CAPUT IV.

DE AEQUATIONUM DIFFERENTIALIUM HOMOGENEARUM RESOLUTIONE.

Problema 85.

492.

Si v aequetur functioni cuicunque binarum quantitatum t et u, ita per tres variabiles x, y et z determinatarum, ut sit

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z,$$

ejus formulas differentiales omnium graduum inde definire.

Solutio.

Cum v sit functio quantitatum

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z,$$

ejus formulae differentiales ex his duabus variabilibus natae innotescent, scilicet

$$\left(\frac{\partial v}{\partial t}\right), \left(\frac{\partial v}{\partial u}\right), \left(\frac{\partial\partial v}{\partial t^2}\right), \left(\frac{\partial\partial v}{\partial t \partial u}\right), \left(\frac{\partial\partial v}{\partial u^2}\right),$$

hinc autem statim colligimus

$$\left(\frac{\partial v}{\partial x}\right) = \alpha \left(\frac{\partial v}{\partial t}\right), \left(\frac{\partial v}{\partial y}\right) = \gamma \left(\frac{\partial v}{\partial u}\right), \left(\frac{\partial v}{\partial z}\right) = \beta \left(\frac{\partial v}{\partial t}\right) + \delta \left(\frac{\partial v}{\partial u}\right),$$

formulas scilicet differentiales primi gradus. Pro formulis autem differentialibus secundi gradus adipiscimur

$$\left(\frac{\partial\partial v}{\partial x^2}\right) = \alpha\alpha \left(\frac{\partial\partial v}{\partial t^2}\right), \left(\frac{\partial\partial v}{\partial y^2}\right) = \gamma\gamma \left(\frac{\partial\partial v}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial v}{\partial z^2}\right) = \beta\beta \left(\frac{\partial\partial v}{\partial t^2}\right) + 2\beta\delta \left(\frac{\partial\partial v}{\partial t \partial u}\right) + \delta\delta \left(\frac{\partial\partial v}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial v}{\partial x \partial y}\right) = \alpha\gamma \left(\frac{\partial\partial v}{\partial t \partial u}\right), \left(\frac{\partial\partial v}{\partial x \partial z}\right) = \alpha\beta \left(\frac{\partial\partial v}{\partial t^2}\right) + \alpha\delta \left(\frac{\partial\partial v}{\partial t \partial u}\right),$$

$$\text{et } \left(\frac{\partial\partial v}{\partial y \partial z}\right) = \beta\gamma \left(\frac{\partial\partial v}{\partial t \partial u}\right) + \gamma\delta \left(\frac{\partial\partial v}{\partial u^2}\right).$$

Simili modo ad tertium gradum ascendimus

$$\left(\frac{\partial^3 v}{\partial x^3}\right) = \alpha^3 \left(\frac{\partial^3 v}{\partial t^3}\right), \quad \left(\frac{\partial^3 v}{\partial y^3}\right) = \gamma^3 \left(\frac{\partial^3 v}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 v}{\partial z^3}\right) = \beta^3 \left(\frac{\partial^3 v}{\partial t^3}\right) + 3\,\beta^2\,\delta \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right) + 3\,\beta\,\delta^2 \left(\frac{\partial^3 v}{\partial t \partial u^2}\right) + \delta^3 \left(\frac{\partial^3 v}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 v}{\partial x^2 \partial y}\right) = \alpha\,\alpha\,\gamma \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right), \quad \left(\frac{\partial^3 v}{\partial x \partial y^2}\right) = \alpha\,\gamma\,\gamma \left(\frac{\partial^3 v}{\partial t \partial u^2}\right),$$

$$\left(\frac{\partial^3 v}{\partial x^2 \partial z}\right) = \alpha\,\alpha\,\beta \left(\frac{\partial^3 v}{\partial t^3}\right) + \alpha\,u\,\delta \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right),$$

$$\left(\frac{\partial^3 v}{\partial y^2 \partial z}\right) = \beta\,\gamma\,\gamma \left(\frac{\partial^3 v}{\partial t \partial u^2}\right) + \gamma\,\gamma\,\delta \left(\frac{\partial^3 v}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 v}{\partial x \partial z^2}\right) = \alpha\,\beta\,\beta \left(\frac{\partial^3 v}{\partial t^3}\right) + 2\,\alpha\,\beta\,\delta \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right) + \alpha\,\delta\,\delta \left(\frac{\partial^3 v}{\partial t \partial u^2}\right),$$

$$\left(\frac{\partial^3 v}{\partial y \partial z^2}\right) = \beta\,\beta\,\gamma \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right) + 2\,\beta\,\gamma\,\delta \left(\frac{\partial^3 v}{\partial t \partial u^2}\right) + \gamma\,\delta\,\delta \left(\frac{\partial^3 v}{\partial u^3}\right),$$

$$\left(\frac{\partial^3 v}{\partial x \partial y \partial z}\right) = \alpha\,\beta\,\gamma \left(\frac{\partial^3 v}{\partial t^2 \partial u}\right) + \alpha\,\gamma\,\delta \left(\frac{\partial^3 v}{\partial t \partial u^2}\right),$$

unde facile patet, quomodo has formulas differentiales ad altiores gradus continuari oporteat.

Scholion 1.

493. Hoc problema fortasse generalius concipi debuisse videbitur, quantitates t et u ita per tres variabiles x, y, z definiendo, ut esset

$$t = \alpha x + \beta y + \gamma z \quad \text{et} \quad u = \delta x + \varepsilon y + \zeta z,$$

verum cum haec hypothesis in eum tantum finem sit facta, ut v fieret functio ipsarum t et u, evidens est tum quoque v spectari posse ut functionem harum duarum quantitatum $\varepsilon t - \beta u$ et $\delta t - \alpha u$, quarum illa ab y haec vero ab x erit libera. Quocirca hypothesis assumta latissime patere est censenda, exceptio tamen forte hinc admittenda videbitur, si fuerit

$$t = x + z \quad \text{et} \quad u = x - z,$$

quia hic ipsius u valor non continetur, verum etiam hoc casu quantitas v ut functio ipsarum $t + u$ et $t - u$ spectata fiet functio ipsarum x et z, qui casus utique in hypothesi continetur, sumtis $\beta = 0$ et $\gamma = 0$.

Scholion 2.

494. Hoc problema ideo praemisi, quia alias aequationes differentiales tractare hic non sustineo, nisi quibus ejusmodi valor satisfaciet, ut v aequetur functioni cuicunque binarum novarum variabilium t et u, quae ab principalibus x, y, z ita pendeant, ut sit quemadmodum assumsi

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z.$$

Hujusmodi autem aequationes, quibus hoc modo satisfieri potest, esse homogeneas, facile patet, ita ut aequatio resolvenda constet nonnisi formulis differentialibus ejusdem gradus, singulis per constantes quantitates multiplicatis, et inter se additis, qua appellatione aequationum homogenearum jam in parte praecedente sum usus. Proposita ergo hujusmodi aequatione homogenea, loco singularum formulam differentialium per elementa ∂x, ∂y, ∂z formatarum substituantur valores hic inventi per elementa ∂t et ∂u formati, et tum singula membra, quatenus certam formulam differentialem ex elementis ∂t et ∂u natam complectuntur, seorsim ad nihilum redigantur; indeque rationes $\frac{\beta}{\alpha}$ et $\frac{\delta}{\gamma}$ determinentur; quandoquidem quaestio non tam circa has ipsas quantitates, quam earum rationes versatur. Quoniam igitur duae tantum res investigationi relinquuntur, si pluribus aequationibus fuerit satisfaciendum, ejusmodi aequationes homogeneae hac ratione resolvi nequeunt, nisi casu quo plures illae aequationes ad duas tantum revocentur, id quod in sequentibus clarius explicabitur.

Problema 86.

495. Proposita aequatione homogenea primi gradus

$$A\left(\frac{\partial v}{\partial x}\right) + B\left(\frac{\partial v}{\partial y}\right) + C\left(\frac{\partial v}{\partial z}\right) = 0,$$

investigare naturam functionis v trium variabilium x, y et z.

Solutio.

Fingatur $v = \Gamma : (t \text{ et } u)$, existente

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z,$$

et facta substitutione ex problemate praecedente aequatio nostra in duas partes dividetur

$$\left(\frac{\partial v}{\partial t}\right)(A\alpha + C\beta) + \left(\frac{\partial v}{\partial u}\right)(B\gamma + C\delta) = 0,$$

quarum utraque seorsim ad nihilum reducta praebet

$$\frac{\beta}{\alpha} = \frac{-A}{C} \text{ et } \frac{\delta}{\gamma} = \frac{-B}{C},$$

unde fit

$$t = Cx - Az \text{ et } u = Cy - Bz.$$

Quare aequationis propositae integrale completum erit

$$v = \Gamma : (\overline{Cx - Az} \text{ et } \overline{Cy - Bz}),$$

quod etiam concinnius ita exhiberi potest

$$v = \Gamma : \left(\frac{x}{A} - \frac{z}{C} \text{ et } \frac{y}{B} - \frac{z}{C}\right).$$

Corollarium 1.

496. Permutandis variabilibus hoc integrale etiam ita exprimi posse evidens est

$$v = \Gamma : \left(\frac{x}{A} - \frac{y}{B} \text{ et } \frac{y}{B} - \frac{z}{C}\right), \text{ vel}$$

$$v = \Gamma : \left(\frac{x}{A} - \frac{y}{B} \text{ et } \frac{x}{A} - \frac{z}{C}\right),$$

quoniam est

$$\frac{x}{A} - \frac{y}{B} = \left(\frac{x}{A} - \frac{z}{C}\right) - \left(\frac{y}{B} - \frac{z}{C}\right).$$

Corollarium 2.

497. Quin etiam constitutis ex aequatione proposita his tribus formulis

$$\frac{x}{A} - \frac{y}{B}, \ \frac{x}{A} - \frac{z}{C}, \ \frac{y}{B} - \frac{z}{C},$$

functio quaecunque ex iis utcunque conflata valorem idoneum pro v suppeditabit. Quoniam enim harum binarum formularum unaquaeque est differentia binarum reliquarum, talis functio duas tantum variabiles complecti est censenda.

Corollarium 3.

498. Perinde est quanam harum trium formarum integralium utamur, quando autem binae novae variabiles t et u inter se fuerint aequales, tum alia est utendum. Veluti si esset $C = 0$, prima forma $v = \Gamma:(z \text{ et } z)$, utpote functio solius z foret inutilis, et integrale completum esset futurum

$$v = \Gamma:\left(\frac{x}{A} - \frac{y}{B} \text{ et } z\right), \text{ seu}$$
$$v = \Gamma:(\overline{Bx - Ay} \text{ et } z).$$

Problema 87.

499. Proposita aequatione homogenea secundi gradus

$$A\left(\frac{\partial\partial v}{\partial x^2}\right) + B\left(\frac{\partial\partial v}{\partial y^2}\right) + C\left(\frac{\partial\partial v}{\partial z^2}\right) + 2D\left(\frac{\partial\partial v}{\partial x \partial y}\right) + 2E\left(\frac{\partial\partial v}{\partial x \partial z}\right)$$
$$+ 2F\left(\frac{\partial\partial v}{\partial y \partial z}\right) = 0,$$

casus investigare, quibus ejus integrale hac forma $\Gamma:(t \text{ et } u)$ exprimi potest, existente

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z.$$

Solutio.

Facta substitutione secundum formulas in problemate 85. traditas, aequatio proposita in tria membra sequentia resolvetur

$$\left.\begin{aligned} &\left(\frac{\partial\partial v}{\partial t^2}\right)(A\alpha\alpha + C\beta\beta + 2E\alpha\beta) \\ &\left(\frac{\partial\partial v}{\partial t \partial u}\right)(2C\beta\delta + 2D\alpha\gamma + 2E\alpha\delta + 2F\beta\gamma) \\ &\left(\frac{\partial\partial v}{\partial u^2}\right)(B\gamma\gamma + C\delta\delta + 2F\gamma\delta) \end{aligned}\right\} = 0,$$

quorum singula seorsim nihilo debent aequari. At primum praebet

$$\frac{\beta}{\alpha} = \frac{-E + \sqrt{(EE - AC)}}{C},$$

ultimum vero

$$\frac{\delta}{\gamma} = \frac{-F + \sqrt{(FF - BC)}}{C},$$

qui valores in media, quae ita referatur

$$\frac{C\beta\delta}{\alpha\gamma} + D + \frac{E\delta}{\gamma} + \frac{F\beta}{\alpha} = 0,$$

substituti suppeditant hanc aequationem

$$EF - CD = \sqrt{(EE - AC)(FF - BC)},$$

qua aequatione conditio inter coëfficientes A, B, C, D, E, F continetur, ut solutio hic applicata locum invenire possit. Haec autem aequatio evoluta dat

$$CCDD - 2CDEF + BCEE + ACFF - ABCC = 0,$$

unde fit

$$C = \frac{2DEF - BEE - AFF}{DD - AB},$$

quia factor C per multiplicationem est ingressus. Quoties autem haec conditio habet locum, ut sit

$$AFF + BEE + CDD = ABC + 2DEF,$$

toties haec expressio algebraica ex aequatione proposita formanda

$$Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz$$

in duos factores potest resolvi, neque ergo aliis casibus solutio hic adhibita locum habere potest, Quo ergo hos casus solutionem admittentes rite evolvamus, ponamus hujus formae factores esse

$$(ax + by + cz)\ (fx + gy + hz),$$

quod ergo eveniet, si fuerit

$$A = af, \qquad B = bg, \qquad C = ch,$$
$$2D = ag + bf, \quad 2E = ah + cf, \quad 2F = bh + cg,$$

unde utique fit

$$AFF + BEE + CDD = ABC + 2\,DEF.$$

Hinc autem pro solutione colligitur

$$\text{vel } \frac{\beta}{\alpha} = \frac{-a}{c}, \quad \text{vel } \frac{\beta}{\alpha} = \frac{-f}{h}, \quad \text{et}$$
$$\text{vel } \frac{\delta}{\gamma} = \frac{-b}{c}, \quad \text{vel } \frac{\delta}{\gamma} = \frac{-g}{h},$$

ubi observari oportet, pro fractionibus $\frac{\beta}{\alpha}$ et $\frac{\delta}{\gamma}$ valores sibi subscriptos conjungi oportere, ita ut sit

$$\text{vel } t = cx - az, \quad \text{et } u = cy - bz,$$
$$\text{vel } t = hx - fz, \quad \text{et } u = hy - gz.$$

Quocirca pro his casibus solutionem admittentibus integrale completum erit

$$v = \Gamma : (\overline{cx - az} \text{ et } \overline{cy - bz}) + \Delta : (\overline{hx - fz} \text{ et } \overline{hy - gz}),$$

seu

$$v = \Gamma : (\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}) + \Delta : (\tfrac{x}{f} - \tfrac{z}{h} \text{ et } \tfrac{y}{g} - \tfrac{z}{h}).$$

Corollarium 1.

500. Hoc ergo modo aliae aequationes homogeneae secundi gradus resolvi nequeunt, nisi quae in hac forma continentur

$$af\left(\frac{\partial\partial v}{\partial x^2}\right) + bg\left(\frac{\partial\partial v}{\partial y^2}\right) + ch\left(\frac{\partial\partial v}{\partial z^2}\right) + (ag + bf)\left(\frac{\partial\partial v}{\partial x\partial y}\right)$$
$$+ (ah + cf)\left(\frac{\partial\partial v}{\partial x\partial z}\right) + (bh + cg)\left(\frac{\partial\partial v}{\partial y\partial z}\right) = 0,$$

tum vero integrale completum erit

$$v = \Gamma : (\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} + \tfrac{z}{c}) + \Delta : (\tfrac{x}{f} - \tfrac{z}{h} \text{ et } \tfrac{y}{g} - \tfrac{z}{h}).$$

Corollarium 2.

501. Quo autem facilius dignoscatur, utrum aequatio quaepiam proposita

$$A\left(\frac{\partial\partial v}{\partial x^2}\right) + B\left(\frac{\partial\partial v}{\partial y^2}\right) + C\left(\frac{\partial\partial v}{\partial z^2}\right) + 2D\left(\frac{\partial\partial v}{\partial x\partial y}\right) + 2E\left(\frac{\partial\partial v}{\partial x\partial z}\right)$$
$$+ 2F\left(\frac{\partial\partial v}{\partial y\partial z}\right) = 0$$

eo reduci possit nec ne? formetur inde haec forma algebraica

$$A\,xx + B\,yy + C\,zz + 2D\,xy + 2E\,xz + 2F\,yz,$$

quae si resolvi patiatur in duos factores rationales

$$(ax + by + cz)\,(fx + gy + hz),$$

ejus integrale completum hinc statim exhiberi potest.

Corollarium 3.

502. Unicus tantum casus quo duo isti factores inter se fiunt aequales, exceptionem postulat, quoniam tum binae functiones inventae in unam coalescerent. Verum ex superioribus colligitur, si hoc eveniat ut sit $f = a$, $g = b$ et $h = c$, integrale completum ita exprimi

$$z = x\,\Gamma : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) + \Delta : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right).$$

Scholion 1.

503. Quibus ergo casibus aequatio homogenea secundi gradus resolutionem admittit, iis quoque in se complectitur duas aequationes homogeneas primi gradus

$$a\left(\frac{\partial v}{\partial x}\right) + b\left(\frac{\partial v}{\partial y}\right) + c\left(\frac{\partial v}{\partial z}\right) = 0, \text{ et}$$
$$f\left(\frac{\partial v}{\partial x}\right) + g\left(\frac{\partial v}{\partial y}\right) + h\left(\frac{\partial v}{\partial z}\right) = 0,$$

quippe quarum utraque illi satisfacit, et harum integralia completa junctim sumta illius integrale completum suppeditant. Hinc alia via aperitur aequationum homogenearum secundi gradus integralia inveniendi, fingendo aequationem primi gradus ipsis satisfacientem

$$a\left(\frac{\partial v}{\partial x}\right) + b\left(\frac{\partial v}{\partial y}\right) + c\left(\frac{\partial v}{\partial z}\right) = 0,$$

tum ex hac per triplicem differentiationem tres novae formentur

$$a\left(\frac{\partial\partial v}{\partial x^2}\right) + b\left(\frac{\partial\partial v}{\partial x\partial y}\right) + c\left(\frac{\partial\partial v}{\partial x\partial z}\right) = 0,$$
$$a\left(\frac{\partial\partial v}{\partial x\partial y}\right) + b\left(\frac{\partial\partial v}{\partial y^2}\right) + c\left(\frac{\partial\partial v}{\partial y\partial z}\right) = 0,$$
$$a\left(\frac{\partial\partial v}{\partial x\partial z}\right) + b\left(\frac{\partial\partial v}{\partial y\partial z}\right) + c\left(\frac{\partial\partial v}{\partial z^2}\right) = 0,$$

quarum prima per f, secunda per g et tertia per h multiplicatae et in unam summam collectae, ipsam illam aequationem generalem producunt, cujus integrale supra exhibuimus. Ea ergo quasi productum ex binis aequationibus homogeneis primi gradus spectari poterit, ex quibus conjunctis integrale completum formatur.

Scholion 2.

504. Infinitae ergo aequationes homogeneae secundi gradus hic excluduntur, quae hoc modo integrationem respuunt, seu ad aequationes primi gradus reduci nequeunt; qui casus exclusi omnes ex hoc criterio agnoscuntur, si non fuerit

$$AFF + BEE + CDD = ABC + 2\,DEF.$$

Hujus generis est ista aequatio $\left(\frac{\partial\partial v}{\partial x\partial y}\right) = \left(\frac{\partial\partial v}{\partial z^2}\right)$, quae ergo tale integrale, cujusmodi hic assumsimus non admittit, neque etiam alia patet via ejus integrale completum investigandi. Integralia autem particularia facile innumera exhiberi possunt, et quae adeo functiones arbitrarias complectuntur, sed tantum unius quantitatis variabilis, quae in praesenti instituto nonnisi integralia particularia constituere sunt censendae. Si enim ponatur

$$v = \Gamma : (\alpha x + \beta y + \gamma z),$$

facta substitutione fieri debet $\alpha\beta = \gamma\gamma$, seu sumto $\gamma = 1$, debet esse $\alpha\beta = 1$; quare innumerabiles adeo hujusmodi formulae conjunctae satisfaciunt, ut sit

$$v = \Gamma : (\tfrac{\alpha}{\beta} x + \tfrac{\beta}{\alpha} y + z) + \Delta : (\tfrac{\gamma}{\delta} x + \tfrac{\delta}{\gamma} y + z)$$
$$+ \Sigma : (\tfrac{\varepsilon}{\zeta} x + \tfrac{\zeta}{\varepsilon} y + z) + \text{etc.}$$

ubi pro α, β, γ, δ, etc. numeros quoscunque accipere licet: quamvis autem infinitae hujusmodi formulae diversae conjunguntur, tamen integrale nonnisi pro particulari haberi potest. Ex quo intelligitur integrationem completam istius aequationis $(\frac{\partial\partial v}{\partial x \partial y}) = (\frac{\partial\partial v}{\partial z^2})$ maximi esse momenti, methodumque eo perveniendi fines analyseos non mediocriter esse prolaturam. Aequationes autem homogeneae tertii gradus multo majorem restrictionem exigunt, ut integratio completa hoc modo succedat; uti sequenti problemate ostendetur.

Problema 88.

505. Aequationum homogenearum tertii gradus eos casus definire, quibus integrale completum per formam assumtam exhiberi, seu ad formam aequationum homogenearum primi gradus reduci potest.

Solutio.

In aequatione homogenea tertii gradus fingatur contineri haec primi gradus

$$a\,(\tfrac{\partial v}{\partial x}) + b\,(\tfrac{\partial v}{\partial y}) + c\,(\tfrac{\partial v}{\partial z}) = 0,$$

quae ut satisfaciat aequationi tertii gradus

$$\left.\begin{aligned} A\,(\tfrac{\partial^3 v}{\partial x^3}) + B\,(\tfrac{\partial^3 v}{\partial y^3}) + C\,(\tfrac{\partial^3 v}{\partial x^3}) + D\,(\tfrac{\partial^3 v}{\partial x^2 \partial y}) + E\,(\tfrac{\partial^3 v}{\partial x \partial y^2}) \\ + F\,(\tfrac{\partial^3 v}{\partial x^2 \partial z}) + G\,(\tfrac{\partial^3 v}{\partial x \partial z^2}) \\ + H\,(\tfrac{\partial^3 v}{\partial y^2 \partial z}) + I\,(\tfrac{\partial^3 v}{\partial y \partial z^2}) \\ + K\,\tfrac{\partial^3 v}{\partial x \partial y \partial z}) \end{aligned}\right\} = 0,$$

necesse est ut expressio haec algebraica

$$Ax^3 + By^3 + Cz^3 + Dxxy + Fxxz + Hyyz + Kxyz$$
$$+ Exyy + Gxzz + Iyzz$$

factorem habeat $ax + by + cz$, nisi autem alter factor denuo in duos simplices sit resolubilis, ad aequationem homogeneam secundi gradus referetur, quae solutionem respuit. Quare ut integratio completa succedat necesse est, istam expressionem tribus constare factoribus simplicibus, qui sint

$$(ax + by + cz) \cdot (fx + gy + hz)\,(kx + my + nz),$$

hincque aequatioins generalis coëfficientes ita se habebunt

$$\begin{array}{lll} A = afk, & D = afm + agk + bfk, & H = bgn + bhm + cgm, \\ B = bgm, & E = agm + bfm + bgk, & I = bhn + cgn + chm, \\ C = chn, & F = afn + ahk + cfk, & K = agn + ahm + bfn \\ & G = ahn + cfn + chk, & \quad + bhk + cfm + cgk, \end{array}$$

ac tum integrale completum erit

$$v = \Gamma : \left(\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}\right) + \Delta : \left(\tfrac{x}{f} - \tfrac{z}{h} \text{ et } \tfrac{y}{g} - \tfrac{z}{h}\right)$$
$$+ \Sigma : \left(\tfrac{x}{k} - \tfrac{z}{n} \text{ et } \tfrac{y}{m} - \tfrac{z}{n}\right),$$

quilibet scilicet factor simplex praebet functionem arbitrariam duarum variabilium.

Corollarium 1.

506. In qualibet harum functionum variabiles x, y, z inter se permutare licet; quin etiam quaelibet quasi ex tribus variabilibus conflata spectari potest, prima nempe ex his

$$\tfrac{x}{a} - \tfrac{y}{b}, \quad \tfrac{y}{b} - \tfrac{z}{c} \quad \text{et} \quad \tfrac{z}{c} - \tfrac{x}{a},$$

similique modo de caeteris.

Corollarium 2.

507. Si duo factores fuerint aequales $f = a$, $g = b$, $h = c$, quo casu duae priores functiones in unam coalescerent, earum loco scribi debet

$$x\Gamma : (\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}) + \Delta : (\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c});$$

at si omnes tres fuerint aequales, ut insuper sit

$$k = a, \quad m = b, \quad n = c,$$

integrale completum erit

$$\begin{aligned} v = xx\,\Gamma &: (\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}) \\ + \; x\,\Delta &: (\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}) \\ + \; \Sigma &: (\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}). \end{aligned}$$

Corollarium 3.

508. Quemadmodum hic duas priores partes per xx et x multiplicavimus, ita eas quoque per yy et y item zz et z multiplicare possemus, perinde enim est quanam variabili hic utamur, dum ne sit ea, quae forte sola post signum functionis occurit, scilicet si esset $a = 0$, et functiones quantitatum x et $\frac{y}{b} - \frac{z}{c}$ capi debeant, tum multiplicatores xx et x excludi deberent.

Scholion 1.

509. Simili modo patet aequationes homogeneas quarti gradus hac methodo resolvi non posse, nisi in quatuor ejusmodi aequationes simplices resolvi, et quasi earum producta spectari queant. Etsi enim hic revera nulla resolutio in factores locum habeat, tamen ex allatis exemplis clare perspicitur, quemadmodum ex aequatione differentiali homogenea cujuscunque gradus expressio algebraica ejusdem gradus ternas variabiles x, y, z involvens debeat formari;

quae si in factores simplices formae $ax + by + cz$ resolvi queat, simul inde aequationis differentialis integrale completum facile exhibebitur, cum quilibet factor functionem duarum variabilium suppeditet, integralis partem constituentem; ita ut etiam haec pars seorsim sumta aequationi differentiali satisfaciat et pro integrali particulari haberi possit. At si illa expressio algebraica ita fuerit comparata, ut factores quidem habeat simplices sed non tot, quot dimensiones, singuli quidem integralia particularia praebebunt, quae autem junctim sumta non integrale completum suppeditabunt. Veluti si proponatur haec aequatio differentialis tertii gradus

$$a\left(\frac{\partial^3 v}{\partial x^2 \partial y}\right) + b\left(\frac{\partial^3 v}{\partial x \partial y^2}\right) - a\left(\frac{\partial^3 v}{\partial x \partial z^2}\right) - b\left(\frac{\partial^3 v}{\partial y \partial z^2}\right) = 0,$$

quia forma algebraica

$$axxy + bxyy - axzz - byzz$$

factorem habet simplicem $ax + by$, illi utique satisfaciet valor $v = \Gamma : (\frac{x}{a} - \frac{y}{b} \text{ et } z)$, pro integrali autem completo adhuc desunt duae functiones arbitrariae, integrale completum hujus aequationis $\left(\frac{\partial\partial v}{\partial x \partial y \partial z}\right) - \left(\frac{\partial\partial v}{\partial z^2}\right) = 0$ continentes, ex qua quippe alter factor $xy - zz$ illius expressionis nascitur. Quoties ergo hae expressiones algebraicae ex aequationibus differentialibus homogeneis altiorum graduum formatae resolutionem in factores, etsi non simplices, admittant; hinc saltem discimus, quomodo earum integratio ad aequationes inferiorum graduum revocari possit, quod in hujusmodi arduis investigationibus sine dubio maximi est momenti.

Scholion 2.

510. Haec sunt quae de functionibus trium variabilium ex data quadam differentialium relatione investigandis proferre potui,

in quibus utique nonnisi prima elementa hujus scientiae continentur, quorum ulterior evolutio sagacitati Geometrarum summo studio est commendanda. Tantum enim abest, ut hae speculationes pro sterilibus sint habendae, ut potius pleraque, quae adhuc in Theoria motus fluidorum desiderantur, ad has Analyseos partes subliniores sint referenda; quarum propterea utilitas neutiquam parti priori calculi integralis postponenda videtur. Eo magis autem hae partes posteriores excoli merentur, quod Theoria fluidorum adeo circa functiones quatuor variabilium versetur, quarum naturam ex aequationibus differentialibus secundi gradus investigari oportet, quam partem ob penuriam materiae ne attingere quidem volui. In hac autem Theoria resolutio hujus aequationis

$$\left(\frac{\partial\partial v}{\partial t^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \left(\frac{\partial\partial v}{\partial y^2}\right) + \left(\frac{\partial\partial v}{\partial z^2}\right)$$

maxime est momenti, ubi litterae x, y, z ternas coordinatas, t vero tempus elapsum exprimunt, harumque quatuor variabilium functio quaeritur, quae loco v substituta illi aequationi satisfaciat. Ex hactenus autem allatis facile colligitur, integrale completum hujus aequationis duas complecti debere functiones arbitrarias, quarum utraque sit functio trium variabilium, aliasque solutiones omnes minus late patentes pro incompletis esse habendas. Facili autem negotio innumeras solutiones particulares exhibere licet, veluti si ponamus

$$v = \Gamma : (\alpha x + \beta y + \gamma z + \delta t),$$

reperitur

$$\delta\delta = \alpha\alpha + \beta\beta + \gamma\gamma,$$

quod cum infinitis modis fieri possit, infinitae hujusmodi functiones additae valorem idoneum pro v exhibebunt. Deinde etiam satisfaciunt isti valores

$$v = \frac{\Gamma : [t \pm \sqrt{(xx + yy + zz)}]}{\sqrt{(xx + yy + zz)}},$$

$$v = \frac{\Gamma : [x \pm \sqrt{(tt - yy - zz)}]}{\sqrt{(tt - yy - zz)}},$$

$$v = \frac{\Gamma : [y \pm \sqrt{(tt - xx - zz)}]}{\sqrt{(tt - xx - zz)}},$$

$$v = \frac{\Gamma : [z \pm \sqrt{(tt - xx - yy)}]}{\sqrt{(tt - xx - yy)}},$$

quorum quidem investigatio jam est difficilior. Cum autem hae functiones tantum sint unius variabilis, integralia maxime particularia exhibent, quae adeo etiamnum forent particularia, si pro v functiones binarum variabilium haberentur, quales autem ne suspicari quidem licet. Quare cum integrale completum duas adeo functiones arbitrarias trium variabilium complecti debeat, facile intelligitur quantopere adhuc ab hoc scopo simus remoti.