHAP. VI.

Of the Properties *of* Integers, *with respect to their* Divisors.

58. As we have seen that some numbers are divisible by certain divisors, while others are not so; it will be proper, in order that we may obtain a more particular knowledge of numbers, that this difference should be carefully observed, both by distinguishing the numbers that are divisible by divisors from those which are not, and by considering the remainder that is left in the division of the latter. For this purpose let us examine the divisors;

2, 3, 4, 5, 6, 7, 8, 9, 10, &c.

59. First let the divisor be 2; the numbers divisible by it, are, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, &c. which, it appears, increase always by two. These numbers, as far as they can be continued, are called *even numbers*. But there are other numbers, viz.

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, &c.

which are uniformly less or greater than the former by unity, and which cannot be divided by 2, without the remainder 1; these are called *odd numbers*.

The even numbers are all comprehended in the general expression $2a$; for they are all obtained by successively substituting for a the integers 1, 2, 3, 4, 5, 6, 7, &c. and hence it follows that the odd numbers are all comprehended in the expression $2a + 1$, because $2a + 1$ is greater by unity than the even number $2a$.

60. In the second place, let the number 3 be the divisor; the numbers divisible by it are,

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, and so on;

which numbers may be represented by the expression $3a$; for $3a$, divided by 3, gives the quotient a without a remainder. All other numbers which we would divide by 3, will give 1 or 2 for a remainder, and are consequently of two kinds. Those which after the division leave the remainder 1, are,

1, 4, 7, 10, 13, 16, 19, &c.

and are contained in the expression $3a + 1$; but the other kind, where the numbers give the remainder 2, are,

2, 5, 8, 11, 14, 17, 20, &c.

which may be generally represented by $3a + 2$; so that all numbers may be expressed either by $3a$, or by $3a + 1$, or by $3a + 2$.

61. Let us now suppose that 4 is the divisor under consideration; then the numbers which it divides are,

4, 8, 12, 16, 20, 24, &c.

which increase uniformly by 4, and are comprehended in the expression $4a$. All other numbers, that is, those which are not divisible by 4, may either leave the remainder 1, or be greater than the former by 1; as,

1, 5, 9, 13, 17, 21, 25, &c.

and consequently may be comprehended in the expression $4a + 1$: or they may give the remainder 2; as,

2, 6, 10, 14, 18, 22, 26, &c.

and be expressed by $4a + 2$; or, lastly, they may give the remainder 3; as,

3, 7, 11, 15, 19, 23, 27, &c.

and may then be represented by the expression $4a + 3$.

All possible integer numbers are contained therefore in one or other of these four expressions;

$4a$, $4a + 1$, $4a + 2$, $4a + 3$.

62. It is also nearly the same when the divisor is 5; for all numbers which can be divided by it are comprehended in the expression $5a$, and those which cannot be divided by 5, are reducible to one of the following expressions:

$5a + 1$, $5a + 2$, $5a + 3$, $5a + 4$;

and in the same manner we may continue, and consider any greater divisor.

63. It is here proper to recollect what has been already said on the resolution of numbers into their simple factors; for every number, among the factors of which is found

2, or 3, or 4, or 5, or 7,

or any other number, will be divisible by those numbers. For example; 60 being equal to $2 \times 2 \times 3 \times 5$, it is evident that 60 is divisible by 2, and by 3, and by 5*.

* There are some numbers which it is easy to perceive whether they are divisors of a given number or not.

1. A given number is divisible by 2, if the last digit is even; it is divisible by 4, if the two last digits are divisible by 4; it is divisible by 8, if the three last digits are divisible by 8; and, in general, it is divisible by 2^n, if the n last digits are divisible by 2^n.

2. A number is divisible by 3, if the sum of the digits is divisible by 3; it may be divided by 6, if, beside this, the last digit is even; it is divisible by 9, if the sum of the digits may be divided by 9.

3. Every number that has the last digit 0 or 5, is divisible by 5.

64. Farther, as the general expression *abcd* is not only divisible by *a*, and *b*, and *c*, and *d*, but also by

ab, *ac*, *ad*, *bc*, *bd*, *cd*, and by
abc, *abd*, *acd*, *bcd*, and lastly by
abcd, that is to say, its own value;

it follows that 60, or $2 \times 2 \times 3 \times 5$, may be divided not only by these simple numbers, but also by those which are composed of any two of them; that is to say, by 4, 6, 10, 15: and also by those which are composed of any three of its simple factors; that is to say, by 12, 20, 30, and lastly also, by 60 itself.

65. When, therefore, we have represented any number, assumed at pleasure, by its simple factors, it will be very easy to exhibit all the numbers by which it is divisible. For we have only, first, to take the simple factors one by one, and then to multiply them together two by two,

4. A number is divisible by 11, when the sum of the first third, fifth, &c. digits is equal to the sum of the second, fourth, sixth, &c. digits.

It would be easy to explain the reason of these rules, and to extend them to the products of the divisors which we have just now considered. Rules might be devised likewise for some other numbers, but the application of them would in general be longer than an actual trial of the division.

For example, I say that the number 53704689213 is divisible by 7, because I find that the sum of the digits of the number 64004245433 is divisible by 7; and this second number is formed, according to a very simple rule, from the remainders found after dividing the component parts of the former number by 7.

Thus, $53704689213 = 50000000000 + 3000000000 + 700000000 + 0 + 4000000 + 600000 + 80000 + 9000 + 200 + 10 + 3$; which being, each of them, divided by 7, will leave the remainders 6, 4, 0, 0, 4, 2, 4, 5, 4, 3, 3, the number here given.

If *a*, *b*, *c*, *d*, *e*, &c. be the digits composing any number, the number itself may be expressed universally thus; $a + 10b + 10^2c + 10^3d + 10^4e$, &c. to $10^n z$; where it is easy to perceive that, if each of the terms a, $10b$, 10^2c, &c. be divisible by n, the number itself $a + 10b + 10^2c$, &c. will also be divisible by n. And, if $\frac{a}{n}$, $\frac{10b}{n}$, $\frac{10^2c}{n}$, &c. leave the remainders p, q, r, &c. it is obvious, that $a + 10b + 10^2c$, &c. will be divisible by n, when $p + q + r$, is divisible by n; which renders the principle of the rule sufficiently clear.

The reader is indebted to that excellent mathematician, the late Professor Bonnycastle, for this satisfactory illustration of M. Bernoulli's note.

three by three, four by four, &c. till we arrive at the number proposed.

66. It must here be particularly observed, that every number is divisible by 1; and also, that every number is divisible by itself; so that every number has at least two factors, or divisors, the number itself, and unity: but every number which has no other divisor than these two, belongs to the class of numbers, which we have before called *simple*, or *prime numbers*.

Except these simple numbers, all other numbers have, beside unity and themselves, other divisors, as may be seen from the following Table, in which are placed under each number all its divisors*.

TABLE.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	3	2	5	2	7	2	3	2	11	2	13	2	3	2	17	2	19	2
			4		3		4	9	5		3		7	5	4		3		4
					6		8		10		4		14	15	8		6		5
											6				16		9		10
											12						18		20
1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	2	6	2	6
P.	P.	P.		P.		P.				P.		P.				P.		P.	

67. Lastly, it ought to be observed that 0, or *nothing*, may be considered as a number which has the property of being divisible by all possible numbers; because by whatever number *a* we divide 0, the quotient is always 0; for it must be remarked, that the multiplication of any number by *nothing* produces nothing, and therefore 0 times *a*, or $0a$, is 0.

* A similar Table for all the divisors of the natural numbers, from 1 to 10000, was published at Leyden, in 1767, by M. Henry Anjema. We have likewise another table of divisors, which goes as far as 100000, but it gives only the least divisor of each number. It is to be found in Harris's Lexicon Technicum, the Encyclopédie, and in M. Lambert's Recueil, which we have quoted in the note to p. 11. In this last work, it is continued as far as 102000. F. T.

CHAP. VII.

Of Fractions *in general*.

68. When a number, as 7, for instance, is said not to be divisible by another number, let us suppose by 3, this only means, that the quotient cannot be expressed by an integer number; but it must not by any means be thought that it is impossible to form an idea of that quotient. Only imagine a line of 7 feet in length; nobody can doubt the possibility of dividing this line into 3 equal parts, and of forming a notion of the length of one of those parts.

69. Since therefore we may form a precise idea of the quotient obtained in similar cases, though that quotient may not be an integer number, this leads us to consider a particular species of numbers, called *fractions*, or *broken numbers*; of which the instance adduced furnishes an illustration. For if we have to divide 7 by 3, we easily conceive the quotient which should result, and express it by $\frac{7}{3}$; placing the divisor under the dividend, and separating the two numbers by a stroke, or line.

70. So, in general, when the number a is to be divided by the number b, we represent the quotient by $\frac{a}{b}$, and call this form of expression *a fraction*. We cannot therefore give a better idea of a fraction $\frac{a}{b}$, than by saying that it expresses the quotient resulting from the division of the upper number by the lower. We must remember also, that in all fractions the lower number is called the *denominator*, and that above the line the *numerator*.

71. In the above fraction $\frac{7}{3}$, which we read *seven thirds*, 7 is the numerator, and 3 the denominator. We must also read $\frac{2}{3}$, two thirds; $\frac{3}{4}$, three fourths; $\frac{3}{8}$, three eighths; $\frac{12}{100}$, twelve hundredths; and $\frac{1}{2}$, one half, &c.

72. In order to obtain a more perfect knowledge of the nature of fractions, we shall begin by considering the case in which the numerator is equal to the denominator, as in $\frac{a}{a}$. Now, since this expresses the quotient obtained by dividing a by a, it is evident that this quotient is exactly unity, and that consequently the fraction $\frac{a}{a}$ is of the same value as 1, or one integer; for the same reason, all the following fractions,

$$\tfrac{2}{2}, \tfrac{3}{3}, \tfrac{4}{4}, \tfrac{5}{5}, \tfrac{6}{6}, \tfrac{7}{7}, \tfrac{8}{8}, \text{\&c.}$$

are equal to one another, each being equal to 1, or one integer.

73. We have seen that a fraction whose numerator is equal to the denominator, is equal to unity. All fractions therefore whose numerators are less than the denominators, have a value less than unity: for if I have a number to divide by another, which is greater than itself, the result must necessarily be less than 1. If we cut a line, for example, two feet long, into three parts, one of those parts will undoubtedly be shorter than a foot: it is evident then, that $\frac{2}{3}$ is less than 1, for the same reason; that is, the numerator 2 is less than the denominator 3.

74. If the numerator, on the contrary, be greater than the denominator, the value of the fraction is greater than unity. Thus $\frac{3}{2}$ is greater than 1, for $\frac{3}{2}$ is equal to $\frac{2}{2}$ together with $\frac{1}{2}$. Now $\frac{2}{2}$ is exactly 1; consequently $\frac{3}{2}$ is equal to $1 + \frac{1}{2}$, that is, to an integer and a half. In the same manner, $\frac{4}{3}$ is equal to $1\frac{1}{3}$, $\frac{5}{3}$ to $1\frac{2}{3}$, and $\frac{7}{3}$ to $2\frac{1}{3}$. And, in general, it is sufficient in such cases to divide the upper number by the lower, and to add to the quotient a fraction, having the remainder for the numerator, and the divisor for the denominator. If the given fraction, for example, were $\frac{43}{12}$, we should have for the quotient 3, and 7 for the remainder; whence we should conclude that $\frac{43}{12}$ is the same as $3\frac{7}{12}$.

75. Thus we see how fractions, whose numerators are greater than the denominators, are resolved into two members; one of which is an integer, and the other a fractional number, having the numerator less than the denominator. Such fractions as contain one or more integers, are called *improper fractions*, to distinguish them from fractions properly so called, which having the numerator less than the denominator, are less than unity, or than an integer.

76. The nature of fractions is frequently considered in another way, which may throw additional light on the subject. If, for example, we consider the fraction $\frac{3}{4}$, it is evident that it is three times greater than $\frac{1}{4}$. Now, this fraction $\frac{1}{4}$ means, that if we divide 1 into 4 equal parts, this will be the value of one of those parts; it is obvious then, that by taking 3 of those parts we shall have the value of the fraction $\frac{3}{4}$.

In the same manner we may consider every other fraction; for example, $\frac{7}{12}$; if we divide unity into 12 equal parts, 7 of those parts will be equal to the fraction proposed.

77. From this manner of considering fractions, the expressions *numerator* and *denominator* are derived. For, as in the preceding fraction $\frac{7}{12}$, the number under the line shews that 12 is the number of parts into which unity is to be divided; and as it may be said to denote, or name, the parts, it has not improperly been called the *denominator*.

Farther, as the upper number, viz. 7, shews that, in order to have the value of the fraction, we must take, or collect, 7 of those parts, and therefore may be said to reckon or number them, it has been thought proper to call the number above the line the *numerator*.

78. As it is easy to understand what $\frac{3}{4}$ is, when we know the signification of $\frac{1}{4}$, we may consider the fractions whose numerator is unity, as the foundation of all others. Such are the fractions,

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \text{ \&c.}$$

and it is observable that these fractions go on continually diminishing; for the more you divide an integer, or the greater the number of parts into which you distribute it, the less does each of those parts become. Thus, $\frac{1}{100}$ is less than $\frac{1}{10}$; $\frac{1}{1000}$ is less than $\frac{1}{100}$; and $\frac{1}{10000}$ is less than $\frac{1}{1000}$, &c.

79. As we have seen that the more we increase the denominator of such fractions the less their values become, it may be asked, whether it is not possible to make the denominator so great that the fraction shall be reduced to nothing? I answer, no; for into whatever number of parts unity (the length of a foot, for instance) is divided; let those parts be ever so small, they will still preserve a certain magnitude, and therefore can never be absolutely reduced to nothing.

80. It is true, if we divide the length of a foot into 1000 parts, those parts will not easily fall under the cognisance of our senses; but view them through a good microscope, and each of them will appear large enough to be still subdivided into 100 parts, and more.

At present, however, we have nothing to do with what depends on ourselves, or with what we are really capable of performing, and what our eyes can perceive; the question is rather what is possible in itself: and, in this sense, it is certain, that however great we suppose the denominator, the fraction will never entirely vanish, or become equal to 0.

81. We can never therefore arrive completely at 0, or nothing, however great the denominator may be; and, consequently, as those fractions must always preserve a certain quantity, we may continue the series of fractions in the 78th article without interruption. This circumstance has introduced the expression, that the denominator must be *infinite*, or infinitely great, in order that the fraction may be reduced to 0, or to nothing; hence the word *infinite* in reality signifies here, that we can never arrive at the end of the series of the above-mentioned *fractions*.

82. To express this idea, according to the sense of it above mentioned, we make use of the sign ∞, which consequently indicates a number infinitely great; and we may therefore say, that this fraction $\frac{1}{\infty}$ is in reality nothing; because a fraction cannot be reduced to nothing, until the denominator has been increased to *infinity*.

83. It is the more necessary to pay attention to this idea of infinity, as it is derived from the first elements of our knowledge, and as it will be of the greatest importance in the following part of this treatise.

We may here deduce from it a few consequences that are extremely curious, and worthy of attention. The fraction $\frac{1}{\infty}$ represents the quotient resulting from the division of the dividend 1 by the divisor ∞. Now, we know, that if we divide the dividend 1 by the quotient $\frac{1}{\infty}$, which is equal to nothing, we obtain again the divisor ∞: hence we acquire a new idea of infinity; and learn that it arises from the division of 1 by 0; so that we are thence authorised in saying, that 1 divided by 0 expresses a number infinitely great, or ∞.

84. It may be necessary also, in this place, to correct the mistake of those who assert, that a number infinitely great is not susceptible of increase. This opinion is inconsistent with the just principles which we have laid down; for $\frac{1}{0}$ signifying a number infinitely great, and $\frac{2}{0}$ being incontestably the double of $\frac{1}{0}$, it is evident that a number, though infinitely great, may still become twice, thrice, or any number of times greater *.

* There appears to be a fallacy in this reasoning, which consists in taking the sign of infinity for infinity itself; and applying the property of fractions in general to a fractional expression, whose denominator bears no assignable relation to unity. It is certain, that infinity may be represented by a series of units (that is, by $\frac{1}{0} = \frac{1}{1-1} = 1 + 1 + 1$, &c.) or by a series of numbers increasing in any given ratio. Now, though any definite part of one infinite series may be the half, the third, &c. of a definite part of another, yet still that part bears no proportion to the whole, and the series can only be said, in that case, to go on to infinity in a different ratio. But, farther, $\frac{2}{0}$, or any other nu-

CHAP. VIII.

Of the Properties *of* Fractions.

85. We have already seen, that each of the fractions,

$$\tfrac{2}{2}, \tfrac{3}{3}, \tfrac{4}{4}, \tfrac{5}{5}, \tfrac{6}{6}, \tfrac{7}{7}, \tfrac{8}{8}, \text{\&c.}$$

makes an integer, and that consequently they are all equal to one another. The same equality prevails in the following fractions,

$$\tfrac{2}{1}, \tfrac{4}{2}, \tfrac{6}{3}, \tfrac{8}{4}, \tfrac{10}{5}, \tfrac{12}{6}, \text{\&c.}$$

each of them making two integers; for the numerator of each, divided by its denominator, gives 2. So all the fractions

$$\tfrac{3}{1}, \tfrac{6}{2}, \tfrac{9}{3}, \tfrac{12}{4}, \tfrac{15}{5}, \tfrac{18}{6}, \text{\&c.}$$

are equal to one another, since 3 is their common value.

86. We may likewise represent the value of any fraction in an infinite variety of ways. For if we multiply both the numerator and the denominator of a fraction by the same number, which may be assumed at pleasure, this fraction will still preserve the same value. For this reason, all the fractions

$$\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \tfrac{5}{10}, \tfrac{6}{12}, \tfrac{7}{14}, \tfrac{8}{16}, \tfrac{9}{18}, \tfrac{10}{20}, \text{\&c.}$$

are equal, the value of each being $\tfrac{1}{2}$. Also,

$$\tfrac{1}{3}, \tfrac{2}{6}, \tfrac{3}{9}, \tfrac{4}{12}, \tfrac{5}{15}, \tfrac{6}{18}, \tfrac{7}{21}, \tfrac{8}{24}, \tfrac{9}{27}, \tfrac{10}{30}, \text{\&c.}$$

are equal fractions, the value of each being $\tfrac{1}{3}$. The fractions

$$\tfrac{2}{3}, \tfrac{4}{6}, \tfrac{8}{12}, \tfrac{10}{15}, \tfrac{12}{18}, \tfrac{14}{21}, \tfrac{16}{24}, \text{\&c.}$$

have likewise all the same value. Hence we may conclude, in general, that the fraction $\frac{a}{b}$ may be represented by any of the following expressions, each of which is equal to $\frac{a}{b}$; viz.

merator, having 0 for its denominator, is, when expanded, precisely the same as $\frac{1}{0}$.

Thus, $\frac{2}{0} = \frac{2}{2-2}$, by division becomes

$$\begin{array}{l} 2-2)2\ (1+1+1, \text{\&c. ad infinitum} \\ \quad\ \ 2-2 \\ \quad\ \ \overline{\quad\ \ 2} \\ \quad\quad\ \ 2-2 \\ \quad\quad\ \ \overline{\quad\ \ 2} \\ \quad\quad\quad\ \ 2-2 \\ \quad\quad\quad\ \ \overline{\quad\ \ 2, \text{\&c.}} \end{array}$$

$$\frac{a}{b}, \frac{2a}{2b}, \frac{3a}{3b}, \frac{4a}{4b}, \frac{5a}{5b}, \frac{6a}{6b}, \frac{7a}{7b}, \text{\&c.}$$

87. To be convinced of this, we have only to write for the value of the fraction $\frac{a}{b}$ a certain letter c, representing by this letter c the quotient of the division of a by b; and to recollect that the multiplication of the quotient c by the divisor b must give the dividend. For since c multiplied by b gives a, it is evident that c multiplied by $2b$ will give $2a$, that c multiplied by $3b$ will give $3a$, and that, in general, c multiplied by mb will give ma. Now, changing this into an example of division, and dividing the product ma by mb, one of the factors, the quotient must be equal to the other factor c; but ma divided by mb gives also the fraction $\frac{ma}{mb}$, which is consequently equal to c; which is what was to be proved: for c having been assumed as the value of the fraction $\frac{a}{b}$, it is evident that this fraction is equal to the fraction $\frac{ma}{mb}$, whatever be the value of m.

88. We have seen that every fraction may be represented in an infinite number of forms, each of which contains the same value; and it is evident that of all these forms, that which is composed of the least numbers, will be most easily understood. For example, we might substitute, instead of $\tfrac{2}{3}$, the following fractions,

$$\tfrac{4}{6}, \tfrac{6}{9}, \tfrac{8}{12}, \tfrac{10}{15}, \tfrac{12}{18}, \text{\&c.}$$

but of all these expressions $\tfrac{2}{3}$ is that of which it is easiest to form an idea. Here therefore a problem arises, how a fraction, such as $\tfrac{8}{12}$, which is not expressed by the least possible numbers, may be reduced to its simplest form, or to *its least terms*; that is to say, in our present example, to $\tfrac{2}{3}$.

89. It will be easy to resolve this problem, if we consider that a fraction still preserves its value, when we multiply both its terms, or its numerator and denominator, by the same number. For from this it also follows, that if we divide the numerator and denominator of a fraction by the same number, the fraction will still preserve the same value. This is made more evident by means of the general expression $\frac{ma}{mb}$; for if we divide both the numerator ma and the denominator mb by the number m, we obtain the fraction $\frac{a}{b}$, which, as was before proved, is equal to $\frac{ma}{mb}$.

90. In order therefore to reduce a given fraction to its least terms, it is required to find a number, by which both the numerator and denominator may be divided. Such a number is called a *common divisor*; and as long as we can find a common divisor to the numerator and the denominator, it is certain that the fraction may be reduced to a lower form; but, on the contrary, when we see that, except unity, no other common divisor can be found, this shews that the fraction is already in its simplest form.

91. To make this more clear, let us consider the fraction $\frac{48}{120}$. We see immediately that both the terms are divisible by 2, and that there results the fraction $\frac{24}{60}$; which may also be divided by 2, and reduced to $\frac{12}{30}$; and as this likewise has 2 for a common divisor, it is evident that it may be reduced to $\frac{6}{15}$. But now we easily perceive, that the numerator and denominator are still divisible by 3; performing this division, therefore, we obtain the fraction $\frac{2}{5}$, which is equal to the fraction proposed, and gives the simplest expression to which it can be reduced; for 2 and 5 have no common divisor but 1, which cannot diminish these numbers any farther.

92. This property of fractions preserving an invariable value, whether we divide or multiply the numerator and denominator by the same number, is of the greatest importance, and is the principal foundation of the doctrine of fractions. For example, we can seldom add together two fractions, or subtract the one from the other, before we have, by means of this property, reduced them to other forms; that is to say, to expressions whose denominators are equal. Of this we shall treat in the following chapter.

93. We will conclude the present, however, by remarking, that all whole numbers may also be represented by fractions. For example, 6 is the same as $\frac{6}{1}$, because 6 divided by 1 makes 6; we may also, in the same manner, express the number 6 by the fractions $\frac{12}{2}$, $\frac{18}{3}$, $\frac{24}{4}$, $\frac{36}{6}$, and an infinite number of others, which have the same value.

QUESTIONS FOR PRACTICE.

1. Reduce $\frac{cx+x^2}{ca^2+a^2x}$ to its lowest terms. *Ans.* $\frac{x}{a^2}$.

2. Reduce $\frac{x^3-b^2x}{x^2+2bx+b^2}$ to its lowest terms. *Ans.* $\frac{x^2-bx}{x+b}$.

3. Reduce $\frac{x^4-b^4}{x^5-b^2x^3}$ to its lowest terms. *Ans.* $\frac{x^2+b^2}{x^3}$.

4. Reduce $\frac{x^2-y^2}{x^4-y^4}$ to its lowest terms. *Ans.* $\frac{1}{x^2+y^2}$.

5. Reduce $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms. *Ans.* $\frac{a^2+x^2}{a-x}$.

6. Reduce $\frac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$ to its lowest terms. *Ans.* $\frac{5a^4+5a^3x}{a^2x+ax^2+x^3}$.

CHAP. IX.

Of the Addition *and* Subtraction *of* Fractions.

94. When fractions have equal denominators, there is no difficulty in adding and subtracting them; for $\frac{2}{7}+\frac{3}{7}$ is equal to $\frac{5}{7}$, and $\frac{4}{7}-\frac{2}{7}$ is equal to $\frac{2}{7}$. In this case, therefore, either for addition or subtraction, we alter only the numerators, and place the common denominator under the line, thus;

$\frac{7}{100}+\frac{9}{100}-\frac{12}{100}-\frac{15}{100}+\frac{20}{100}$ is equal to $\frac{9}{100}$;

$\frac{24}{50}-\frac{7}{50}-\frac{12}{50}+\frac{31}{50}$ is equal to $\frac{36}{50}$, or $\frac{18}{25}$;

$\frac{16}{20}-\frac{3}{20}-\frac{11}{20}+\frac{14}{20}$ is equal to $\frac{16}{20}$, or $\frac{4}{5}$;

also $\frac{1}{3}+\frac{2}{3}$ is equal to $\frac{3}{3}$, or 1, that is to say, an integer; and $\frac{2}{4}-\frac{3}{4}-\frac{1}{4}+\frac{2}{4}$ is equal to $\frac{0}{4}$, that is to say, nothing, or 0.

95. But when fractions have not equal denominators, we can always change them into other fractions that have the same denominator. For example, when it is proposed to add together the fractions $\frac{1}{2}$ and $\frac{1}{3}$, we must consider that $\frac{1}{2}$ is the same as $\frac{3}{6}$, and that $\frac{1}{3}$ is equivalent to $\frac{2}{6}$; we have therefore, instead of the two fractions proposed, $\frac{3}{6}+\frac{2}{6}$, the sum of which is $\frac{5}{6}$. And if the two fractions were united by the sign *minus* as $\frac{1}{2}-\frac{1}{3}$, we should have $\frac{3}{6}-\frac{2}{6}$, or $\frac{1}{6}$.

As another example, let the fractions proposed be $\frac{3}{4}+\frac{5}{8}$. Here, since $\frac{3}{4}$ is the same as $\frac{6}{8}$, this value may be substituted for $\frac{3}{4}$, and we may then say $\frac{6}{8}+\frac{5}{8}$ makes $\frac{11}{8}$, or $1\frac{3}{8}$.

Suppose farther, that the sum of $\frac{1}{3}$ and $\frac{1}{4}$ were required, I say that it is $\frac{7}{12}$; for $\frac{1}{3}=\frac{4}{12}$, and $\frac{1}{4}=\frac{3}{12}$: therefore $\frac{4}{12}+\frac{3}{12}=\frac{7}{12}$.

96. We may have a greater number of fractions to reduce

to a common denominator; for example, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$. In this case, the whole depends on finding a number that shall be divisible by all the denominators of those fractions. In this instance, 60 is the number which has that property, and which consequently becomes the common denominator. We shall therefore have $\frac{30}{60}$, instead of $\frac{1}{2}$; $\frac{40}{60}$, instead of $\frac{2}{3}$; $\frac{45}{60}$, instead of $\frac{3}{4}$; $\frac{48}{60}$, instead of $\frac{4}{5}$; and $\frac{50}{60}$, instead of $\frac{5}{6}$. If now it be required to add together all these fractions $\frac{30}{60}$, $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$, and $\frac{50}{60}$; we have only to add all the numerators, and under the sum place the common denominator 60; that is to say, we shall have $\frac{213}{60}$, or 3 integers, and the fractional remainder $\frac{11}{20}$.

97. The whole of this operation consists, as we before stated, in changing fractions, whose denominators are unequal, into others whose denominators are equal. In order, therefore, to perform it generally, let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions proposed. First, multiply the two terms of the first fraction by d, and we shall have the fraction $\frac{ad}{bd}$ equal to $\frac{a}{b}$; next multiply the two terms of the second fraction by b, and we shall have an equivalent value of it expressed by $\frac{bc}{bd}$; thus the two denominators are become equal. Now, if the sum of the two proposed fractions be required, we may immediately answer that it is $\frac{ad+bc}{bd}$; and if their difference be asked, we say that it is $\frac{ad-bc}{bd}$. If the fractions $\frac{5}{8}$ and $\frac{7}{9}$, for example, were proposed, we should obtain, in their stead, $\frac{45}{72}$ and $\frac{56}{72}$; of which the sum is $\frac{101}{72}$ and the difference $\frac{11}{72}$*.

98. To this part of the subject belongs also the question, Of two proposed fractions which is the greater or the less?

* The rule for reducing fractions to a common denominator may be concisely expressed thus. Multiply each numerator into every denominator except its own, for a new numerator, and all the denominators together for the common denominator. When this operation has been performed, it will appear that the numerator and denominator of each fraction have been multiplied by the same quantity, and consequently retain the same value.

for, to resolve this, we have only to reduce the two fractions to the same denominator. Let us take, for example, the two fractions $\frac{2}{3}$ and $\frac{5}{7}$; when reduced to the same denominator, the first becomes $\frac{14}{21}$, and the second $\frac{15}{21}$, where it is evident that the second, or $\frac{5}{7}$, is the greater, and exceeds the former by $\frac{1}{21}$.

Again, if the fractions $\frac{3}{5}$ and $\frac{5}{8}$ be proposed, we shall have to substitute for them $\frac{24}{40}$ and $\frac{25}{40}$; whence we may conclude that $\frac{5}{8}$ exceeds $\frac{3}{5}$, but only by $\frac{1}{40}$.

99. When it is required to subtract a fraction from an integer, it is sufficient to change one of the units of that integer into a fraction, which has the same denominator as that which is to be subtracted; then in the rest of the operation there is no difficulty. If it be required, for example, to subtract $\frac{2}{3}$ from 1, we write $\frac{3}{3}$ instead of 1, and say that $\frac{2}{3}$ taken from $\frac{3}{3}$ leaves the remainder $\frac{1}{3}$. So $\frac{5}{12}$, subtracted from 1, leaves $\frac{7}{12}$.

If it were required to subtract $\frac{3}{4}$ from 2, we should write 1 and $\frac{4}{4}$ instead of 2, and should then immediately see that after the subtraction there must remain $1\frac{1}{4}$.

100. It happens also sometimes, that having added two or more fractions together, we obtain more than an integer; that is to say, a numerator greater than the denominator: this is a case which has already occurred, and deserves attention.

We found, for example [Article 96], that the sum of the five fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$ was $\frac{213}{60}$, and remarked that the value of this sum was $3\frac{33}{60}$ or $3\frac{11}{20}$. Likewise, $\frac{2}{3}+\frac{3}{4}$, or $\frac{8}{12}+\frac{9}{12}$, makes $\frac{17}{12}$, or $1\frac{5}{12}$. We have therefore only to perform the actual division of the numerator by the denominator, to see how many integers there are for the quotient, and to set down the remainder.

Nearly the same must be done to add together numbers compounded of integers and fractions; we first add the fractions, and if the sum produces one or more integers, these are added to the other integers. If it be proposed, for example, to add $3\frac{1}{2}$ and $2\frac{2}{3}$; we first take the sum of $\frac{1}{2}$ and $\frac{2}{3}$, or of $\frac{3}{6}$ and $\frac{4}{6}$, which is $\frac{7}{6}$, or $1\frac{1}{6}$; and thus we find the total sum to be $6\frac{1}{6}$.

QUESTIONS FOR PRACTICE.

1. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to a common denominator.

Ans. $\frac{2cx}{ac}$ and $\frac{ab}{ac}$.

2. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to a common denominator.

Ans. $\frac{ac}{bc}$ and $\frac{ab+b^2}{bc}$.

3. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d to fractions having a common denominator.

Ans. $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$ and $\frac{6acd}{6ac}$.

4. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a+\frac{2x}{a}$ to a common denominator.

Ans. $\frac{9a}{12a}$, $\frac{8ax}{12a}$, and $\frac{12a^2+24x}{12a}$.

5. Reduce $\frac{1}{2}$, $\frac{a^2}{3}$, and $\frac{x^2+a^2}{x+a}$, to a common denominator.

Ans. $\frac{3x+3a}{6x+6a}$, $\frac{2a^2x+2a^3}{6x+6a}$, $\frac{6x^2+6a^2}{6x+6a}$.

6. Reduce $\frac{b}{2a^2}$, $\frac{c}{2a}$, and $\frac{d}{a}$, to a common denominator.

Ans. $\frac{2a^2b}{4a^4}$, $\frac{2a^3c}{4a^4}$, and $\frac{4a^3d}{4a^4}$; or $\frac{b}{2a^2}$, $\frac{ac}{2a^2}$, and $\frac{2ad}{2a^2}$.

CHAP. X.

Of the Multiplication *and* Division *of* Fractions.

101. The rule for the multiplication of a fraction by an integer, or whole number, is to multiply the numerator only by the given number, and not to change the denominator: thus,

2 times, or twice $\frac{1}{2}$ makes $\frac{2}{2}$, or 1 integer;
2 times, or twice $\frac{1}{3}$ makes $\frac{2}{3}$; and
3 times, or thrice $\frac{1}{6}$ makes $\frac{3}{6}$, or $\frac{1}{2}$;
4 times $\frac{5}{12}$ makes $\frac{20}{12}$, or $1\frac{8}{12}$, or $1\frac{2}{3}$.

But, instead of this rule, we may use that of dividing the denominator by the given integer, which is preferable, when it can be done, because it shortens the operation. Let it be required, for example, to multiply $\frac{8}{9}$ by 3; if we multiply the numerator by the given integer we obtain $\frac{24}{9}$, which

product we must reduce to $\frac{8}{3}$. But if we do not change the numerator, and divide the denominator by the integer, we find immediately $\frac{8}{3}$, or $2\frac{2}{3}$, for the given product; and, in the same manner, $\frac{13}{24}$ multiplied by 6 gives $\frac{13}{4}$, or $3\frac{1}{4}$.

102. In general, therefore, the product of the multiplication of a fraction $\frac{a}{b}$ by c is $\frac{ac}{b}$; and here it may be remarked, when the integer is exactly equal to the denominator, that the product must be equal to the numerator.

So that $\begin{cases} \frac{1}{2} \text{ taken twice, gives } 1; \\ \frac{2}{3} \text{ taken thrice, gives } 2; \\ \frac{3}{4} \text{ taken four times, gives } 3. \end{cases}$

And, in general, if we multiply the fraction $\frac{a}{b}$ by the number b, the product must be a, as we have already shewn; for since $\frac{a}{b}$ expresses the quotient resulting from the division of the dividend a by the divisor b, and because it has been demonstrated that the quotient multiplied by the divisor will give the dividend, it is evident that $\frac{a}{b}$ multiplied by b must produce a.

103. Having thus shewn how a fraction is to be multiplied by an integer; let us now consider also how a fraction is to be divided by an integer. This inquiry is necessary, before we proceed to the multiplication of fractions by fractions. It is evident, if we have to divide the fraction $\frac{2}{3}$ by 2, that the result must be $\frac{1}{3}$; and that the quotient of $\frac{6}{7}$ divided by 3 is $\frac{2}{7}$. The rule therefore is, to divide the numerator by the integer without changing the denominator. Thus:

$\frac{12}{25}$ divided by 2 gives $\frac{6}{25}$;
$\frac{12}{25}$ divided by 3 gives $\frac{4}{25}$; and
$\frac{12}{25}$ divided by 4 gives $\frac{3}{25}$; &c.

104. This rule may be easily practised, provided the numerator be divisible by the number proposed; but very often it is not: it must therefore be observed, that a fraction may be transformed into an infinite number of other expressions, and in that number there must be some, by which the numerator might be divided by the given integer. If it were required, for example, to divide $\frac{3}{4}$ by 2, we should change the fraction into $\frac{6}{8}$, and then dividing the numerator by 2, we should immediately have $\frac{3}{8}$ for the quotient sought.

In general, if it be proposed to divide the fraction $\frac{a}{b}$ by c, we change it into $\frac{ac}{bc}$, and then dividing the numerator ac by c, write $\frac{a}{bc}$ for the quotient sought.

105. When therefore a fraction $\frac{a}{b}$ is to be divided by an integer c, we have only to multiply the denominator by that number, and leave the numerator as it is. Thus $\frac{5}{8}$ divided by 3 gives $\frac{5}{24}$, and $\frac{9}{16}$ divided by 5 gives $\frac{9}{80}$.

This operation becomes easier, when the numerator itself is divisible by the integer, as we have supposed in article 103. For example, $\frac{9}{16}$ divided by 3 would give, according to our last rule, $\frac{9}{48}$; but by the first rule, which is applicable here, we obtain $\frac{3}{16}$, an expression equivalent to $\frac{9}{48}$, but more simple.

106. We shall now be able to understand how one fraction $\frac{a}{b}$ may be multiplied by another fraction $\frac{c}{d}$. For this purpose, we have only to consider that $\frac{c}{d}$ means that c is divided by d; and on this principle we shall first multiply the fraction $\frac{a}{b}$ by c, which produces the result $\frac{ac}{b}$; after which we shall divide by d, which gives $\frac{ac}{bd}$.

Hence the following rule for multiplying fractions. Multiply the numerators together for a numerator, and the denominators together for a denominator.

Thus $\frac{1}{2}$ by $\frac{2}{3}$ gives the product $\frac{2}{6}$, or $\frac{1}{3}$;
$\frac{2}{3}$ by $\frac{4}{5}$ makes $\frac{8}{15}$;
$\frac{3}{4}$ by $\frac{5}{12}$ produces $\frac{15}{48}$, or $\frac{5}{16}$; &c.

107. It now remains to shew how one fraction may be divided by another. Here we remark first, that if the two fractions have the same number for a denominator, the division takes place only with respect to the numerators; for it is evident, that $\frac{3}{13}$ are contained as many times in $\frac{9}{13}$ as 3 is contained in 9, that is to say, three times; and, in the same manner, in order to divide $\frac{8}{13}$ by $\frac{9}{13}$, we have only to divide 8 by 9, which gives $\frac{8}{9}$. We shall also have $\frac{6}{20}$ in $\frac{18}{20}$, 3 times; $\frac{7}{100}$ in $\frac{49}{100}$, 7 times; $\frac{7}{25}$ in $\frac{6}{25}$, $\frac{6}{7}$, &c.

108. But when the fractions have not equal denominators, we must have recourse to the method already mentioned for reducing them to a common denominator. Let there be, for example, the fraction $\frac{a}{b}$ to be divided by the fraction $\frac{c}{d}$. We first reduce them to the same denominator, and there results $\frac{ad}{bd}$ to be divided by $\frac{cb}{db}$; it is now evident that the quotient must be represented simply by the division of ad by bc; which gives $\frac{ad}{bc}$.

Hence the following rule: Multiply the numerator of the dividend by the denominator of the divisor, and the denominator of the dividend by the numerator of the divisor; then the first product will be the numerator of the quotient, and the second will be its denominator.

109. Applying this rule to the division of $\frac{5}{8}$ by $\frac{2}{3}$, we shall have the quotient $\frac{15}{16}$; also the division of $\frac{3}{4}$ by $\frac{1}{2}$ will give $\frac{6}{4}$, or $\frac{3}{2}$, or $1\frac{1}{2}$; and $\frac{25}{48}$ by $\frac{5}{6}$ will give $\frac{150}{240}$, or $\frac{5}{8}$.

110. This rule for division is often expressed in a manner that is more easily remembered, as follows: Invert the terms of the divisor, so that the denominator may be in the place of the numerator, and the latter be written under the line; then multiply the fraction, which is the dividend by this inverted fraction, and the product will be the quotient sought. Thus, $\frac{3}{4}$ divided by $\frac{1}{2}$ is the same as $\frac{3}{4}$ multiplied by $\frac{2}{1}$, which makes $\frac{6}{4}$, or $1\frac{1}{2}$. Also $\frac{5}{8}$ divided by $\frac{2}{3}$ is the same as $\frac{5}{8}$ multiplied by $\frac{3}{2}$, which is $\frac{15}{16}$; or $\frac{25}{48}$ divided by $\frac{5}{6}$ gives the same as $\frac{25}{48}$ multiplied by $\frac{6}{5}$, the product of which is $\frac{150}{240}$, or $\frac{5}{8}$.

We see then, in general, that to divide by the fraction $\frac{1}{2}$ is the same as to multiply by $\frac{2}{1}$, or 2; and that dividing by $\frac{1}{3}$ amounts to multiplying by $\frac{3}{1}$, or by 3, &c.

111. The number 100 divided by $\frac{1}{2}$ will give 200; and 1000 divided by $\frac{1}{3}$ will give 3000. Farther, if it were required to divide 1 by $\frac{1}{1000}$, the quotient would be 1000; and dividing 1 by $\frac{1}{100000}$, the quotient is 100000. This enables us to conceive that, when any number is divided by 0, the result must be a number indefinitely great; for even the division of 1 by the small fraction $\frac{1}{1000000000}$ gives for the quotient the very great number 1000000000.

112. Every number, when divided by itself, producing unity, it is evident that a fraction divided by itself must also give 1 for the quotient; and the same follows from our rule: for, in order to divide $\frac{3}{4}$ by $\frac{3}{4}$, we must multiply $\frac{3}{4}$ by $\frac{4}{3}$, in

which case we obtain $\frac{12}{12}$, or 1; and if it be required to divide $\frac{a}{b}$ by $\frac{a}{b}$, we multiply $\frac{a}{b}$ by $\frac{b}{a}$; where the product $\frac{ab}{ab}$ is also equal to 1.

113. We have still to explain an expression which is frequently used. It may be asked, for example, what is the half of $\frac{3}{4}$? This means, that we must multiply $\frac{3}{4}$ by $\frac{1}{2}$. So likewise, if the value of $\frac{2}{3}$ of $\frac{5}{8}$ were required, we should multiply $\frac{5}{8}$ by $\frac{2}{3}$, which produces $\frac{10}{24}$; and $\frac{3}{4}$ of $\frac{9}{16}$ is the same as $\frac{9}{16}$ multiplied by $\frac{3}{4}$, which produces $\frac{27}{64}$.

114. Lastly, we must here observe, with respect to the signs + and −, the same rules that we before laid down for integers. Thus $+\frac{1}{2}$ multiplied by $-\frac{1}{3}$, makes $-\frac{1}{6}$; and $-\frac{2}{3}$ multiplied by $-\frac{4}{5}$, gives $+\frac{8}{15}$. Farther $-\frac{5}{8}$ divided by $+\frac{2}{3}$, gives $-\frac{15}{16}$; and $-\frac{3}{4}$ divided by $-\frac{3}{4}$, gives $+\frac{12}{12}$, or +1.

QUESTIONS FOR PRACTICE.

1. Required the product of $\frac{x}{6}$ and $\frac{2x}{9}$. *Ans.* $\frac{x^2}{27}$.

2. Required the product of $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$. *Ans.* $\frac{4x^3}{21}$.

3. Required the product of $\frac{x}{a}$ and $\frac{x+a}{a+c}$. *Ans.* $\frac{x^2+ax}{a^2+ac}$.

4. Required the product of $\frac{3x}{2}$ and $\frac{3a}{b}$. *Ans.* $\frac{9ax}{2b}$.

5. Required the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$. *Ans.* $\frac{3x^3}{5a}$.

6. Required the product of $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$. *Ans.* $9ax$.

7. Required the product of $b+\frac{bx}{a}$ and $\frac{a}{x}$.

Ans. $\frac{ab+bx}{x}$.

8. Required the product of $\frac{x^2-b^2}{bc}$ and $\frac{x^2+b^2}{b+c}$.

Ans. $\frac{x^4-b^4}{b^2c+bc^2}$.

9. Required the product of x, $\frac{x+1}{a}$, and $\frac{x-1}{a+b}$.

Ans. $\frac{x^3-x}{a^2+ab}$.

10. Required the quotient of $\frac{x}{3}$ divided by $\frac{2x}{9}$. *Ans.* $1\frac{1}{2}$.

11. Required the quotient of $\frac{2a}{b}$ divided by $\frac{4c}{d}$.

Ans. $\frac{ad}{2bc}$.

12. Required the quotient of $\frac{x+a}{2x-2b}$ divided by $\frac{x+b}{5x+a}$.

Ans. $\frac{5x^2+6ax+a^2}{2x^2-2b^2}$.

13. Required the quotient of $\frac{2x^2}{a^3+x^3}$ divided by $\frac{x}{x+a}$.

Ans. $\frac{2x^2+2ax}{x^3+a^3}$.

14. Required the quotient of $\frac{7x}{5}$ divided by $\frac{12}{13}$. *Ans.* $\frac{91x}{60}$.

15. Required the quotient of $\frac{4x^2}{7}$ divided by $5x$. *Ans.* $\frac{4x}{35}$.

16. Required the quotient of $\frac{x+1}{6}$ divided by $\frac{2x}{3}$.

Ans. $\frac{x+1}{4x}$.

17. Required the quotient of $\frac{x-b}{8cd}$ divided by $\frac{3cx}{4d}$.

Ans. $\frac{x-b}{6c^2x}$.

18. Required the quotient of $\frac{x^4-b^4}{x^2-2bx+b^2}$ divided by $\frac{x^2+bx}{x-b}$.

Ans. $x+\frac{b^2}{x}$.

CHAP. XI.

Of Square Numbers.

115. The product of a number, when multiplied by itself, is called *a square;* and, for this reason, the number, considered in relation to such a product, is called *a square root.* For example, when we multiply 12 by 12, the product 144 is a square, of which the root is 12.

The origin of this term is borrowed from geometry, which teaches us that the contents of a square are found by multiplying its side by itself.

116. Square numbers are found therefore by multiplication; that is to say, by multiplying the root by itself: thus, 1 is the square of 1, since 1 multiplied by 1 makes 1; likewise, 4 is the square of 2; and 9 the square of 3; 2 also is the root of 4, and 3 is the root of 9.

We shall begin by considering the squares of natural numbers; and for this purpose shall give the following small Table, on the first line of which several numbers, or roots, are ranged, and on the second their squares *.

Numbers.	1	2	3	4	5	6	7	8	9	10	11	12	13
Squares.	1	4	9	16	25	36	49	64	81	100	121	144	169

117. Here it will be readily perceived that the series of square numbers thus arranged has a singular property; namely, that if each of them be subtracted from that which immediately follows, the remainders always increase by 2, and form this series;

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, &c.

which is that of the odd numbers.

118. The squares of fractions are found in the same manner, by multiplying any given fraction by itself. For example, the square of $\frac{1}{2}$ is $\frac{1}{4}$,

$$\text{The square of} \left\{\begin{array}{l} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{4} \\ \frac{3}{4} \end{array}\right\} \text{is} \begin{array}{l} \frac{1}{9}; \\ \frac{4}{9}; \\ \frac{1}{16}; \\ \frac{9}{16}, \text{ \&c.} \end{array}$$

We have only therefore to divide the square of the numerator by the square of the denominator, and the fraction which expresses that division will be the square of the given fraction; thus, $\frac{25}{64}$ is the square of $\frac{5}{8}$; and reciprocally, $\frac{5}{8}$ is the root of $\frac{25}{64}$.

119. When the square of a mixed number, or a number composed of an integer and a fraction, is required, we have only to reduce it to a single fraction, and then take the square of that fraction. Let it be required, for example, to find the square of $2\frac{1}{2}$; we first express this number by $\frac{5}{2}$, and taking the square of that fraction, we have $\frac{25}{4}$, or $6\frac{1}{4}$, for the value of the square of $2\frac{1}{2}$. Also to obtain the square of $3\frac{1}{4}$, we say $3\frac{1}{4}$ is equal to $\frac{13}{4}$; therefore its square is equal to $\frac{169}{16}$, or to $10\frac{9}{16}$. The squares of the numbers between 3 and 4, supposing them to increase by one fourth, are as follows:

Numbers.	3	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	4
Squares.	9	$10\frac{9}{16}$	$12\frac{1}{4}$	$14\frac{1}{16}$	16

From this small Table we may infer, that if a root contain a fraction, its square also contains one. Let the root, for example, be $1\frac{5}{12}$; its square is $\frac{289}{144}$, or $2\frac{1}{144}$; that is to say, a little greater than the integer 2.

120. Let us now proceed to general expressions. First, when the root is a, the square must be aa; if the root be $2a$, the square is $4aa$; which shews that by doubling the root, the square becomes 4 times greater; also, if the root be $3a$, the square is $9aa$; and if the root be $4a$, the square is $16aa$. Farther, if the root be ab, the square is $aabb$; and if the root be abc, the square is $aabbcc$; or $a^2b^2c^2$.

121. Thus, when the root is composed of two, or more factors, we multiply their squares together; and reciprocally, if a square be composed of two, or more factors, of which each is a square, we have only to multiply together the roots of those squares, to obtain the complete root of the square proposed. Thus, 2304 is equal to $4 \times 16 \times 36$, the square root of which is $2 \times 4 \times 6$, or 48; and 48 is found to be the true square root of 2304, because 48×48 gives 2304.

122. Let us now consider what must be observed on this subject with regard to the signs $+$ and $-$. First, it is

* We have very complete tables for the squares of natural numbers, published under the title "Tetragonometria Tabularia, &c. Auct. J. Jobo Ludolfo, Amstelodami, 1690, in 4to." These Tables are continued from 1 to 100000, not only for finding those squares, but also the products of any two numbers less than 100000; not to mention several other uses, which are explained in the introduction to the work. F. T.

evident that if the root have the sign $+$, that is to say, if it be a positive number, its square must necessarily be a positive number also, because $+$ multiplied by $+$ makes $+$: hence the square of $+a$ will be $+aa$: but if the root be a negative number, as $-a$, the square is still positive, for it is $+aa$. We may therefore conclude, that $+aa$ is the square both of $+a$ and of $-a$, and that consequently every square has two roots, one positive, and the other negative. The square root of 25, for example, is both $+5$ and -5, because -5 multiplied by -5 gives 25, as well as $+5$ by $+5$.

CHAP. XII.

Of Square Roots, *and of* Irrational Numbers *resulting from them.*

123. What we have said in the preceding chapter amounts to this; that the square root of a given number is that number whose square is equal to the given number; and that we may put before those roots either the positive, or the negative sign.

124. So that when a square number is given, provided we retain in our memory a sufficient number of square numbers, it is easy to find its root. If 196, for example, be the given number, we know that its square root is 14.

Fractions, likewise, are easily managed in the same way. It is evident, for example, that $\frac{5}{7}$ is the square root of $\frac{25}{49}$; to be convinced of which, we have only to take the square root of the numerator and that of the denominator.

If the number proposed be a mixed number, as $12\frac{1}{4}$, we reduce it to a single fraction, which, in this case, will be $\frac{49}{4}$; and from this we immediately perceive that $\frac{7}{2}$, or $3\frac{1}{2}$, must be the square root of $12\frac{1}{4}$.

125. But when the given number is not a square, as 12, for example, it is not possible to extract its square root; or to find a number, which, multiplied by itself, will give the product 12. We know, however, that the square root of 12 must be greater than 3, because 3×3 produces only 9; and less than 4, because 4×4 produces 16, which is more than 12; we know also, that this root is less than $3\frac{1}{2}$, for we have seen that the square of $3\frac{1}{2}$, or $\frac{7}{2}$, is $12\frac{1}{4}$; and we may approach still nearer to this root, by comparing it with $3\frac{7}{15}$; for the square of $3\frac{7}{15}$, or of $\frac{52}{15}$, is $\frac{2704}{225}$, or $12\frac{4}{225}$; so that this

fraction is still greater than the root required, though but very little so, as the difference of the two squares is only $\frac{4}{225}$.

126. We may suppose that as $3\frac{1}{2}$ and $3\frac{7}{15}$ are numbers greater than the root of 12, it might be possible to add to 3 a fraction a little less than $\frac{7}{15}$, and precisely such, that the square of the sum would be equal to 12.

Let us therefore try with $3\frac{3}{7}$, since $\frac{3}{7}$ is a little less than $\frac{7}{15}$. Now $3\frac{3}{7}$ is equal to $\frac{24}{7}$, the square of which is $\frac{576}{49}$, and consequently less by $\frac{12}{49}$ than 12, which may be expressed by $\frac{588}{49}$. It is, therefore, proved that $3\frac{3}{7}$ is less, and that $3\frac{7}{15}$ is greater than the root required. Let us then try a number a little greater than $3\frac{3}{7}$, but yet less than $3\frac{7}{15}$; for example, $3\frac{5}{11}$; this number, which is equal to $\frac{38}{11}$, has for its square $\frac{1444}{121}$; and by reducing 12 to this denominator, we obtain $\frac{1452}{121}$, which shews that $3\frac{5}{11}$ is still less than the root of 12, viz. by $\frac{8}{121}$; let us therefore substitute for $\frac{5}{11}$ the fraction $\frac{6}{13}$, which is a little greater, and see what will be the result of the comparison of the square of $3\frac{6}{13}$, with the proposed number 12. Here the square of $3\frac{6}{13}$ is $\frac{2025}{169}$; and 12 reduced to the same denominator is $\frac{2028}{169}$; so that $3\frac{6}{13}$ is still too small, though only by $\frac{3}{169}$, whilst $3\frac{7}{15}$ has been found too great.

127. It is evident, therefore, that whatever fraction is joined to 3, the square of that sum must always contain a fraction, and can never be exactly equal to the integer 12. Thus, although we know that the square root of 12 is greater than $3\frac{6}{13}$, and less than $3\frac{7}{15}$, yet we are unable to assign an intermediate fraction between these two, which, at the same time, if added to 3, would express exactly the square root of 12; but notwithstanding this, we are not to assert that the square root of 12 is absolutely and in itself indeterminate: it only follows from what has been said, that this root, though it necessarily has a determinate magnitude, cannot be expressed by fractions.

128. There is therefore a sort of numbers, which cannot be assigned by fractions, but which are nevertheless determinate quantities; as, for instance, the square root of 12: and we call this new species of numbers, *irrational numbers*. They occur whenever we endeavour to find the square root of a number which is not a square; thus, 2 not being a perfect square, the square root of 2, or the number which, multiplied by itself, would produce 2, is an irrational quantity. These numbers are also called *surd quantities*, or *incommensurables*.

129. These irrational quantities, though they cannot be expressed by fractions, are nevertheless magnitudes of which we may form an accurate idea; since, however concealed

the square root of 12, for example, may appear, we are not ignorant that it must be a number, which, when multiplied by itself, would exactly produce 12; and this property is sufficient to give us an idea of the number, because it is in our power to approximate towards its value continually.

130. As we are therefore sufficiently acquainted with the nature of irrational numbers, under our present consideration, a particular sign has been agreed on to express the square roots of all numbers that are not perfect squares; which sign is written thus $\surd$, and is read *square root*. Thus, $\surd 12$ represents the square root of 12, or the number which, multiplied by itself, produces 12; and $\surd 2$ represents the square root of 2; $\surd 3$ the square root of 3; $\surd\frac{2}{3}$ that of $\frac{2}{3}$; and, in general, $\surd a$ represents the square root of the number a. Whenever, therefore, we would express the square root of a number, which is not a square, we need only make use of the mark $\surd$ by placing it before the number.

131. The explanation which we have given of irrational numbers will readily enable us to apply to them the known methods of calculation. For knowing that the square root of 2, multiplied by itself, must produce 2; we know also, that the multiplication of $\surd 2$ by $\surd 2$ must necessarily produce 2; that, in the same manner, the multiplication of $\surd 3$ by $\surd 3$ must give 3; that $\surd 5$ by $\surd 5$ makes 5; that $\surd\frac{2}{3}$ by $\surd\frac{2}{3}$ makes $\frac{2}{3}$; and, in general, that $\surd a$ multiplied by $\surd a$ produces a.

132. But when it is required to multiply $\surd a$ by $\surd b$, the product is $\surd ab$; for we have already shewn, that if a square has two or more factors, its root must be composed of the roots of those factors; we therefore find the square root of the product ab, which is $\surd ab$, by multiplying the square root of a, or $\surd a$, by the square root of b, or $\surd b$; &c. It is evident from this, that if b were equal to a, we should have $\surd aa$ for the product of $\surd a$ by $\surd b$. But $\surd aa$ is evidently a, since aa is the square of a.

133. In division, if it were required, for example, to divide $\surd a$, by $\surd b$, we obtain $\surd\frac{a}{b}$; and, in this instance, the irrationality may vanish in the quotient. Thus, having to divide $\surd 18$ by $\surd 8$, the quotient is $\surd\frac{18}{8}$, which is reduced to $\surd\frac{9}{4}$, and consequently to $\frac{3}{2}$, because $\frac{9}{4}$ is the square of $\frac{3}{2}$.

134. When the number before which we have placed the radical sign $\surd$, is itself a square, its root is expressed in the usual way; thus, $\surd 4$ is the same as 2; $\surd 9$ is the same as 3; $\surd 36$ the same as 6; and $\surd 12\frac{1}{4}$, the same as $\frac{7}{2}$, or $3\frac{1}{2}$. In these instances, the irrationality is only apparent, and vanishes of course.

135. It is easy also to multiply irrational numbers by ordinary numbers; thus, for example, 2 multiplied by $\surd 5$ makes $2\surd 5$; and 3 times $\surd 2$ makes $3\surd 2$. In the second example, however, as 3 is equal to $\surd 9$, we may also express 3 times $\surd 2$ by $\surd 9$ multiplied by $\surd 2$, or by $\surd 18$; also $2\surd a$ is the same as $\surd 4a$, and $3\surd a$ the same as $\surd 9a$; and, in general, $b\surd a$ has the same value as the square root of bba, or $\surd bba$: whence we infer reciprocally, that when the number which is preceded by the radical sign contains a square, we may take the root of that square, and put it before the sign, as we should do in writing $b\surd a$ instead of $\surd bba$. After this, the following reductions will be easily understood:

$$\left.\begin{matrix}\surd 8, \text{ or } \surd(2.4)\\ \surd 12, \text{ or } \surd(3.4)\\ \surd 18, \text{ or } \surd(2.9)\\ \surd 24, \text{ or } \surd(6.4)\\ \surd 32, \text{ or } \surd(2.16)\\ \surd 75, \text{ or } \surd(3.25)\end{matrix}\right\}\text{is equal to}\left\{\begin{matrix}2\surd 2\\ 2\surd 3\\ 3\surd 2\\ 2\surd 6\\ 4\surd 2\\ 5\surd 3\end{matrix}\right.$$

and so on.

136. Division is founded on the same principles; as $\surd a$ divided by $\surd b$ gives $\frac{\surd a}{\surd b}$, or $\surd\frac{a}{b}$. In the same manner,

$$\left.\begin{matrix}\frac{\surd 8}{\surd 2}\\ \frac{\surd 18}{\surd 2}\\ \frac{\surd 12}{\surd 3}\end{matrix}\right\}\text{is equal to}\left\{\begin{matrix}\surd\frac{8}{2}, \text{ or } \surd 4, \text{ or } 2\\ \surd\frac{18}{2}, \text{ or } \surd 9, \text{ or } 3\\ \surd\frac{12}{3}, \text{ or } \surd 4, \text{ or } 2.\end{matrix}\right.$$

Farther,
$$\left.\begin{matrix}\frac{2}{\surd 2}\\ \frac{3}{\surd 3}\\ \frac{12}{\surd 6}\end{matrix}\right\}\text{is equal to}\left\{\begin{matrix}\frac{\surd 4}{\surd 2}, \text{ or } \surd\frac{4}{2}, \text{ or } \surd 2,\\ \frac{\surd 9}{\surd 3}, \text{ or } \surd\frac{9}{3}, \text{ or } \surd 3.\\ \frac{\surd 144}{\surd 6}, \text{ or } \surd\frac{144}{6}, \text{ or } \surd 24,\end{matrix}\right.$$

or $\surd(6 \times 4)$, or lastly $2\surd 6$.

137. There is nothing in particular to be observed in addition and subtraction, because we only connect the numbers by the signs + and −: for example, $\surd 2$ added to $\surd 3$ is

written $\sqrt{2} + \sqrt{3}$; and $\sqrt{3}$ subtracted from $\sqrt{5}$ is written $\sqrt{5} - \sqrt{3}$.

138. We may observe lastly, that in order to distinguish the irrational numbers, we call all other numbers, both integral and fractional, *rational numbers;* so that, whenever we speak of rational numbers, we understand integers, or fractions.

CHAP. XIII.

Of Impossible, *or* Imaginary Quantities, *which arise from the same source.*

139. We have already seen that the squares of numbers, negative as well as positive, are always positive, or affected by the sign $+$; having shewn that $-a$ multiplied by $-a$ gives $+aa$, the same as the product of $+a$ by $+a$: wherefore, in the preceding chapter, we supposed that all the numbers, of which it was required to extract the square roots, were positive.

140. When it is required, therefore, to extract the root of a negative number, a great difficulty arises; since there is no assignable number, the square of which would be a negative quantity. Suppose, for example, that we wished to extract the root of -4; we here require such a number as, when multiplied by itself, would produce -4: now, this number is neither $+2$ nor -2, because the square both of $+2$ and of -2, is $+4$, and not -4.

141. We must therefore conclude, that the square root of a negative number cannot be either a positive number or a negative number, since the squares of negative numbers also take the sign *plus:* consequently, the root in question must belong to an entirely distinct species of numbers; since it cannot be ranked either among positive or among negative numbers.

142. Now, we before remarked, that positive numbers are all greater than nothing, or 0, and that negative numbers are all less than nothing, or 0; so that whatever exceeds 0 is expressed by positive numbers, and whatever is less than 0 is expressed by negative numbers. The square roots of negative numbers, therefore, are neither greater nor less than nothing; yet we cannot say, that they are 0; for 0 multiplied by 0 produces 0, and consequently does not give a negative number.

143. And, since all numbers which it is possible to conceive, are either greater or less than 0, or are 0 itself, it is evident that we cannot rank the square root of a negative number amongst possible numbers, and we must therefore say that it is an impossible quantity. In this manner we are led to the idea of numbers, which from their nature are impossible; and therefore they are usually called *imaginary quantities*, because they exist merely in the imagination.

144. All such expressions, as $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$, &c. are consequently impossible, or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing; which necessarily constitutes them imaginary, or impossible.

145. But notwithstanding this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by $\sqrt{-4}$ is meant a number which, multiplied by itself, produces -4; for this reason also, nothing prevents us from making use of these imaginary numbers, and employing them in calculation.

146. The first idea that occurs on the present subject is, that the square of $\sqrt{-3}$, for example, or the product of $\sqrt{-3}$ by $\sqrt{-3}$, must be -3; that the product of $\sqrt{-1}$ by $\sqrt{-1}$, is -1; and, in general, that by multiplying $\sqrt{-a}$ by $\sqrt{-a}$, or by taking the square of $\sqrt{-a}$ we obtain $-a$.

147. Now, as $-a$ is equal to $+a$ multiplied by -1, and as the square root of a product is found by multiplying together the roots of its factors, it follows that the root of a times -1, or $\sqrt{-a}$, is equal to $\sqrt{a}$ multiplied by $\sqrt{-1}$; but $\sqrt{a}$ is a possible or real number, consequently the whole impossibility of an imaginary quantity may be always reduced to $\sqrt{-1}$; for this reason, $\sqrt{-4}$ is equal to $\sqrt{4}$ multiplied by $\sqrt{-1}$, or equal to $2\sqrt{-1}$, because $\sqrt{4}$ is equal to 2; likewise -9 is reduced to $\sqrt{9} \times \sqrt{-1}$, or $3\sqrt{-1}$; and $\sqrt{-16}$ is equal to $4\sqrt{-1}$.

148. Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$; and $\sqrt{4}$, or 2, for the value of the product of $\sqrt{-1}$ by $\sqrt{-4}$. Thus we see that two imaginary numbers, multiplied together, produce a real, or possible one.

But, on the contrary, a possible number, multiplied by an impossible number, gives always an imaginary product: thus, $\sqrt{-3}$ by $\sqrt{+5}$, gives $\sqrt{-15}$.

149. It is the same with regard to division; for $\sqrt{a}$ divided by $\sqrt{b}$ making $\sqrt{\frac{a}{b}}$, it is evident that $\sqrt{-4}$ divided by $\sqrt{-1}$ will make $\sqrt{+4}$, or 2; that $\sqrt{+3}$ divided by $\sqrt{-3}$ will give $\sqrt{-1}$; and that 1 divided by $\sqrt{-1}$ gives $\sqrt{\frac{+1}{-1}}$, or $\sqrt{-1}$; because 1 is equal to $\sqrt{+1}$.

150. We have before observed, that the square root of any number has always two values, one positive and the other negative; that $\sqrt{4}$, for example, is both $+2$ and -2, and that, in general, we may take $-\sqrt{a}$ as well as $+\sqrt{a}$ for the square root of a. This remark applies also to imaginary numbers; the square root of $-a$ is both $+\sqrt{-a}$ and $-\sqrt{-a}$; but we must not confound the signs $+$ and $-$, which are before the radical sign $\sqrt{}$, with the sign which comes after it.

151. It remains for us to remove any doubt, which may be entertained concerning the utility of the numbers of which we have been speaking; for those numbers being impossible, it would not be surprising if they were thought entirely useless, and the object only of an unfounded speculation. This, however, would be a mistake; for the calculation of imaginary quantities is of the greatest importance, as questions frequently arise, of which we cannot immediately say whether they include any thing real and possible, or not; but when the solution of such a question leads to imaginary numbers, we are certain that what is required is impossible.

In order to illustrate what we have said by an example, suppose it were proposed to divide the number 12 into two such parts, that the product of those parts may be 40. If we resolve this question by the ordinary rules, we find for the parts sought $6+\sqrt{-4}$ and $6-\sqrt{-4}$; but these numbers being imaginary, we conclude, that it is impossible to resolve the question.

The difference will be easily perceived, if we suppose the question had been to divide 12 into two parts which multiplied together would produce 35; for it is evident that those parts must be 7 and 5.

CHAP. XIV.

Of Cubic Numbers.

152. When a number has been multiplied twice by itself, or, which is the same thing, when the square of a number has been multiplied once more by that number, we obtain a product which is called a *cube*, or a *cubic number*. Thus, the cube of a is aaa, since it is the product obtained by multiplying a by itself, or by a, and that square aa again by a.

The cubes of the natural numbers, therefore, succeed each other in the following order *:

Numbers	1	2	3	4	5	6	7	8	9	10
Cubes	1	8	27	64	125	216	343	512	729	1000

153. If we consider the differences of those cubes, as we did of the squares, by subtracting each cube from that which comes after it, we obtain the following series of numbers:

7, 19, 37, 61, 91, 127, 169, 217, 271.

Where we do not at first observe any regularity in them; but if we take the respective differences of these numbers, we find the following series:

12, 18, 24, 30, 36, 42, 48, 54, 60;

in which the terms, it is evident, increase always by 6.

154. After the definition we have given of a cube, it will not be difficult to find the cubes of fractional numbers; thus, $\frac{1}{8}$ is the cube of $\frac{1}{2}$; $\frac{1}{27}$ is the cube of $\frac{1}{3}$; and $\frac{8}{27}$ is the cube of $\frac{2}{3}$. In the same manner, we have only to take the cube of the numerator and that of the denominator separately, and we shall have $\frac{27}{64}$ for the cube of $\frac{3}{4}$.

155. If it be required to find the cube of a mixed number, we must first reduce it to a single fraction, and then proceed in the manner that has been described. To find, for example, the cube of $1\frac{1}{2}$, we must take that of $\frac{3}{2}$, which

* We are indebted to a mathematician of the name of J. Paul Buchner, for Tables published at Nuremberg in 1701, in which are to be found the cubes, as well as the squares, of all numbers from 1 to 12000. F. T.

is $\frac{27}{8}$, or $3\frac{3}{8}$; also the cube of $1\frac{1}{4}$, or of the single fraction $\frac{5}{4}$, is $\frac{125}{64}$, or $1\frac{61}{64}$; and the cube of $3\frac{1}{4}$, or of $\frac{13}{4}$, is $\frac{2197}{64}$, or $34\frac{21}{64}$.

156. Since aaa is the cube of a, that of ab will be $aaabbb$; whence we see, that if a number has two or more factors, we may find its cube by multiplying together the cubes of those factors. For example, as 12 is equal to 3×4, we multiply the cube of 3, which is 27, by the cube of 4, which is 64, and we obtain 1728, the cube of 12; and farther, the cube of $2a$ is $8aaa$, and consequently 8 times greater than the cube of a: likewise, the cube of $3a$ is $27aaa$; that is to say, 27 times greater than the cube of a.

157. Let us attend here also to the signs $+$ and $-$. It is evident that the cube of a positive number $+a$ must also be positive, that is $+aaa$; but if it be required to cube a negative number $-a$, it is found by first taking the square, which is $+aa$, and then multiplying, according to the rule, this square by $-a$, which gives for the cube required $-aaa$. In this respect, therefore, it is not the same with cubic numbers as with squares, since the latter are always positive: whereas the cube of -1 is -1, that of -2 is -8, that of -3 is -27, and so on.

CHAP. XV.

Of Cube Roots, *and of* Irrational Numbers *resulting from them.*

158. As we can, in the manner already explained, find the cube of a given number, so, when a number is proposed, we may also reciprocally find a number, which, multiplied twice by itself, will produce that number. The number here sought is called, with relation to the other, *the cube root;* so that the cube root of a given number is the number whose cube is equal to that given number.

159. It is easy therefore to determine the cube root, when the number proposed is a real cube, such as in the examples in the last chapter; for we easily perceive that the cube root of 1 is 1; that of 8 is 2; that of 27 is 3; that of 64 is 4, and so on. And, in the same manner, the cube root of -27 is -3; and that of -125 is -5.

Farther, if the proposed number be a fraction, as $\frac{8}{27}$, the

cube root of it must be $\frac{2}{3}$; and that of $\frac{64}{343}$ is $\frac{4}{7}$. Lastly, the cube root of a mixed number, such as $2\frac{10}{27}$ must be $\frac{4}{3}$, or $1\frac{1}{3}$; because $2\frac{10}{27}$ is equal to $\frac{64}{27}$.

160. But if the proposed number be not a cube, its cube root cannot be expressed either in integers, or in fractional numbers. For example, 43 is not a cubic number; therefore it is impossible to assign any number, either integer or fractional, whose cube shall be exactly 43. We may however affirm, that the cube root of that number is greater than 3, since the cube of 3 is only 27; and less than 4, because the cube of 4 is 64: we know, therefore, that the cube root required is necessarily contained between the numbers 3 and 4.

161. Since the cube root of 43 is greater than 3, if we add a fraction to 3, it is certain that we may approximate still nearer and nearer to the true value of this root: but we can never assign the number which expresses the value exactly; because the cube of a mixed number can never be perfectly equal to an integer, such as 43. If we were to suppose, for example, $3\frac{1}{2}$, or $\frac{7}{2}$ to be the cube root required, the error would be $\frac{1}{8}$; for the cube of $\frac{7}{2}$ is only $\frac{343}{8}$, or $42\frac{7}{8}$.

162. This therefore shews, that the cube root of 43 cannot be expressed in any way, either by integers or by fractions. However, we have a distinct idea of the magnitude of this root; and therefore we use, in order to represent it, the sign $\sqrt[3]{}$, which we place before the proposed number, and which is read *cube root*, to distinguish it from the square root, which is often called simply *the root;* thus $\sqrt[3]{43}$ means the cube root of 43; that is to say, the number whose cube is 43, or which, multiplied by itself, and then by itself again, produces 43.

163. Now, it is evident that such expressions cannot belong to rational quantities, but that they rather form a particular species of irrational quantities. They have nothing in common with square roots, and it is not possible to express such a cube root by a square root; as, for example, by $\sqrt{12}$; for the square of $\sqrt{12}$ being 12, its cube will be $12\sqrt{12}$, consequently still irrational, and therefore it cannot be equal to 43.

164. If the proposed number be a real cube, our expressions become rational. Thus, $\sqrt[3]{1}$ is equal to 1; $\sqrt[3]{8}$ is equal to 2; $\sqrt[3]{27}$ is equal to 3; and, generally, $\sqrt[3]{aaa}$ is equal to a.

165. If it were proposed to multiply one cube root, $\sqrt[3]{a}$, by another, $\sqrt[3]{b}$, the product must be $\sqrt[3]{ab}$; for we know that

the cube root of a product ab is found by multiplying together the cube roots of the factors. Hence, also, if we divide $\sqrt[3]{a}$ by $\sqrt[3]{b}$, the quotient will be $\sqrt[3]{\frac{a}{b}}$.

166. We farther perceive, that $2\sqrt[3]{a}$ is equal to $\sqrt[3]{8a}$, because 2 is equivalent to $\sqrt[3]{8}$; that $3\sqrt[3]{a}$ is equal to $\sqrt[3]{27a}$, $b\sqrt[3]{a}$ is equal to $\sqrt[3]{abbb}$; and, reciprocally, if the number under the radical sign has a factor which is a cube, we may make it disappear by placing its cube root before the sign; for example, instead of $\sqrt[3]{64a}$ we may write $4\sqrt[3]{a}$; and $5\sqrt[3]{a}$ instead of $\sqrt[3]{125a}$: hence $\sqrt[3]{16}$ is equal to $2\sqrt[3]{2}$, because 16 is equal to 8×2.

167. When a number proposed is negative, its cube root is not subject to the same difficulties that occurred in treating of square roots; for, since the cubes of negative numbers are negative, it follows that the cube roots of negative numbers are also negative; thus $\sqrt[3]{-8}$ is equal to -2, and $\sqrt[3]{-27}$ to -3. It follows also, that $\sqrt[3]{-12}$ is the same as $-\sqrt[3]{12}$, and that $\sqrt[3]{-a}$ may be expressed by $-\sqrt[3]{a}$. Whence we see that the sign $-$, when it is found after the sign of the cube root, might also have been placed before it. We are not therefore led here to impossible, or imaginary numbers, which happened in considering the square roots of negative numbers.

CHAP. XVI.

Of Powers *in general.*

168. The product which we obtain by multiplying a number once, or several times by itself, is called *a power*. Thus, a square which arises from the multiplication of a number by itself, and a cube which we obtain by multiplying a number twice by itself, are powers. We say also in the former case, that the number is raised to the second degree, or to the second power; and in the latter, that the number is raised to the third degree, or to the third power.

169. We distinguish those powers from one another by the number of times that the given number has been multiplied by itself. For example, a square is called the second

power, because a certain given number has been multiplied by itself; and if a number has been multiplied twice by itself we call the product the third power, which therefore means the same as the cube; also if we multiply a number three times by itself we obtain its fourth power, or what is commonly called the *biquadrate*: and thus it will be easy to understand what is meant by the fifth, sixth, seventh, &c. power of a number. I shall only add, that powers, after the fourth degree, cease to have any other but these numeral distinctions.

170. To illustrate this still better, we may observe, in the first place, that the powers of 1 remain always the same; because, whatever number of times we multiply 1 by itself, the product is found to be always 1. We shall therefore begin by representing the powers of 2 and of 3, which succeed each other as in the following order:

Powers.	Of the number 2.	Of the number 3.
1st	2	3
2d	4	9
3d	8	27
4th	16	81
5th	32	243
6th	64	729
7th	128	2187
8th	256	6561
9th	512	19683
10th	1024	59049
11th	2048	177147
12th	4096	531441
13th	8192	1594323
14th	16384	4782969
15th	32768	14348907
16th	65536	43046721
17th	131072	129140163
18th	262144	387420489

But the powers of the number 10 are the most remarkable: for on these powers the system of our arithmetic is founded. A few of them ranged in order, and beginning with the first power, are as follow:

1st 2d 3d 4th 5th 6th
10, 100, 1000, 10000, 100000, 1000000, &c.

171. In order to illustrate this subject, and to consider it in a more general manner, we may observe, that the

powers of any number, a, succeed each other in the following order:

1st 2d 3d 4th 5th 6th
a, aa, aaa, $aaaa$, $aaaaa$, $aaaaaa$, &c.

But we soon feel the inconvenience attending this manner of writing the powers, which consists in the necessity of repeating the same letter very often, to express high powers; and the reader also would have no less trouble, if he were obliged to count all the letters, to know what power is intended to be represented. The hundredth power, for example, could not be conveniently written in this manner; and it would be equally difficult to read it.

172. To avoid this inconvenience, a much more commodious method of expressing such powers has been devised, which, from its extensive use, deserves to be carefully explained. Thus, for example, to express the hundredth power, we simply write the number 100 above the quantity, whose hundredth power we would express, and a little towards the right-hand; thus a^{100} represents a raised to the 100th power, or the hundredth power of a. It must be observed, also, that the name *exponent* is given to the number written above that whose power, or degree, it represents, which, in the present instance, is 100.

173. In the same manner, a^2 signifies a raised to the 2d power, or the second power of a, which we represent sometimes also by aa, because both these expressions are written and understood with equal facility; but to express the cube, or the third power aaa, we write a^3, according to the rule, that we may occupy less room; so a^4 signifies the fourth, a^5 the fifth, and a^6 the sixth power of a.

174. In a word, the different powers of a will be represented by a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, &c. Hence we see that in this manner we might very properly have written a^1 instead of a for the first term, to shew the order of the series more clearly. In fact, a^1 is no more than a, as this unit shews that the letter a is to be written only once. Such a series of powers is called also a geométrical progression, because each term is by one-time, or term, greater than the preceding.

175. As in this series of powers each term is found by multiplying the preceding term by a, which increases the exponent by 1; so when any term is given, we may also find the preceding term, if we divide by a, because this diminishes the exponent by 1. This shews that the term which precedes the first term a^1 must necessarily be

$\frac{a}{a}$, or 1; and, if we proceed according to the exponents, we immediately conclude, that the term which precedes the first must be a^0; and hence we deduce this remarkable property, that a^0 is always equal to 1, however great or small the value of the number a may be, and even when a is nothing; that is to say, a^0 is equal to 1.

176. We may also continue our series of powers in a retrograde order, and that in two different ways; first, by dividing always by a; and secondly, by diminishing the exponent by unity; and it is evident that, whether we follow the one or the other, the terms are still perfectly equal. This decreasing series is represented in both forms in the following Table, which must be read backwards, or from right to left.

1st.	$\frac{1}{aaaaaa}$	$\frac{1}{aaaaa}$	$\frac{1}{aaaa}$	$\frac{1}{aaa}$	$\frac{1}{aa}$	$\frac{1}{a}$	1	a
	$\frac{1}{a^6}$	$\frac{1}{a^5}$	$\frac{1}{a^4}$	$\frac{1}{a^3}$	$\frac{1}{a^2}$	$\frac{1}{a^1}$		
2d.	a^{-6}	a^{-5}	a^{-4}	a^{-3}	a^{-2}	a^{-1}	a^0	a^1

177. We are now come to the knowledge of powers whose exponents are negative, and are enabled to assign the precise value of those powers. Thus, from what has been said, it appears that

$$\left.\begin{array}{l} a^0 \\ a^{-1} \\ a^{-2} \\ a^{-3} \\ a^{-4} \end{array}\right\} \text{is equal to} \left\{\begin{array}{l} 1 \\ \frac{1}{a} \\ \frac{1}{aa} \text{ or } \frac{1}{a^2}, \\ \frac{1}{a^3} \\ \frac{1}{a^4}, \text{ \&c.} \end{array}\right.$$

178. It will also be easy, from the foregoing notation, to find the powers of a product, ab; for they must evidently be ab, or a^1b^1, a^2b^2, a^3b^3, a^4b^4, a^5b^5, &c. and the powers of fractions will be found in the same manner; for example, those of $\frac{a}{b}$ are

$$\frac{a^1}{b^1}, \frac{a^2}{b^2}, \frac{a^3}{b^3}, \frac{a^4}{b^4}, \frac{a^5}{b^5}, \frac{a^6}{b^6}, \frac{a^7}{b^7}, \text{ \&c.}$$

179. Lastly, we have to consider the powers of negative numbers. Suppose the given number to be $-a$; then its powers will form the following series:

$$-a, +a^2, -a^3, +a^4, -a^5, +a^6, \&c,$$

Where we may observe, that those powers only become negative, whose exponents are odd numbers, and that, on the contrary, all the powers, which have an even number for the exponent, are positive. So that the third, fifth, seventh, ninth, &c. powers have all the sign $-$; and the second, fourth, sixth, eighth, &c. powers are affected by the sign $+$.

CHAP. XVII.

Of the Calculation *of* Powers.

180. We have nothing particular to observe with regard to the *Addition* and *Subtraction* of powers; for we only represent those operations by means of the signs $+$ and $-$, when the powers are different. For example, $a^3 + a^2$ is the sum of the second and third powers of a; and $a^5 - a^4$ is what remains when we subtract the fourth power of a from the fifth; and neither of these results can be abridged: but when we have powers of the same kind or degree, it is evidently unnecessary to connect them by signs; as $a^3 + a^3$ becomes $2a^3$, &c.

181. But in the *Multiplication* of powers, several circumstances require attention.

First, when it is required to multiply any power of a by a, we obtain the succeeding power; that is to say, the power whose exponent is greater by an unit. Thus, a^2, multiplied by a, produces a^3; and a^3, multiplied by a, produces a^4. In the same manner, when it is required to multiply by a the powers of any number represented by a, having negative exponents, we have only to add 1 to the exponent. Thus, a^{-1} multiplied by a produces a^0, or 1; which is made more evident by considering that a^{-1} is equal to $\frac{1}{a}$, and that the product of $\frac{1}{a}$ by a being $\frac{a}{a}$, it is consequently equal to 1; likewise a^{-2} multiplied by a, produces a^{-1}, or $\frac{1}{a}$; and a^{-10} multiplied by a, gives a^{-9}, and so on. [See Art. 175, 176.]

182. Next, if it be required to multiply any power of a by a^2, or the second power, I say that the exponent becomes greater by 2. Thus, the product of a^2 by a^2 is a^4; that of a^2 by a^3 is a^5; that of a^4 by a^2 is a^6; and, more generally, a^n multiplied by a^2 makes a^{n+2}. With regard to negative exponents, we shall have a^1, or a, for the product of a^{-1} by a^2; for a^{-1} being equal to $\frac{1}{a}$, it is the same as if we had divided aa by a; consequently, the product required is $\frac{aa}{a}$, or a; also a^{-2}, multiplied by a^2, produces a^0, or 1; and a^{-3}, multiplied by a^2, produces a^{-1}.

183. It is no less evident, that to multiply any power of a by a^3, we must increase its exponent by three units; and that, consequently, the product of a^n by a^3 is a^{n+3}. And whenever it is required to multiply together two powers of a, the product will be also a power of a, and such that its exponent will be the sum of those of the two given powers. For example, a^4 multiplied by a^5 will make a^9, and a^{12} multiplied by a^7 will produce a^{19}, &c.

184. From these considerations we may easily determine the highest powers. To find, for instance, the twenty-fourth power of 2, I multiply the twelfth power by the twelfth power, because 2^{24} is equal to $2^{12} \times 2^{12}$. Now, we have already seen that 2^{12} is 4096; I say therefore that the number 16777216, or the product of 4096 by 4096, expresses the power required, namely, 2^{24}.

185. Let us now proceed to division. We shall remark, in the first place, that to divide a power of a by a, we must subtract 1 from the exponent, or diminish it by unity; thus, a^5 divided by a gives a^4; and a^0, or 1, divided by a, is equal to a^{-1} or $\frac{1}{a}$; also a^{-3} divided by a, gives a^{-4}.

186. If we have to divide a given power of a by a^2, we must diminish the exponent by 2; and if by a^3, we must subtract 3 units from the exponent of the power proposed; and, in general, whatever power of a it is required to divide by any other power of a, the rule is always to subtract the exponent of the second from the exponent of the first of those powers: thus a^{15} divided by a^7 will give a^8; a^6 divided by a^7 will give a^{-1}; and a^{-3} divided by a^4 will give a^{-7}.

187. From what has been said, it is easy to understand

the method of finding the powers of powers, this being done by multiplication. When we seek, for example, the square, or the second power of a^3, we find a^6; and in the same manner we find a^{12} for the third power, or the cube, of a^4. To obtain the square of a power, we have only to double its exponent; for its cube, we must triple the exponent; and so on. Thus, the square of a^n is a^{2n}; the cube of a^n is a^{3n}; the seventh power of a^n is a^{7n}, &c.

188. The square of a^2, or the square of the square of a, being a^4, we see why the fourth power is called the *biquadrate:* also, the square of a^3 being a^6, the sixth power has received the name of *the square-cubed.*

Lastly, the cube of a^3 being a^9, we call the ninth power the *cubo-cube:* after this, no other denominations of this kind have been introduced for powers; and, indeed, the two last are very little used.

CHAP. XVIII.

Of Roots, *with relation to* Powers *in general.*

189. Since the square root of a given number is a number, whose square is equal to that given number; and since the cube root of a given number is a number, whose cube is equal to that given number; it follows that any number whatever being given, we may always suppose such roots of it, that the fourth, or the fifth, or any other power of them, respectively, may be equal to the given number. To distinguish these different kinds of roots better, we shall call the square root, *the second root;* and the cube root, *the third root;* because according to this denomination we may call *the fourth root,* that whose biquadrate is equal to a given number; and *the fifth root,* that whose fifth power is equal to a given number, &c.

190. As the square, or second root, is marked by the sign $\sqrt{\ }$, and the cubic, or third root, by the sign $\sqrt[3]{\ }$, so the fourth root is represented by the sign $\sqrt[4]{\ }$; the fifth root by the sign $\sqrt[5]{\ }$; and so on. It is evident that, according to this method of expression, the sign of the square root ought to be $\sqrt[2]{\ }$: but as of all roots this occurs most frequently, it has been agreed, for the sake of brevity, to omit the number 2 as the sign of this root. So that when the radical sign has no num-

ber prefixed to it, this always shews that the square root is meant.

191. To explain this matter still better, we shall here exhibit the different roots of the number a, with their respective values:

$$\left.\begin{array}{l}\sqrt{a}\\ \sqrt[3]{a}\\ \sqrt[4]{a}\\ \sqrt[5]{a}\\ \sqrt[6]{a}\end{array}\right\}\text{ is the }\left\{\begin{array}{l}\text{2d}\\ \text{3d}\\ \text{4th}\\ \text{5th}\\ \text{6th}\end{array}\right\}\text{ root of }\left\{\begin{array}{l}a,\\ a,\\ a,\\ a,\\ a,\text{ and so on.}\end{array}\right.$$

So that, conversely,

$$\left.\begin{array}{l}\text{The 2d}\\ \text{The 3d}\\ \text{The 4th}\\ \text{The 5th}\\ \text{The 6th}\end{array}\right\}\text{ power of }\left\{\begin{array}{l}\sqrt{a}\\ \sqrt[3]{a}\\ \sqrt[4]{a}\\ \sqrt[5]{a}\\ \sqrt[6]{a}\end{array}\right\}\text{ is equal to }\left\{\begin{array}{l}a,\\ a,\\ a,\\ a,\\ a,\text{ and so on.}\end{array}\right.$$

192. Whether the number a therefore be great or small, we know what value to affix to all these roots of different degrees.

It must be remarked also, that if we substitute unity for a, all those roots remain constantly 1; because all the powers of 1 have unity for their value. If the number a be greater than 1, all its roots will also exceed unity. Lastly, if that number be less than 1, all its roots will also be less than unity.

193. When the number a is positive, we know from what was before said of the square and cube roots, that all the other roots may also be determined, and will be real and possible numbers.

But if the number a be negative, its second, fourth, sixth, and all its even roots, become impossible, or imaginary numbers; because all the powers of an even order, whether of positive or of negative numbers, are affected by the sign $+$: whereas the third, fifth, seventh, and all its odd roots, become negative, but rational; because the odd powers of negative numbers are also negative.

194. We have here also an inexhaustible source of new kinds of surds, or irrational quantities; for whenever the number a is not really such a power, as some one of the foregoing indices represents, or seems to require, it is impossible to express that root either in whole numbers or in fractions; and, consequently, it must be classed among the numbers which are called irrational.

CHAP. XIX.

Of the Method of representing Irrational Numbers *by* Fractional Exponents.

195. We have shewn in the preceding chapter, that the square of any power is found by doubling the exponent of that power; or that, in general, the square, or the second power, of a^n, is a^{2n}; and the converse also follows, viz. that the square root of the power a^{2n} is a^n, which is found by taking half the exponent of that power, or dividing it by 2.

196. Thus, the square root of a^2 is a^1, or a; that of a^4 is a^2; that of a^6 is a^3; and so on: and, as this is general, the square root of a^3 must necessarily be $a^{\frac{3}{2}}$, and that of a^5 must be $a^{\frac{5}{2}}$; consequently, we shall in the same manner have $a^{\frac{1}{2}}$ for the square root of a^1. Whence we see that $a^{\frac{1}{2}}$ is equal to $\sqrt{a}$; which new method of representing the square root demands particular attention.

197. We have also shewn, that, to find the cube of a power, as a^n, we must multiply its exponent by 3, and consequently that cube is a^{3n}.

Hence, conversely, when it is required to find the third, or cube root, of the power a^{3n}, we have only to divide that exponent by 3, and may therefore with certainty conclude, that the root required is a: consequently a^1, or a, is the cube root of a^3; a^2 is the cube root of a^6; a^3 of a^9; and so on.

198. There is nothing to prevent us from applying the same reasoning to those cases, in which the exponent is not divisible by 3, or from concluding that the cube root of a^2 is $a^{\frac{2}{3}}$, and that the cube root of a^4 is $a^{\frac{4}{3}}$, or $a^{1\frac{1}{3}}$; consequently, the third, or cube root of a, or a^1, must be $a^{\frac{1}{3}}$: whence also, it appears, that $a^{\frac{1}{3}}$ is the same as $\sqrt[3]{a}$.

199. It is the same with roots of a higher degree: thus, the fourth root of a will be $a^{\frac{1}{4}}$, which expression has the same value as $\sqrt[4]{a}$; the fifth root of a will be $a^{\frac{1}{5}}$, which is consequently equivalent to $\sqrt[5]{a}$; and the same observation may be extended to all roots of a higher degree.

200. We may therefore entirely reject the radical signs at present made use of, and employ in their stead the fractional exponents which we have just explained: but as we have been long accustomed to those signs, and meet with them in most books of Algebra, it might be wrong to banish them entirely from calculation; there is, however, sufficient reason also to employ, as is now frequently done, the other method of notation, because it manifestly corresponds with the nature of the thing. In fact, we see immediately that $a^{\frac{1}{2}}$ is the square root of a, because we know that the square of $a^{\frac{1}{2}}$, that is to say, $a^{\frac{1}{2}}$ multiplied by $a^{\frac{1}{2}}$, is equal to a^1, or a.

201. What has been now said is sufficient to shew how we are to understand all other fractional exponents that may occur. If we have, for example, $a^{\frac{4}{3}}$, this means, that we must first take the fourth power of a, and then extract its cube, or third root; so that $a^{\frac{4}{3}}$ is the same as the common expression $\sqrt[3]{a^4}$. Hence, to find the value of $a^{\frac{3}{4}}$, we must first take the cube, or the third power of a, which is a^3, and then extract the fourth root of that power; so that $a^{\frac{3}{4}}$ is the same as $\sqrt[4]{a^3}$, and $a^{\frac{4}{5}}$ is equal to $\sqrt[5]{a^4}$, &c.

202. When the fraction which represents the exponent exceeds unity, we may express the value of the given quantity in another way: for instance, suppose it to be $a^{\frac{5}{2}}$; this quantity is equivalent to $a^{2\frac{1}{2}}$, which is the product of a^2 by $a^{\frac{1}{2}}$: now $a^{\frac{1}{2}}$ being equal to $\sqrt{a}$, it is evident that $a^{\frac{5}{2}}$ is equal to $a^2\sqrt{a}$: also $a^{\frac{10}{3}}$, or $a^{3\frac{1}{3}}$, is equal to $a^3\sqrt[3]{a}$; and $a^{\frac{15}{4}}$, that is, $a^{3\frac{3}{4}}$, expresses $a^3\sqrt[4]{a^3}$. These examples are sufficient to illustrate the great utility of fractional exponents.

203. Their use extends also to fractional numbers: for if there be given $\frac{1}{\sqrt{a}}$, we know that this quantity is equal to $\frac{1}{a^{\frac{1}{2}}}$; and we have seen already that a fraction of the form $\frac{1}{a^n}$ may be expressed by a^{-n}; so that instead of $\frac{1}{\sqrt{a}}$ we may use the expresssion $a^{-\frac{1}{2}}$; and, in the same man-

ner, $\frac{1}{\sqrt[3]{a}}$ is equal to $a^{-\frac{1}{3}}$. Again, if the quantity $\frac{a^2}{\sqrt[4]{a^3}}$ be proposed; let it be transformed into this, $\frac{a^2}{a^{\frac{3}{4}}}$, which is the product of a^2 by $a^{-\frac{3}{4}}$; now this product is equivalent to $a^{\frac{5}{4}}$, or to $a^{1\frac{1}{4}}$, or lastly, to $a\sqrt[4]{a}$. Practice will render similar reductions easy.

204. We shall observe, in the last place, that each root may be represented in a variety of ways; for $\sqrt{a}$ being the same as $a^{\frac{1}{2}}$, and $\frac{1}{2}$ being transformable into the fractions, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, &c. it is evident that $\sqrt{a}$ is equal to $\sqrt[4]{a^2}$, or to $\sqrt[6]{a^3}$, or to $\sqrt[8]{a^4}$, and so on. In the same manner, $\sqrt[3]{a}$, which is equal to $a^{\frac{1}{3}}$, will be equal to $\sqrt[6]{a^2}$, or to $\sqrt[9]{a^3}$, or to $\sqrt[12]{a^4}$. Hence also we see that the number a, or a^1, might be represented by the following radical expressions:

$$\sqrt[2]{a^2}, \sqrt[3]{a^3}, \sqrt[4]{a^4}, \sqrt[5]{a^5}, \text{\&c.}$$

205. This property is of great use in multiplication and division; for if we have, for example, to multiply $\sqrt[2]{a}$ by $\sqrt[3]{a}$, we write $\sqrt[6]{a^3}$ for $\sqrt[2]{a}$, and $\sqrt[6]{a^2}$ instead of $\sqrt[3]{a}$; so that in this manner we obtain the same radical sign for both, and the multiplication being now performed, gives the product $\sqrt[6]{a^5}$. The same result is also deduced from $a^{\frac{1}{2}+\frac{1}{3}}$, which is the product of $a^{\frac{1}{2}}$ multiplied by $a^{\frac{1}{3}}$; for $\frac{1}{2}+\frac{1}{3}$ is $\frac{5}{6}$, and consequently the product required is $a^{\frac{5}{6}}$, or $\sqrt[6]{a^5}$.

On the contrary, if it were required to divide $\sqrt[2]{a}$, or $a^{\frac{1}{2}}$, by $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, we should have for the quotient $a^{\frac{1}{2}-\frac{1}{3}}$, or $a^{\frac{3}{6}-\frac{2}{6}}$, that is to say, $a^{\frac{1}{6}}$, or $\sqrt[6]{a}$.

QUESTIONS FOR PRACTICE RESPECTING SURDS.

1. Reduce 6 to the form of $\sqrt{5}$. *Ans.* $\sqrt{36}$.
2. Reduce $a + b$ to the form of $\sqrt{bc}$. *Ans.* $\sqrt{(aa + 2ab + bb)}$.
3. Reduce $\frac{a}{b\sqrt{c}}$ to the form of $\sqrt{d}$. *Ans.* $\sqrt{\frac{aa}{bbc}}$.
4. Reduce a^1 and $b^{\frac{3}{2}}$ to the common index $\frac{1}{3}$. *Ans.* $\left.a^{6}\right|^{\frac{1}{3}}$, and $\left.b^{\frac{9}{2}}\right|^{\frac{1}{3}}$.

5. Reduce $\sqrt{48}$ to its simplest form. *Ans.* $4\sqrt{3}$.
6. Reduce $\sqrt{(a^3x - a^2x^2)}$ to its simplest form. *Ans.* $a\sqrt{(ax - x^2)}$.
7. Reduce $\sqrt[3]{\frac{27a^3b^3}{8b-8a}}$ to its simplest form. *Ans.* $\frac{3ab}{2}\sqrt[3]{\frac{a}{b-a}}$.
8. Add $\sqrt{6}$ to $2\sqrt{6}$; and $\sqrt{8}$ to $\sqrt{50}$. *Ans.* $3\sqrt{6}$; and $7\sqrt{2}$.
9. Add $\sqrt{4a}$ and $\sqrt[4]{a^6}$ together. *Ans.* $(a+2)\sqrt{a}$.
10. Add $\left.\frac{b}{c}\right|^{\frac{1}{2}}$ and $\left.\frac{c}{b}\right|^{\frac{3}{2}}$ together. *Ans.* $\frac{b^2+c^2}{b\sqrt{bc}}$.
11. Subtract $\sqrt{4a}$ from $\sqrt[4]{a^6}$. *Ans.* $(a-2)\sqrt{a}$.
12. Subtract $\left.\frac{c}{b}\right|^{\frac{3}{2}}$ from $\left.\frac{b}{c}\right|^{\frac{1}{2}}$. *Ans.* $\frac{b^2-c^2}{b}\sqrt{\frac{1}{bc}}$.
13. Multiply $\sqrt{\frac{2ab}{3c}}$ by $\sqrt{\frac{9ad}{2b}}$. *Ans.* $\frac{3a^2d}{c}$.
14. Multiply $\sqrt{d}$ by $\sqrt[3]{ab}$. *Ans.* $\sqrt[6]{(a^2b^2d^3)}$.
15. Multiply $\sqrt{(4a - 3x)}$ by $2a$. *Ans.* $\sqrt{(16a^3 - 12a^2x)}$.
16. Multiply $\frac{a}{2b}\sqrt{(a-x)}$ by $(c-d)\sqrt{ax}$. *Ans.* $\frac{ac-ad}{2b}\sqrt{(a^2x - ax^2)}$.
17. Divide $a^{\frac{2}{3}}$ by $a^{\frac{1}{4}}$; and $a^{\frac{1}{n}}$ by $a^{\frac{1}{m}}$. *Ans.* $a^{\frac{5}{12}}$; and $a^{\frac{m-n}{mn}}$.
18. Divide $\frac{ac-ad}{2b}\sqrt{(a^2x - ax^2)}$ by $\frac{a}{2b}\sqrt{(a-x)}$. *Ans.* $(c-d)\sqrt{ax}$.
19. Divide $a^2 - ad - b + d\sqrt{b}$ by $a - \sqrt{b}$. *Ans.* $a + \sqrt{b} - d$.
20. What is the cube of $\sqrt{2}$? *Ans.* $\sqrt{8}$.
21. What is the square of $3\sqrt[3]{bc^2}$? *Ans.* $9c\sqrt[3]{b^2c}$.
22. What is the fourth power of $\frac{a}{2b}\sqrt{\frac{2a}{c-b}}$? *Ans.* $\frac{a^6}{4b^4(c^2-2bc+b^2)}$.
23. What is the square of $3+\sqrt{5}$? *Ans.* $14+6\sqrt{5}$.
24. What is the square root of a^3? *Ans.* $a^{\frac{3}{2}}$; or $\sqrt{a^3}$.
25. What is the cube root of $\sqrt{(a^2 - x^2)}$? *Ans.* $\sqrt[6]{(a^2 - x^2)}$.

26. What multiplier will render $a+\sqrt{3}$ rational?

Ans. $a-\sqrt{3}$.

27. What multiplier will render $\sqrt{a}-\sqrt{b}$ rational?

Ans. $\sqrt{a}+\sqrt{b}$.

28. What multiplier will render the denominator of the fraction $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$ rational?

Ans. $\sqrt{7}-\sqrt{3}$.

CHAP. XX.

Of the different Methods *of* Calculation, *and of their mutual* Connexion.

206. Hitherto we have only explained the different methods of calculation: namely, addition, subtraction, multiplication, and division; the involution of powers, and the extraction of roots. It will not be improper, therefore, in this place, to trace back the origin of these different methods, and to explain the connexion which subsists among them; in order that we may satisfy ourselves whether it be possible or not for other operations of the same kind to exist. This inquiry will throw new light on the subjects which we have considered.

In prosecuting this design, we shall make use of a new character, which may be employed instead of the expression that has been so often repeated, *is equal to*; this sign is $=$, which is read *is equal to*: thus, when I write $a = b$, this means that a is equal to b: so, for example, $3 \times 5 = 15$.

207. The first mode of calculation that presents itself to the mind, is undoubtedly addition, by which we add two numbers together and find their sum: let therefore a and b be the two given numbers, and let their sum be expressed by the letter c, then we shall have $a + b = c$; so that when we know the two numbers a and b, addition teaches us to find the number c.

208. Preserving this comparison $a + b = c$, let us reverse the question by asking, how we are to find the number b, when we know the numbers a and c.

It is here required therefore to know what number must be added to a, in order that the sum may be the number c: suppose, for example, $a = 3$ and $c = 8$; so that we must have $3 + b = 8$; then b will evidently be found by subtracting 3 from 8: and, in general, to find b, we must subtract a from c, whence arises $b = c - a$; for by adding a to both sides again, we have $b + a = c - a + a$, that is to say, $= c$, as we supposed.

209. Subtraction therefore takes place, when we invert the question which gives rise to addition. But the number which it is required to subtract may happen to be greater than that from which it is to be subtracted; as, for example, if it were required to subtract 9 from 5: this instance therefore furnishes us with the idea of a new kind of numbers, which we call negative numbers, because $5 - 9 = -4$.

210. When several numbers are to be added together, which are all equal, their sum is found by multiplication, and is called a product. Thus, ab means the product arising from the multiplication of a by b, or from the addition of the number a, b number of times; and if we represent this product by the letter c, we shall have $ab = c$; thus multiplication teaches us how to determine the number c, when the numbers a and b are known.

211. Let us now propose the following question: the numbers a and c being known, to find the number b. Suppose, for example, $a = 3$, and $c = 15$; so that $3b = 15$, and let us inquire by what number 3 must be multiplied, in order that the product may be 15; for the question proposed is reduced to this. This is a case of division; and the number required is found by dividing 15 by 3; and, in general, the number b is found by dividing c by a; from which results the equation $b = \frac{c}{a}$.

212. Now, as it frequently happens that the number c cannot be really divided by the number a, while the letter b must however have a determinate value, another new kind of numbers present themselves, which are called *fractions*. For example, suppose $a = 4$, and $c = 3$, so that $4b = 3$; then it is evident that b cannot be an integer, but a fraction, and that we shall have $b = \frac{3}{4}$.

213. We have seen that multiplication arises from addition; that is to say, from the addition of several equal quantities: and if we now proceed farther, we shall perceive that, from the multiplication of several equal quantities together, powers are derived; which powers are represented in a general manner by the expression a^b. This signifies that the number a must be multiplied as many times by itself, *minus* 1, as is indicated by the number b. And we know from what has been already said, that, in the present in-

stance, a is called the root, b the exponent, and a^b the power.

214. Farther, if we represent this power also by the letter c, we have $a^b = c$, an equation in which three letters a, b, c, are found; and we have shewn in treating of powers, how to find the power itself, that is, the letter c, when a root a and its exponent b are given. Suppose, for example, $a = 5$, and $b = 3$, so that $c = 5^3$: then it is evident that we must take the third power of 5, which is 125, so that in this case $c = 125$.

215. We have now seen how to determine the power c, by means of the root a and the exponent b; but if we wish to reverse the question, we shall find that this may be done in two ways, and that there are two different cases to be considered: for if two of these three numbers a, b, c, were given, and it were required to find the third, we should immediately perceive that this question would admit of three different suppositions, and consequently of three solutions. We have considered the case in which a and b were the given numbers, we may therefore suppose farther that c and a, or c and b, are known, and that it is required to determine the third letter. But, before we proceed any farther, let us point out a very essential distinction between involution and the two operations which lead to it. When, in addition, we reversed the question, it could be done only in one way; it was a matter of indifference whether we took c and a, or c and b, for the given numbers, because we might indifferently write $a + b$, or $b + a$; and it was also the same with multiplication; we could at pleasure take the letters a and b for each other, the equation $ab = c$ being exactly the same as $ba = c$: but in the calculation of powers, the same thing does not take place, and we can by no means write b^a instead of a^b; as a single example will be sufficient to illustrate: for let $a = 5$, and $b = 3$; then we shall have $a^b = 5^3 = 125$; but $b^a = 3^5 = 243$: which are two very different results.

216. It is evident then, that we may propose two questions more: one, to find the root a by means of the given power c, and the exponent b; the other, to find the exponent b, supposing the power c and the root a to be known.

217. It may be said, indeed, that the former of these questions has been resolved in the chapter on the extraction of roots; since if $b = 2$, for example, and $a^2 = c$, we know by this means, that a is a number whose square is equal to c, and consequently that $a = \sqrt{c}$. In the same manner, if $b = 3$ and $a^3 = c$, we know that the cube of a must be equal to the given number c, and consequently that $a = \sqrt[3]{c}$. It is therefore easy to conclude, generally, from this, how to determine the letter a by means of the letters c and b; for we must necessarily have $a = \sqrt[b]{c}$.

218. We have already remarked also the consequence which follows, when the given number is not a real power; a case which very frequently occurs; namely, that then the required root, a, can neither be expressed by integers, nor by fractions; yet since this root must necessarily have a determinate value, the same consideration led us to a new kind of numbers, which, as we observed, are called *surds*, or *irrational* numbers; and which we have seen are divisible into an infinite number of different sorts, on account of the great variety of roots. Lastly, by the same inquiry, we were led to the knowledge of another particular kind of numbers, which have been called *imaginary numbers*.

219. It remains now to consider the second question, which was to determine the exponent; the power c, and the root a, both being known. On this question, which has not yet occurred, is founded the important theory of Logarithms, the use of which is so extensive through the whole compass of mathematics, that scarcely any long calculation can be carried on without their assistance; and we shall find, in the following chapter, for which we reserve this theory, that it will lead us to another kind of numbers entirely new, as they cannot be ranked among the irrational numbers before mentioned.

CHAP. XXI.

Of Logarithms *in general.*

220. Resuming the equation $a^b = c$, we shall begin by remarking that, in the doctrine of Logarithms, we assume for the root a, a certain number taken at pleasure, and suppose this root to preserve invariably its assumed value. This being laid down, we take the exponent b such, that the power a^b becomes equal to a given number c; in which case this exponent b is said to be the *logarithm* of the number c. To express this, we shall use the letter L. or the initial letters *log*. Thus, by $b = \text{L}.\ c$, or $b = log.\ c$,

we mean that b is equal to the logarithm of the number c, or that the logarithm of c is b.

221. We see then, that the value of the root a being once established, the logarithm of any number, c, is nothing more than the exponent of that power of a, which is equal to c: so that c being $= a^b$, b is the logarithm of the power a^b. If, for the present, we suppose $b = 1$, we have 1 for the logarithm of a^1, and consequently $log.\ a = 1$; but if we suppose $b = 2$, we have 2 for the logarithm of a^2; that is to say, $log.\ a^2 = 2$, and we may, in the same manner, obtain $log.\ a^3 = 3$; $log.\ a^4 = 4$; $log.\ a^5 = 5$, and so on.

222. If we make $b = 0$, it is evident that 0 will be the logarithm of a^0; but $a^0 = 1$; consequently $log.\ 1 = 0$, whatever be the value of the root a.

Suppose $b = -1$, then -1 will be the logarithm of a^{-1}; but $a^{-1} = \frac{1}{a}$; so that we have $log.\ \frac{1}{a} = -1$, and in the same manner, we shall have $log.\ \frac{1}{a^2} = -2$; $log.\ \frac{1}{a^3} = -3$; $log.\ \frac{1}{a^4} = -4$, &c.

223. It is evident, then, how we may represent the logarithms of all the powers of a, and even those of fractions, which have unity for the numerator, and for the denominator a power of a. We see also, that in all those cases the logarithms are integers; but it must be observed, that if b were a fraction, it would be the logarithm of an irrational number: if we suppose, for example, $b = \frac{1}{2}$, it follows, that $\frac{1}{2}$ is the logarithm of $a^{\frac{1}{2}}$, or of $\sqrt{a}$; consequently we have also $log.\ \sqrt{a} = \frac{1}{2}$; and we shall find, in the same manner, that $log.\ \sqrt[3]{a} = \frac{1}{3}$, $log.\ \sqrt[4]{a} = \frac{1}{4}$, &c.

224. But if it be required to find the logarithm of another number c, it will be readily perceived, that it can neither be an integer, nor a fraction; yet there must be such an exponent b, that the power a^b may become equal to the number proposed; we have therefore $b = log.\ c$; and generally, $a^{L.c} = c$.

225. Let us now consider another number d, whose logarithm has been represented in a similar manner by $log.\ d$; so that $a^{L.d} = d$. Here if we multiply this expression by the preceding one $a^{L.c} = c$, we shall have $a^{L.c+L.d} = cd$; hence, *the exponent is always the logarithm of the power*; consequently, $log.\ c + log.\ d = log.\ cd$. But if, instead of multiplying, we divide the former expression by the latter, we shall obtain $a^{L.c-L.d} = \frac{c}{d}$; and, consequently, $log.\ c - log.\ d = log.\ \frac{c}{d}$.

226. This leads us to the two principal properties of logarithms, which are contained in the equations $log.\ c + log.\ d = log.\ cd$, and $log.\ c - log.\ d = log.\ \frac{c}{d}$. The former of these equations teaches us, that the logarithm of a product, as cd, is found by adding together the logarithms of the factors; and the latter shews us this property, namely, that the logarithm of a fraction may be determined by subtracting the logarithm of the denominator from that of the numerator.

227. It also follows from this, that when it is required to multiply, or divide, two numbers by one another, we have only to add, or subtract, their logarithms; and this is what constitutes the singular utility of logarithms in calculation: for it is evidently much easier to add, or subtract, than to multiply, or divide, particularly when the question involves large numbers.

228. Logarithms are attended with still greater advantages, in the involution of powers, and in the extraction of roots; for if $d = c$, we have, by the first property, $log.\ c + log.\ c = log.\ cc$, or c^2; consequently, $log.\ cc = 2\ log.\ c$; and, in the same manner, we obtain $log.\ c^3 = 3\ log.\ c$; $log.\ c^4 = 4\ log.\ c$; and, generally, $log.\ c^n = n\ log.\ c$. If we now substitute fractional numbers for n, we shall have, for example, $log.\ c^{\frac{1}{2}}$, that is to say, $log.\ \sqrt{c}$, $= \frac{1}{2} log.\ c$; and lastly, if we suppose n to represent negative numbers, we shall have $log.\ c^{-1}$, or $log.\ \frac{1}{c}$, $= -\ log.\ c$; $log.\ c^{-2}$, or $log.\ \frac{1}{c^2}$, $= -2 log.\ c$, and so on; which follows not only from the equation $log.\ c^n = n\ log.\ c$, but also from $log.\ 1 = 0$, as we have already seen.

229. If therefore we had Tables, in which logarithms were calculated for all numbers, we might certainly derive from them very great assistance in performing the most prolix calculations; such, for instance, as require frequent multiplications, divisions, involutions, and extractions of roots: for, in such Tables, we should have not only the logarithms of all numbers, but also the numbers answering to all logarithms. If it were required, for example, to find the square root of the number c, we must first find the loga-

rithm of c, that is, *log.* c, and next taking the half of that logarithm, or $\frac{1}{2}$*log.* c, we should have the logarithm of the square root required: we have therefore only to look in the Tables for the number answering to that logarithm, in order to obtain the root required.

230. We have already seen, that the numbers, 1, 2, 3, 4, 5, 6, &c. that is to say, all positive numbers, are logarithms of the root a, and of its positive powers; consequently, logarithms of numbers greater than unity: and, on the contrary, that the negative numbers, as -1, -2, &c. are logarithms of the fractions $\frac{1}{a}$, $\frac{1}{a^2}$, &c. which are less than unity, but yet greater than nothing.

Hence, it follows, that, if the logarithm be positive, the number is always greater than unity: but if the logarithm be negative, the number is always less than unity, and yet greater than 0; consequently, we cannot express the logarithms of negative numbers: we must therefore conclude, that the logarithms of negative numbers are impossible, and that they belong to the class of imaginary quantities.

231. In order to illustrate this more fully, it will be proper to fix on a determinate number for the root a. Let us make choice of that, on which the common Logarithmic Tables are formed, that is, the number 10, which has been preferred, because it is the foundation of our Arithmetic. But it is evident that any other number, provided it were greater than unity, would answer the same purpose: and the reason why we cannot suppose $a =$ unity, or 1, is manifest; because all the powers a^b would then be constantly equal to unity, and could never become equal to another given number, c.

CHAP. XXII.

Of the Logarithmic Tables *now in use.*

232. In those Tables, as we have already mentioned, we begin with the supposition, that the root a is $= 10$; so that the logarithm of any number, c, is the exponent to which we must raise the number 10, in order that the power resulting from it may be equal to the number c; or if we denote the logarithm of c by $L.c$, we shall always have $10^{L.c} = c$.

233. We have already observed, that the logarithm of the number 1 is always 0; and we have also $10^0 = 1$; consequently, *log.* $1 = 0$; *log.* $10 = 1$; *log.* $100 = 2$; *log.* $1000 = 3$; *log.* $10000 = 4$; *log.* $100000 = 5$; *log.* $1000000 = 6$. Farther, *log.* $\frac{1}{10} = -1$; *log.* $\frac{1}{100} = -2$; *log.* $\frac{1}{1000} = -3$; *log.* $\frac{1}{10000} = -4$; *log.* $\frac{1}{100000} = -5$; *log.* $\frac{1}{1000000} = -6$.

234. The logarithms of the principal numbers, therefore, are easily determined; but it is much more difficult to find the logarithms of all the other intervening numbers; and yet they must be inserted in the Tables. This however is not the place to lay down all the rules that are necessary for such an inquiry; we shall therefore at present content ourselves with a general view only of the subject.

235. First, since *log.* $1 = 0$, and *log.* $10 = 1$, it is evident that the logarithms of all numbers between 1 and 10 must be included between 0 and unity; and, consequently, be greater than 0, and less than 1. It will therefore be sufficient to consider the single number 2; the logarithm of which is certainly greater than 0, but less than unity: and if we represent this logarithm by the letter x, so that *log.* $2 = x$, the value of that letter must be such as to give exactly $10^x = 2$.

We easily perceive, also, that x must be considerably less than $\frac{1}{2}$, or which amounts to the same thing, $10^{\frac{1}{2}}$ is greater than 2; for if we square both sides, the square of $10^{\frac{1}{2}} = 10$, and the square of $2 = 4$. Now, this latter is much less than the former: and, in the same manner, we see that x is also less than $\frac{1}{3}$; that is to say, $10^{\frac{1}{3}}$ is greater than 2: for the cube of $10^{\frac{1}{3}}$ is 10, and that of 2 is only 8. But, on the contrary, by making $x = \frac{1}{4}$, we give it too small a value; because the fourth power of $10^{\frac{1}{4}}$ being 10, and that of 2 being 16, it is evident that $10^{\frac{1}{4}}$ is less than 2. Thus, we see that x, or the *log.* 2, is less than $\frac{1}{3}$, but greater than $\frac{1}{4}$: and, in the same manner, we may determine, with respect to every fraction contained between $\frac{1}{4}$ and $\frac{1}{3}$, whether it be too great or too small.

In making trial, for example, with $\frac{2}{7}$, which is less than $\frac{1}{3}$, and greater than $\frac{1}{4}$, 10^x, or $10^{\frac{2}{7}}$, ought to be $= 2$; or the seventh power of $10^{\frac{2}{7}}$, that is to say, 10^2, or 100, ought to be equal to the seventh power of 2, or 128; which is consequently greater than 100. We see, therefore, that $\frac{2}{7}$ is less than *log.* 2, and that *log.* 2, which was found less than $\frac{1}{3}$, is however greater than $\frac{2}{7}$.

Let us try another fraction, which, in consequence of what we have already found, must be contained between $\frac{2}{7}$ and $\frac{1}{3}$. Such a fraction between these limits is $\frac{3}{10}$; and it is therefore required to find whether $10^{\frac{3}{10}} = 2$; if this be the case, the tenth powers of those numbers are also equal: but the tenth power of $10^{\frac{3}{10}}$ is $10^3 = 1000$, and the tenth power of 2 is 1024; we conclude therefore, that $10^{\frac{3}{10}}$ is less than 2, and, consequently, that $\frac{3}{10}$ is too small a fraction, and therefore the *log.* 2, though less than $\frac{1}{3}$, is yet greater than $\frac{3}{10}$.

236. This discussion serves to prove, that *log.* 2 has a determinate value, since we know that it is certainly greater than $\frac{3}{10}$, but less than $\frac{1}{3}$; we shall not however proceed any farther in this investigation at present. Being therefore still ignorant of its true value, we shall represent it by x, so that $log.\ 2 = x$; and endeavour to shew how, if it were known, we could deduce from it the logarithms of an infinity of other numbers. For this purpose, we shall make use of the equation already mentioned, namely, $log.\ cd = log.\ c + log.\ d$, which comprehends the property, that the logarithm of a product is found by adding together the logarithms of the factors.

237. First, as $log.\ 2 = x$, and $log.\ 10 = 1$, we shall have

$log.\ 20 = x + 1$, $\quad log.\ 200 = x + 2$
$log.\ 2000 = x + 3$, $\quad log.\ 20000 = x + 4$
$log.\ 200000 = x + 5$, $\quad log.\ 2000000 = x + 6$, &c.

238. Farther, as $log.\ c^2 = 2\ log.\ c$, and $log.\ c^3 = 3\ log.\ c$, and $log.\ c^4 = 4\ log.\ c$, &c. we have

$log.\ 4 = 2x$; $log.\ 8 = 3x$; $log.\ 16 = 4x$; $log.\ 32 = 5x$; $log.\ 64 = 6x$, &c. Hence we find also, that

$log.\ 40 = 2x + 1$, $\quad log.\ 400 = 2x + 2$
$log.\ 4000 = 2x + 3$, $\quad log.\ 40000 = 2x + 4$, &c.
$log.\ 80 = 3x + 1$, $\quad log.\ 800 = 3x + 2$
$log.\ 8000 = 3x = 3$, $\quad log.\ 80000 = 3x + 4$, &c.
$log.\ 160 = 4x + 1$, $\quad log.\ 1600 = 4x + 2$
$log.\ 16000 = 4x + 3$, $\quad log.\ 160000 = 4x + 4$, &c.

239. Let us resume also the other fundamental equation, $log.\ \frac{c}{d} = log.\ c - log.\ d$, and let us suppose $c = 10$, and $d = 2$; since $log.\ 10 = 1$, and $log.\ 2 = x$, we shall have $log.\ \frac{10}{2}$, or $log.\ 5 = 1 - x$, and shall deduce from hence the following equations:

$log.\ 50 = 2 - x$, $\quad log.\ 500 = 3 - x$
$log.\ 5000 = 4 - x$, $\quad log.\ 50000 = 5 - x$, &c.
$log.\ 25 = 2 - 2x$, $\quad log.\ 125 = 3 - 3x$
$log.\ 625 = 4 - 4x$, $\quad log.\ 3125 = 5 - 5x$, &c.
$log.\ 250 = 3 - 2x$, $\quad log.\ 2500 = 4 - 2x$
$log.\ 25000 = 5 - 2x$, $\quad log.\ 250000 = 6 - 2x$, &c.
$log.\ 1250 = 4 - 3x$, $\quad log.\ 12500 = 5 - 3x$
$log.\ 125000 = 6 - 3x$, $\quad log.\ 1250000 = 7 - 3x$, &c.
$log.\ 6250 = 5 - 4x$, $\quad log.\ 62500 = 6 - 4x$
$log.\ 625000 = 7 - 4x$, $\quad log.\ 6250000 = 8 - 4x$, &c.

and so on.

240. If we knew the logarithm of 3, this would be the means also of determining a number of other logarithms; as appears from the following examples. Let the *log.* 3 be represented by the letter y: then,

$log.\ 30 = y + 1$, $\quad log.\ 300 = y + 2$
$log.\ 3000 = y + 3$, $\quad log.\ 30000 = y + 4$, &c.

$log.\ 9 = 2y$, $log.\ 27 = 3y$, $log.\ 81 = 4y$, &c. we shall have also,

$log.\ 6 = x + y$, $log.\ 12 = 2x + y$, $log.\ 18 = x + 2y$,
$log.\ 15 = log.\ 3 + log.\ 5 = y + 1 - x$.

241. We have already seen that all numbers arise from the multiplication of prime numbers. If therefore we only knew the logarithms of all the prime numbers, we could find the logarithms of all the other numbers by simple additions. The number 210, for example, being formed by the factors 2, 3, 5, 7, its logarithm will be $log.\ 2 + log.\ 3 + log.\ 5 + log.\ 7$. In the same manner, since $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$, we have $log.\ 360 = 3\ log.\ 2 + 2\ log.\ 3 + log.\ 5$. It is evident, therefore, that by means of the logarithms of the prime numbers, we may determine those of all others; and that we must first apply to the determination of the former, if we would construct Tables of Logarithms.

CHAP. XXIII.

Of the Method *of expressing* Logarithms.

242. We have seen that the logarithm of 2 is greater than $\frac{3}{10}$, and less than $\frac{1}{3}$, and that, consequently, the exponent of 10 must fall between those two fractions, in order that the power may become 2. Now, although we know this, yet

whatever fraction we assume on this condition, the power resulting from it will be always an irrational number, greater or less than 2; and, consequently, the logarithm of 2 cannot be accurately expressed by such a fraction: therefore we must content ourselves with determining the value of that logarithm by such an approximation as may render the error of little or no importance; for which purpose, we employ what are called *decimal fractions*, the nature and properties of which ought to be explained as clearly as possible.

243. It is well known that, in the ordinary way of writing numbers by means of the ten figures, or characters,

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9,$$

the first figure on the right alone has its natural signification; that the figures in the second place have ten times the value which they would have had in the first; that the figures in the third place have a hundred times the value; and those in the fourth a thousand times, and so on: so that as they advance towards the left, they acquire a value ten times greater than they had in the preceding rank. Thus, in the number 1765, the figure 5 is in the first place on the right, and is just equal to 5; in the second place is 6; but this figure, instead of 6, represents 10×6, or 60: the figure 7 is in the third place, and represents 100×7, or 700; and lastly, the 1, which is in the fourth row, becomes 1000; so that we read the given number thus;

One thousand, seven hundred, and sixty-five.

244. As the value of figures becomes always ten times greater, as we go from the right towards the left, and as it consequently becomes continually ten times less as we go from the left towards the right; we may, in conformity with this law, advance still farther towards the right, and obtain figures whose value will continue to become ten times less than in the preceding place: but it must be observed, that the place where the figures have their natural value is marked by a point. So that if we meet, for example, with the number 36·54892, it is to be understood in this manner: the figure 6, in the first place, has its natural value; and the figure 3, which is in the second place to the left, means 30. But the figure 5, which comes after the point, expresses only $\frac{5}{10}$; and the 4 is equal only to $\frac{4}{100}$; the figure 8 is equal to $\frac{8}{1000}$; the figure 9 is equal to $\frac{9}{10000}$; and the figure 2 is equal to $\frac{2}{100000}$. We see then, that the more those figures advance towards the right, the more their

values diminish, and at last, those values become so small, that they may be considered as nothing*.

245. This is the kind of numbers which we call *decimal fractions*, and in this manner logarithms are represented in the Tables. The logarithm of 2, for example, is expressed by 0·3010300; in which we see, 1st. That since there is 0 before the point, this logarithm does not contain an integer; 2dly, that its value is $\frac{3}{10} + \frac{0}{100} + \frac{1}{1000} + \frac{0}{10000} + \frac{3}{100000} + \frac{0}{1000000} + \frac{0}{10000000}$. We might have left out the two last ciphers, but they serve to shew that the logarithm in question contains none of those parts, which have 1000000 and 10000000 for the denominator. It is however to be understood, that, by continuing the series, we might have found still smaller parts; but with regard to these, they are neglected, on account of their extreme minuteness.

246. The logarithm of 3 is expressed in the Table by 0·4771213; we see, therefore, that it contains no integer, and that it is composed of the following fractions: $\frac{4}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{1}{10000} + \frac{2}{100000} + \frac{1}{1000000} + \frac{3}{10000000}$. But we must not suppose that the logarithm is thus expressed with the utmost exactness; we are only certain that the error is less than $\frac{1}{10000000}$; which is certainly so small, that it may very well be neglected in most calculations.

247. According to this method of expressing logarithms, that of 1 must be represented by 0·0000000, since it is really $= 0$: the logarithm of 10 is 1·0000000, where it evidently is exactly $= 1$: the logarithm of 100 is 2·0000000, or 2. And hence we may conclude, that the logarithms of all numbers, which are included between 10 and 100, and

* The operations of arithmetic are performed with decimal fractions in the same manner nearly, as with whole numbers; some precautions only are necessary, after the operation, to place the point properly, which separates the whole numbers from the decimals. On this subject, we may consult almost any of the treatises on arithmetic. In the multiplication of these fractions, when the multiplicand and multiplier contain a great number of decimals, the operation would become too long, and would give the result much more exact than is for the most part necessary; but it may be simplified by a method, which is not to be found in many authors, and which is pointed out by M. Marie in his edition of the mathematical lessons of M. de la Caille, where he likewise explains a similar method for the division of decimals. F. T.

The method alluded to in this note is clearly explained in Bonnycastle's Arithmetic.

consequently composed of two figures, are comprehended between 1 and 2, and therefore must be expressed by 1 *plus* a decimal fraction, as *log*. 50 = 1·6989700; its value therefore is unity, *plus* $\frac{6}{10} + \frac{9}{100} + \frac{8}{1000} + \frac{9}{10000} + \frac{7}{100000}$: and it will be also easily perceived, that the logarithms of numbers, between 100 and 1000, are expressed by the integer 2 with a decimal fraction: those of numbers between 1000 and 10000, by 3 *plus* a decimal fraction: those of numbers between 10000 and 100000, by 4 integers *plus* a decimal fraction, and so on. Thus, the *log*. 800, for example, is 2·9030900; that of 2290 is 3·3598355, &c.

248. On the other hand, the logarithms of numbers which are less than 10, or expressed by a single figure, do not contain an integer, and for this reason we find 0 before the point: so that we have two parts to consider in a logarithm. First, that which precedes the point, or the integral part; and the other, the decimal fractions that are to be added to the former. The integral part of a logarithm, which is usually called the *characteristic*, is easily determined from what we have said in the preceding article. Thus, it is 0, for all the numbers which have but *one figure*; it is 1, for those which have *two*; it is 2, for those which have *three*; and, in general, it is always one less than the number of figures. If therefore the logarithm of 1766 be required, we already know that the first part, or that of the integers, is necessarily 3.

249. So reciprocally, we know at the first sight of the integer part of a logarithm, how many figures compose the number answering to that logarithm; since the number of those figures always exceed the integer part of the logarithm by unity. Suppose, for example, the number answering to the logarithm 6·4771213 were required, we know immediately that that number must have seven figures, and be greater than 1000000. And in fact this number is 3000000; for *log*. 3000000 = *log*. 3 + *log*. 1000000. Now *log*. 3 = 0·4771213, and *log*. 1000000 = 6, and the sum of those two logarithms is 6·4771213.

250. The principal consideration therefore with respect to each logarithm is, the decimal fraction which follows the point, and even that, when once known, serves for several numbers. In order to prove this, let us consider the logarithm of the number 365; its first part is undoubtedly 2; with respect to the other, or the decimal fraction, let us at present represent it by the letter x; we shall have *log*. 365 $= 2 + x$; then multiplying continually by 10, we shall have *log*. 3650 $= 3 + x$; *log*. 36500 $= 4 + x$; *log*. 365000 $= 5 + x$, and so on.

But we can also go back, and continually divide by 10; which will give us *log*. 36·5 $= 1 + x$; *log*. 3·65 $= 0 + x$; *log*. 0·365 $= -1 + x$; *log*. 0·0365 $= -2 + x$; *log*. 0·00365 $= -3 + x$, and so on.

251. All those numbers then which arise from the figures 365, whether preceded, or followed, by ciphers, have always the same decimal fraction for the second part of the logarithm: and the whole difference lies in the integer before the point, which, as we have seen, may become negative; namely, when the number proposed is less than 1. Now, as ordinary calculators find a difficulty in managing negative numbers, it is usual, in those cases, to increase the integers of the logarithm by 10, that is, to write 10 instead of 0 before the point; so that instead of −1 we have 9; instead of −2 we have 8; instead of −3 we have 7, &c.; but then we must remember, that the characteristic has been taken ten units too great, and by no means suppose that the number consists of 10, 9, or 8 figures. It is likewise easy to conceive, that, if in the case we speak of, this characteristic be less than 10, we must write the figures of the number after a point, to shew that they are decimals: for example, if the characteristic be 9, we must begin at the first place after a point; if it be 8, we must also place a cipher in the first row, and not begin to write the figures till the second: thus 9·5622929 would be the logarithm of 0·365, and 8·5622929 the log. of 0·0365. But this manner of writing logarithms is principally employed in Tables of sines.

252. In the common Tables, the decimals of logarithms are usually carried to seven places of figures, the last of which consequently represents the $\frac{1}{10000000}$ part, and we are sure that they are never erroneous by the whole of this part, and that therefore the error cannot be of any importance. There are, however, calculations in which we require still greater exactness; and then we employ the large Tables of Vlacq, where the logarithms are calculated to ten decimal places *.

* The most valuable set of tables we are acquainted with are those published by Dr. Hutton, late Professor of Mathematics at the Royal Military Academy, Woolwich, under the title of, "Mathematical Tables; containing common, hyperbolic, and logistic logarithms. Also sines, tangents, &c. to which is prefixed a large and original history of the discoveries and writings relating to those subjects."

253. As the first part, or characteristic of a logarithm, is subject to no difficulty, it is seldom expressed in the Tables; the second part only is written, or the seven figures of the decimal fraction. There is a set of English Tables in which we find the logarithms of all numbers from 1 to 100000, and even those of greater numbers; for small additional Tables shew what is to be added to the logarithms, in proportion to the figures, which the proposed numbers have more than those in the Tables. We easily find, for example, the logarithm of 379456, by means of that of 37945 and the small Tables of which we speak *.

254. From what has been said, it will easily be perceived, how we are to obtain from the Tables the number corresponding to any logarithm which may occur. Thus, in multiplying the numbers 343 and 2401; since we must add

* The English Tables spoken of in the text are those which were published by Sherwin in the beginning of the last century, and have been several times reprinted; they are likewise to be found in the tables of Gardener, which are commonly made use of by astronomers, and which have been reprinted at Avignon. With respect to these Tables it is proper to remark, that as they do not carry logarithms farther than seven places, independently of the characteristic, we cannot use them with perfect exactness except on numbers that do not exceed six digits; but when we employ the great Tables of Vlacq, which carry the logarithms as far as ten decimal places, we may, by taking the proportional parts, work, without error, upon numbers that have as many as nine digits. The reason of what we have said, and the method of employing these Tables in operations upon still greater numbers, is well explained in Saunderson's "Elements of Algebra," Book IX. Part II.

It is farther to be observed, that these Tables only give the logarithms answering to given numbers, so that when we wish to get the numbers answering to given logarithms, it is seldom that we find in the Tables the precise logarithms that are given, and we are for the most part under the necessity of seeking for these numbers in an indirect way, by the method of interpolation. In order to supply this defect, another set of Tables was published at London in 1742, under the title of "The Antilogarithmic Canon, &c. by James Dodson." He has arranged the decimals of logarithms from 0,0001 to 1,0000, and opposite to them, in order, the corresponding numbers carried as far as eleven places. He has likewise given the proportional parts necessary for determining the numbers, which answer to the intermediate logarithms that are not to be found in the Table. F. T.

together the logarithms of those numbers, the calculation will be as follows:

$$\left.\begin{array}{l} log.\ 343 = 2{\cdot}5352941 \\ log.\ 2401 = 3{\cdot}3803922 \end{array}\right\}\text{added}$$

$$5{\cdot}9156863 \text{ their sum}$$

$$log.\ 823540 = 5{\cdot}9156847 \text{ nearest tabular log.}$$

$$16 \text{ difference,}$$

which in the Table of Differences answers to 3; this therefore being used instead of the cipher, gives 823543 for the product sought: for the sum is the logarithm of the product required; and its characteristic 5 shews that the product is composed of 6 figures; which are found as above.

255. But it is in the extraction of roots that logarithms are of the greatest service; we shall therefore give an example of the manner in which they are used in calculations of this kind. Suppose, for example, it were required to extract the square root of 10. Here we have only to divide the logarithm of 10, which is 1·0000000 by 2; and the quotient 0·5000000 is the logarithm of the root required. Now, the number in the Tables which answers to that logarithm is 3·16228, the square of which is very nearly equal to 10, being only one hundred thousandth part too great *.

* In the same manner, we may extract any other root, by dividing the log. of the number by the denominator of the index of the root to be extracted; that is, to extract the cube root, divide the log. by 3, the fourth root by 4, and so on for any other extraction. For example, if the 5th root of 2 were required, the log. of 2 is 0·3010300: therefore

$$5)0{\cdot}3010300$$

0·0602060 is the log. of the root, which by the Tables is found to correspond to 1·1497; and hence we have $\sqrt[5]{2} = 1{\cdot}1497$. When the index, or characteristic of the log. is negative, and not divisible by the denominator of the index of the root to be extracted; then as many units must be borrowed as will make it exactly divisible, carrying those units to the next figure, as in common division.