SUPPLEMENTUM

CONTINENS

EVOLUTIONEM CASUUM SINGULARIUM CIRCA INTEGRATIONEM

AEQUATIONUM DIFFERENTIALIUM.

EVOLUTIO

CASUUM PRORSUS SINGULARIUM CIRCA INTEGRATIONEM AEQUATIONUM DIFFERENTIALIUM.

1.

Cum adhuc plurimae atque inter se maxime discrepantes methodi sint in medium allatae, aequationes differentiales integrandi, quaestio exoritur summi sane momenti, an non unica detur eaque aequabilis methodus, cujus ope omnes illae diversae aequationes differentiales, quas etiamnum resolvere licuit, integrari queant? nullum enim est dubium quin inventio talis methodi maxima incrementa in universam Analysin esset allatura. Pluribus Geometris quidem separatio binarum variabilium hujusmodi methodum suppeditare est visa, cum omnes aequationum differentialium integrationes vel hac ratione sint integratae, vel eo facile possint revocari. Praeterquam autem quod haec methodus substitutionibus absolvitur, quae plerumque non minorem sagacitatem postulant, quam id ipsum quod quaeritur, ac nonnunquam soli casui deberi videntur, haec methodus etiam neutiquam extenditur ad aequationes differentiales secundi altiorumque graduum; et qui tales aequationes adhuc tractaverunt, longe alia artificia in subsidium vocare sunt coacti. Quamobrem separationem variabilium nequaquam tanquam methodum uniformem ac latissime patentem spectare licet, quae omnes integrationes, quae adhuc successerunt, in se complectatur.

2. Talem autem methodum universalem jam pridem mihi equidem indicasse videor, dum ostendi proposita quacunque aequatione differentiali sive primi sive altioris gradus, semper dari ejusmodi quantitatem, per quam si aequatio multiplicetur, evadat integrabilis, ita ut hoc modo nulla plane substitutione alibi anxie quaerenda sit opus. Ex quo non dubito, hanc methodum aequationes differentiales ope multiplicationum ad integrabilitatem revocandi, tanquam latissime patentem atque naturae maxime convenientem pronunciare; cum nulla integratio adhuc sit expedita, quae hoc modo non facile absolvi possit. Cum scilicet omnis aequatio differentialis primi gradus in hac forma $P\partial x + Q\partial y = 0$ contineatur, denotantibus litteris P et Q functiones quascunque binarum variabilium x et y, semper datur ejusmodi multiplicator M itidem functio quaedam ambarum variabilium x et y, ut facta multiplicatione haec forma $MP\partial x + MQ\partial y$ fiat integrabilis; cujus propterea integrale quantitati constanti arbitrariae aequatum exhibebit aequationem integralem aequationis differentialis propositae $P\partial x + Q\partial y = 0$, quae eadem ratio quoque in aequationibus differentialibus altiorum graduum locum habet. Verum hoc argumentum hic fusius exponere non est animus; sed potius praestantiam hujus methodi prae separatione variabilium etiam ejusmodi casibus quibus id minime videatur, simulque summam ejus utilitatem hic declarare constitui.

3. Quoties scilicet in aequatione differentiali variabiles x et y jam sunt separatae, totum negotium vulgo ut jam confectum spectari solet, quandoquidem hujus aequationis

$$X\partial x + Y\partial y = 0,$$

ubi X denotat functionem solius x et Y solius y, integrale in promtu est

$$\int X\partial x + \int Y\partial y = \text{Const.}$$

Interim tamen saepenumero usu venire potest, ut hoc pacto neutiquam forma integralis simplicissima obtineatur, vel ea demum per plures ambages inde derivari debeat. Veluti ex hac aequatione

$$\frac{\partial x}{x} + \frac{\partial y}{y} = 0,$$

primo elicitur integrale logarithmicum

$$lx + ly = la,$$

unde quidem statim se prodit algebraicum $xy = a$. Verum ex hac forma

$$\frac{\partial x}{aa + xx} + \frac{\partial y}{aa + yy} = 0,$$

integratio solita praebet

$$\text{Ang. tang. } x + \text{Ang. tang. } y = \text{Const.}$$

unde non tam facile forma integralis algebraica $\frac{x + y}{aa - xy} = C$ deducitur. Ac proposita hac forma

$$\frac{\partial x}{\sqrt{(\alpha + \beta x + \gamma xx)}} + \frac{\partial y}{\sqrt{(\alpha + \beta y + \gamma yy)}} = 0,$$

in genere ne patet quidem, utrum utraque pars integralis arcu circulari an logarithmo exprimatur. Interim tamen ejus integrale ita algebraice exhiberi potest

$$CC(x + y)^2 + 2\gamma Cxy + \beta C(x + y) + 2\alpha C + \tfrac{1}{4}\beta\beta - \alpha\gamma = 0,$$

quae certe forma simplicissima nonnisi per plures ambages ex integrali transcendente derivatur.

4. His quidem casibus perspicitur, quomodo reductionem ad formam algebraicam institui oporteat, sed ante aliquot annos ejusmodi integrationes protuli, in quibus ne hoc quidem ullo modo praestari potest. Veluti si proposita sit haec aequatio

$$\frac{\partial x}{\sqrt{(1 + x^4)}} + \frac{\partial y}{\sqrt{(1 + y^4)}} = 0,$$

integrationem neque per logarithmos neque arcus circulares expedire licet, ut inde deinceps simili ratione aequatio algebraica colligi posset: interim tamen ostendi hujus integrale idque adeo completum hoc modo algebraice exprimi

$$0 = 2C + (CC - 1)(xx + yy) - 2(1 + CC)\,xy + 2\,Cxxyy,$$

ubi C denotat constantem per integrationem ingressam. Quin etiam hujus aequationis multo latius patentis

$$\frac{\partial x}{\sqrt{(\alpha + 2\beta x + \gamma xx + 2\delta x^3 + \varepsilon x^4)}} + \frac{\partial y}{\sqrt{(\alpha + 2\beta y + \gamma yy + 2\delta y + \varepsilon y^4)}} = 0$$

integrale completum est

$$\begin{aligned} 0 = 2\alpha C + \beta\beta - \alpha\gamma + 2\big(\beta C - \alpha\delta(x+y) + (CC - \alpha\varepsilon)(xx+yy)\big) \\ + 2(\gamma C - CC - \alpha\varepsilon - \beta\delta)\,xy + 2(\delta C - \beta\varepsilon)\,xy\,(x+y) \\ + (2\varepsilon C + \delta\delta - \gamma\varepsilon)\,xxyy, \end{aligned}$$

denotante C item constantem quantitatem arbitrariam per integrationem inventam. His igitur casibus perspicuum est separationem variabilium, qua aequationes differentiales sunt praeditae, nihil plane juvare ad integralia earum forma algebraica contenta eruenda, ex quo merito ejusmodi methodus desideratur, cujus beneficio haec integralia statim ex aequationibus differentialibus investigari potuissent, in quo negotio certe omnes ingenii vires tentasse non pigebit.

5. Observavi igitur hunc scopum ope multiplicatorum idoneorum obtineri posse, quibus aequationes differentiales multiplicatae ita integrabiles evadant, ut integralia statim algebraice expressa prodeant. Quod quo clarius perspiciatur ab aequatione primum proposita $\frac{\partial x}{x} + \frac{\partial y}{y} = 0$ exordiar, quae per xy multiplicata statim praebet $y\partial x + x\partial y = C$. Hoc ergo modo sublata separatione aequatio in aliam transformatur, quae integrationem admittit, ex quo intelligitur methodum ope multiplicatorum integrandi id

praestare, quod a separatione variabilium immediate exspectari nequeat. Idem evenit in aequatione $\frac{m\,\partial x}{x} + \frac{n\,\partial y}{y} = 0$, quae per $x^m y^n$ multiplicata integrale praebet $x^m y^n = C$, dum ex ipsa aequatione proposita statim ad logarithmos fuisset perventum. Simili modo si haec aequatio separata

$$\frac{\partial x}{1 + xx} + \frac{\partial y}{1 + yy} = 0$$

multiplicetur in $\frac{(1 + xx)(1 + yy)}{(x + y)^2}$, aequatio resultans

$$\frac{\partial x.(1 + yy) + \partial y\,(1 + xx)}{(x + y)^2} = 0$$

integrationem jam sponte admittit, praebetque integrata

$$\frac{-1 + xy}{x + y} = \text{Const. seu } \frac{x + y}{1 - xy} = a.$$

Hanc vero aequationem

$$\frac{2\partial x}{1 + xx} + \frac{\partial y}{1 + yy} = 0$$

multiplicari convenit in $\frac{(xx + 1)^2\,(1 + yy)}{(2xy + xx - 1)^2}$, ut prodeat

$$\frac{2\partial x\,(1 + xx)\,(1 + yy) + \partial y\,(xx + 1)^2}{(2xy + xx - 1)^2} = 0,$$

cujus integrale reperitur

$$\frac{xxy - 2x - y}{2xy + xx - 1} = \text{Const. seu } \frac{2x + y - xxy}{2xy + xx - 1} = a.$$

6. Contra haec exempla, quibus integralia algebraica sine subsidio separationis sunt eruta, objicietur, multiplicatores negotium hoc conficientes ex ipsis integralibus illis transcendentibus, ad quae separatio variabilium immediate perducit esse conclusos, iisque adeo praestantiam methodi per multiplicatores procedentis neutiquam probari. Cui quidem objectioni primum respondeo, priora exempla statim ab inventis integrationis principiis simili modo fuisse expedita, antequam integratio per logarithmos erat explorata, quae

ergo nullum subsidium eo attulisse est censenda. Tum vero quamvis concedam, in posterioribus exemplis integrationem per arcus circulares multiplicatores illos idoneos commode suppeditasse, id tamen in ipsa evolutione minus cernitur, eademque integratio sine dubio inveniri potuisset, antequam constaret formulae $\frac{\partial x}{1+xx}$ integrale esse arcum circuli tangenti x respondentem. Verum aequatio supra allata

$$\frac{\partial x}{\sqrt{(1+x^4)}} + \frac{\partial y}{\sqrt{(1+y^4)}} = 0,$$

cujus integrale completum algebraice exhibere licet, nulli amplius dubio locum relinquit, cum enim neutrius partis integrale ne concessis quidem logarithmis vel arcubus circularibus exhiberi possit, ejusque forma ad genus quantitatum transcendentium etiamnum incognitum sit referenda, haec certe nullum auxilium ad integrale algebraicum inveniendum attulisse censeri potest. Atque hoc multo magis de aequatione illa latius patente in §. 4. proposita est tenendum, quippe cujus integratio omnino singularis ex principiis longe diversissimis a me est eruta.

7. Methodus autem, qua tum sum usus, tantopere est abscondita, ut vix ulla via ad eadem integralia perducens patere videatur, et cum separatio variabilium nihil plane eo contulisset, vix etiam quicquam ab altera methodo ad multiplicatores adstricta sperari posse videbatur, propterea quod tum ipse adhuc in ea opinione versabar, per multiplicatores nihil praestari posse, nisi quatenus separatio variabilium eodem manuducat; quandoquidem quaestio differentialia tantum primi gradus implicaret. Deinceps autem re diligentius considerata perspexi, quoties aequationis cujusque differentialis integrale completum exhibere licet, ex eo vicissim semper ejusmodi multiplicatorem elici posse, per quem si aequatio differentialis multiplicetur, non solum fiat integrabilis, sed etiam integrata id ipsum integrale, quod jam erat cognitum, reproducere de-

beat; ad hoc autem omnino necesse est ut integrale completum sit exploratum, dum ex integralibus particularibus nihil plane pro hoc scopo concludere licet. Si enim proposita sit aequatio differentialis

$$P\partial x + Q\partial y = 0,$$

cujus integrale completum, undecunque sit cognitum, constabit id aequatione, quae praeter binas variabiles x et y et quantitates constantes in ipsa aequatione differentiali contentas insuper quantitatem constantem novam prorsus ab arbitrio nostro pendentem complectetur. Quae si littera C indicetur, eruatur ejus valor ex aequatione integrali, ac reperiatur $C = V$, eritque V certa quaedam functio ipsarum x et y; tum autem hac aequatione differentiata $0 = \partial V$, differentiale ∂V necessario ita formulam differentialem $P\partial x + Q\partial y$ continere debet, ut sit

$$\partial V = M(P\partial x + Q\partial y),$$

ex qua forma multiplicator M, ad hoc integrale $C = V$ perducens, sponte se offert.

8. Quo haec operatio aliquot exemplis illustretur, sumatur primo haec aequatio

$$\frac{m\partial x}{x} + \frac{n\partial y}{y} = 0,$$

cujus integrale completum cum sit $x^m y^n = C$, instituta differentiatione prodit

$$0 = mx^{m-1} y^n \partial x + nx^m y^{n-1} \partial y, \text{ seu}$$

$$0 = x^m y^n \left(\frac{m\partial x}{x} + \frac{n\partial y}{y}\right),$$

unde patet, multiplicatorem ad hoc integrale ducentem esse $x^m y^n$.

Deinde cum hujus aequationis

$$\frac{\partial x}{1 + xx} + \frac{\partial y}{1 + yy} = 0$$

integrale completum sit

$$1 - xy = C(x + y),$$

valor constantis arbitrariae hinc fit $C = \frac{1 - xy}{x + y}$, cujus differentiatio praebet

$$0 = \frac{-\partial x\,(1 + yy) - \partial y\,(1 + xx)}{(x + y)^2}, \quad \text{seu}$$

$$0 = \frac{(1 + xx)(1 + yy)}{(x + y)^2}\left(\frac{\partial x}{1 + xx} + \frac{\partial y}{1 + yy}\right),$$

unde multiplicator quaesitus est $= \frac{(1 + xx)(1 + yy)}{(x + y)^2}$.

Proposita porro sit haec aequatio

$$\frac{\partial x}{\sqrt{(\alpha + 2\beta x + \gamma xx)}} + \frac{\partial y}{\sqrt{(\alpha + 2\beta y + \gamma yy)}} = 0,$$

cujus integrale completum

$$CC\,(x - y)^2 - 2C\,(\alpha + \beta x + \beta y + \gamma xy) + \beta\beta - \alpha\gamma = 0$$

dat primo

$$C = \frac{+\alpha + \beta(x + y) + \gamma xy + \sqrt{\left[\alpha\alpha + 2\alpha\beta(x + y) {+\alpha\gamma(xx + yy) \atop +4\beta\beta xy} {+4\beta\gamma xy(x + y) \atop +\gamma\gamma xxyy}\right]}}{(x - y)^2},$$

seu

$$C = \frac{+\alpha + \beta(x + y) + \gamma xy + \sqrt{(\alpha + 2\beta x + \gamma xx)(\alpha + 2\beta y + \gamma yy)}}{(x - y)^2}$$

vel concinnius

$$\frac{\beta\beta - \alpha\gamma}{C} = + \alpha + \beta\,(x + y) + \gamma xy$$
$$+ \sqrt{(\alpha + 2\beta x + \gamma xx)(\alpha + 2\beta y + \gamma yy)},$$

unde differentiando fit

$$0 = + \partial x\,(\beta + \gamma y) + \partial y\,(\beta + \gamma x)$$
$$+ \frac{\partial x\,(\beta + \gamma x)\sqrt{(\alpha + 2\beta y + \gamma yy)}}{\sqrt{(\alpha + 2\beta x + \gamma xx)}} + \frac{\partial y\,(\beta + \gamma y)\sqrt{(\alpha + 2\beta x + \gamma xx)}}{\sqrt{(\alpha + 2\beta y + \gamma yy)}},$$

hincque colligitur multiplicator quaesitus

$$M = (\beta + \gamma x)\sqrt{(\alpha + 2\beta y + \gamma yy)} + (\beta + \gamma y)\sqrt{(\alpha + 2\beta x + \gamma xx)}.$$

9. Simili modo pro aequatione magis complexa

$$\frac{\partial x}{\sqrt{(\alpha + 2\beta x + \gamma xx + 2\delta x^3 + \varepsilon x^4)}} + \frac{\partial y}{\sqrt{(\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4)}} = 0,$$

ex ejus integrali completo supra exhibito multiplicator idoneus M investigari poterit, ex quo si statim fuisset cognitus, idem hoc integrale immediate elici potuisset. Verum hic opus multo majus molior, quod autem primo conatu neutiquam ad finem perducere licebit; ex quo satis mihi equidem praestitisse videbor, si saltem primo quasi lineamenta novae atque maxime desiderandae methodi adumbravero, cujus ope, proposita hujusmodi aequatione differentiali, multiplicator idoneus eam reddens integrabilem inveniri queat. Ac primo quidem in hoc negotio plurimum observasse juvabit, si unicus hujusmodi multiplicator innotuerit, ex eo facile infinitos alios idem officium praestantes erui posse. Quodsi enim multiplicator M aequationem differentialem

$$P\partial x + Q\partial y = 0$$

integrabilem reddat, ita ut sit

$$\int M\,(P\partial x + Q\partial y) = V,$$

ideoque aequatio integralis $V = C$, quoniam formula

$$\partial V = M\,(P\partial x + Q\partial y)$$

per functionem quamcunque quantitatis V multiplicata perinde manet integrabilis, perspicuum est hanc formam $Mf:V$, quaecunque functio ipsius V pro $f:V$ accipiatur, semper multiplicatorem idoneum praebere, cum sit

$$(P\partial x + Q\partial y)\,Mf:V = \partial V f:V,$$

ideoque integrabile. Inter infinitos igitur hos multiplicatores idoneos quovis casu eum eligi conveniet, qui negotium facillime conficiat, et integrale si fuerit algebraicum forma simplicissima exhibeat. Etiamsi enim integrale revera sit algebraicum, omnino fieri potest, ut id ne suspicari quidem liceat, nisi multiplicator idoneus in usum vocetur, quemadmodum superiora exempla abunde declarant.

10. Sit ergo aequatio differentialis proposita hujus formae

$$\frac{\partial x}{X} + \frac{\partial y}{Y} = 0,$$

in qua X sit functio x et Y solius y; atque investigari oporteat ejusmodi multiplicatorem M, quo illa aequatio algebraice integrabilis reddatur, siquidem fieri potest: quod cum raro eveniat, vicissim assumta multiplicatoris forma M indagasse juvabit functiones X et Y. Sit primo multiplicator

$$M = \frac{XY}{(\alpha + \beta x + \gamma y)^2},$$

ut integrabilis esse debeat haec forma

$$\frac{Y\partial x + X\partial y}{(\alpha + \beta x + \gamma y)^2} = 0.$$

Hinc sumta y constante colligitur integrale

$$\frac{-Y}{\beta(\alpha + \beta x + \gamma y)} + \Gamma : y,$$

sumta autem x constante prodit

$$\frac{-X}{\gamma(\alpha + \beta x + \gamma y)} + \Delta : x,$$

quam ambas formas inter se aequales esse oportet; unde fit

$$-\gamma Y + \beta\gamma(\alpha + \beta x + \gamma y)\Gamma : y = -\beta X + \beta\gamma(\alpha + \beta x + \gamma y)\Delta : x,$$

seu

$$\beta X - \gamma Y = \beta\gamma(\alpha + \beta x + \gamma y)(\Delta : x - \Gamma : y),$$

sicque patet functiones $\Delta : x$ et $\Gamma : y$ ita comparatas esse debere, ut evoluto posteriori membro termini, qui simul x et y continerent, se mutuo tollant. Ex quo intelligitur fore

$$\Delta : x = m\beta x + \text{Const. et } \Gamma : y = m\gamma y + \text{Const.}$$

Statuamus ergo

$$\Delta : x - \Gamma : y = m\beta x + m\gamma y + n, \quad \text{fietque}$$

$$\beta X - \gamma Y = \beta\gamma \left\{ \begin{array}{llll} m\beta\beta xx - m\gamma\gamma yy + n\beta x & + n\gamma y & + n\alpha \\ & + m\alpha\beta x - m\alpha\gamma y & + f \\ & & - f \end{array} \right\},$$

unde colligimus

$$X = \gamma (m\,\beta\beta\,xx + \beta\,(m\alpha + n)\,x + f + \tfrac{1}{2}\,n\alpha),$$
$$Y = \beta\,(m\gamma\gamma\,yy + \gamma\,(m\alpha - n)\,y + f - \tfrac{1}{2}\,n\alpha),$$

et integralis aequatio algebraica erit

$$m\gamma y - \frac{m\gamma\gamma yy - \gamma\,(m\alpha - n)\,y - f + \frac{1}{2}\,n\alpha}{\alpha + \beta x + \gamma y} = \text{Const.}$$

seu

$$m\beta\gamma xy + n\gamma y - f + \tfrac{1}{2}\,n\alpha = C\,(\alpha + \beta x + \gamma y),$$

vel loco C scribendo $C + \frac{1}{2}\,n$, erit concinnius

$$m\beta\gamma xy - \tfrac{1}{2}\,n\beta x + \tfrac{1}{2}\,n\gamma y - f = C\,(\alpha + \beta x + \gamma y).$$

11. Videamus jam sub quibus conditionibus haec forma aequationis generalis ista ratione integrabilis evadat

$$\frac{h\partial x}{Axx + Bx + C} + \frac{k\partial y}{Dyy + Ey + F} = 0.$$

Comparatione ergo cum valoribus inventis instituta colligitur

$A = hm\beta\beta\gamma$,	$D = km\beta\gamma\gamma$,
$B = h\beta\gamma\,(m\alpha + n)$,	$E = k\beta\gamma\,(m\alpha - n)$,
$C = h\gamma\,(f + \frac{1}{2}\,n\alpha)$,	$F = k\beta\,(f - \frac{1}{2}\,n\alpha)$.

Quoniam hic totum negotium ad rationes litterarum reducitur, sumtis pro primis aequalitatibus

$$\beta = Ak \quad \text{et} \quad \gamma = Dh,$$

concluduntur reliquae

$m = \frac{1}{ADhhkk}$, $\alpha = \frac{Bk + Eh}{2}$, $n = \frac{Bk - Eh}{2ADhhkk}$ et $f = \frac{ACkk + DFhh}{2AD\,hhkk}$,

praeterea vero haec conditio requiritur, ut sit

$$\frac{4AC - BB}{hh} = \frac{4DF - EE}{kk},$$

quae si habuerit locum, multiplicator idoneus erit

$$M = \frac{(Axx + Bx + C)(Dyy + Ey + F)}{hk\,[\frac{1}{2}(Bk + Eh) + Akx + Dhy]^2},$$

et aequatio integralis inde resultans erit per hk multiplicando

$$xy - \frac{(Bk - Eh)\,x}{4Dh} + \frac{(Bk - Eh)\,y}{4Ak} - \frac{ACkk - DFhh}{2ADhk}$$
$$= G\,[\tfrac{1}{2}(Bk + Eh) + Akx + Dhy],$$

quae immutata constante arbitraria G ad hanc formam revocatur

$$(x + \tfrac{B}{2A} - GDh)(y + \tfrac{E}{2D} - GAk) = GGADhk$$
$$+ \frac{(4AC - BB)\,kk + (4DF - EE)\,hh}{8ADhk},$$

seu

$$\left(\frac{2Ax + B}{h} + G\right)\left(\frac{2Dy + E}{k} + G\right) = GG + \frac{4AC - BB}{2hh} + \frac{4DF - EE}{2kk}.$$

12. En ergo Theorema minime spernendum, etiamsi ejus veritas ex aliis principiis satis manifesta esse queat.

Si haec aequatio differentialis

$$\frac{h\partial x}{Axx + Bx + C} + \frac{k\partial y}{Dyy + Ey + F} = 0$$

ita fuerit comparata, ut sit

$$\frac{4AC - BB}{hh} = \frac{4DF - EE}{kk},$$

tum ejus integrale completum erit algebraicum, atque hac aequatione expressum

$$\left(\frac{2Ax + B}{h}\right)\left(\frac{2Dy + E}{k}\right) + G\left(\frac{2Ax + B}{h} + \frac{2Dy + E}{k}\right) = \frac{4AC - BB}{2hh} + \frac{4DF - EE}{2kk},$$

ubi G denotat constantem arbitrariam per integrationem invectam Hoc vero integrale invenitur si aequatio proposita ducatur in hunc multiplicatorem

$$\frac{(Axx + Bx + C)(Dyy + Ey + F)}{\left(\frac{2Ax + B}{h} + \frac{2Dy + E}{k}\right)^2}.$$

13. Quemadmodum multiplicatori M tribuimus formam

$$\frac{XY}{(\alpha + \beta x + \gamma y)^2},$$

ita etiam formis magis complicatis uti licebit, quod quidem in genere praestari nequit. Evolvamus autem multiplicatorem

$$M = \frac{XY}{(\alpha + \beta x + \gamma y + \delta xy)^2},$$

ut haec aequatio integrabilis sit efficienda

$$\frac{Y\partial x + X\partial y}{(\alpha + \beta x + \gamma y + \delta xy)^2} = 0,$$

cujus integratio ad hanc perducit aequationem

$$\frac{-Y}{(\beta + \delta y)(\alpha + \beta x + \gamma y + \delta xy)} + \Gamma : y = \frac{-X}{(\gamma + \delta x)(\alpha + \beta x + \gamma y + \delta xy)} + \Delta : x,$$

quae transformatur in hanc

$$\frac{X}{\gamma + \delta x} - \frac{Y}{\beta + \delta y} = (\alpha + \beta x + \gamma y + \delta xy)(\Delta : x - \Gamma : y),$$

ubi evidens est, statui debere

$$\Delta : x = \frac{\zeta x + \eta}{\gamma + \delta x} \quad \text{et} \quad \Gamma : y = \frac{\zeta y + \theta}{\beta + \delta y},$$

ut nulli termini occurrant qui utramque variabilem simul complectantur: hinc ergo fit

$$\frac{X}{\gamma + \delta x} - \frac{Y}{\alpha + \delta y} = \begin{matrix} + \eta y \\ + f \end{matrix} + \frac{(\alpha + \beta x)(\zeta x + \eta)}{\gamma + \delta x} \begin{matrix} - \theta x \\ - f \end{matrix} - \frac{(\alpha + \gamma y)(\zeta y + \theta)}{\beta + \delta y},$$

unde concludimus

$$X = (\alpha + \beta x)(\zeta x + \eta) - (\gamma + \delta x)(\theta x + f),$$
$$Y = (\alpha + \gamma y)(\zeta y + \theta) - (\beta + \delta y)(\eta y + f),$$

sive evolvendo

$$X = (\beta\zeta - \delta\theta)\,xx + (\alpha\zeta + \beta\eta - \gamma\theta - \delta f)\,x + \alpha\eta - \gamma f,$$
$$Y = (\gamma\zeta - \delta\eta)\,yy + (\alpha\zeta + \gamma\theta - \beta\eta - \delta f)\,y + \alpha\theta - \beta f,$$

et aequatio integralis erit

$$\frac{\zeta x + \eta}{\gamma + \delta x} - \frac{X}{(\gamma + \delta x)(\alpha + \beta x + \gamma y + \delta xy)} = \text{Const.}$$

quae loco X substituto valore invento abit in hanc formam

$$\frac{\zeta xy + \eta y + \theta x + f}{\alpha + \beta x + \gamma y + \delta xy} = \text{Const.}$$

14. Transferamus haec iterum ad formam

$$\frac{h\partial x}{Axx + Bx + C} + \frac{k\partial y}{Dyy + Ey + F} = 0,$$

ac fieri oportet

$$A = h(\beta\zeta - \delta\theta), \qquad D = k(\gamma\zeta - \delta\eta),$$
$$B = h(\alpha\zeta + \beta\eta - \gamma\theta - \delta f), \qquad E = k(\alpha\zeta + \gamma\theta - \beta\eta - \delta f),$$
$$C = h(\alpha\eta - \gamma f), \qquad F = k(\alpha\theta - \beta f).$$

Primae aequationes praebent

$$\theta = \frac{\beta\zeta}{\delta} - \frac{A}{\delta h}, \quad \eta = \frac{\gamma\zeta}{\delta} - \frac{D}{\delta k};$$

secundae vero

$$f = \frac{\alpha\zeta}{\delta} - \frac{Bk - Eh}{2\delta hk} \quad \text{et} \quad \delta = \frac{2A\gamma k - 2D\beta h}{Bk - Eh},$$

unde ex tertiis colligitur

$$\frac{2Ck\,(A\gamma k - D\beta h)}{Bk - Eh} = \frac{\gamma}{2}\,(Bk + Eh) - D\alpha h,$$
$$\frac{2Fh\,(A\gamma k - D\beta h)}{Bk - Eh} = \frac{\beta}{2}\,(Bk + Eh) - A\alpha k.$$

Hinc α elidendo fit

$$\frac{2\,(ACkk - DFhh)\,(Ak\gamma - Dh\beta)}{Bk - Eh} = \tfrac{1}{2}\,(Ak\gamma - Dh\beta)\,(Bk + Eh),$$

unde cum esse nequeat

$$Ak\gamma - Dh\beta = 0,$$

quia alioquin fieret $\delta = 0$, et quantitates θ, η, f infinitae, tum vero quod praecique est notandum, aequatio integralis prodiret Const. $=$ Const. quo ergo casu nihil indicaretur, necesse est ut sit

$$4(ACkk - DFhh) = BBkk - EEhh, \text{ seu}$$

$$\frac{4AC - BB}{hh} = \frac{4DF - EE}{kk}, \text{ ut ante.}$$

Quod autem hic maxime animadverti meretur, est, quod etsi tres litterae β, γ et ζ manent indefinitae, aequatio tamen integralis a praecedente nonnisi quantitate constante discrepat; prodit enim

$$\frac{2\zeta hk}{Bk - Eh} + \frac{k(2Ax + B) + h(2Dy + E)}{2(Ak\gamma - Dh\beta)xy + (Bk - Eh)(\beta x + \gamma y) + 2(Ck\beta - Fh\gamma)} = \text{Const.}$$

seu

$$\frac{\gamma ky(2Ax + B) + \beta k(Bx + 2C) - \beta hx(2Dy + E) - \gamma h(Ey + 2F)}{k(2Ax + B) + h(2Dy + E)} = \text{Const.}$$

quae forma, quomodocunque accipiantur litterae β et γ, semper veram aequationem integralem exhibet. Quod cum minus sit perspicuum, ostendissi sufficiet, ambas partes β et γ involventes seorsim sumtas eandem relationem inter x et y definire. Constitutis enim his duabus aequationibus

$$\frac{2Akxy + Bky - Ehy - 2Fh}{2Akx + 2Dhy + Bk + Eh} = \text{Const.}$$

$$\frac{-2Dhxy - Ehx + Bkx + 2Ck}{2Akx + 2Dhy + Bk + Eh} = \text{Const.}$$

multiplicetur prior per Dh posterior per Ak, fietque summa

$$\frac{Ak(Bk - Eh)x + Dh(Bk - Eh)y + 2ACkk - 2DFhh}{2Akx + 2Dhy + Bk + Eh},$$

cujus valor utique est constans $= \frac{Bk - Eh}{2}$, propterea quod

$$\frac{2ACkk - 2DFhh}{Bk - Eh} = \frac{Bk - Eh}{2},$$

unde patet propositum.

15. Progredior nunc ad formam aequationum magis arduam, quae sit

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

sitque multiplicator eam reddens integrabilem

$$M = P\sqrt{X} + Q\sqrt{Y},$$

ita ut aequatio integrationem admittens sit

$$P\partial x + Q\partial y + \frac{Q\partial x\sqrt{Y}}{\sqrt{X}} + \frac{P\partial y\sqrt{X}}{\sqrt{Y}} = 0,$$

cujus utrumque membrum seorsim integrabile sit oportet. Pro priore ergo erit $(\frac{\partial P}{\partial y}) = (\frac{\partial Q}{\partial x})$, posterioris vero integrale statuatur $2V\sqrt{XY}$, unde colligitur

$$Q = 2X\,(\frac{\partial V}{\partial x}) + V\,.\,\frac{\partial X}{\partial x} \text{ et}$$
$$P = 2Y\,(\frac{\partial V}{\partial y}) + V\,.\,\frac{\partial Y}{\partial y},$$

et ob priorem conditionem

$$2Y\,(\frac{\partial\partial V}{\partial y^2}) + \frac{3\partial Y}{\partial y}\,(\frac{\partial V}{\partial y}) + V.\frac{\partial\partial Y}{\partial y^2} = 2X\,(\frac{\partial\partial V}{\partial x^2}) + \frac{3\partial X}{\partial x}\,(\frac{\partial V}{\partial x}) + V.\frac{\partial\partial X}{\partial x^2},$$

ex qua aequatione, si loco V sumserimus certam functionem ipsarum x et y, dispiciendum est, quomodo idonei valores pro functionibus X et Y obtineantur.

16. Demus primo ipsi V valorem constantem puta $V = 1$, ac pervenimus ad hanc conditionem

$$\frac{\partial\partial Y}{\partial y^2} = \frac{\partial\partial X}{\partial x^2},$$

quae aequalitas subsistere nequit, nisi utrumque membrum seorsim aequetur quantitati constanti, quae sit $= 2a$, unde colligemus

$$X = axx + bx + c \text{ et } Y = ayy + dy + e,$$

hincque porro

$$P = \frac{\partial Y}{\partial y} = 2ay + d \text{ et } Q = \frac{\partial X}{\partial x} = 2ax + b,$$

unde aequatio integralis completa colligitur

$$2axy + dx + by + 2\sqrt{XY} = \text{Const.}$$

Quocirca ista aequatio differentialis

$$\frac{\partial x}{\sqrt{(axx + bx + c)}} + \frac{\partial y}{\sqrt{(ayy + dy + e)}} = 0$$

integrabilis redditur ope multiplicatoris

$$M = (2ay + d)\sqrt{(axx + bx + c)} + (2ax + b)\sqrt{(ayy + dy + e)},$$

ac tum integrale completum reperietur

$$2axy + dx + by + 2\sqrt{(axx + bx + c)(ayy + dy + e)} = C,$$

seu sublata irrationalitate

$$CC - 2C(2axy + dx + by) = (4ae - dd)xx + (4ac - bb)yy + 4bex + 4cdy + 4ce.$$

Haec autem aequatio differentialis multo latius patet illa quam initio §. 3. attuleram.

17. Tribuamus nunc ipsi V hunc valorem

$$V = \frac{1}{(\alpha + \beta x - \gamma y)^2},$$

si enim loco exponentis 2 indefinitum sumsissem, mox potuisset, hanc potestatem accipi debuisse. Erit ergo

$$\left(\frac{\partial V}{\partial x}\right) = \frac{-2\beta}{(\alpha + \beta x + \gamma y)^3}, \quad \left(\frac{\partial V}{\partial y}\right) = \frac{-2\gamma}{(\alpha + \beta x + \gamma y)^3},$$

$$\left(\frac{\partial\partial V}{\partial x^2}\right) = \frac{6\beta\beta}{(\alpha + \beta x + \gamma y)^4}, \text{ et } \left(\frac{\partial\partial V}{\partial y^2}\right) = \frac{6\gamma\gamma}{(\alpha + \beta x + \gamma y)^4}.$$

His autem valoribus substitutis sequentes oriuntur binae formae

$$12\beta\beta X - \frac{6\beta\partial X}{\partial x}(\alpha + \beta x + \gamma y) + \frac{\partial\partial X}{\partial x^2}(\alpha + \beta x + \gamma y)^2$$

$$= 12\gamma\gamma Y - \frac{6\gamma\partial Y}{\partial y}(\alpha + \beta x + \gamma y) + \frac{\partial\partial Y}{\partial y^2}(\alpha + \beta x + \gamma y)^2,$$

quia igitur in priore y in altera x non ultra duas dimensiones assurgit, evidens est in formulis

$$\frac{\partial\partial X}{\partial x^2} \text{ et } \frac{\partial\partial Y}{\partial y^2},$$

variabiles x et y totidem dimensiones habere debere, quia alioquin termini ex x et y mixti utrinque aequales fieri non possent. Cum ergo ipsae functiones X et Y ad quartum gradum sint ascensurae, ponamus

$$\mathrm{X} = \mathrm{A}x^4 + 2\,\mathrm{B}x^3 + \mathrm{C}x^2 + 2\,\mathrm{D}x + \mathrm{E} \quad \text{et}$$
$$\mathrm{Y} = \mathfrak{A}y^4 + 2\,\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\,\mathfrak{D}y + \mathfrak{E}.$$

Facta jam substitutione pro priori parte prodit

$$\begin{array}{lllll}
12\,\beta\beta\mathrm{A}x^4 & +\,24\,\beta\beta\mathrm{B}x^3 & +\,12\,\beta\beta\mathrm{C}xx & +\,24\,\beta\beta\mathrm{D}x & +\,12\,\beta\beta\mathrm{E} \\
-\,24\,\beta\beta\mathrm{A} & -\,36\,\beta\beta\mathrm{B} & -\,12\,\beta\beta\mathrm{C} & -\,12\,\beta\beta\mathrm{D} & -\,12\,\alpha\beta\mathrm{D} \\
+\,12\,\beta\beta\mathrm{A} & -\,24\,\alpha\beta\mathrm{A} & -\,36\,\alpha\beta\mathrm{B} & -\,12\,\alpha\beta\mathrm{C} & +\,2\,\alpha\alpha\mathrm{C} \\
 & +\,12\,\beta\beta\mathrm{B} & +\,2\,\beta\beta\mathrm{C} & +\,4\,\alpha\beta\mathrm{C} & \\
 & +\,24\,\alpha\beta\mathrm{A} & +\,24\,\alpha\beta\mathrm{B} & +\,12\,\alpha\alpha\mathrm{B} & \\
 & & +\,12\,\alpha\alpha\mathrm{A} & &
\end{array}$$

$$\begin{array}{llll}
-\,24\,\beta\gamma\mathrm{A}x^3y & -\,36\,\beta\gamma\mathrm{B}y^2y & -\,12\,\beta\gamma\mathrm{C}xy & -\,12\,\beta\gamma\mathrm{D}y \\
+\,24\,\beta\gamma\mathrm{A} & +\,24\,\beta\gamma\mathrm{B} & +\,4\,\beta\gamma\mathrm{C} & +\,4\,\alpha\gamma\mathrm{C} \\
 & +\,24\,\alpha\gamma\mathrm{A} & +\,24\,\alpha\gamma\mathrm{B} &
\end{array}$$

$$+\,12\,\gamma\gamma\mathrm{A}xxyy + 12\,\gamma\gamma\mathrm{B}xyy + 2\,\gamma\gamma\mathrm{C}yy,$$

qui termini in ordinem disponentur

$$\begin{aligned}
&12\,\gamma\gamma\mathrm{A}xxyy + 12\,\gamma\gamma\mathrm{B}xyy + 12\,\gamma\,(2\,\alpha\mathrm{A} - \beta\mathrm{B})\,xxy \\
&\quad + 2\gamma\gamma\mathrm{C}yy + 8\,\gamma\,(3\,\alpha\mathrm{B} - \beta\mathrm{C})\,xy + 2\,(6\,\alpha\alpha\mathrm{A} - 6\,\alpha\beta\mathrm{B} + \beta\beta\mathrm{C})\,xx \\
&\quad + 4\,\gamma\,(\alpha\mathrm{C} - 3\,\beta\mathrm{D})\,y + 4\,(3\,\alpha\alpha\mathrm{B} - 2\,\alpha\beta\mathrm{C} + 3\,\beta\beta\mathrm{D})\,x \\
&\quad + (2\,\alpha\alpha\mathrm{C} - 6\,\alpha\beta\mathrm{D} + 6\,\beta\beta\mathrm{E}).
\end{aligned}$$

Simili vero modo altera pars erit

$$\begin{aligned}
&12\,\beta\beta\mathfrak{A}xxyy + 12\,\beta\beta\mathfrak{B}xxy + 12\,\beta\,(2\,\alpha\mathfrak{A} - \gamma\mathfrak{B})\,xyy + 2\,\beta\beta\mathfrak{C}xx \\
&\quad + 8\,\beta\,(3\,\alpha\mathfrak{B} - \gamma\mathfrak{C})\,xy + 2\,(6\,\alpha\alpha\mathfrak{A} - 6\,\alpha\gamma\mathfrak{B} + \gamma\gamma\mathfrak{C})\,yy \\
&\quad + 4\,\beta\,(\alpha\mathfrak{C} - 3\,\gamma\mathfrak{D})\,x + 4\,(3\,\alpha\alpha\mathfrak{B} - 2\,\alpha\gamma\mathfrak{C} + 3\,\gamma\gamma\mathfrak{D})\,y \\
&\quad + 2\,(\alpha\alpha\mathfrak{C} - 6\,\alpha\gamma\mathfrak{D} + 6\,\gamma\gamma\mathfrak{E}).
\end{aligned}$$

18. Coaequentur nunc inter se termini homologi utriusque formae, et sequentibus aequationibus erit satisfaciendum

$$\begin{array}{l|l}
xxyy & \gamma\gamma A = \beta\beta\mathfrak{A}, \\
xxy & 2\alpha\gamma A - \beta\gamma B = \beta\beta\mathfrak{B}, \\
xyy & \gamma\gamma B = 2\alpha\beta\mathfrak{A} - \beta\gamma\mathfrak{B}, \\
xx & 6\alpha\alpha A - 6\alpha\beta B + \beta\beta C = \beta\beta\mathfrak{C}, \\
yy & \gamma\gamma C = 6\alpha\alpha\mathfrak{A} - 6\alpha\gamma\mathfrak{B} + \gamma\gamma\mathfrak{C}, \\
xy & 3\alpha\gamma\beta - \beta\gamma C = 3\alpha\beta\mathfrak{B} - \beta\gamma\mathfrak{C}, \\
x & 3\alpha\alpha B - 2\alpha\beta C + 3\beta\beta D = \alpha\beta\mathfrak{C} - 3\beta\gamma\mathfrak{D}, \\
y & \alpha\gamma C - 3\beta\gamma D = 3\alpha\alpha\mathfrak{B} - 2\alpha\gamma\mathfrak{C} + 3\gamma\gamma\mathfrak{D}, \\
1 & \alpha\alpha C - 6\alpha\beta D + 6\beta\beta E = \alpha\alpha\mathfrak{C} - 6\alpha\gamma\mathfrak{D} + 6\gamma\gamma\mathfrak{E}.
\end{array}$$

Tres autem primae aequationes tantum duas dant determinationes

$$\beta = \frac{2\alpha A\sqrt{\mathfrak{A}}}{B\sqrt{\mathfrak{A}} + \mathfrak{B}\sqrt{A}} \quad \text{et} \quad \gamma = \frac{2\alpha\mathfrak{A}\sqrt{A}}{B\sqrt{\mathfrak{A}} + \mathfrak{B}\sqrt{A}},$$

quarta et quinta itidem unicam determinationem suppeditant

$$C - \mathfrak{C} = \frac{3(\mathfrak{A}BB - A\mathfrak{B}\mathfrak{B})}{2A\mathfrak{A}} = \tfrac{3}{2}\left(\frac{BB}{A} - \frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}\right),$$

quae eadem quoque ex sexta sequitur. Statuatur ergo

$$C = \frac{3BB}{2A} + n \quad \text{et} \quad \mathfrak{C} = \frac{3\mathfrak{B}\mathfrak{B}}{2\mathfrak{A}} + n,$$

septima et octava etiam unicam determinationem involvunt

$$\frac{D\sqrt{A} + \mathfrak{D}\sqrt{\mathfrak{A}}}{B\sqrt{\mathfrak{A}} + \mathfrak{B}\sqrt{A}} = \frac{A\mathfrak{B}\mathfrak{B} + \mathfrak{A}BB - B\mathfrak{B}\sqrt{A\mathfrak{A}} + 2nA\mathfrak{A}}{4A\mathfrak{A}\sqrt{A\mathfrak{A}}}, \quad \text{vel}$$

$$D\sqrt{A} + \mathfrak{D}\sqrt{\mathfrak{A}} = \frac{B^3}{4A\sqrt{A}} + \frac{\mathfrak{B}^3}{4\mathfrak{A}\sqrt{\mathfrak{A}}} + \frac{nB}{2\sqrt{A}} + \frac{n\mathfrak{B}}{2\sqrt{\mathfrak{A}}},$$

qui valores in ultima aequatione substituti praebent

$$24(AE - \mathfrak{A}\mathfrak{E}) = + \frac{3B^4}{2AA} + \frac{6nBB}{A} + \frac{12mB}{\sqrt{A}} - \frac{3\mathfrak{B}^4}{2\mathfrak{A}\mathfrak{A}} - \frac{6n\mathfrak{B}\mathfrak{B}}{\mathfrak{A}} + \frac{12m\mathfrak{B}}{\sqrt{\mathfrak{A}}},$$

quare commode statui licebit

$$\mathrm{E} = \frac{\mathrm{B}^4}{16\,\mathrm{A}^3} + \frac{n\,\mathrm{B}\,\mathrm{B}}{4\,\mathrm{A}\mathrm{A}} + \frac{m\,\mathrm{B}}{2\,\mathrm{A}\sqrt{\mathrm{A}}} + \frac{l}{\mathrm{A}},$$

$$\mathfrak{E} = \frac{\mathfrak{B}^4}{16\,\mathfrak{A}^3} + \frac{n\,\mathfrak{B}\mathfrak{B}}{4\,\mathfrak{A}\mathfrak{A}} + \frac{m\,\mathfrak{B}}{2\,\mathfrak{A}\sqrt{\mathfrak{A}}} + \frac{l}{\mathfrak{A}}.$$

19. Cum autem sumserimus $\mathrm{V} = \frac{1}{(\alpha+\beta x+\gamma y)^2}$, erit

$$\mathrm{Q} = \frac{-4\beta(\mathrm{A}x^4+2\mathrm{B}x^3+\mathrm{C}xx+2\,\mathrm{D}x+\mathrm{E})}{(\alpha+\beta x+\gamma y)^3} + \frac{2(2\,\mathrm{A}x^3+3\,\mathrm{B}xx+\mathrm{C}x+\mathrm{D})}{(\alpha+\beta x+\gamma y)^2},$$

$$\mathrm{P} = \frac{-4\gamma(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}{(\alpha+\beta x+\gamma y)^3} + \frac{2(2\,\mathfrak{A}y^3+3\,\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})}{(\alpha+\beta x+\gamma y)^2},$$

sive

$$\mathrm{Q} = \frac{2\gamma y(2\mathrm{A}x^3+3\mathrm{B}xx+\mathrm{C}x+\mathrm{D})+2(2\alpha\mathrm{A}-\beta\mathrm{B})x^3+2(3\alpha\mathrm{B}-\beta\mathrm{C})xx+2(\alpha\mathrm{C}-3\beta\mathrm{D})x+2(\alpha\mathrm{D}-2\beta\mathrm{E})}{(\alpha+\beta x+\gamma y)^3},$$

$$\mathrm{P} = \frac{2\beta x(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})+2(2\alpha\mathfrak{A}-\gamma\mathfrak{B})y^3+2(3\alpha\mathfrak{B}-\gamma\mathfrak{C})yy+2(\alpha\mathfrak{C}-3\gamma\mathfrak{D})y+2(\alpha\mathfrak{D}-2\gamma\mathfrak{E})}{(\alpha+\beta x+\gamma y)^3},$$

unde investigari oportet integrale formulae $\mathrm{P}\partial x + \mathrm{Q}\partial y$, ad quod si deinceps addatur $\frac{2\sqrt{\mathrm{XY}}}{(\alpha+\beta x+\gamma y)^2}$, aggregatum quantitati constanti aequatum exhibebit integrale completum aequationis

$$\frac{\partial x}{\sqrt{\mathrm{X}}} + \frac{\partial y}{\sqrt{\mathrm{Y}}} = 0.$$

Pro illo autem integrali inveniendo, ex prioribus valoribus pro P et Q exhibitis, notetur fore separatim

$$\int \mathrm{Q}\partial y = \frac{2\beta(\mathrm{A}x^4+2\mathrm{B}x^3+\mathrm{C}xx+2\mathrm{D}x+\mathrm{E})}{\gamma(\alpha+\beta x+\gamma y)^2} - \frac{2(2\mathrm{A}x^3+3\mathrm{B}xx+\mathrm{C}x+\mathrm{D})}{\gamma(\alpha+\beta x+\gamma y)} + \Gamma : x,$$

$$\int \mathrm{P}\partial x = \frac{2\gamma(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}{\beta(\alpha+\beta x+\gamma y)^2} - \frac{2(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})}{\beta(\alpha+\beta x+\gamma y)} + \Delta : y,$$

quae duae expressiones aequales esse debent: quem in finem ponatur

$$\Gamma : x = \frac{2(\mathrm{A}xx+\mathrm{B}x+\mathrm{N})}{\beta\gamma} \text{ et } \Delta : y = \frac{2(\mathfrak{A}yy+\mathfrak{B}y+\mathfrak{N})}{\beta\gamma},$$

fietque

$\frac{1}{2}\beta\gamma\,(\alpha+\beta x+\gamma y)^2 \int Q\partial y$	$\frac{1}{2}\beta\gamma\,(\alpha+\beta x+\gamma y)^2 \int P\partial x$
$+A\gamma\gamma xxyy$	$+\mathfrak{A}\beta\beta xxyy$
$+B\gamma\gamma xyy$	$+\beta\,(2\mathfrak{A}\alpha-\mathfrak{B}\gamma)\,xyy$
$+\gamma\,(2A\alpha-B\beta)\,xxy$	$+\mathfrak{B}\beta\beta xxy$
$+N\gamma\gamma yy$	$+(\mathfrak{A}\alpha\alpha-\mathfrak{B}\alpha\gamma+\mathfrak{N}\gamma\gamma)\,yy$
$+(A\alpha\alpha-B\alpha\beta+N\beta\beta)\,xx$	$+\mathfrak{N}\beta\beta xx$
$+\gamma\,(2B\alpha-C\beta+2N\beta)\,xy$	$+\beta\,(2\mathfrak{B}\alpha-\mathfrak{C}\gamma+2\mathfrak{N}\gamma)\,xy$
$+\gamma\,(2N\alpha-D\beta)\,y$	$+(\mathfrak{B}\alpha\alpha-\mathfrak{C}\alpha\gamma+\mathfrak{D}\gamma\gamma+2\mathfrak{N}\alpha\gamma)y$
$+(B\alpha\alpha-C\alpha\beta+D\beta\beta+2N\alpha\beta)x$	$+\beta\,(2\mathfrak{N}\alpha-\mathfrak{D}\gamma)\,x$
$+E\beta\beta-D\alpha\beta+N\alpha\alpha$	$+\mathfrak{E}\gamma\gamma-\mathfrak{D}x\gamma+\mathfrak{N}\alpha\alpha.$

20. Hae conditiones cum praecedentibus (§. 18.) perfecte conveniunt, si modo sumatur

$$N=\tfrac{1}{6}C \quad\text{et}\quad \mathfrak{N}=\tfrac{1}{6}\mathfrak{C}.$$

Dividamus singulos terminos per 6γ, ut prodeat valor formulae

$$\tfrac{1}{2}\,(\alpha+\beta x+\gamma y)^2 \int Q\partial y,$$

qui substitutis valoribus ante inventis reperietur

$$\begin{aligned} & xxyy\sqrt{A\mathfrak{A}}+Bxyy\sqrt{\tfrac{\mathfrak{A}}{A}}+\mathfrak{B}xxy\sqrt{\tfrac{A}{\mathfrak{A}}}+\tfrac{1}{6}Cyy\sqrt{\tfrac{\mathfrak{A}}{A}}+\tfrac{1}{6}\mathfrak{C}xx\sqrt{\tfrac{A}{\mathfrak{A}}} \\ & +\left(\tfrac{B\mathfrak{B}}{\sqrt{A\mathfrak{A}}}-\tfrac{2}{3}n\right)xy+\left(\tfrac{BB\mathfrak{B}}{4A\sqrt{A\mathfrak{A}}}-\tfrac{nB}{3A}+\tfrac{n\mathfrak{B}}{6\sqrt{A\mathfrak{A}}}-\tfrac{m}{2\sqrt{A}}\right)y \\ & \qquad +\left(\tfrac{B\mathfrak{B}\mathfrak{B}}{4\mathfrak{A}\sqrt{A\mathfrak{A}}}-\tfrac{n\mathfrak{B}}{3\mathfrak{A}}+\tfrac{nB}{6\sqrt{A\mathfrak{A}}}-\tfrac{m}{2\sqrt{\mathfrak{A}}}\right)x \\ & +\tfrac{BB\mathfrak{B}\mathfrak{B}}{16A\mathfrak{A}\sqrt{A\mathfrak{A}}}+\tfrac{n(B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A})^2}{24A\mathfrak{A}\sqrt{A\mathfrak{A}}}-\tfrac{nB\mathfrak{B}}{4A\mathfrak{A}}+\tfrac{m(B\sqrt{\mathfrak{A}}-\mathfrak{B}\sqrt{A})}{4A\mathfrak{A}}+\tfrac{I}{\sqrt{A\mathfrak{A}}}. \end{aligned}$$

Sit haec forma brevitatis gratia $=S$, eritque integrale completum

$$\frac{S+\sqrt{XY}}{(\alpha+\beta x+\gamma y)^2}=\text{Const. seu}$$

$$S+\sqrt{XY}=\text{Const.}\,(B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}+2Ax\sqrt{\mathfrak{A}}+2\mathfrak{A}y\sqrt{A})^2,$$

quod etiam hac forma concinniori exhiberi potest

$$S + \sqrt{XY} = \text{Const.} \left(\frac{B}{\sqrt{A}} + \frac{\mathfrak{B}}{\sqrt{\mathfrak{A}}} + 2x\sqrt{A} + 2y\sqrt{\mathfrak{A}}\right)^2.$$

Quare dum functiones X et Y conditionibus ante definitis sint praeditae, hoc modo habebitur integrale completum aequationis differentialis

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0.$$

21. Haec investigatio aliquanto generalius institui potest tribuendo ipsi V talem valorem $\frac{1}{(\alpha + \beta x + \gamma y + \delta xy)^2}$, quo facilius autem calculi molestias superare queamus observo, dummodo variabiles x et y quantitate constante augeantur vel minuantur, eum ad hanc formam $\frac{1}{(a + xy)^2}$ reduci posse: expedito autem calculo restitutio facile instituetur. Considerabo ergo hanc aequationis differentialis formam

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

quam integrabilem reddi assumo ope multiplicatoris $P\sqrt{X} + Q\sqrt{Y}$, ut integrari debeat haec formula

$$P\partial x + Q\partial y + \frac{Q\partial x\sqrt{Y}}{\sqrt{X}} + \frac{P\partial y\sqrt{X}}{\sqrt{Y}} = 0.$$

Statuatur partis posterioris integrale $= 2V\sqrt{XY}$, fietque ut vidimus

$$Q = 2X\left(\frac{\partial V}{\partial x}\right) + V.\frac{\partial X}{\partial x}, \quad \text{et} \quad P = 2Y\left(\frac{\partial V}{\partial y}\right) + V.\frac{\partial Y}{\partial y}.$$

Sit igitur $V = \frac{1}{(a + xy)^2}$, ideoque

$$\left(\frac{\partial V}{\partial x}\right) = \frac{-2y}{(a + xy)^3}, \quad \text{et} \quad \left(\frac{\partial V}{\partial y}\right) = \frac{-2x}{(a + xy)^3},$$

ita ut habeamus

$$Q = \frac{-4Xy}{(a + xy)^3} + \frac{\partial X}{\partial x}\,\frac{1}{(a + xy)^2}, \quad \text{et}$$
$$P = \frac{-4Yx}{(a + xy)^3} + \frac{\partial Y}{\partial y}\,\frac{1}{(a + xy)^2}.$$

Nunc autem effici debet ut formula $P\partial x + Q\partial y$ integrationem admittat, hunc in finem duplici modo ejus integrale capiatur, dum vel y vel x constans accipitur, sicque obtinebimus

$$\int P\partial x = \frac{4Y}{yy(a+xy)} - \frac{2aY}{yy(a+xy)^2} - \frac{\partial Y}{y\partial y}\cdot\frac{1}{a+xy} + \frac{\Gamma:y}{yy},$$

$$\int Q\partial y = \frac{4X}{xx(a+xy)} - \frac{2aX}{xx(a+xy)^2} - \frac{\partial X}{x\partial x}\cdot\frac{1}{a+xy} + \frac{\Delta:x}{xx},$$

quas duas formas inter se aequales reddi oportet. Multiplicando ergo per $xxyy(a+xy)^2$ habebimus

$$4xxY(a+xy) - 2axxY - \frac{xxy\partial Y}{\partial y}(a+xy) + xx(a+xy)^2\Gamma:y$$

$$= 4yyX(a+xy) - 2ayyX - \frac{xyy\partial X}{\partial x}(a+xy) + yy(a+xy)^2\Delta:x,$$

unde fingamus

$$X = Ax^4 + 2Bx^3 + Cxx + 2Dx + E,\quad \Delta:x = Lxx + Mx + N,$$
$$Y = \mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E},\quad \Gamma:y = \mathfrak{L}yy + \mathfrak{M}y + \mathfrak{N},$$

ut fiat $\frac{\partial X}{\partial x} = 4Ax^3 + 6Bxx + 2Cx + 2D$ et

$$\frac{\partial Y}{\partial y} = 4\mathfrak{A}y^3 + 6\mathfrak{B}yy + 2\mathfrak{C}y + 2\mathfrak{D},$$

Hinc nostrae expressiones induent has formas

$xxyy(a+xy)^2\int Q\partial y$	$xxyy(a+xy)^2\int P\partial x$
$+ Lx^4y^4$	$+ \mathfrak{L}x^4y^4$
$+ Mx^3y^4$	$+ 2\mathfrak{B}x^3y^4$
$+ 2Bx^4y^3$	$+ \mathfrak{M}x^4y^3$
$+ Nxxy^4$	$- 2a\mathfrak{A}xxy^4$
$+ 2(C+aL)x^3y^3$	$+ 2(\mathfrak{C}+a\mathfrak{L})x^3y^3$
$- 2aAx^4y^2$	$+ \mathfrak{N}x^4y^2$
$+ 2(3D+aM)xxy^3$	$- 2a\mathfrak{B}xxy^3$
$- 2aBx^3y^2$	$+ 2(3\mathfrak{D}+a\mathfrak{M})x^3y^2$
$+ aaLxxyy$	$+ aa\mathfrak{L}xxyy$
$+ 2(2E+aN)xy^3$	$+ 0xy^3$
$+ 0x^3y$	$+ 2(2\mathfrak{E}+a\mathfrak{N})x^3y$
$+ (2aD+aaM)xyy$	$+ 0xyy$
$+ 0xxy$	$+ (2a\mathfrak{D}+aa\mathfrak{M})xxy$
$+ (2aE+aaN)yy$	$+ 0yy$
$+ 0xx$	$+ (2a\mathfrak{E}+aa\mathfrak{N})xx.$

22. Harum formarum coaequatio suppeditat sequentes determinationes

$$\mathfrak{L} = \mathrm{L},\ \mathrm{M} = 2\mathfrak{B},\ \mathfrak{M} = 2\mathrm{B},\ \mathrm{N} = -2a\mathfrak{A},\ \mathfrak{N} = -2a\mathrm{A},$$
$$\mathfrak{C} = \mathrm{C},\ \mathrm{D} = -a\mathfrak{B},\ \mathfrak{D} = -a\mathrm{B},\ \mathrm{E} = aa\mathfrak{A},\ \mathfrak{E} = aa\mathrm{A},$$

ita ut habeatur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(\mathrm{A}x^4 + 2\mathrm{B}x^3 + \mathrm{C}xx + 2\mathrm{D}x + \mathrm{E})}} + \frac{\partial y}{\sqrt{(\frac{\mathrm{E}}{aa}y^4 - \frac{2\mathrm{D}}{a}y^3 + \mathrm{C}yy - 2a\mathrm{B}y + aa\mathrm{A})}} = 0,$$

cujus integrale completum est

$$\frac{2\mathrm{B}xxy - \frac{2\mathrm{D}}{a}xyy - 2a\mathrm{A}xx - \frac{2\mathrm{E}}{a}yy + 2\mathrm{C}xy - 2a\mathrm{B}x + 2\mathrm{D}y + 2\sqrt{\mathrm{XY}}}{(a + xy)^2} = \mathrm{Const.}$$

Hic observo si ponamus $y = \frac{-a}{z}$, prodire aequationem initio allatam

$$\frac{\partial x}{\sqrt{(\mathrm{A}x^4 + 2\mathrm{B}x^3 + \mathrm{C}xx + 2\mathrm{D}x + \mathrm{E})}} + \frac{\partial z}{\sqrt{(\mathrm{A}z^4 + 2\mathrm{B}z^3 + \mathrm{C}zz + 2\mathrm{D}z + \mathrm{E})}} = 0,$$

cujus propterea integrale nunc etiam per principia integrationis maxime naturalia assignari potest, cum antea methodo admodum indirecta eo fuissem deductus. Integrale quippe est

$$\mathrm{A}xxzz + \mathrm{B}xz(x+z) + \mathrm{C}xz + \mathrm{D}(x+z) + \mathrm{E} + \mathrm{G}(x-z)^2 =$$
$$\sqrt{(\mathrm{A}x^4 + 2\mathrm{B}x^3 + \mathrm{C}xx + 2\mathrm{D}x + \mathrm{E})(\mathrm{A}z^4 + 2\mathrm{B}z^3 + \mathrm{C}zz + 2\mathrm{D}z + \mathrm{E})},$$

quae ab irrationalitate liberata induit hanc formam

$$\mathrm{GG}(x-z)^2 + 2\mathrm{G}[\mathrm{A}xxzz + \mathrm{B}xz(x+z) + \mathrm{C}xz + \mathrm{D}(x+z) + \mathrm{E}]$$
$$+ (\mathrm{BB} - \mathrm{AC})xxzz - 2\mathrm{AD}xz(x+z) - \mathrm{AE}(x+z)^2 - 2\mathrm{BD}xz$$
$$- 2\mathrm{BE}(x+z) + \mathrm{DD} - \mathrm{CE} = 0,$$

quae aequatio in hanc formam reducta cum superiori convenit

$$(2\,AG + BB - AC)\,xxzz + 2\,(BG - AD)\,xz\,(x + z)$$
$$+ (GG - AE)\,(x + z)^2 - 2\,(2\,GG + BD - CG)\,xz$$
$$+ 2\,(DG - BE)\,(x + z) + 2\,EG + DD - CE = 0.$$

23. Si nunc scrutari velimus, sub quibus conditionibus haec aequatio differentialis integrationem admittat

$$\frac{\partial x}{\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)}} + \frac{\partial y}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E})}} = 0,$$

concipiamus hanc nasci ex illa ponendo $z = \frac{fy + g}{hy + k}$, ita ut aequatio integralis futura sit

$$(2\,AG + BB - AC)\,xx\,(fy + g)^2$$
$$+ 2\,(BG - AD)\,x\,(fy + g)\,(hxy + kx + fy + g)$$
$$+ (GG - AE)\,(hxy + kx + fy + g)^2$$
$$- 2\,(2\,GG - CG + BD)\,x\,(fy + g)\,(hy + k)$$
$$+ 2\,(DG - BE)\,(hy + k)\,(hxy + kx + fy + g)$$
$$+ (2\,EG + DD - CE)\,(hy + k)^2 = 0.$$

At vero coëfficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, ex his quantitatibus f, g, h, k, ita definiuntur, ut sit

$$\mathfrak{A}\,(fk - gh)^2 = Af^4 + 2Bf^3h + Cffhh + 2Dfh^3 + Eh^4$$
$$\mathfrak{B}\,(fk - gh)^2 = 2Af^3g + Bff\,(3gh + fk) + Cfh(fk + gh) + Dhh\,(3fk + gh) + 2Eh^3k$$
$$\mathfrak{C}\,(fk - gh)^2 = 6Af^2g^2 + 6Bfg\,(fk + gh) + C\,(fk + gh)^2 + 6Dhk\,(fk + gh) + 6Ehhkk + 2Cfghk$$
$$\mathfrak{D}\,(fk - gh)^2 = 2Afg^3 + Bgg\,(gh + 3fk) + Cgk\,(fk + gh) + Dkk\,(fk + 3gh) + 2Ehk^3$$
$$\mathfrak{E}\,(fk - gh)^2 = Ag^4 + 2Bg^3k + Cggkk + 2Dgk^3 + Ek^4.$$

24. Videamus autem quousque problema in genere agressi calculum expedire queamus. Sit igitur proposita aequatio

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

quae per $P\sqrt{Y} + Q\sqrt{Y}$ multiplicata fiat integrabilis, sitque integrale

$$\int(P\partial x + Q\partial y) + \frac{2\sqrt{XY}}{(\alpha+\beta x+\gamma y+\delta xy)^2} = \text{Const.}$$

eritque ut vidimus

$$Q = \frac{-4X(\beta+\delta y)}{(\alpha+\beta x+\gamma y+\delta xy)^3} + \frac{\partial X}{\partial x(\alpha+\beta x+\gamma y+\delta xy)^2},$$

$$P = \frac{-4Y(\gamma+\delta x)}{(\alpha+\beta x+\gamma y+\delta xy)^3} + \frac{\partial Y}{\partial y(\alpha+\beta x+\gamma y+\delta xy)^2},$$

unde colligimus

$$\begin{aligned}(\gamma+\delta x)^2(\alpha+\beta x+\gamma y+\delta xy)^2\int Q\partial y &= 2(\beta\gamma-\alpha\delta)X \\ &+ [4\delta X - (\gamma+\delta x)\tfrac{\partial X}{\partial x}](\alpha+\beta x+\gamma y+\delta xy) \\ &+ (\alpha+\beta x+\gamma y+\delta xy)^2\,\Delta : x,\end{aligned}$$

similique modo

$$\begin{aligned}(\beta+\delta y)^2(\alpha+\beta x+\gamma y+\delta xy)^2\int P\partial x &= 2(\beta\gamma-\alpha\delta)Y \\ &+ [4\delta Y - (\beta+\delta y)\tfrac{\partial Y}{\partial y}](\alpha+\beta x+\gamma y+\delta xy) \\ &+ (\alpha+\beta x+\gamma y+\delta xy)^2\,\Gamma : y,\end{aligned}$$

quae duae formae ad consensum perduci debent, ita ut prima per $(\gamma+\delta x)^2$, altera vero per $(\beta+\delta y)^2$ divisa eandem functionem exhibeant. Quamobrem necesse est ut prior per $(\gamma+\delta x)^2$, posterior per $(\beta+\delta y)^2$ divisionem admittat, cui ergo requisitó ante omnia est satisfaciendum.

25. Evolvamus priorem valorem, partibus ab y pendentibus distinguendis

$$\begin{aligned}\text{I.}\quad & 2(\beta\gamma-\alpha\delta)X + 4\delta(\alpha+\beta x)X - (\alpha+\beta x)(\gamma+\delta x)\tfrac{\partial X}{\partial x} \\ & + (\alpha+\beta x)^2\,\Delta : x,\\ \text{II.}\quad & -y(\gamma+\delta x)[4\delta Y + (\gamma+\delta X)\tfrac{\partial X}{\partial x} + 2(\alpha+\beta x)\,\Delta : a],\\ \text{III.}\quad & +yy(\gamma+\delta x)^2\,\Delta : x,\end{aligned}$$

quae expressio per $(\gamma + \delta x)^2$ divisibilis esse debet; cum ergo tertia pars sponte sit divisibilis, pro secunda ponamus

$$(\alpha + \beta x)\Delta : x + 2\delta X = (\gamma + \delta x)R,$$

et prima pars erit

$$2(\beta\gamma - \alpha\delta)X + 2\delta(\alpha + \beta x)X + (\alpha + \beta x)(\gamma + \delta x)R \\ - (\alpha + \beta x)(\gamma + \delta x)\frac{\partial X}{\partial x},$$

quae redit ad hanc formam

$$(\gamma + \delta x)\left[2\beta X + (\alpha + \beta x)R - (\alpha + \beta x)\frac{\partial X}{\partial x}\right];$$

ita ut

$$2\beta X + (\alpha + \beta x)\left(R - \frac{\partial X}{\partial x}\right),$$

adhuc divisionem per $\gamma + \delta x$ admittere debeat. Cui conditioni satisfit sumendo

$$R = \frac{\beta}{\delta}\Delta : x - \frac{(\alpha + \beta x)}{\delta}\Delta' : x + (\gamma + \delta x)S,$$

unde fit

$$X = \frac{\beta\gamma - \alpha\delta}{2\delta\delta}\Delta : x - \frac{(\alpha + \beta x)(\gamma + \delta x)}{2\delta\delta}\Delta' : x + \frac{(\gamma + \delta x)^2}{2\delta}S$$

ideoque prima pars erit

$$(\gamma + \delta x)^2\left(\frac{\beta}{\delta}R - \frac{(\alpha + \beta x)\partial R}{2\delta\partial x}\right) + \tfrac{1}{2}(\alpha + \beta x)(\gamma + \delta x)^2 S,$$

sive

$$(\gamma + \delta x)^2\left\{\begin{array}{l}\frac{\beta\beta}{\delta\delta}\Delta : x - \frac{\beta(\alpha + \beta x)}{\delta\delta}\Delta' : x + \frac{(\alpha + \beta x)^2}{2\delta\delta}\Delta'' : x \\ \qquad + \frac{\beta(\gamma + \delta x)}{\delta}S - \frac{(\alpha + \beta x)(\gamma + \delta x)}{2\delta}\cdot\frac{\partial S}{\partial x}\end{array}\right\}$$

deinde secunda

$$y(\gamma + \delta x)^2\left\{\begin{array}{l}\frac{2\beta}{\delta}\Delta : x - \frac{(\alpha + \beta x)}{\delta}\Delta' : x + \frac{(\alpha + \beta x)(\gamma + \delta x)}{2\delta\delta}\Delta'' : x \\ \qquad + (\gamma + \delta x)S - \frac{(\gamma + \delta x)^2}{2\delta}\cdot\frac{\partial S}{\partial x}\end{array}\right\}$$

ac tertia

$$yy(\gamma + \delta x)^2\Delta : x.$$

Quocirca formulae

$$(\alpha + \beta x + \gamma y + \delta xy)^2 \int Q\partial y$$

valor erit

$$\tfrac{\beta\beta}{\delta\delta}\Delta : x + \tfrac{2\beta}{\delta} y\Delta : x + yy\Delta : x - \tfrac{\beta(\alpha+\beta x)}{\delta\delta}\Delta' : x - \tfrac{(\alpha+\beta x)}{\delta} y\Delta' : x$$
$$+ \tfrac{(\alpha+\beta x)^2}{2\delta\delta}\Delta'' : x + \tfrac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta\delta} y\Delta'' : x$$
$$+ \tfrac{\beta}{\delta}(\gamma+\delta x)\,S + (\gamma+\delta x)\,yS - \tfrac{(\alpha+\beta x)(\gamma+\delta x)}{2\delta}\cdot\tfrac{\partial S}{\partial x}$$
$$- \tfrac{(\gamma+\delta x)^2}{2\delta}\,y\cdot\tfrac{\partial S}{\partial x},$$

seu ita concinnius expressus

$$\tfrac{(\beta+\delta y)^2}{\delta\delta}\Delta : x - \tfrac{(\alpha+\beta x)(\beta+\delta y)}{\delta\delta}\Delta' : x + \tfrac{(\alpha+\beta x)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta\delta}\Delta'' : x$$
$$+ \tfrac{(\gamma+\delta x)(\beta+\delta y)}{\delta}\,S - \tfrac{(\gamma+\delta x)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta}\cdot\tfrac{\partial S}{\partial x},$$

cui alter aequalis fieri debet, qui est

$$\tfrac{(\gamma+\delta x)^2}{\delta\delta}\Gamma : y - \tfrac{(\alpha+\gamma y)(\gamma+\delta x)}{\delta\delta}\Gamma' : y + \tfrac{(\alpha+\gamma y)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta\delta}\Gamma'' : y$$
$$+ \tfrac{(\beta+\delta y)(\gamma+\delta x)}{\delta}\,\mathfrak{S} - \tfrac{(\beta+\delta y)(\alpha+\beta x+\gamma y+\delta xy)}{2\delta}\cdot\tfrac{\partial\mathfrak{S}}{\partial y}.$$

26. Quodsi jam ponamus

$$\Delta : x = \delta\delta\,(Axx + 2\,Bx + C) \quad \text{et} \quad S = \delta\,(Dxx + 2\,Ex + F)$$

item

$$\Gamma : y = \delta\delta\,(\mathfrak{A}yy + 2\,\mathfrak{B}y + \mathfrak{C}) \quad \text{et} \quad \mathfrak{S} = \delta\,(\mathfrak{D}yy + 2\mathfrak{E}y + \mathfrak{F}),$$

reperientur nostrae expressiones ita evolutae

$(\alpha+\beta x+\gamma y+\delta xy)^2\int Q\partial y$	$(\alpha+\beta x+\gamma y+\delta xy)^2\int P\partial x$
$+\delta\delta\,Axxyy$	$+\delta\delta\mathfrak{A}xxyy$
$+2\,\delta\delta\,Bxyy$	$+\delta(\gamma\mathfrak{A}-\beta\mathfrak{D}+\delta\mathfrak{E})\,xyy$
$+\delta\,(\beta A-\gamma D+\delta E)\,xxy$	$+2\delta\delta\,\mathfrak{B}\,xxy$
$+\delta\delta\,Cyy$	$+\delta(\gamma\mathfrak{E}-\alpha\mathfrak{D})\,yy$
$+\delta\,(\beta E-\alpha D)\,xx$	$+\delta\delta\,\mathfrak{C}xx$
$+[2\beta\delta B+(\beta\gamma-\alpha\delta)A-\gamma\gamma D+\delta\delta F]\,xy$	$+[2\gamma\delta\mathfrak{B}+(\beta\gamma-\alpha\delta)\mathfrak{A}-\beta\beta\mathfrak{D}+\delta\delta\mathfrak{F}]\,xy$
$+(\alpha\gamma A-2\alpha\delta B+2\beta\delta C-\gamma\gamma E+\gamma\delta F)\,y$	$+[\gamma\delta\mathfrak{F}+(\beta\gamma-\alpha\delta)\,\mathfrak{E}-\alpha\beta\mathfrak{D})\,y$
$+[\beta\delta F+(\beta\gamma-\alpha\delta)\,E-\alpha\gamma D]\,x$	$+(\alpha\beta\mathfrak{A}-2\alpha\delta\mathfrak{B}+2\gamma\delta\mathfrak{C}+\beta\beta\mathfrak{E}+\beta\delta\mathfrak{F})\,x$
$+\alpha\alpha A-2\alpha\beta B+\beta\beta C-\alpha\gamma E+\beta\gamma F$	$+\alpha\alpha\mathfrak{A}-2\alpha\gamma\mathfrak{B}+\gamma\gamma\mathfrak{C}-\alpha\beta\mathfrak{E}+\beta\gamma\mathfrak{F}$

unde nonnisi sequentes sex determinationes deducuntur

$$\mathfrak{A} = A,$$
$$\mathfrak{B} = \frac{\beta A - \gamma D}{2\delta} + \tfrac{1}{2} E,$$
$$\mathfrak{C} = \frac{\beta E + \alpha D}{\delta},$$
$$\mathfrak{D} = \frac{2\gamma\delta B - \gamma\gamma A - \delta\delta C}{\alpha\delta - \beta\gamma},$$
$$\mathfrak{E} = \frac{2\alpha\delta B - \alpha\gamma A - \beta\delta C}{\alpha\delta - \beta\gamma},$$
$$\mathfrak{F} = F - \frac{\gamma E}{\delta} - \frac{\alpha\beta\gamma A + 2\alpha\beta\delta B - \beta\beta\delta C}{\delta(\alpha\delta - \beta\gamma)},$$

his enim omnibus illis conditionibus satisfit. Sic igitur omnes litterae A, B, C, D, E, F, una cum $\alpha, \beta, \gamma, \delta$, arbitrio nostro manent relictae, ex quibus deinde colligitur functio

$$\begin{aligned} 2X = {} & \delta\delta D x^4 + 2\delta(\delta E + \gamma D - \beta A) x^3 \\ & + [\delta\delta F + 4\gamma\delta E + \gamma\gamma D - 2\beta\delta B - (\beta\gamma + 3\alpha\delta) A] xx \\ & + 2(\gamma\delta F + \gamma\gamma E - \alpha\gamma A - 2\alpha\delta B) x \\ & + \gamma\gamma F - 2\alpha\gamma B + (\beta\gamma - \alpha\delta) C. \end{aligned}$$

27. Hunc autem calculum ulterius non prosequor, cum nunc quidem sufficiat methodum directam et rei naturae conformem aperuisse, quae ad easdem integrationes omnino singulares, quas olim ex longe aliis principiis erueram, perducat. In augmentum igitur hujus scientiae plurimum intererit istam novam methodum omni studio penitius scrutari. Hunc in finem adhuc observo, aliam multiplicatoris formam adhiberi posse, cujus ope talis forma

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0$$

integrabilis reddi queat. Statuatur scilicet multiplicator $M = P + Q\sqrt{XY}$, ut integrabilis esse debeat haec forma

$$\frac{P\partial x}{\sqrt{X}} + Q\partial y\sqrt{X} + \frac{P\partial y}{\sqrt{Y}} + Q\partial x\sqrt{Y} = 0.$$

Fingatur prioris partis integrale $= 2R\sqrt{X}$, posterioris vero $= 2S\sqrt{Y}$, ut integrale completum sit

$$R\sqrt{X} + S\sqrt{Y} = \text{Const.}$$

et facta evolutione reperitur

$$P = \frac{R\partial X}{\partial x} + 2X\left(\frac{\partial R}{\partial x}\right); \quad P = \frac{S\partial Y}{\partial y} + 2Y\left(\frac{\partial S}{\partial y}\right);$$

$$Q = 2\left(\frac{\partial R}{\partial y}\right); \qquad Q = 2\left(\frac{\partial S}{\partial x}\right).$$

Cum igitur debeat esse $\left(\frac{\partial R}{\partial y}\right) = \left(\frac{\partial S}{\partial x}\right)$, manifestum est formulam $R\partial x + S\partial y$ integrabilem esse debere. Non autem opus est, ut ea algebraicum habeat integrale, sed sufficit ut integrationis charactere sit praedita.

28. Sumatur enim

$$R = \frac{y}{\alpha + \beta xy + \gamma xxyy} \quad \text{et} \quad S = \frac{x}{\alpha + \beta xy + \gamma xxyy},$$

eritque

$$Q = \frac{2\alpha - 2\gamma xxyy}{(\alpha + \beta xy + \gamma xxyy)^2} \quad \text{et}$$

$$P = \frac{y\partial X}{\partial x(\alpha + \beta xy + \gamma xxyy)} - \frac{2Xyy(\beta + 2\gamma xy)}{(\alpha + \beta xy + \gamma xxyy)^2},$$

simulque

$$P = \frac{x\partial Y}{\partial y(\alpha + \beta xy + \gamma xxyy)} - \frac{2Yxx(\beta + 2\gamma xy)}{(\alpha + \beta xy + \gamma xxyy)^2},$$

ita ut habeatur

$$(\alpha + \beta xy + \gamma xxyy)^2 P$$
$$= \frac{y\partial X}{\partial x}(\alpha + \beta xy + \gamma xxyy) - 2yyX(\beta + 2\gamma xy)$$
$$= \frac{x\partial Y}{\partial y}(\alpha + \beta xy + \gamma xxyy) - 2xxY(\beta + 2\gamma xy).$$

Statuatur

$$X = Ax^4 + 2Bx^3 + Cxx + 2Dx + E$$

itemque

$$Y = \mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E},$$

ac duo illi valores inter se aequandi postulare deprehenduntur, ut sit

$$\beta = 0; \quad B = 0; \quad \mathfrak{B} = 0; \quad D = 0 \quad \text{et} \quad \mathfrak{D} = 0;$$

tum vero ii fient

$$\text{I.} = -2\gamma C x^3 y^3 + 4\alpha A x^3 y - 4\gamma E x y^3 + 2\alpha C x y,$$
$$\text{II.} = -2\gamma \mathfrak{C} x^3 y^3 + 4\alpha \mathfrak{A} x y^3 - 4\gamma \mathfrak{E} x^3 y + 2\alpha \mathfrak{C} x y,$$

unde colligitur

$$\mathfrak{C} = C; \quad \frac{\alpha}{\gamma} = \frac{-\mathfrak{E}}{A} = \frac{-E}{\mathfrak{A}} \quad \text{seu} \quad \mathfrak{A}\mathfrak{E} = AE.$$

Erit ergo

$$X = Ax^4 + Cxx - \frac{\alpha}{\gamma}\mathfrak{A}; \quad Y = \mathfrak{A}y^4 + Cyy - \frac{\alpha}{\gamma}A;$$

et aequationis

$$\frac{\partial x}{\sqrt{(Ax^4 + Cxx - \frac{\alpha}{\gamma}\mathfrak{A})}} + \frac{\partial y}{\sqrt{(\mathfrak{A}y^4 + Cyy - \frac{\alpha}{\gamma}A)}} = 0$$

integrale completum erit

$$y\sqrt{(Ax^4 + Cxx - \frac{\alpha}{\gamma}\mathfrak{A})} + x\sqrt{(\mathfrak{A}y^4 + Cyy - \frac{\alpha}{\gamma}A)}$$
$$= \text{Const.} \times (\alpha + \gamma xxyy).$$

29. Ex his exemplis facile intelligitur, fere novum adhuc analyseos genus desiderari, quo hujusmodi operationes certo ordine institui atque ulterius extendi queant, a quo quidem scopo adhuc longissime sumus remoti. Interim tamen ea, quae hactenus exposui maximi momenti esse videntur, ad universalitatem principii integrandi initio memorati stabiliendam, cum adeo ejus beneficio per multiplicatores idoneos eae integrationes, quae maxime arduae et cognita principia transcendentes erant visae, expediri queant. Mihi quidem cum primum in eas incidissem, nulla alia via eo deducere videbatur praeter eam, qua tum eram usus; nondum enim animadverteram semper, quoties cujuscunque aequationis differentialis integrale completum constaret, ex eo multiplicatorem, quo illa inte-

grabilis reddatur, concludi posse; quae conclusio, si integrale tantum fuisset particulare, neutiquam valuisset. Quamobrem integrationum illarum particularium, quas olim simul ex eodem principio alieno eram consecutus, longe aliter est ratio comparata, neque adhuc perspicere licet, quomodo methodo quadam directa et naturali ad easdem perveniri queat.

30. Eo magis igitur operae erit pretium, indolem harum integrationum particularium accuratius examinari, quod quidem contemplatione casus simplicissimi fiet. Hujus igitur aequationis differentialis

$$\partial x \sqrt{(1 + xx)} + \partial y \sqrt{(1 + yy)} + ny\partial x + nx\partial y = 0$$

integrale particulare inveneram esse

$$xx + yy + 2xy \sqrt{(1 + nn)} = nn,$$

similiaque integralia innumerabilia etiam inveni pro ejusmodi aequationibus differentialibus, quae neque a logarithmis neque a circuli quadratura pendent: quare haec aequatio ita spectetur, quasi non per logarithmos integrari posset. Hic igitur primo quaeritur, qua via directa hoc integrale particulare ex forma differentiali concludi queat? deinde quomodo aequatio differentialis comparata esse debeat, ut tale integrale particulare exhiberi queat? Ad has ergo quaestiones primum observo, aequationem algebraicam esse integrale completum istius aequationis differentialis

$$\frac{\partial x}{\sqrt{(1 + xx)}} + \frac{\partial y}{\sqrt{(1 + yy)}} = 0,$$

tum vero ex illa sequi

$$x + y \sqrt{(1 + nn)} = n \sqrt{(1 + yy)} \quad \text{et}$$
$$y + x \sqrt{(1 + nn)} = n \sqrt{(1 + xx)},$$

ita ut tam $\sqrt{(1 + xx)}$ quam $\sqrt{(1 + yy)}$ rationaliter per x et y exprimi queat. Cum igitur hinc sit differentiando

$$\frac{x\partial x}{\sqrt{(1+xx)}} = \frac{\partial y + \partial x\sqrt{(1+nn)}}{n} \quad \text{et} \quad \frac{y\partial y}{\sqrt{(1+yy)}} = \frac{\partial x + \partial y\sqrt{(1+nn)}}{n},$$

si harum formarum multipla quaecunque ad illam

$$\frac{\partial x}{\sqrt{(1+xx)}} + \frac{\partial y}{\sqrt{(1+yy)}} = 0$$

addantur, semper prodire aequationem differentialem, cui aequatio algebraica particulariter saltem satisfaciat. In genere ergo hujus aequationis differentialis

$$\frac{\partial x + Px\partial x}{\sqrt{(1+xx)}} + \frac{\partial y + Qy\partial y}{\sqrt{(1+yy)}} = \frac{P\partial y + Q\partial x + (P\partial x + Q\partial y)\sqrt{(1+nn)}}{n}$$

integrale particulare erit

$$xx + yy + 2xy\sqrt{(1+nn)} = nn.$$

Sit jam $P = x$ et $Q = y$, ac satisfiet huic aequationi

$$\partial x\sqrt{(1+xx)} + \partial y\sqrt{(1+yy)} = \frac{x\partial y + y\partial x + (x\partial x + y\partial y)\sqrt{(1+nn)}}{n},$$

ex integrali vero fit

$$x\partial x + y\partial y = -(x\partial y + y\partial x)\sqrt{(1+nn)},$$

ita ut habeatur haec aequatio differentialis

$$\partial x\sqrt{(1+xx)} + \partial y\sqrt{(1+yy)} + nx\partial y + ny\partial x = 0,$$

cui ergo integrale supra datum particulariter convenit.

31. Transferamus jam haec ad casus latius patentes, et postquam hujus aequationis

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0$$

inventum fuerit integrale completum, quod sit $W = \text{Const.}$ notetur hinc semper utrumque valorem radicalem $\sqrt{X}$ et $\sqrt{Y}$ per functiones rationales ipsarum x et y definiri. Sit ergo

$$\sqrt{X} = R \quad \text{et} \quad \sqrt{Y} = S,$$

ideoque

$$\frac{\partial X}{\sqrt{X}} = 2\partial R \quad \text{et} \quad \frac{\partial Y}{\sqrt{Y}} = 2\partial S.$$

Sit jam P functio ipsius x et Q ipsius y, hincque conflatur ista aequatio

$$\frac{\partial x + P\partial X}{\sqrt{X}} + \frac{\partial y + Q\partial Y}{\sqrt{Y}} - 2P\partial R - 2Q\partial S = 0,$$

cui aequatio algebraica W = Const. certe particulariter satisfacit. Hinc si P et Q ita accipiantur, ut formula $P\partial R + Q\partial S$ integrationem admittat, cujus integrale sit = V, orietur aequatio transcendens

$$\int \frac{\partial x + P\partial X}{\sqrt{X}} + \int \frac{\partial y + Q\partial Y}{\sqrt{Y}} - 2V = \text{Const.}$$

cui aequationi W = Const. seu valoribus inde deductis, $\sqrt{X} = R$ vel $\sqrt{Y} = S$ particulariter satisfacit. Tale ergo ratiocinium viam ad hujusmodi integrationes particulares alioquin inventu difficillimas patefacere videtur.