CAPUT V.

TRANSFORMATIO SINGULARIS EARUNDEM AEQUATIONUM.

Problema 56.

349.

Proposita hac aequatione

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + Rz,$$

in qua P, Q, R sint functiones ipsius x tantum, eam ope substitutionis

$$z = M\left(\frac{\partial v}{\partial x}\right) + Nv,$$

ubi quoque sint M et N functiones ipsius x tantum, in aliam ejusdem formae transmutare ut prodeat

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + Hv,$$

existentibus F, G, H functionibus solius x.

Solutio.

Quia quantitates M et N ab y sunt immunes, erit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = M\left(\frac{\partial^3 v}{\partial x \partial y}\right) + N\left(\frac{\partial\partial v}{\partial y^2}\right),$$

quae forma per aequationem, quam tandem resultare assumimus, abit in hanc

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = MF\left(\frac{\partial^3 v}{\partial x^3}\right) \begin{array}{l} + \frac{M\partial F}{\partial x} \\ + MG \\ + NF \end{array} \left(\frac{\partial\partial v}{\partial x^2}\right) \begin{array}{l} + \frac{M\partial G}{\partial x} \\ + MH \\ + NG \end{array} \left(\frac{\partial v}{\partial x}\right) \begin{array}{l} + \frac{M\partial H}{\partial x} \\ + NH \end{array} v.$$

Deinde vero pro altero aequationis propositae membro nostra substitutio praebet

$$\left(\frac{\partial z}{\partial x}\right) = M\left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{\partial M}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial N}{\partial x} v$$
$$+ N$$

hincque porro

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = M\left(\frac{\partial^3 v}{\partial x^3}\right) + \left(\frac{2\partial M}{\partial x} + N\right)\left(\frac{\partial\partial v}{\partial x^2}\right)$$
$$+ \left(\frac{\partial\partial M}{\partial x^2} + \frac{2\partial N}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + \frac{\partial\partial N}{\partial x^2} v.$$

Cum nunc sit per hypothesin

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + Rz,$$

si hic valores modo inventi substituantur, singulaque membra $\left(\frac{\partial^3 v}{\partial x^3}\right)$, $\left(\frac{\partial\partial v}{\partial x^2}\right)$, $\left(\frac{\partial v}{\partial x}\right)$ et v seorsim ad nihilum redigantur, quatuor sequentes aequationes orientur, scilicet

ex	colligitur aequatio
$\left(\frac{\partial^3 v}{\partial x^3}\right)$	$MF = MP$
$\left(\frac{\partial\partial v}{\partial x^2}\right)$	$\frac{M\partial F}{\partial x} + MG + NF = \left(\frac{2\partial M}{\partial x} + N\right)P + MQ$
$\left(\frac{\partial v}{\partial x}\right)$	$\frac{M\partial G}{\partial x} + MH + NG = \left(\frac{\partial\partial M}{\partial x^2} + \frac{2\partial N}{\partial x}\right)P + \left(\frac{\partial M}{\partial x} + N\right)Q + MR$
v	$\frac{M\partial H}{\partial x} + NH = \frac{\partial\partial N}{\partial x^2}P + \left(\frac{\partial N}{\partial x}\right)Q + NR,$

ex quibus commodissime primo quaeruntur P, Q et R. Verum prima dat statim $P = F$, unde secunda fit

$$\frac{M\partial F - 2F\partial M}{M\partial x} + G = Q.$$

Ex binis ultimis autem eliminando R colligitur

$$\frac{M(N\partial G - M\partial H)}{\partial x} + NNG = \left(\frac{N\partial\partial M - M\partial\partial N}{\partial x^2} + \frac{2N\partial N}{\partial x}\right)F$$
$$+ \left(\frac{N\partial M - M\partial N}{\partial x} + NN\right)Q,$$

et illum valorem pro Q substituendo

$$0 = \frac{MM\partial H}{\partial x} - \frac{MN\partial G}{\partial x} + \frac{(N\partial\partial M - M\partial\partial N)}{\partial x^2} F + \frac{2NF\partial M}{\partial x}$$
$$+ \frac{N\partial M - M\partial N}{\partial x} G + \frac{(N\partial M - M\partial N)}{\partial x^2} \partial F + \frac{NN\partial F}{\partial x}$$
$$- \frac{2F\partial M\,(N\partial M - M\partial N)}{M\partial x^2} - \frac{2NNF\partial M}{M\partial x}$$

quae aequatio per $\frac{\partial x}{MM}$ multiplicata commode integrabilis redditur, inveniturque integrale

$$C = H - \frac{N}{M} G + \frac{N\partial M - M\partial N}{MM\partial x} F + \frac{NNF}{MM}.$$

Quod si ergo brevitatis gratia ponamus $N = Ms$, erit

$$C = H - Gs - F\frac{\partial s}{\partial x} + Fss, \text{ seu}$$
$$\partial s + \frac{G}{F} s\partial x - ss\partial x + \frac{(C - H)\partial x}{F} = 0.$$

Sive jam hinc definiatur quantitas $s = \frac{N}{M}$, sive una functionum F, G et H, pro ipsa aequatione proposita litterae P, Q et R, ita determinabuntur, ut sit

$$\text{I.} \quad P = F$$
$$\text{II.} \quad Q = G + \frac{\partial F}{\partial x} - \frac{2F\partial M}{M\partial x},$$

et ex ultima aequatione derivatur

$$R = H + \frac{M\partial H}{N\partial x} - \frac{F\partial\partial N}{N\partial x^2} - \frac{\partial N}{N\partial x}\left(G + \frac{\partial F}{\partial x} - \frac{2F\partial M}{M\partial x}\right),$$

qui valor ob $N = Ms$ evadit

$$R = H + \frac{\partial H}{s\partial x} - \frac{G\partial s}{s\partial x} - \frac{G\partial M}{M\partial x} - \frac{F\partial\partial s}{s\partial x^2} - \frac{F\partial\partial M}{M\partial x^2}$$
$$+ \frac{2F\partial M^2}{MM\partial x^2} - \frac{\partial F\partial s}{s\partial x^2} - \frac{\partial F\partial M}{M\partial x^2},$$

et cum aequatio inventa, si differentietur, det

$$0 = \partial H - G\partial s - s\partial G - \frac{F\partial\partial s}{\partial x} - \frac{\partial F\partial s}{\partial x} + 2Fs\partial s + ss\partial F,$$

obtinebimus

$$\text{III.} \quad R = H - \frac{G\partial M}{M\partial x} + \frac{\partial G}{\partial x} - \frac{F\partial\partial M}{M\partial x^2} - \frac{2F\partial s}{\partial x},$$
$$+ \frac{2F\partial M^2}{MM\partial x^2} - \frac{s\partial F}{\partial x} - \frac{\partial F\partial M}{M\partial x^2},$$

unde si aequatio

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + Hv$$

resolutionem admittat, etiam resolutio succedet hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + Rz,$$

cum sit

$$z = M\left(\frac{\partial v}{\partial x}\right) + Nv = M\left[sv + \left(\frac{\partial v}{\partial x}\right)\right].$$

Corollarium 1.

350. Si ponatur $M = 1$, ut fiat $z = sv + \left(\frac{\partial v}{\partial x}\right)$, erit

$$P = F,\ Q = G + \frac{\partial F}{\partial x};\ \text{et}\ R = H + \frac{\partial G}{\partial x} - \frac{2F\partial s - s\partial F}{\partial x},$$

neque hoc modo usus istius reductionis restringitur; quoniam si deinceps loco z ponatur Mz, etiam aequationis hinc ortae resolutio est in promtu.

Corollarium 2.

351. Quoties ergo aequationis

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + Hv$$

resolutio est in potestate, toties etiam hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = F\left(\frac{\partial\partial z}{\partial x^2}\right) + \left(G + \frac{\partial F}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right) + \left(H + \frac{\partial G}{\partial x} - \frac{2F\partial s - s\partial F}{\partial x}\right)z$$

resolutio succedit, si modo capiatur s ex hac aequatione

$$F\partial s + Gs\partial x - Fss\partial x + (C - H)\partial x = 0,$$

tum enim erit $z = sv + \left(\frac{\partial v}{\partial x}\right)$. Sunt autem litterae F, G, H functiones ipsius x tantum.

Scholion.

352. Haec reductio methodum maxime naturalem suppeditare videtur ejusmodi integrationes perficiendi, quae simul functio-

num differentialia involvunt. Si enim aequationis pro v datae integrale sit $v = \Phi : t$, existente t functione ipsarum x et y, ob

$$\partial v = \partial t \Phi' : t, \text{ erit } \left(\frac{\partial v}{\partial x}\right) = \left(\frac{\partial t}{\partial x}\right) \Phi' : t$$

et aequationis inde derivatae pro z habebimus integrale

$$z = s\Phi : t + \left(\frac{\partial t}{\partial x}\right) \Phi' : t.$$

Deinde si fuerit generalius $v = u\Phi : t$, fiet

$$z = su\Phi : t + \left(\frac{\partial u}{\partial x}\right) \Phi : t + u \left(\frac{\partial t}{\partial x}\right) \Phi' : t,$$

unde ratio perspicitur ad ejusmodi aequationes perveniendi, quarum integralia praeter functionem $\Phi : t$ etiam functiones ex ejus differentiatione natas $\Phi' : t$, atque adeo etiam sequentes $\Phi'' : t$, $\Phi''' : t$, etc. complectantur. Quamobrem operae pretium erit hanc reductionem accuratius evolvere.

Problema 57.

353. Concessa resolutione hujus aequationis

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{n}{xx} v,$$

invenire aliam aequationem hujus formae

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + Rz,$$

pro qua sit

$$z = sv + \left(\frac{\partial v}{\partial x}\right).$$

Solutio.

Facta comparatione cum praecedente problemate habemus

$$F = 1, \quad G = \frac{m}{x} \text{ et } H = \frac{n}{xx},$$

unde quantitatem s ex hac aequatione definiri oportet

$$\partial s + \frac{ms\partial x}{x} - ss\partial x + \left(f - \frac{n}{xx}\right)\partial x = 0,$$

qua inventa ob $\frac{\partial G}{\partial x} = -\frac{m}{xx}$, aequatio quaesita erit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \left(\frac{n-m}{xx} - \frac{s\partial s}{\partial x}\right) z,$$

seu loco ∂s valore inde substituto

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \left(2f - \frac{m-n}{xx} + \frac{2ms}{x} - 2ss\right) z,$$

pro qua est

$$z = sv + \left(\frac{\partial v}{\partial x}\right).$$

I. Ponamus primo quantitatem constantem $f = 0$, ut sit

$$\partial s + \frac{ms\partial x}{x} - ss\partial x - \frac{n\partial x}{xx} = 0,$$

cujus integrale particulare est $s = \frac{\alpha}{x}$, existente

$$-\alpha + m\alpha - \alpha\alpha - n = 0, \text{ seu } \alpha\alpha - (m-1)\alpha + n = 0,$$

ex quo ob $\frac{\partial s}{\partial x} = \frac{-\alpha}{xx}$, oritur haec aequatio

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{2\alpha - m + n}{xx} z,$$

pro qua est

$$z = \frac{\alpha}{x} v + \left(\frac{\partial v}{\partial x}\right),$$

seu exclusa $n = \alpha(m - 1 - \alpha)$, si constet resolutio hujus aequationis

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{\alpha(m-1-\alpha)}{xx} v,$$

pro hac

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{(\alpha-1)(m-\alpha)}{xx} z,$$

erit

$$z = \frac{\alpha}{x} v + \left(\frac{\partial v}{\partial x}\right).$$

II. Maneat $f = 0$, et quaeramus pro s valorem completum ponendo $s = \frac{\alpha}{x} + \frac{1}{t}$, fietque ob

$$n = (m - 1)\alpha - \alpha\alpha,\quad \partial t + \frac{(2\alpha - m)t\partial x}{x} + \partial x = 0,$$

quae per $x^{2\alpha - m}$ multiplicata et integrata praebet

$$t = \frac{cx^{m-2\alpha}}{2\alpha - m + 1} - \frac{x}{2\alpha - m + 1},$$

hincque

$$s = \frac{\alpha c x^{m-2\alpha-1} + \alpha - m + 1}{x(cx^{m-2\alpha-1} - 1)} = \frac{\alpha}{x} + \frac{2\alpha - m + 1}{x(cx^{m-2\alpha-1} - 1)},$$

unde fit

$$\frac{\partial s}{\partial x} = \frac{-\alpha}{xx} + \frac{(m - 2\alpha - 1)(m - 2\alpha)}{xx(cx^{m-2\alpha-1} - 1)} + \frac{(m - 2\alpha - 1)^2}{xx(cx^{m-2\alpha-1} - 1)^2}.$$

Hic praecipue notetur casus $c = 0$, quo fit

$$s = \frac{m - \alpha - 1}{x} \quad \text{et} \quad \frac{\partial s}{\partial x} = \frac{-m + \alpha + 1}{xx},$$

ita ut data aequatione

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{\alpha(m - 1 - \alpha)}{xx} v,$$

pro hac aequatione

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{(\alpha + 1)(m - 2 - \alpha)}{xx} z,$$

futurum sit

$$z = \frac{m - \alpha - 1}{x} v + \left(\frac{\partial v}{\partial x}\right).$$

Pro generali autem valore sit $m - 2\alpha - 1 = \beta$, ut habeatur

$$s = \frac{\alpha}{x} - \frac{\beta}{x(cx^{\beta} - 1)} \quad \text{et}$$

$$\frac{\partial s}{\partial x} = \frac{-\alpha}{xx} + \frac{\beta(\beta + 1)}{xx(cx^{\beta} - 1)} + \frac{\beta\beta}{xx(cx^{\beta} - 1)^2},$$

unde si detur haec aequatio

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{2\alpha + \beta + 1}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{\alpha(\alpha + \beta)}{xx} v,$$

ejus ope resolvetur haec

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{2\alpha+\beta+1}{x}\left(\frac{\partial z}{\partial x}\right)$$
$$+\left[(\alpha-1)(\alpha+\beta+1) - \frac{2\beta(\beta+1)}{cx^\beta-1} - \frac{2\beta\beta}{(cx^\beta-1)^2}\right]\frac{z}{xx},$$

cum sit

$$z = \left(\alpha - \frac{\beta}{cx^\beta-1}\right)\frac{v}{x} + \left(\frac{\partial v}{\partial x}\right).$$

III. Rationem quoque habeamus constantis f, ponamusque $f = \frac{1}{aa}$, ut facto $n = \alpha(m-1-\alpha)$ habeamus

$$\partial s + \frac{ms\partial x}{x} - ss\partial x - \frac{\alpha(m-1-\alpha)\partial x}{xx} + \frac{\partial x}{aa} = 0,$$

quae posito $s = \frac{\alpha}{x} + \frac{1}{t}$ abit in

$$\partial t - \frac{(m-2\alpha)t\partial x}{x} + \partial x = \frac{tt}{aa}\partial x.$$

Sit $m - 2\alpha = \gamma$, ut aequatio data sit

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{2\alpha+\gamma}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{\alpha(\alpha+\gamma-1)}{xx}v,$$

et inventa quantitate s prodeat haec aequatio

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{2\alpha+\gamma}{x}\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\alpha\alpha-3\alpha+\alpha\gamma-\gamma}{xx} - \frac{2\partial s}{\partial x}\right)z,$$

seu

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{2\alpha+\gamma}{x}\left(\frac{\partial z}{\partial x}\right) + \left(\frac{(\alpha-1)(\alpha+\gamma)}{xx} + \frac{2\partial t}{tt\partial x}\right)z,$$

pro qua est

$$z = \left(\frac{\alpha}{x} + \frac{1}{t}\right)v + \left(\frac{\partial v}{\partial x}\right),$$

ubi totum negotium ad inventionem quantitatis t redit ex aequatione

$$\partial t - \frac{\gamma t\partial x}{x} + \partial x = \frac{tt}{aa}\partial x.$$

Hunc in finem statuatur $t = a - \frac{aa\partial u}{u\partial x}$, ac reperitur

$$\frac{\partial\partial u}{\partial x^2} - \frac{\gamma\partial u}{x\partial x} - \frac{2\partial u}{a\partial x} + \frac{\gamma u}{ax} = 0,$$

cujus duplex resolutio datur, altera ponendo

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

existente

$$B = \frac{\gamma A}{\gamma a}, \quad C = \frac{(\gamma - 2)B}{2(\gamma - 1)a}, \quad D = \frac{(\gamma - 4)C}{3(\gamma - 2)a}, \quad E = \frac{(\gamma - 6)D}{4(\gamma - 3)a}, \text{ etc.}$$

altera vero ponendo

$$u = Ax^{\gamma+1} + Bx^{\gamma+2} + Cx^{\gamma+3} + Dx^{\gamma+4} + Ex^{\gamma+5} + \text{etc.}$$

ubi

$$B = \frac{(\gamma+2)A}{(\gamma+2)a}, \quad C = \frac{(\gamma+4)B}{2(\gamma+3)a}, \quad D = \frac{(\gamma+6)C}{3(\gamma+4)a},$$

$$E = \frac{(\gamma+8)D}{4(\gamma+5)a}, \quad \text{etc.}$$

quarum illa abrumpitur, si sit γ numerus integer par positivus, haec vero si negativus. Qui valores etsi sunt particulares, tamen supra jam ostendimus, quomodo inde valores completi sint eliciendi.

Corollarium 1.

354. Supra autem vidimus, (§. 333.) hanc aequationem

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{2m}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{(m+i)(m-i-1)}{xx}\, v,$$

esse integrabilem, si sit i numerus integer quicunque, unde colligimus hanc aequationem

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial v}{\partial x}\right) + \frac{\alpha(m-1-\alpha)}{xx}\, v$$

integrationem admittere, quoties fuerit vel $\alpha = \frac{1}{2}m + i$ vel $\alpha = \frac{1}{2}m - i - 1$, seu $m - 2\alpha$ numerus integer par sive positivus sive negativus, qui casus ob $m - 2\alpha = \gamma$ cum casibus integrabilitatis, pro valore generali ipsius s inveniendo, congruunt.

Corollarium 2.

355. Quando autem ex hac aequatione functionem v definire licet, tum etiam hae duae sequentes aequationes illi similes resolvi poterunt

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{(\alpha-1)(m-\alpha)}{xx} z \quad \text{et}$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{m}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{(\alpha+2)(m-\alpha-1)}{xx} z,$$

cum pro illa sit

$$z = \frac{\alpha}{x} v + \left(\frac{\partial v}{\partial x}\right),$$

pro hac vero

$$z = \frac{m-\alpha-1}{x} v + \left(\frac{\partial v}{\partial x}\right).$$

Corollarium 3.

356. Praeterea vero etiam aequationes alius generis, ub postremus terminus non est formae $\frac{n}{xx} z$, resolvi possunt, qui inveniuntur, si quantitatis s valor generalius investigatur, atque adeo constantis f ratio habetur.

Exemplum 1.

357. *Proposita aequatione* $\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right)$, *pro qua est*

$$v = \pi : (x+y) + \Phi : (x-y),$$

invenire aequationes magis complicatas, quae hujus ope integrari queant.

Cum hic sit $F = 1$, $G = 0$ et $H = 0$, resolvatur haec aequatio

$$\partial s - ss\partial x + C\partial x = 0,$$

et hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2\partial s}{\partial x} z$$

integrale erit

$$z = sv + \left(\frac{\partial v}{\partial x}\right).$$

Sumta autem primo constante $C = 0$, fit $\frac{\partial s}{ss} = \partial x$ et $\frac{1}{s} = c - x$ seu $s = \frac{1}{c-x}$, atque $\frac{\partial s}{\partial x} = \frac{1}{(c-x)^2}$, ubi quidem sine ulla restrictione poni potest $c = 0$, ut hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2}{xx} z,$$

integrale sit

$$z = -\frac{1}{x}[\pi:(x+y) + \Phi:(x-y)] + \pi':(x+y) + \Phi':(x-y).$$

Sit deinde $C = aa$, et ob $\partial s = \partial x(ss - aa)$ fiet

$$x = \frac{1}{2a} l \frac{s-a}{s+a}, \text{ hincque}$$

$$\frac{s-a}{s+a} = Ae^{2ax} \text{ et } s = \frac{a(1 + Ae^{2ax})}{1 - Ae^{2ax}}, \text{ unde}$$

$$\frac{\partial s}{\partial x} = \frac{4Aaae^{2ax}}{(1 - Ae^{2ax})^2},$$

et aequationes

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{8Aaae^{2ax}}{(1 - Ae^{2ax})^2} z$$

integrale est

$$z = \frac{a(1 + Ae^{2ax})}{1 - Ae^{2ax}} v + \left(\frac{\partial v}{\partial x}\right).$$

Sit tandem $C = -aa$, et ob $\partial s = \partial x(aa + ss)$ fit

$$ax + b = \text{Ang. tang. } \frac{s}{a},$$

hincque

$$s = a \text{ tang. } (ax + b) \text{ et } \frac{\partial s}{\partial x} = \frac{aa}{\cos.(ax + b)^2},$$

quocirca hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2aa}{\cos.(ax + b)^2} z$$

integrale est

$$z = \frac{a \sin.(ax+b)}{\cos.(ax+b)} v + \left(\frac{\partial v}{\partial x}\right).$$

Exemplum 2.

358. *Proposita aequatione*

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right) - \frac{2}{xx} v,$$

cujus integrale constat, invenire alias ejus ope integrabiles.

Pro hoc casu habemus

$$\partial s - ss\partial x + \left(C + \frac{2}{xx}\right)\partial x = 0,$$

qua resoluta erit hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - 2\left(\frac{1}{xx} + \frac{\partial s}{\partial x}\right) z$$

integrale

$$z = sv + \left(\frac{\partial v}{\partial x}\right).$$

I. Sit primo $C = 0$, et ex aequatione

$$\partial s - ss\partial x - \frac{2\partial x}{xx} = 0$$

fit particulariter $s = \frac{1}{x}$ vel $s = -\frac{2}{x}$. Ponatur ergo $s = \frac{1}{x} + \frac{1}{t}$, eritque

$$\partial t + \frac{2t\partial x}{x} + \partial x = 0, \text{ hinc}$$

$$txx + \tfrac{1}{3}x^3 = \tfrac{1}{3}a^3.$$

Ergo

$$t = \frac{a^3 - x^3}{3xx} \text{ et } s = \frac{a^3 + 2x^3}{x(a^3 - x^3)},$$

ideoque

$$\frac{\partial s}{\partial x} + \frac{1}{xx} = \frac{3x(2a^3 + x^3)}{(a^3 - x^3)^2},$$

unde hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{6x(2a^3 + x^3)}{(a^3 - x^3)^2} z$$

posito $C = -aa - nau$

seu $\frac{\partial\partial u}{\partial\omega^2} + \frac{2\partial u}{\partial\omega}\,\text{tang.}\,\omega + nu = 0,$

ob $\partial x = \frac{\partial\omega}{a},$

cujus resolutio non parum ardua videtur, inter complures autem modos eam tractandi hic ad institutum maxime idoneus videtur. Fingatur

$$u = A\cos.\lambda\omega + B\cos.(\lambda+2)\omega + C\cos.(\lambda+4)\omega + \text{etc.}$$

eritque

$$\frac{\partial u}{\partial\omega} = -\lambda A\sin.\lambda\omega - (\lambda+2)B\sin.(\lambda+2)\omega$$
$$-(\lambda+4)C\sin.(\lambda+4)\omega - \text{etc.}$$
$$\frac{\partial\partial u}{\partial\omega^2} = -\lambda\lambda A\cos.\lambda\omega - (\lambda+2)^2 B\cos.(\lambda+2)\omega$$
$$-(\lambda+4)^2 C\cos.(\lambda+4)\omega - \text{etc.}$$

et aequatio hac forma repraesentata

$$\frac{2\partial\partial u}{\partial\omega^2}\cos.\omega + \frac{4\partial u}{\partial\omega}\sin.\omega + 2nu\cos.\omega = 0 \text{ dabit}$$

$$0 = -\lambda\lambda A\cos.(\lambda-1)\omega - (\lambda+2)^2 B\cos.(\lambda+1)\omega - (\lambda+4)^2 C\cos.(\lambda+3)\omega - \text{etc.}$$

	$-\lambda\lambda A$	$-(\lambda+2)^2 B$
$-2\lambda A$	$-2(\lambda+2)B$	$-2(\lambda+4)C$
	$+2\lambda A$	$+2(\lambda+2)B$
$+nA$	$+nB$	$+nC$
	$+nA$	$+nB$

unde λ ita capi oportet ut sit

$$\lambda\lambda + 2\lambda = n, \text{ seu } \lambda = -1 \pm \sqrt{(n+1)},$$

duplexque pro λ habeatur valor. Praeterea vero secundus terminus ob $n = \lambda\lambda + 2\lambda$ praebet $B = \frac{\lambda}{\lambda+2}A$, tertius vero commode dat $C = 0$, unde et sequentes omnes evanescunt.

Sumamus $n = mm - 1$, ut sit

$$\lambda = -1 \pm m \text{ et } B = \frac{-1 \pm m}{1 \pm m}A;$$

atque integrale completum concludi videtur

$$u = 2m \cos. m\omega \cos. \omega + 2 \sin. m\omega \sin. \omega;$$

ideoque

$$\frac{1}{(m \cos. m\omega + \sin. m\omega \tang. \omega)^2} = A - \int \frac{d\omega}{(m \cos. m\omega + \sin. m\omega \tang. \omega)^2},$$

cujus postremi membri integrale deprehenditur

$$\frac{-m \tang. m\omega + \tang. \omega}{m(mm-1)(m + \tang. m\omega \tang. \omega)} = \frac{-m \sin. m\omega + \tang. \omega \cos. m\omega}{m(mm-1)(m \cos. m\omega + \sin. m\omega \tang. \omega)},$$

ita ut sit

$$\frac{1}{(m \cos. m\omega + \sin. m\omega \tang. \omega)^2} = A + \frac{\cos. m\omega \tang. \omega - m \sin. m\omega}{m(mm-1)(m \cos. m\omega + \sin. m\omega \tang. \omega)},$$

seu

$$\frac{1}{t} = \frac{m(mm-1)}{[C(m \cos. m\omega + \sin. m\omega \tang. \omega) + \cos. m\omega \tang. \omega - m \sin. m\omega](m \cos. m\omega + \sin. m\omega \tang. \omega)]},$$

cui addatur

$$\Theta = -\tang. \omega + \frac{(mm-1) \sin. m\omega}{m \cos. m\omega + \sin. m\omega \tang. \omega},$$

ut prodeat $\frac{s}{a}$, eritque

$$\frac{s}{a} = -\tang. \omega + \frac{(mm-1)(C \sin. m\omega + \cos. m\omega)}{C(m \cos. m\omega + \sin. m\omega \tang. \omega) + \cos. m\omega \tang. \omega - m \sin. m\omega},$$

seu

$$\frac{s}{a} = \frac{(mm-1-\tang. \omega^2)(C \sin. m\omega + \cos. m\omega) - m \tang. \omega (C \cos. m\omega - \sin. m\omega)}{C(m \cos. m\omega + \sin. m\omega \tang. \omega) + \cos. m\omega \tang. \omega - m \sin. m\omega}.$$

Corollarium 1.

361. Hic praecipue notandum est, hujus aequationis

$$\frac{\partial\partial u}{\partial\omega^2} + \frac{2\partial u}{\partial\omega} \tang. \omega + (mm-1)u = 0.$$

integrale particulare esse

$$u = m \cos. m\omega \cos. \omega + \sin. m\omega \sin. \omega,$$

aliud vero integrale particulare reperitur simili modo

$$u = m \sin. m\omega \cos. \omega - \cos. m\omega \sin. \omega,$$

unde concluditur completum

$$u = A(m \cos. m\omega \cos. \omega + \sin. m\omega \sin. \omega)$$
$$+ B(m \sin. m\omega \cos. \omega - \cos. m\omega \sin. \omega).$$

Corollarium 2.

362. Si hic ponatur

$$A = C \cos. \alpha \text{ et } B = - C \sin. \alpha,$$

hoc integrale completum ad hanc formam redigitur

$$u = C [m \cos. (m \omega + \alpha) \cos. \omega + \sin. (m \omega + \alpha) \sin. \omega],$$

quod quidem ex integrali particulari primum invento statim concludi potuisset, cum ibi loco anguli $m\,\omega$ scribere liceat $m\,\omega + \alpha$.

Corollarium 3.

363. Hinc multo facilius reperitur valor

$$\frac{s}{a} = - \text{tang.}\, \omega - \frac{\partial u}{u\, \partial \omega}, \text{ cum enim sit}$$

$$\frac{\partial u}{\partial \omega} = - C (m\,m - 1) \sin. (m \omega + \alpha) \cos. \omega, \text{ erit}$$

$$\frac{s}{a} = - \text{tang.}\, \omega + \frac{(m\,m - 1) \sin. (m \omega + \alpha) \cos. \omega}{m \cos. (m \omega + \alpha) \cos. \omega + \sin. (m \omega + \alpha) \sin. \omega},$$

hincque

$$\frac{\partial s}{a\, \partial \omega} = \frac{\partial s}{a\, a\, \partial x} = \frac{-1}{\cos. \omega^2} + \frac{(m\,m - 1) [m^2 \cos. \omega^2 - \sin. (m \omega + \alpha)^2]}{[m \cos. (m \omega + \alpha) \cos. \omega + \sin. (m \omega + \alpha) \sin. \omega]^2},$$

et aequatio, cujus integrationem invenimus, erit

$$\left(\frac{\partial \partial z}{\partial y^2}\right) = \left(\frac{\partial \partial z}{\partial x^2}\right) - \frac{2 (m\,m - 1)\, a\, a [m^2 \cos. \omega^2 - \sin. (m \omega + \alpha)^2]}{[m \cos. (m \omega + \alpha) \cos. \omega + \sin. (m \omega + \alpha) \sin. \omega]^2},$$

ejusque integrale colligitur

$$z = \frac{m\, a\, a [m \sin. (m \omega + \alpha) \sin. \omega + \cos. (m \omega + \alpha) \cos. \omega]}{m \cos. (m \omega + \alpha) \cos. \omega + \sin. (m \omega + \alpha) \sin. \omega} [\pi : (x + y) + \Phi : (x - y)]$$

$$+ \frac{(m\,m - 1)\, a \sin. (m \omega + \alpha) \cos. \omega}{m \cos. (m \omega + \alpha) \cos. \omega + \sin. (m \omega + \alpha) \sin. \omega} [\pi' : (x + y) + \Phi' : (x - y)$$

$$+ \pi'' : (x + y) + \Phi'' : (x - y)],$$

existente $\omega = a\,x + b$.

Scholion 1.

364. Omnino memoratu digna est integratio hujus aequationis

$$\frac{\partial\partial u}{\partial\omega^2} + \frac{2\partial u}{\partial\omega}\,\text{tang.}\,\omega + (mm - 1)\,u = 0,$$

unde occasionem carpo, hanc aequationem generaliorem tractandi

$$\frac{\partial\partial u}{\partial\omega^2} + \frac{2f\partial u}{\partial\omega}\,\text{tang.}\,\omega + g\,u = 0,$$

quam primum observo posito

$$\frac{\partial u}{u} = -(2f+1)\,\partial\omega\,\text{tang.}\,\omega + \frac{\partial v}{v},\ \text{ut sit}$$

$$u = \cos.\,\omega^{2f+1}\,v,$$

abire in hanc formam

$$\frac{\partial\partial v}{\partial\omega^2} - \frac{2(f+1)\partial v}{\partial\omega}\,\text{tang.}\,\omega + (g - 2f - 1)\,v = 0,$$

ita ut si illa integrabilis existat casu $f = n$, integrabilis quoque sit casu $f = -n - 1$. Jam pro illa aequatione ponatur

$$u = A\sin.\lambda\omega + B\sin.(\lambda+2)\omega + C\sin.(\lambda+4)\omega + D\sin.(\lambda+6)\omega + \text{etc.}$$

et facta substitutione in aequatione

$$\frac{2\partial\partial u}{\partial\omega^2}\cos.\omega + \frac{4f\partial u}{\partial\omega}\sin.\omega + 2\,g\,u\cos.\omega = 0,$$

reperitur

$-\lambda\lambda A\,\text{si.}(\lambda-1)\omega$	$-(\lambda+2)^2 B\,\text{si.}(\lambda+1)\omega$	$-(\lambda+4)^2 C\,\text{si.}(\lambda+3)\omega$	$-(\lambda+6)^2 D\,\text{si.}(\lambda+5)\omega$
$-2\lambda Af$	$-\lambda\lambda A$	$-(\lambda+2)^2 B$	$-(\lambda+4)^2 C$
$+Ag$	$+2\lambda Af$	$+2(\lambda+2)Bf$	$+2(\lambda+4)Cf$
	$-2(\lambda+2)Bf$	$-2(\lambda+4)Cf$	$-2(\lambda+6)Df$
	$+Ag$	$+Bg$	$+Cg$
	$+Bg$	$+Cg$	$+Dg$

Oportet ergo sit $g = \lambda\lambda + 2\lambda f$, tum vero coëfficientes assumti ita determinantur

$$B = \frac{\lambda f A}{\lambda+f+1},\quad C = \frac{(\lambda+1)(f-1)B}{2(\lambda+f+2)},\quad D = \frac{(\lambda+2)(f-2)C}{3(\lambda+f+3)},\ \text{etc.}$$

Statuamus ergo $g = mm - ff$, ut fiat $\lambda = m - f$, et aequationes nostrae sint

$$\frac{\partial\partial u}{\partial\omega^2} + \frac{2f\partial u}{\partial\omega}\,\text{tang.}\,\omega + (mm - ff)\,u = 0\ \text{et}$$

$$\frac{\partial\partial v}{\partial\omega^2} - \frac{2(f+1)\partial v}{\partial\omega}\operatorname{tang.}\omega + [m\,m - (f+1)^2]\,v = 0,$$

existente

$$u = v\cos.\omega^{2f+1} \text{ seu } v = \frac{u}{\cos.\omega^{2f+1}}.$$

Quoniam nunc series nostra abrumpitur, quoties est f numerus integer, percurramus casus simpliciores.

I. Sit $f = 0$, erit

$$\lambda = m \text{ et } B = 0,\ C = 0,\ \text{etc.}$$

ideoque

$$u = A\sin.m\,\omega \text{ et } v = \frac{A\sin.m\,\omega}{\cos.\omega}.$$

II. Sit $f = 1$, erit

$$\lambda = m - 1 \text{ et } B = \frac{(m-1)A}{m+1},\ C = 0,\ \text{etc.}$$

ergo

$$\frac{u}{a} = (m+1)\sin.(m-1)\omega + (m-1)\sin.(m+1)\omega, \text{ et } v = \frac{u}{\cos.\omega^3},$$

seu $\frac{u}{a} = m\sin.m\,\omega\cos.\omega - \cos.m\,\omega\sin.\omega$.

III. Sit $f = 2$, erit $\lambda = m - 2$, et

$$B = \frac{2(m-2)A}{m+1},\ C = \frac{(m-1)B}{2(m+2)} = \frac{(m-1)(m-2)A}{(m+1)(m+2)},\ D = 0,\ \text{etc.}$$

hinc

$$\frac{u}{a} = (m+1)(m+2)\sin.(m-2)\omega + 2(m-2)(m+2)\sin.m\,\omega$$
$$- (m-1)(m-2)\sin.(m+2)\omega,$$

indeque $v = \frac{u}{\cos.\omega^5}$ seu

$$\frac{u}{2a} = (m\,m-2)\sin.m\,\omega\cos.2\,\omega + 2(m\,m-4)\sin.m\,\omega$$
$$- 3\,m\cos.m\,\omega\sin.2\,\omega.$$

IV. Sit $f = 3$, erit $\lambda = m - 3$, et

$$B = \frac{3(m-3)A}{m+1},\ C = \frac{2(m-2)B}{2(m+2)},\ D = \frac{(m+1)C}{3(m+3)},\ E = 0,\ \text{etc.}$$

Ergo

$$\frac{u}{a} = +(m+1)(m+2)(m+3)\sin.(m-3)\omega + 3(m+2)(mm-9)\sin.(m-1)\omega$$
$$+(m-1)(m-2)(m-3)\sin.(m+3)\omega + 3(m-2)(mm-9)\sin.(m+1)\omega$$

existente $u = \frac{u}{\cos.\omega^7}$.

V. Sit $f = 4$, erit $\lambda = m - 4$, ac reperitur

$$\frac{u}{a} = +(m+1)(m+2)(m+3)(m+4)\sin.(m-4)\omega + 4(m+2)(m+3)(mm-16)\sin.(m-2)\omega$$
$$+(m-1)(m-2)(m-3)(m-4)\sin.(m+4)\omega + 4(m-2)(m-3)(mm-16)\sin.(m+2)\omega$$
$$+6(mm-9)(mm-16)\sin.m\omega.$$

existente $v = \frac{u}{\cos.\omega^9}$,

unde ratio progressionis per se est manifesta. Notari autem convenit si posuissemus

$$0 = A\cos.\lambda\omega + B\cos.(\lambda+2)\omega + C\cos.(\lambda+4)\omega + \text{etc.}$$

easdem coëfficientium determinationes proditeras fuisse, ex qua hi duo valores conjuncti integrale completum exhibebunt: quod etiam ex forma inventa colligitur, si modo loco anguli $m\omega$ generalius scribatur $m\omega + \alpha$.

Scholion 2.

365. Pluribus autem aliis modis eadem aequatio

$$\frac{\partial\partial u}{\partial\omega^2} + \frac{2f\partial u}{\partial\omega}\text{tang.}\omega + gu = 0$$

tractari, et ejus integrale per series exprimi potest, unde alii casus integrabilitatis obtinentur. Ad hoc primum notetur, posito $u = \sin.\omega^\lambda$ fore

$$\frac{\partial u}{\partial\omega} = \lambda\sin\,\omega^{\lambda-1}\cos.\omega, \text{ hincque}$$
$$\frac{\partial u}{\partial\omega}\text{tang.}\omega = \lambda\sin.\omega^\lambda, \text{ et}$$
$$\frac{\partial\partial u}{\partial\omega^2} = \lambda(\lambda-1)\sin.\omega^{\lambda-2}\cos.\omega^2 - \lambda\sin.\omega^\lambda$$
$$= \lambda(\lambda-1)\sin.\omega^{\lambda-2} - \lambda\lambda\sin.\omega^\lambda.$$

Hinc si ponamus

$$u = A \sin.\omega^{\lambda} + B \sin.\omega^{\lambda+2} + C \sin.\omega^{\lambda+4} + D \sin.\omega^{\lambda+6} + \text{etc.}$$

facta substitutione adipiscimur

$$\begin{array}{llll} 0 = \lambda(\lambda-1)\,A\,\mathrm{si.}\omega^{\lambda-2} & + (\lambda+2)(\lambda+1)\,B\,\mathrm{si.}\omega^{\lambda} & + (\lambda+4)(\lambda+3)\,C\,\mathrm{si.}\omega^{\lambda+2} & + \text{etc} \\ & -\lambda\lambda A & -(\lambda+2)^2 B & \\ & +2\lambda f A & +2(\lambda+2) f B & \\ & +gA & +gB & \end{array}$$

unde sumi oportet vel $\lambda = 0$ vel $\lambda = 1$, tum vero erit

$$B = \frac{\lambda\lambda - 2\lambda f - g}{(\lambda+1)(\lambda+2)} A, \quad C = \frac{(\lambda+2)^2 - 2(\lambda+2) f - g}{(\lambda+3)(\lambda+4)} B, \text{ etc.}$$

hinc duo casus evolvi convenit

$\lambda = 0$,	$\lambda = 1$,
$B = \frac{-g}{1.2} A$,	$B = \frac{1 - 2f - g}{2.3} A$,
$C = \frac{4 - 4f - g}{3.4} B$,	$C = \frac{9 - 6f - g}{4.5} B$,
$D = \frac{16 - 8f - g}{5.6} C$,	$D = \frac{25 - 10f - g}{6.7} C$,
$E = \frac{36 - 12f - g}{7.8} D$,	$E = \frac{49 - 14f - g}{8.9} D$,
etc.	etc.

Integratio ergo succedit, quoties fuerit $g = ii - 2if$ denotante i numerum integrum positivum. Quare cum posito $u = v \cos.\omega^{2f+1}$ aequatio transformata sit

$$\frac{\partial\partial v}{\partial\omega^2} - \frac{2(f+1)\,\partial v}{\partial\omega} \tang.\omega + (g - 2f - 1) v = 0,$$

haec ideoque et illa erit integrabilis, quoties fuerit

$$g = (i+1)^2 + 2(i+1) f,$$

quos binos casus ita uno complecti licet, ut integratio succedat, dum sit $g = ii \pm 2if$.

Scholion 3.

366. Eidem aequationi adhuc inhaerens, cum posito $u = \cos.\omega^{\lambda}$, sit

$$\frac{\partial u}{\partial\omega} = -\lambda \cos.\omega^{\lambda-1} \sin.\omega, \text{ ideoque}$$

$$\frac{\partial u}{\partial\omega} \tang.\omega = -\lambda \cos.\omega^{\lambda-2} + \lambda \cos.\omega^{\lambda}, \text{ et}$$

$$\frac{\partial\partial u}{\partial\omega^2} = \lambda(\lambda-1) \cos.\omega^{\lambda-2} - \lambda\lambda \cos.\omega^{\lambda},$$

statuo

$$u = A\cos.\omega^{\lambda} + B\cos.\omega^{\lambda+2} + C\cos.\omega^{\lambda+4} + D\cos.\omega^{\lambda+6} + \text{etc.}$$

et facta substitutione orietur

$$\begin{array}{llll} 0 = \lambda(\lambda-1)A\cos.\omega^{\lambda-2} & + (\lambda+2)(\lambda+1)B\cos.\omega^{\lambda} & + (\lambda+4)(\lambda+3)C\cos.\omega^{\lambda+2} & + \text{etc.} \\ \quad -2\lambda f A & -\lambda\lambda A & -(\lambda+2)^2 B & \\ & -2(\lambda+2)fB & -2(\lambda+4)fC & \\ & -2\lambda f A & +2(\lambda+2)fB & \\ & +gA & +gB & \end{array}$$

Oportet ergo sit vel $\lambda = 0$ vel $\lambda = 2f + 1$, tum vero

$$B = \frac{\lambda\lambda - 2\lambda f - g}{(\lambda+2)(\lambda+1+2f)} A, \quad C = \frac{(\lambda+2)^2 - 2(\lambda+2)f - g}{(\lambda+4)(\lambda+3-2f)} B, \text{ etc.}$$

et ambo casus ita se habebunt

$\lambda = 0$,	$\lambda = 2f + 1$
$B = \frac{-g}{2(1+2f)} A$,	$B = \frac{1+2f-g}{2(2f+3)} A$,
$C = \frac{4-4f-g}{4(3-2f)} B$,	$C = \frac{9+6f-g}{4(2f+5)} B$,
$D = \frac{16-8f-g}{6(5-2f)} C$,	$D = \frac{25+10f-g}{6(2f+7)} C$,
etc.	etc.

Ex priori integratio succedit si $g = 4ii - 4if$, ex posteriori si $g = (2i+1)^2 + 2(2i+1)f$, qui casus cum iis, qui ex transformata nascuntur juncti, eodem redeunt ac in §. praec. inventi. Omnes ergo hactenus inventi integrabilitatis casus huc revocantur, ut posito $g = mm - ff$, sit vel $f = \pm i$, vel $m = i \pm f$, hoc est vel $f = \pm i$, vel $f = \pm i \pm m$. Caeterum hi posteriores casus etiam ex prima resolutione (§. 364) sequuntur, ubi series quoque abrumpitur si $\lambda = -i$, ideoque $g = mm - ff = ii - 2if$, ergo $i - f = \pm m$, et transformatione in subsidium vocata $f = \pm i \pm m$. Contra vero casus primo inventi in resolutionibus posterioribus non occurrunt.

Problema 59.

367. Concessa hujus aequationis integratione

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + Hv,$$

invenire aequationem hujus formae

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + R\,z,$$

pro qua sit

$$z = \left(\frac{\partial\partial v}{\partial x^2}\right) + r\left(\frac{\partial v}{\partial x}\right) + s\,v,$$

ubi F, G, H; P, Q, R; et r, s sunt functiones ipsius x tantum.

Solutio.

Cum sit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial^4 v}{\partial x^2\partial y^2}\right) + r\left(\frac{\partial^3 v}{\partial x\partial y^2}\right) + s\left(\frac{\partial\partial v}{\partial y^2}\right), \text{ ob}$$

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + H\,v, \text{ erit}$$

$$\begin{array}{lllll} \left(\frac{\partial^3 v}{\partial x\partial y^2}\right) = F\left(\frac{\partial^3 v}{\partial x^3}\right) & + \frac{\partial F}{\partial x}\left(\frac{\partial\partial v}{\partial x^2}\right) & + \frac{\partial G}{\partial x}\left(\frac{\partial v}{\partial x}\right) & + \frac{\partial H}{\partial x}\,v, \text{ et} \\ & + G & + H & \end{array}$$

$$\begin{array}{llllll} \left(\frac{\partial^4 v}{\partial x^2\partial y^2}\right) = F\left(\frac{\partial^4 v}{\partial x^4}\right) & + \frac{2\partial F}{\partial x}\left(\frac{\partial^3 v}{\partial x^3}\right) & + \frac{\partial\partial F}{\partial x^2}\left(\frac{\partial\partial v}{\partial x^2}\right) & + \frac{\partial\partial G}{\partial x^2}\left(\frac{\partial v}{\partial x}\right) & + \frac{\partial\partial H}{\partial x^2}\,v. \\ & + G & + \frac{2\partial G}{\partial x} & + \frac{2\partial H}{\partial x} & \\ & & + H & & \end{array}$$

Deinde vero ob

$$z = \left(\frac{\partial\partial v}{\partial x^2}\right) + r\left(\frac{\partial v}{\partial x}\right) + s\,v, \text{ fit}$$

$$\begin{array}{lllll} \left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial^3 v}{\partial x^3}\right) & + r\left(\frac{\partial\partial v}{\partial x^2}\right) & + \frac{\partial r}{\partial x}\left(\frac{\partial v}{\partial x}\right) & + \frac{\partial s}{\partial x}\,v, \text{ et} \\ & & + s & \end{array}$$

$$\begin{array}{llllll} \left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial^4 v}{\partial x^4}\right) & + r\left(\frac{\partial^3 v}{\partial x^3}\right) & + \frac{2\partial r}{\partial x}\left(\frac{\partial\partial v}{\partial x^2}\right) & + \frac{\partial\partial r}{\partial x^2}\left(\frac{\partial v}{\partial x}\right) & + \frac{\partial\partial s}{\partial x^2}\,v. \\ & & + s & + \frac{2\partial s}{\partial x} & \end{array}$$

His jam substitutis necesse est, ut omnes termini affecti per

$$\left(\frac{\partial^4 v}{\partial x^4}\right), \left(\frac{\partial^3 v}{\partial x^3}\right), \left(\frac{\partial\partial v}{\partial x^2}\right), \left(\frac{\partial v}{\partial x}\right), \text{ et } v$$

seorsim evanescant unde sequentes resultant aequationes

ex	
$\left(\frac{\partial^4 v}{\partial x^4}\right)$	I. $F = P$,
$\left(\frac{\partial^3 v}{\partial x^3}\right)$	II. $G + \frac{2\partial F}{\partial x} + Fr = Pr + Q$,
$\left(\frac{\partial\partial v}{\partial x^2}\right)$	III. $H + \frac{2\partial G}{\partial x} + \frac{\partial\partial F}{\partial x^2} + Gr + \frac{r\partial F}{\partial x} + Fs = P\left(s + \frac{2\partial r}{\partial x}\right) + Qr + R$,
$\left(\frac{\partial v}{\partial x}\right)$	IV. $\frac{2\partial H}{\partial x} + \frac{\partial\partial G}{\partial x^2} + Hr + \frac{r\partial G}{\partial x} + Gs = P\left(\frac{2\partial s}{\partial x} + \frac{\partial\partial r}{\partial x^2}\right) + Q\left(s + \frac{\partial r}{\partial x}\right) + Rr$,
v	V. $\frac{\partial\partial H}{\partial x^2} + \frac{r\partial H}{\partial x} + Hs = P\frac{\partial\partial s}{\partial x^2} + Q\frac{\partial s}{\partial x} + Rs$.

Ex prima fit $P = F$, ex secunda $Q = G + \frac{2\partial F}{\partial x}$, et tertia

$$R = H + \frac{2\partial G}{\partial x} + \frac{\partial\partial F}{\partial x^2} - \frac{r\partial F - 2F\partial r}{\partial x},$$

qui valores in binis ultimis substituti praebent

$$\frac{2\partial H}{\partial x} + \frac{\partial\partial G}{\partial x^2} - \frac{r\partial G - G\partial r}{\partial x} - \frac{r\partial\partial F}{\partial x^2} - \frac{2\partial F\partial r}{\partial x^2} - \frac{2s\partial F - 2F\partial s}{\partial x}$$
$$+ \frac{rr\partial F + 2Fr\partial r}{\partial x} - \frac{F\partial\partial r}{\partial x^2} = 0 \text{ et}$$
$$\frac{\partial\partial H}{\partial x^2} + \frac{r\partial H}{\partial x} - \frac{s\partial\partial F - 2\partial F\partial s - F\partial\partial s}{\partial x^2} - \frac{2s\partial G - G\partial s}{\partial x}$$
$$+ \frac{s(r\partial F + 2F\partial r)}{\partial x} = 0,$$

quarum illa sponte est integrabilis, praebens

$$2H + \frac{\partial G}{\partial x} - Gr - \frac{r\partial F - F\partial r}{\partial x} - 2Fs + Frr = A;$$

deinde binis illis aequationibus ita repraesentatis

$$-\frac{\partial\partial.Fr}{\partial x^2} - \frac{2\partial.Fs}{\partial x} + \frac{\partial.Frr}{\partial x} + \frac{\partial\partial G}{\partial x^2} - \frac{\partial.Gr}{\partial x} + \frac{2\partial H}{\partial x} = 0,$$
$$-\frac{\partial\partial.Fs}{\partial x^2} + \frac{s}{r}\cdot\frac{\partial.Frr}{\partial x} - \frac{2s\partial G - G\partial s}{\partial x} + \frac{r\partial H}{\partial x} + \frac{\partial\partial H}{\partial x^2} = 0,$$

vel adeo hoc modo

$$\frac{\partial\partial.(G - Fr)}{\partial x} - \partial.r(G - Fr) + 2\partial.(H - Fs) = 0,$$
$$\frac{\partial\partial.(H - Fs)}{\partial x} + 2Fs\,dr + rs\partial F - G\partial s - 2s\partial G + r\partial H = 0,$$

ultima vero ita repraesentari potest

$$\frac{\partial\partial(H - Fs)}{\partial x} - 2s\partial.(G - Fr) - \partial s(G - Fr) + r\partial.(H - Fs) = 0.$$

Quod si jam prior per $H - Fs$ haec vero per $-(G - Fr)$ multiplicetur, summa fit

$$\frac{(H-Fs)\partial\partial.(G-Fr)-(G-Fr)\partial\partial.(H-Fs)}{\partial x} - (G-Fr)(H-Fs)\partial r = 0.$$
$$+ 2(H-Fs)\partial.(H-Fs) - r(H-Fs)\partial.(G-Fr)$$
$$+ 2s(G-Fr)\partial.(G-Fr) + (G-Fr)^2\partial s - r(G-Fr)\partial.(H-Fs)$$

cujus integrale manifesto est

$$\frac{(H-Fs)\partial.(G-Fr)-(G-Fr)\partial.(H-Fs)}{\partial x} + (H+Fs)^2 + (G-Fr)^2 s - (G-Fr)(H-Fs)r = B;$$

integrale autem prius inventum est

$$\frac{\partial.(G-Fr)}{\partial x} - (G-Fr)r + 2(H-Fs) = A,$$

quae per $H - Fs$ multiplicata et ab illa subtracta relinquit

$$-\frac{(G-Fr)\partial.(H-Fs)}{\partial x} - (H-Fs)^2 + (G-Fr)^2 s = B - A(H-Fs),$$

sicque habentur duae aequationes simpliciter differentiales, ex quibus binas quantitates r et s definiri oportet, quibus cognitis etiam functiones P, Q et R innotescunt.

Corollarium 1.

368. Si sit $F = 1$, $G = 0$ et $H = 0$, aequationes inventae erunt

$$-\frac{\partial r}{\partial x} + rr - 2s = a \quad \text{et} \quad \frac{s\partial r - r\partial s}{\partial x} + ss = b,$$

unde ∂x eliminando fit

$$\frac{r\partial s - s\partial r}{\partial r} = \frac{b - ss}{a - 2s - rr}, \quad \text{seu} \quad \frac{r\partial s}{\partial r} = \frac{b + as + ss - rrs}{a + 2s - rr},$$

cujus resolutio in genere vix suscipienda videtur. Sumtis autem constantibus $a = 0$ et $b = 0$, aequatio $\frac{r\partial s}{\partial r} = \frac{ss - rrs}{2s - rr}$, posito $s = rrt$, transit in

$$\frac{r\partial t + 2t\partial r}{\partial r} = \frac{tt - t}{2t - 1}, \quad \text{seu} \quad \frac{r\partial t}{\partial r} = \frac{-3tt + t}{2t - 1},$$

unde fit

$$\frac{\partial r}{r} = \frac{\partial t(1-2t)}{t(3t-1)} = \frac{-\partial t}{t} + \frac{\partial t}{3t-1}, \text{ et}$$

$$r = \frac{\alpha \sqrt[3]{(3t-1)}}{t}, \text{ hinc}$$

$$s = \frac{\alpha\alpha \sqrt[3]{(3t-1)^2}}{t}.$$

Corollarium 2.

369. Pro eodem casu singulari ponamus $3t - 1 = u^3$, ut fiat

$$r = \frac{3\alpha u}{1+u^3} \text{ et } s = \frac{3\alpha\alpha uu}{1+u^3}.$$

Jam ob $\alpha = 0$ est

$$\partial x = \frac{\partial r}{rr-2s} = \frac{\partial r}{rr(1-2t)} = \frac{3\partial r}{rr(1-2u^3)} \text{ et}$$

$$\frac{\partial r}{rr} = \frac{\partial u}{3\alpha uu} - \frac{2u\partial u}{3\alpha} = \frac{\partial u(1-2u^3)}{3\alpha uu},$$

ita ut sit

$$\partial x = \frac{\partial u}{\alpha uu}, \text{ hincque}$$

$$\frac{1}{u} = \beta - \alpha x \text{ et } u = \frac{1}{\beta - \alpha x};$$

ubi quidem salva generalitate sumi potest

$$\beta = 0 \text{ et } u = \frac{-1}{\alpha x},$$

unde fit

$$r = \frac{-3xx}{x^3+c^3} \text{ facto}$$

$$\alpha = -\frac{1}{c} \text{ et } s = \frac{3x}{x^3+c^3}.$$

Tandem ergo colligitur

$$P = 1, \; Q = 0 \text{ et } R = -\frac{2\partial r}{\partial x} = -\frac{6x(2c^3-x^3)}{(c^3+x3)^2}.$$

Corollarium 3.

370. Proposita ergo aequatione $\left(\frac{\partial\partial v}{\partial y^2}\right) = \left(\frac{\partial\partial v}{\partial x^2}\right)$, cujus integrale est

$$v = \Gamma : (x + y) + \Delta : (x - y),$$

hujus aequationis integrale assignari poterit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{6x(2c^3 - x^3)}{(c^3 + x^3)^2} z,$$

est enim

$$z = \left(\frac{\partial\partial v}{\partial x^2}\right) - \frac{3xx}{c^3 + x^3}\left(\frac{\partial v}{\partial x}\right) + \frac{3x}{c^3 + x^3} v.$$

Scholion 1.

371. Haec pro casu

$$F = 1,\ G = 0 \text{ et } H = 0,$$

multo facilius atque generalius computari possunt pro quocunque valore quantitatis a, dum sit $b = 0$, tum enim altera aequatio statim dat

$$\partial x = \frac{r\partial s - s\partial r}{ss}, \text{ hincque}$$

$$x = \frac{-r}{s} \text{ et } s = \frac{-r}{x},$$

ex quo prima aequatio hanc induit formam

$$\frac{\partial r}{\partial x} - rr - \frac{2r}{x} + a = 0.$$

Ponamus $r = \frac{a}{t}$, fiet

$$dt + \frac{2t\partial x}{x} - tt\partial x + a\partial x = 0,$$

cui particulariter satisfacit

$$t = \sqrt{a} + \frac{1}{x}.$$

Statuatur ergo

$$t = \sqrt{a} + \frac{1}{x} + \frac{1}{u},$$

ac prodit

$$\partial u + \partial x + 2u\,\partial x\sqrt{a} = 0,$$

quae per $e^{2x\sqrt{a}}$ multiplicata et integrata praebet

$$e^{2x\sqrt{a}}\,u + \frac{1}{2\sqrt{a}}\,e^{2x\sqrt{a}} = \frac{n}{2\sqrt{a}},$$

ideoque

$$\frac{1}{u} = \frac{2\,e^{2x\sqrt{a}}\sqrt{a}}{n - e^{2x\sqrt{a}}} = \frac{2\sqrt{a}}{n\,e^{-2x\sqrt{a}} - 1},$$

$$t = \frac{1}{x} + \frac{n\,e^{-2x\sqrt{a}} + 1}{n\,e^{-2x\sqrt{a}} - 1}\sqrt{a} = \frac{1}{x} + \frac{n + e^{2x\sqrt{a}}}{n - e^{2x\sqrt{a}}}\sqrt{a} \text{ et}$$

$$r = \frac{a\,x\,(n - e^{2x\sqrt{a}})}{n\,(x\sqrt{a} + 1) + e^{2x\sqrt{a}}\,(x\sqrt{a} - 1)},$$

ac propterea

$$s = \frac{-a\,(n - e^{2x\sqrt{a}})}{n\,(x\sqrt{a} + 1) + e^{2x\sqrt{a}}\,(x\sqrt{a} - 1)},$$

tum vero postremo

$$\mathrm{P} = -1,\ \mathrm{Q} = 0 \text{ et } \mathrm{R} = -\frac{2\partial r}{\partial x} = -2rr - \frac{4r}{x} + 2a,$$

seu

$$\mathrm{R} = \frac{-2\,a\,(nn - 4\,n\,a\,x\,x\,e^{2x\sqrt{a}} - 2\,n\,e^{2x\sqrt{a}} + e^{4x\sqrt{a}})}{[n\,(x\sqrt{a} + 1) + e^{2x\sqrt{a}}\,(x\sqrt{a} - 1)]^2}$$

$$= \frac{-2\,a\,(n - e^{2x\sqrt{a}})^2 + 8\,n\,a\,x\,x\,e^{2x\sqrt{a}}}{[n\,(x\sqrt{a} + 1) + e^{2x\sqrt{a}}\,(x\sqrt{a} - 1)]^2}.$$

Si jam sumatur a evanescens et $n = 1 + \frac{2}{3}\,a\,c^3\sqrt{a}$, formulae ante inventae resultant. At si a sit quantitas negativa puta $a = -m^2$, capiaturque $n = \frac{\alpha\sqrt{-1} + \beta}{\alpha\sqrt{-1} - \beta}$, reperitur

$$r = \frac{-mmx\,(\beta\cos.mx + \alpha\sin.mx)}{\beta\cos.mx + \alpha\sin.mx - mx(\alpha\cos.mx - \beta\sin.mx)} = \frac{-mmx\cos.(mx+\gamma)}{\cos.(mx+\gamma) - mx\sin.(mx+\gamma)}$$

et

$$s = \frac{mm\cos.(mx+\gamma)}{\cos.(mx+\gamma) - mx\sin.(mx+\gamma)},$$

indeque

$$R = \frac{2mm[\cos.(mx+\gamma)^2 + mmxx]}{[\cos.(mx+\gamma) - mx\sin.(mx+\gamma)]^2}.$$

Quantitas R reducitur ad hanc

$$R = \frac{8naaxx - 2a(ne^{-x\sqrt{a}} - e^{x\sqrt{a}})^2}{[n(1+x\sqrt{a})e^{-x\sqrt{a}} - (1-x\sqrt{a})e^{x\sqrt{a}}]^2},$$

quae forma sumto a valde parvo abit in

$$R = \frac{8naaxx - 2a[n-1-(n+1)x\sqrt{a} + \frac{(n-1)}{2}axx - \frac{(n+1)}{6}ax^3\sqrt{a} + \text{etc.}]^2}{[n-1-\frac{1}{2}(n-1)axx+\frac{1}{3}(n+1)ax^3\sqrt{a}]^2}.$$

Statuatur $n = 1 + \beta a\sqrt{a}$, ut sit

$$n-1 = \beta a\sqrt{a} \text{ et } n+1 = 2 = \beta a\sqrt{a}, \text{ erit}$$

$$R = \frac{8naaxx - 2a(\beta a\sqrt{a} - 2x\sqrt{a} - \beta aax + \frac{\beta aaxx\sqrt{a}}{2} - \frac{1}{3}ax^3\sqrt{a})^2}{(\beta a\sqrt{a} - \frac{1}{2}\beta aaxx\sqrt{a} + \frac{2}{3}ax^3\sqrt{a})^2},$$

ubi numerator fit

$$8aaxx + 8\beta a^3xx\sqrt{a} - 2a(\beta\beta a^5 - 4\beta aax - 2\beta\beta a^3x\sqrt{a})$$
$$+ 4axx + \tfrac{4}{3}aax^4,$$

ubi cum termini per aa affecti se destruant, retineantur ii soli qui per a^3 sunt affecti, erit idem in denominatore observato

$$R = \frac{8\beta a^3x - \frac{8}{3}a^3x^4}{a^3(\beta+\frac{2}{3}x^3)^2} = \frac{8x(\beta - \frac{1}{3}x^3)}{(\beta+\frac{2}{3}x^3)^2},$$

quae jam facile ad formam

$$R = \frac{6x(2c^3 - x^3)}{(c^3+x^3)^2}$$

reducitur, sumendo

$$3\beta = 2c^3, \text{ ut sit } \beta = \tfrac{2}{3}c^3.$$

Quare hic casus oritur, sumendo a evanescens et

$$n = 1 + \tfrac{2}{3}c^3a\sqrt{a}.$$

Scholion 2.

372. Cum evolutio solutionis inventae sit difficillima, neque ulla via pateat, quomodo ambae quantitates incognitae r et s ex binis aequationibus erutis definiri queant, in scientiae incrementum haud parum juvabit observasse, idem problema per repetitionem transformationis in primo problemate hujus capitis quoque solvi posse, neque proinde usu carebit has duas solutiones inter se comparasse. Proposita ergo aequatione

$$\left(\frac{\partial\partial v}{\partial y^2}\right) = F\left(\frac{\partial\partial v}{\partial x^2}\right) + G\left(\frac{\partial v}{\partial x}\right) + H v,$$

ponamus primo

$$u = \left(\frac{\partial v}{\partial x}\right) + p v,$$

ac p ex hac aequatione determinetur

$$F\,\partial p + G p\,\partial x - F p p\,\partial x + (C - H)\,\partial x = 0,$$

ac tum ista resultabit aequatio

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = F\left(\frac{\partial\partial z}{\partial x^2}\right) + \left(G + \frac{\partial F}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) + \left(H + \frac{\partial G}{\partial x} - \frac{2F\partial p - p\partial F}{\partial x}\right) u.$$

Nunc pro hac aequatione porro transformando, statuamus simili modo

$$z = \left(\frac{\partial u}{\partial x}\right) + q u,$$

ita ut sit quoque

$$z = \left(\frac{\partial\partial v}{\partial x^2}\right) + (p + q)\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial p}{\partial x} + p q\right) v,$$

et quantitate q ex hac aequatione definita

$$F\,\partial q + \left(G + \frac{\partial F}{\partial x}\right) q\,\partial x - F q q\,\partial x + \left(D - H - \frac{\partial G}{\partial x} + \frac{2F\partial p + p\partial F}{\partial x}\right)\partial x = 0,$$

orietur haec aequatio

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial\partial z}{\partial x^2}\right) + Q\left(\frac{\partial z}{\partial x}\right) + R z,$$

cujus quantitates P, Q, R ita se habent

$$P = F,\ Q = G + \frac{2\partial F}{\partial x} \text{ et}$$
$$R = H + \frac{2\partial G}{\partial x} - \frac{2F\partial p - p\partial F}{\partial x} + \frac{\partial\partial F}{\partial x^2} + \frac{2F\partial q - q\partial F}{\partial x}.$$

Cum hac ergo solutione convenire debet ea, quam postremum problema suppeditavit, in quo cum statim posuerimus

$$z = \left(\frac{\partial\partial v}{\partial x^2}\right) + r\left(\frac{\partial v}{\partial x}\right) + s\,v,$$

erit utique

$$r = p + q \text{ et } s = \frac{dp}{\partial x} + p\,q,$$

unde quidem statim valores pro P, Q et R manifesto prodeunt iidem. Verum multo minus apparet, si pro r et s isti valores per p et q substituantur, tum istas binas aequationes

$$\frac{\partial.(G - Fr)}{\partial x} - (G - F\,r)\,r + 2\,(H - F\,s) = A \text{ et}$$
$$\frac{(G - Fr)\partial.(H - Fs)}{\partial x} + (H - F\,s)^2 - (G - F\,r)^2\,s - A\,(H - F\,s) = B,$$

ad eas quas ante invenimus reduci

$$\frac{F\partial p}{\partial x} + G\,p - F\,p\,p - H + C = 0 \text{ et}$$
$$\frac{F\partial q}{\partial x} + \left(G + \frac{\partial F}{\partial x}\right) q - F\,q\,q - H - \frac{\partial G}{\partial x} + \frac{2F\partial p + p\partial F}{\partial x} + D = 0,$$

ita ut hae constantes C et D ad illas A et B certam teneant relationem. Interim patet has postremas aequationes multo esse simpliciores, dum prior duas tantum variabiles p et x complectitur, indeque p per x, cujus F, G et H sunt functiones datae, determinari debet, qua inventa quantitatem q simili modo ex altera aequatione elici oportet. Verum in ambabus superioribus aequationibus binae variabiles r et s ita inter se sunt permixtae, ut nulla methodus eas resolvendi, vel adeo ad aequationem inter duas tantum variabiles perveniendi, habeatur. Cum igitur certum sit priores solutu difficillimas ad posteriores multo faciliores ope substitutionum assignatarum perduci posse, sine dubio methodus hanc reductionem efficiendi haud contemnenda subsidia in Analysin esse allatura videtur.

seu

$$\frac{\partial.(Fr-G)}{\partial x} + q(G - Fr) - H + Fs + D = 0,$$

hincque

$$D = \frac{\partial.(G-Fr)}{\partial x} - q(G - Fr) + H - Fs,$$

ex quibus concluditur

$$CD = \frac{(H-Fs)\,\partial.(G-Fr)}{\partial x} - q(G - Fr)(H - Fs) + (H - Fs)^2$$
$$- \frac{p(G-Fr)\partial.(G-Fr)}{\partial x} + pq(G - Fr)^2 - p(G - Fr)(H - Fs).$$

Ex secunda vero habemus

$$B = \frac{(G-Fr)\partial.(H-Fs)}{\partial x} - \frac{(H-Fs)\partial.(G-Fr)}{\partial x} - (H - Fs)^2$$
$$+ (G - Fr)(H - Fs)r - (G - Fr)^2 s,$$

quibus expressionibus conjunctis fit

$$\frac{CD+B}{G-Fr} = \frac{\partial.(H-Fs)}{\partial x} - \frac{p\,\partial.(G-Fr)}{\partial x} - \frac{\partial p(G-Fr)}{\partial x}$$
$$= \frac{\partial.(H-Fs) - \partial.p(G-Fr)}{\partial x} = 0,$$

siquidem est

$$C = H - Fs - p(G - Fr),$$

ex quo etiam in genere est

$$B = -CD \text{ et } A = C + D.$$

Interim tamen hinc non perspicitur, quomodo ex aequationibus I. et II. binae reliquae III. et IV. derivari queant.

Scholion 4.

374. Omnibus his diligenter pensitatis manifestum fiet, totum negotium ope substitutionis satis simplicis confici posse. Quod quo facilius ostendatur, ponamus brevitatis causa

$$G - Fr = R \text{ et } H - Fs = S,$$

ut habeantur hae duae aequationes

$$\text{I.}\quad A = \frac{\partial R}{\partial x} - \frac{GR}{F} + \frac{RR}{F} + 2S,$$

$$\text{II.}\quad B = \frac{R\partial S - S\partial R}{\partial x} - \frac{HRR}{F} + \frac{GRS}{F} - SS,$$

ex quibus duas quantitates R et S erui oporteat, dum F, G, H sunt functiones quaecunque ipsius x, at A et B quantitates constantes. Ad hoc adhibeatur ista substitutio $S = C + Rp$ ita adornanda, ut binae illae aequationes coalescant in unam, in qua praeter x unica insit nova variabilis p, deinceps per methodos cognitas investiganda. Hinc ob

$\partial S = R\partial p + p\partial R$ habebitur

$$\text{I.}\quad A = \frac{\partial R}{\partial x} - \frac{GR}{F} + \frac{RR}{F} + 2C + 2Rp,$$

$$\text{II.}\quad B = \frac{RR\partial p}{\partial x} - \frac{C\partial R}{\partial x} - \frac{HRR}{F} + \frac{CGR}{F} + \frac{GRRp}{F} - CC + 2CRp - RRpp,$$

unde primo eliminando ∂R, concluditur

$$B + AC = \frac{RR\partial p}{\partial x} + \frac{CRR}{F} + CC - \frac{HRR}{F} - RRpp,$$

dummodo ergo constantem C ita assumamus, ut sit $CC = B + AC$, per divisionem etiam ipsa quantitas R tolletur, resultabitque haec aequatio

$$0 = \frac{\partial p}{\partial x} + \frac{C}{F} - \frac{H}{F} - pp,$$

cujus resolutio ad methodos magis cognitas pertinet. Cum igitur ista methodus maximi sit momenti, sequens problema, etiamsi ad primam partem calculi integralis sit referendum, hic adjicere operae pretium videtur.

Problema 60.

375. Propositis hujusmodi duabus aequationibus differentialibus

$$\text{I.}\quad 0 = \frac{\partial y}{\partial x} + F + Gy + Hz + Iyy + Kyz + Lzz,$$

$$\text{II.}\quad 0 = \frac{y\partial z - z\partial y}{\partial x} + P + Qy + Rz + Syy + Tyz + Vzz,$$

ubi F, G, H, etc. P, Q, R, etc. sint functiones ipsius x, methodum exponere has aequationes, siquidem fieri licet, resolvendi.

Solutio.

Methodus indicata in hoc consistit, ut ope substitutionis $z = a + yv$ ex illis aequationibus una elici queat duas tantum variabiles x et v implicans. Quoniam igitur est

$$y\partial z - z\partial y = yy\partial v - a\partial y,$$

ex I $\times a +$ II. nascitur haec aequatio

$$0 = \frac{yy\partial v}{\partial x} + P + Qy + Rz + Syy + Tyz + Vzz$$
$$+ aF + aGy + aHz + aIyy + aKyz + aLzz,$$

quae, loco z substituto valore $a + yv$, ita exhibeatur secundum potestates ipsius y

$$0 = \frac{yy\partial v}{\partial x} + y^0 [P + aF + a(R + aH) + aa(V + aL)]$$
$$+ y^1 [Q + aG + v(R + aH) + a(T + aK) + 2av(V + aL)]$$
$$+ y^2 [S + aI + v(T + aK) + vv(V + aL)],$$

nunceque efficiendum est, ut tota aequatio per yy dividi queat ideoque partes per y^0 et y^1 affectae evanescant. Ex parte ergo y^0 fieri oportet

$$P + aF + a(R + aH) + aa(V + aL) = 0,$$

ex parte autem y^1, quia v est nova variabilis in calculum inducta, hae duae conditiones nascuntur

$$Q + aG + a(T + aK) = 0 \text{ et}$$
$$R + aH + 2a(V + aL) = 0,$$

unde prima dabit

$$P + aF - aa(V + aL) = 0.$$

Conditiones ad istam reductionem requisitae sunt hae tres

$$\text{I. } P + aF - aa(V + aL) = 0,$$
$$\text{II. } Q + aG + a(T + aK) = 0,$$
$$\text{III. } R + aH + 2a(V + aL) = 0,$$

unde vel P, Q et R, vel F, G et H commode definiuntur.

His autem conditionibus stabilitis, totum negotium ad resolutionem hujus aequationis revocatur

$$0 = \frac{\partial v}{\partial x} + S + aI + v(T + aK) + vv(V + aL),$$

quae duas tantum continet variabiles x et v, ex qua v per x determinari oportet. Cum deinde posito $z = a + yv$ prima aequatio induat hanc formam

$$0 = \frac{\partial y}{\partial x} + F + aH + aaL + y(G + Hv + aK + 2aLv)$$
$$+ yy(I + Kv + Lvv),$$

secunda vero istam

$$0 = \frac{yy\partial v}{\partial x} - \frac{a\partial y}{\partial x} + P + aR + aaV + y(Q + Rv + aT + 2aVv)$$
$$+ yy(S + Tv + Vvv),$$

seu hinc superiorem per yy multiplicatam subtrahendo

$$0 = \frac{-a\partial y}{\partial x} + P + aR + aaV + y(Q + Rv + aT + 2aVv)$$
$$- yy(Ia + aKv + aLvv),$$

quae quidem cum illa congruit, ut natura rei postulat.

Corollarium 1.

376. Si ergo hujusmodi binae aequationes fuerint propositae

$$0 = \frac{\partial y}{\partial x} + F + Gy + Hz + Iyy + Kyz + Lzz,$$
$$0 = \frac{y\partial z - z\partial y}{\partial x} - aF - aGy - aHz + Syy + Tyz + Vzz$$
$$+ a^3L - aaKy - 2aaLz$$
$$+ aaV - aTy - 2aVz,$$

facto $z = a + yv$, primo resolvi debet haec aequatio

$$0 = \frac{\partial v}{\partial x} + \mathrm{S} + a\mathrm{I} + v(\mathrm{T} + a\mathrm{K}) + vv(\mathrm{V} + a\mathrm{L}),$$

unde definita v per x, hanc aequationem tractari oportet

$$0 = \frac{\partial y}{\partial x} + \mathrm{F} + a\mathrm{H} + aa\mathrm{L} + y(\mathrm{G} + a\mathrm{K}) + yy(\mathrm{I} + \mathrm{K}v + \mathrm{L}vv)$$
$$+ vy(\mathrm{H} + 2a\mathrm{L}),$$

quo facto habebitur quoque $z = a + vy$.

Corollarium 2.

377. Si $\mathrm{F} = \mathrm{A}$, $\mathrm{K} = 0$, $\mathrm{L} = 0$, $\mathrm{H} = -2b$, $\mathrm{V} = b$ et $\mathrm{T} = -\mathrm{G}$, casus supra §. 374. tractatus resultat harum aequationum

$$0 = \frac{\partial y}{\partial x} + \mathrm{A} + \mathrm{G}y - 2bz + \mathrm{I}yy,$$

$$0 = \frac{y\partial z - z\partial y}{\partial x} - a\mathrm{A} + \mathrm{S}yy - \mathrm{G}yz + bzz,$$
$$+ aab,$$

ubi G, I et S sunt functiones quaecunque ipsius x, et resolutio ita se habet, ut posito $x = a + yv$, hae aequationes successive debeant expediri

$$0 = \frac{\partial v}{\partial x} + \mathrm{S} + a\mathrm{I} - \mathrm{G}v + bvv \text{ et}$$

$$0 = \frac{\partial y}{\partial x} + \mathrm{A} - 2ab + y(\mathrm{G} - 2bv) + \mathrm{I}yy.$$

Corollarium 3.

378. Evidens est postremam aequationem nulla laborare difficultate, etiam in genere dum sit

$$\mathrm{F} + a\mathrm{H} + aa\mathrm{L} = 0,$$

prioris autem solutio in promtu est, si sit vel $\mathrm{S} + a\mathrm{T} = 0$, vel $\mathrm{V} = a\mathrm{L} = 0$.