CAPUT IV.

ALIA METHODUS PECULIARIS HUJUSMODI AEQUATIONES INTEGRANDI.

Problema 52.

322.

Si aequatio proposita hanc habuerit formam

$$(x+y)^2 \left(\frac{\partial\partial z}{\partial x \partial y}\right) + m(x+y)\left(\frac{\partial z}{\partial x}\right) + m(x+y)\left(\frac{\partial z}{\partial y}\right) + nz = 0,$$

ejus integrale completum investigare.

Solutio.

Cum hic binae variabiles x et y aequaliter insint, ponatur primo

$$z = A(x+y)^\lambda \, f : x + B(x+y)^{\lambda+1} \, f' : x + C(x+y)^{\lambda+2} f'' : x$$
$$+ D(x+y)^{\lambda+3} \, f''' : x \text{ etc.}$$

ubi pro faciliori substitutione notetur, posito $v = (x+y)^\mu \, F : x$ fore

$$\left(\frac{\partial v}{\partial x}\right) = \mu(x+y)^{\mu-1} \, F : x + (x+y)^\mu \, F' : x,$$
$$\left(\frac{\partial v}{\partial y}\right) = \mu(x+y)^{\mu-1} \, F : x,$$
$$\left(\frac{\partial\partial v}{\partial x \partial y}\right) = \mu(\mu-1)(x+y)^{\mu-2} \, F : x + \mu(x+y)^{\mu-1} \, F' : x.$$

Facta ergo substitutione obtinebimus hanc aequationem

$$\begin{array}{llll}
0 = nA(x+y)^\lambda f : x & + nB(x+y)^{\lambda+1} f' : x & + nC(x+y)^{\lambda+2} f'' : x & + \text{etc.} \\
\quad + 2m\lambda A & \quad + mA & \quad + mB & \\
\quad + \lambda(\lambda-1)A & \quad + 2m(\lambda+1)B & \quad + 2m(\lambda+2)C & \\
 & \quad + \lambda A & \quad + (\lambda+1)B & \\
 & \quad + (\lambda+1)\lambda B & \quad + (\lambda+2)(\lambda+1)C, &
\end{array}$$

ubi totum negotium ad coëfficientium A, B, C, D, etc. determinationem revocatur; facile autem erat praevidere, forma superiori assumta potestates ipsius $(x + y)$ in singulis membris pares esse prodituras: Fieri igitur necesse est

$$n + 2m\lambda + \lambda\lambda - \lambda = 0,$$
$$(n + 2m\lambda + 2m + \lambda\lambda + \lambda)\,B + (m + \lambda)\,A = 0,$$
$$(n + 2m\lambda + 4m + \lambda\lambda + 3\lambda + 2)\,C + (m + \lambda + 1)\,B = 0,$$
$$(n + 2m\lambda + 6m + \lambda\lambda + 5\lambda + 6)\,D + (m + \lambda + 2)\,C = 0,$$
etc.

quae determinationes ope primae $n + 2m\lambda + \lambda\lambda - \lambda = 0$ ita commodius exprimuntur:

$$B = -\frac{(m+\lambda)\,A}{2(m+\lambda)},$$
$$C = -\frac{(m+\lambda+1)\,B}{2(2m+2\lambda+1)},$$
$$D = -\frac{(m+\lambda+2)\,C}{3(2m+2\lambda+2)},$$
$$E = -\frac{(m+\lambda+3)\,D}{4(2m+2\lambda+3)},$$
$$F = -\frac{(m+\lambda+4)\,E}{5(2m+2\lambda+4)},$$
$$G = -\frac{(m+\lambda+5)\,F}{6(2m+2\lambda+5)},$$
$$H = -\frac{(m+\lambda+6)\,G}{7(2m+2\lambda+6)},$$
etc.

unde lex progressionis est manifesta. At pro exponente λ duplicem eruimus valorem

$$\lambda = \tfrac{1}{2} - m \pm \sqrt{(\tfrac{1}{4} - m - n + mm)},$$

quorum utrumque aeque pro λ accipere licet. Hic autem praecipue notandi sunt casus, quibus series assumta abrumpitur, quod fit, quoties $m + \lambda + i = 0$, denotante i numerum quemcunque integrum positivum cyphra non exclusa. Hoc ergo evenit quoties fuerit

$$\tfrac{1}{2} + i \pm \sqrt{(\tfrac{1}{4} - m - n + mm)} = 0,$$

id quod fieri nequit nisi $\frac{1}{4} - m - n + mm$ fuerit quadratum. Inventa autem hujusmodi serie sive finita sive in infinitum excurrente, alia similis pro functionibus ipsius y reperitur; unde valor ipsius z ita reperietur expressus

$$z = A\,(x+y)^{\lambda}\,(f:x+F:y) + B\,(x+y)^{\lambda+1}\,(f':x+F':y)$$
$$+ C\,(x+y)^{\lambda+2}(f'':x+F'':y) + D\,(x+y)^{\lambda+3}(f''':x+F''':y)$$
$$+ E\,(x+y)^{\lambda+4}(f^{IV}:x+F^{IV}:y) + F\,(x+y)^{\lambda+5}(f^{V}:x+F^{V}:y),$$
$$+ \text{etc.},$$

ubi cum binae functiones arbitrariae adsint, id certum est signum, hanc formam esse integrale completum aequationis propositae.

Corollarium 1.

323. Si fuerit $\lambda = -m$, hoc est $n - mm + m = 0$, seu $n = mm - m$, integrale ex unico membro constabit ob $B = 0$, eritque integrale

$$z = A\,(x+y)^{-m}\,(f:x+F:y).$$

Corollarium 2.

324. Integrale autem duo membra continebit, si

$$\lambda = -m-1 \text{ vel } n = mm - m - 2 = (m+1)(m-2);$$

tum erit $B = -\frac{1}{2}A$ et integrale erit

$$z = (x+y)^{-m-1}(f:x+F:y) - \tfrac{1}{2}(x+y)^{-m}(f':x+F':y).$$

Corollarium 3.

325. Integrale tribus terminis constabit, si $\lambda = -m-2$, vel $n = (m+2)(m-3)$; tum erit

$$B = -\tfrac{1}{2}A, \text{ et } C = -\tfrac{1}{6}B = +\tfrac{1}{12}A,$$

integrale vero

$$z = (x+y)^{-m-2}(f:x+F:y) - \tfrac{1}{2}(x+y)^{-m-1}(f':x+F':y)$$
$$+ \tfrac{1}{12}(x+y)^{-m}(f'':x+F'':y).$$

Corollarium 4.

326. Ex quatuor autem membris integrale constabit, si fuerit $\lambda = -m - 3$, seu $n = (m+3)(m-4)$; tum autem erit

$$B = -\tfrac{1}{2}A,\ C = -\tfrac{1}{5}B = +\tfrac{1}{10}A,\ D = -\tfrac{1}{12}C = -\tfrac{1}{120}A,$$

et integrale

$$z = (x+y)^{-m-3}(f:x + F:y) - \tfrac{1}{2}(x+y)^{-m-2}(f':x+F':y)$$
$$+ \tfrac{1}{10}(x+y)^{-m-1}(f'':x+F'':y) - \tfrac{1}{120}(x+y)^{-m}(f''':x+F''':y).$$

Scholion.

327. Quod si in genere ponamus $\lambda + m = -i$, erit $n = (m+i)(m-i-1)$, tum vero

$$B = -\tfrac{1}{2}A,\ C = -\frac{(i-1)B}{2(2i-1)},\ D = -\frac{(i-2)C}{3(2i-2)},\ E = -\frac{(i-3)D}{4(2i-3)},$$

unde fit omnes ad primum reducendo

$$B = -\tfrac{1}{2}A,\ C = \frac{(i-1)}{2.2(2i-1)}A,\ D = \frac{-(i-2)}{2.2.2.3(2i-1)}A,$$
$$E = \frac{+(i-2)(i-3)}{2.2.2.3.4(2i-1)(2i-3)}A,\ F = \frac{-(i-3)(i-4)}{2^4.3.4.6(2i-1)(2i-3)}A, \text{ etc.}$$

qui ita se habent

	A	B	C	D	E	F
$i = 1$	1	$-\frac{1}{2}$	0	0	0	0
$i = 2$	1	$-\frac{1}{2}$	$\frac{1}{12}$	0	0	0
$i = 3$	1	$-\frac{1}{2}$	$\frac{2}{20}$	$-\frac{1}{120}$	0	0
$i = 4$	1	$-\frac{1}{2}$	$\frac{3}{28}$	$-\frac{2}{7.24}$	$\frac{2}{96.7.5}$	0
$i = 5$	1	$-\frac{1}{2}$	$\frac{4}{36}$	$-\frac{3}{9.24}$	$\frac{3.2}{96.9.7}$	$\frac{2.1}{960.9.7}$
$i = 6$	1	$-\frac{1}{2}$	$\frac{4}{44}$	$-\frac{4}{11.24}$	$\frac{4.3}{96.11.9}$	$\frac{3.2}{960.11.9}$

ita hujus aequationis

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \frac{m}{x+y}\left(\frac{\partial z}{\partial x}\right) + \frac{m}{x+y}\left(\frac{\partial z}{\partial y}\right) + \frac{(m+i)(m-i-1)}{(x+y)^2}z = 0,$$

integrale completum erit

$$\begin{aligned} z = &+ (x+y)^{-m-i}\,(\mathrm{f}:x + \mathrm{F}:y) \\ &- \frac{i}{2i}\,(x+y)^{-m-i+1}\,(\mathrm{f}':x + \mathrm{F}':y) \\ &+ \frac{i(i-1)}{2i\,.\,2(2i-1)}\,(x+y)^{-m-i+2}\,(\mathrm{f}'':x + \mathrm{F}'':y) \\ &- \frac{i(i-1)(i-2)}{2i\,.\,2(2i-1)\,.\,3(2i-2)}\,(x+y)^{-m-i+3}\,(\mathrm{f}''':x + \mathrm{F}''':y) \\ &+ \frac{i(i-1)(i-2)(i-3)}{2i\,.\,2(2i-1)\,.\,3(2i-2)\,.\,4(2i-3)}\,(x+y)^{-m-i+4}\,(\mathrm{f}^{\mathrm{IV}}:x + \mathrm{F}^{\mathrm{IV}}:y) \\ &- \frac{i(i-1)(i-2)(i-3)(i-4)}{2i\,.\,2(2i-1)\,.\,3(2i-2)\,.\,4(2i-3)\,.\,5(2i-4)}\,(x+y)^{-m-i+5}\,(\mathrm{f}^{\mathrm{V}}:x + \mathrm{F}^{\mathrm{V}}:y) \\ &+ \text{etc.} \end{aligned}$$

quae forma quoties i fuerit numerus integer positivus, finito constat terminorum numero: secus autem in infinitum excurrit. Imprimis autem ista integratio hoc habet singulare, quod non solum ipsas functiones arbitrarias $\mathrm{f}:x$ et $\mathrm{F}:y$ complectatur, sed etiam earum formulas differentiales.

Exemplum.

328. *Si occurrat ista aequatio*

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) + \frac{m}{x+y}\left(\frac{\partial z}{\partial x}\right) + \frac{m}{x+y}\left(\frac{\partial z}{\partial y}\right) = 0,$$

definire casus, quibus ejus integrale per formam finitam exhiberi potest.

Cum hic sit $n = (m+i)(m-i-1) = 0$, sumendo pro i numeros integros positivos, duo ordines habebuntur casuum, quibus integratio succedit, alter quo est $m = -i$, alter quo $m = i+1$, ita ut in genere integratio finita locum habeat, quoties m fuerit numerus integer sive positivus sive negativus. Primo ergo si sit $m = -i$, erit

$$z = 1\,(\mathrm{f}:x + \mathrm{F}:y) - \frac{i}{2i}(x+y)(\mathrm{f}':x + \mathrm{F}':y)$$
$$+ \tfrac{1}{2}\cdot\frac{i(i-1)}{2i(2i-1)}(x+y)^2(\mathrm{f}'':x + \mathrm{F}'':y)$$
$$- \tfrac{1}{6}\cdot\frac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^3(\mathrm{f}''':x + \mathrm{F}''':y)$$
$$+ \tfrac{1}{24}\cdot\frac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^4(\mathrm{f}^{\mathrm{IV}}:x + \mathrm{F}^{\mathrm{IV}}:y)$$
$$- \text{etc.}$$

Deinde si sit $m = i + 1$, erit

$$(x+y)^{2i+1} z = 1\,(\mathrm{f}:x + \mathrm{F}:y) - \frac{i}{2i}(x+y)(\mathrm{f}':x + \mathrm{F}':y)$$
$$+ \tfrac{1}{2}\cdot\frac{i(i-1)}{2i(2i-1)}(x+y)^2(\mathrm{f}'':x + \mathrm{F}'':y)$$
$$- \tfrac{1}{6}\cdot\frac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^3(\mathrm{f}''':x + \mathrm{F}''':y)$$
$$+ \tfrac{1}{24}\cdot\frac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^4(\mathrm{f}^{\mathrm{IV}}:x + \mathrm{F}^{\mathrm{IV}}:y)$$
$$- \text{etc.}$$

utrinque scilicet eadem habetur expressio, cui casu priori ipsa quantitas z, posteriori quantitas $(x+y)^{2i+1}z$ aequatur. Ad singulos hos casus distinctius evolvendos ponamus

$\mathrm{A} = (\mathrm{f}:x + \mathrm{F}:y)$,

$\mathrm{B} = (\mathrm{f}:x + \mathrm{F}:y) - \frac{1}{2}(x+y)(\mathrm{f}':x + \mathrm{F}':y)$,

$\mathrm{C} = (\mathrm{f}:x + \mathrm{F}:y) - \frac{2}{4}(x+y)(\mathrm{f}':x + \mathrm{F}':y) + \frac{1}{4\cdot 3}(x+y)^2(\mathrm{f}'':x + \mathrm{F}'':y)$,

$\mathrm{D} = (\mathrm{f}:x + \mathrm{F}:y) - \frac{3}{6}(x+y)(\mathrm{f}':x + \mathrm{F}':y) + \frac{3}{6\cdot 5}(x+y)^2(\mathrm{f}'':x + \mathrm{F}'':y)$
$\qquad - \frac{1}{6\cdot 5\cdot 4}(x+y)^3(\mathrm{f}''':x + \mathrm{F}''':y)$, etc.

vel posito brevitatis gratia

$$\mathfrak{A} = \mathrm{f}:x + \mathrm{F}:y,$$
$$\mathfrak{B} = (x+y)(\mathrm{f}':x + \mathrm{F}':y),$$
$$\mathfrak{C} = (x+y)^2(\mathrm{f}'':x + \mathrm{F}'':y),$$
$$\mathfrak{D} = (x+y)^3(\mathrm{f}''':x + \mathrm{F}''':y),$$
$$\mathfrak{E} = (x+y)^4(\mathrm{f}^{\mathrm{IV}}:x + \mathrm{F}^{\mathrm{IV}}:y),$$
etc.

sit

$$A = \mathfrak{A},$$
$$B = \mathfrak{A} - \tfrac{1}{2}\mathfrak{B},$$
$$C = \mathfrak{A} - \tfrac{2}{4}\mathfrak{B} + \tfrac{1}{4 \cdot 3}\mathfrak{C},$$
$$D = \mathfrak{A} - \tfrac{3}{6}\mathfrak{B} + \tfrac{3}{6 \cdot 5}\mathfrak{C} - \tfrac{1}{6 \cdot 5 \cdot 4}\mathfrak{D},$$
$$E = \mathfrak{A} - \tfrac{4}{8}\mathfrak{B} + \tfrac{6}{8 \cdot 7}\mathfrak{C} - \tfrac{4}{8 \cdot 7 \cdot 6}\mathfrak{D} + \tfrac{1}{8 \cdot 7 \cdot 6 \cdot 5}\mathfrak{E},$$
$$F = \mathfrak{A} - \tfrac{5}{10}\mathfrak{B} + \tfrac{10}{10 \cdot 9}\mathfrak{C} - \tfrac{10}{10 \cdot 9 \cdot 8}\mathfrak{D} + \tfrac{5}{10 \cdot 9 \cdot 8 \cdot 7}\mathfrak{E} - \tfrac{1}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}\mathfrak{F},$$
$$G = \mathfrak{A} - \tfrac{6}{12}\mathfrak{B} + \tfrac{15}{12 \cdot 11}\mathfrak{C} - \tfrac{20}{12 \cdot 11 \cdot 10}\mathfrak{D} + \tfrac{15}{12 \cdot 11 \cdot 10 \cdot 9}\mathfrak{E} - \tfrac{6}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}\mathfrak{F}$$
$$+ \tfrac{1}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}\mathfrak{G},$$

etc.

Quibus valoribus inventis, erit pro duplici ordine,

si		si	
$m = 0,$	$z = A,$	$m = 1,\ (x+y)$	$z = A,$
$m = -1,$	$z = B,$	$m = 2,\ (x+y)^3$	$z = B,$
$m = -2,$	$z = C,$	$m = 3,\ (x+y)^5$	$z = C,$
$m = -3,$	$z = D,$	$m = 4,\ (x+y)^7$	$z = D,$
$m = -4,$	$z = E,$	$m = 5,\ (x+y)^9$	$z = E,$
$m = -5,$	$z = F,$	$m = 6,\ (x+y)^{11}$	$z = F,$
$m = -6,$	$z = G,$	$m = 7,\ (x+y)^{13}$	$z = G,$
etc.		etc.	

Scholion.

329. Si pro i sumatur numerus negativus, expressio in infinitum excurrit. Sit enim $i = -k$, et ex formula prima erit $m = k$, ideoque

$$z = \mathfrak{A} - \frac{k}{2k}\mathfrak{B} + \frac{1}{2} \cdot \frac{k(k+1)}{2k(2k+1)}\mathfrak{C} - \frac{1}{6} \cdot \frac{k(k+1)(k+2)}{2k(2k+1)(2k+2)}\mathfrak{D} + \text{etc.}$$

Pro eodem autem casu $m = k$ altera forma ob $i = k - 1$ dat

$$(x+y)^{2k-1}z=\mathfrak{A}-\frac{(k-1)}{2k-2}\mathfrak{B}+\frac{1}{2}\cdot\frac{(k-1)(k-2)}{(2k-2)(2k-3)}\mathfrak{C}$$
$$-\frac{1}{6}\cdot\frac{(k-1)(k-2)(k-3)}{(2k-2)(2k-3)(2k-4)}\mathfrak{D}+\text{etc.}$$

quae autem formae non absolute aequales sunt censendae, sed in altera functiones $f:x$ et $F:y$ alias formas habebunt, ut nihilominus ambae aeque satisfaciant. Casu quidem $k=\frac{1}{2}$, ambae conveniunt perfecte: ponamus autem $k=0$, ut prior det

$$z=\mathfrak{A}=f:x+F:y,$$

at posterior praebet

$$\frac{z}{x+y}=\mathfrak{A}-\frac{1}{2}\mathfrak{B}+\frac{1}{6}\mathfrak{C}-\frac{1}{24}\mathfrak{D}+\frac{1}{120}\mathfrak{E}-\text{etc.}$$

Quarum consensus ut appareat, sit in hac posteriori

$$f:x=ax^3 \text{ et } F:y=by^2, \text{ erit}$$
$$\mathfrak{A}=ax^3+by^2,\quad \mathfrak{B}=(x+y)(3axx+2by),$$
$$\mathfrak{C}=(x+y)^2(6ax+2b),\quad \mathfrak{D}=(x+y)^3\,6a,$$

at reliquae partes evanescunt. Obtinebimus ergo ex posteriori

$$z=(x+y)(ax^3+by^2)-\tfrac{1}{2}(x+y)^2(3axx+2by)$$
$$+\tfrac{1}{3}(x+y)^3(3ax+b)-\tfrac{1}{4}(x+y)^4a,$$

quae evoluta praebet

$$\tfrac{1}{4}ax^4-ay^4+\tfrac{1}{3}bx^3+\tfrac{1}{3}by^3=z,$$

quae forma utique in priori $z=f:x+F:y$ continetur. Consensus ergo binarum illarum formarum generalium eo magis est notatu dignus.

Problema 53.

330. Invenire casus, quibus haec aequatio generalis

$$\left(\frac{\partial\partial z}{\partial y^2}\right)-QQ\left(\frac{\partial\partial z}{\partial x^2}\right)+R\left(\frac{\partial z}{\partial y}\right)+S\left(\frac{\partial z}{\partial x}\right)+Tz=0$$

ad formam praecedentem reduci, ideoque iisdem casibus integrari potest.

Solutio.

Introducendo binas novas variabiles t et u, ut sit quemadmodum reductio §. 319. adhibita, ubi $P = 0$ et $V = 0$, declarat

$$t = \int p\,(\partial x + Q\partial y) \text{ et } u = \int q\,(\partial x - Q\partial y),$$

si ponamus ad abbreviandum

$$M = S + QR + \left(\frac{\partial Q}{\partial y}\right) + Q\left(\frac{\partial Q}{\partial x}\right),$$

$$N = S - QR - \left(\frac{\partial Q}{\partial y}\right) + Q\left(\frac{\partial Q}{\partial x}\right),$$

prodibit haec aequatio

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right) - \frac{M}{4QQq}\left(\frac{\partial z}{\partial t}\right) - \frac{N}{4QQp}\left(\frac{\partial z}{\partial u}\right) - \frac{T}{4QQpq}z = 0,$$

quam ergo ad hanc formam revocari oportet

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial t}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial u}\right) + \frac{n}{(t+u)^2}z = 0,$$

cujus casus integrabilitatis ante designavimus; scilicet quoties fuerit $n = (m+i)(m-i-1)$, denotante i numerum integrum quemcunque positivum cyphra non exclusa. Ad hoc ergo necesse est ut fiat

$$M = \frac{-4mQQq}{t+u},\; N = \frac{-4mQQp}{t+u} \text{ et } T = \frac{-4nQQpq}{(t+u)^2}.$$

Quia autem hic integrabilitatis formularum t et u ratio haberi debet, sumamus $Q = \frac{\Phi' : y}{\pi' : x}$, sitque

$$p = a\pi' : x \text{ et } q = b\pi' : x,$$

eritque

$$t = a\pi : x + a\Phi : y \text{ et } u = b\pi : x - b\Phi : y.$$

Hinc si

$$M + N = 2S + 2Q\left(\frac{\partial Q}{\partial x}\right) = \frac{-4m(a+b)QQ\pi' : x}{t+u} \text{ et}$$

$$M - N = 2QR + 2\left(\frac{\partial Q}{\partial y}\right) = \frac{4m(a-b)QQ\pi' : x}{t+u},$$

ideoque

$$R = \frac{2m(a-b)Q\pi':x}{t+u} - \frac{1}{Q}\left(\frac{\partial Q}{\partial y}\right),$$
$$S = \frac{-2m(a+b)QQ\pi':x}{t+u} - Q\left(\frac{\partial Q}{\partial x}\right), \text{ et}$$
$$T = \frac{-4nabQQ\pi':x.\pi':x}{(t+u)^2} = \frac{-4nab\Phi':y.\Phi':y}{(t+u)^2},$$

ob $Q = \frac{\Phi':y}{\pi':x}$; unde est

$$\left(\frac{\partial Q}{\partial y}\right) = \frac{\Phi'':y}{\pi':x} \text{ et } \left(\frac{\partial Q}{\partial x}\right) = \frac{-\pi'':x.\Phi':y}{\pi':x.\pi':x} \text{ et}$$
$$t+u = (a+b)\,\pi:x + (a-b)\,\Phi:y.$$

Ideoque habebimus

$$R = \frac{2m(a-b)\Phi':y}{t+u} - \frac{\Phi'':y}{\Phi':y} \text{ et}$$
$$\frac{S}{QQ} = \frac{-2m(a+b)\pi':x}{t+u} + \frac{\pi'':x}{\pi':x}.$$

Quo aequatio fiat simplicior, duo casus praecipue sunt considerandi, alter ubi $b = a$, alter ubi $b = -a$. Priori est $t + u = 2a\pi:x$ aequatio nostra erit

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - \left(\frac{\Phi':y}{\pi':x}\right)^2\left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{\Phi'':y}{\Phi':y}\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\Phi':y}{\pi':x}\right)^2\left(\frac{\pi'':x}{\pi':x} - \frac{2m\pi':x}{\pi:x}\right)\left(\frac{\partial z}{\partial x}\right)$$
$$- \frac{n\Phi':y.\Phi':y}{\pi:x.\pi:x}\,z = 0.$$

Altero vero casu $b = -a$ fit $t + u = 2a\Phi:y$ et

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - \left(\frac{\Phi':y}{\pi':x}\right)^2\left(\frac{\partial\partial z}{\partial x^2}\right) + \left(\frac{2m\Phi':y}{\Phi:y} - \frac{\Phi'':y}{\Phi':y}\right)\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\Phi':y}{\pi':x}\right)^2 \cdot \frac{\pi'':x}{\pi':x}\left(\frac{\partial z}{\partial x}\right)$$
$$+ \frac{n\Phi':y.\Phi':y}{\Phi:y.\Phi:y}\,z = 0,$$

quae ambae aequationes integrationem admittunt casibus

$$n = (m+i)\,(m-i-1).$$

Corollarium 1.

831. Aequationes postremo inventae a se invicem non differunt, nisi quod binae variabiles x et y invicem permutantur, unde sufficit alterutram solam considerasse. Prior autem transformatur ponendo

$$t = \pi:x + \Phi:y \text{ et } u = \pi:x - \Phi:y,$$

posterior vero ponendo

$$t = \pi : x + \Phi : y \text{ et } u = \Phi : y - \pi : x.$$

Corollarium 2.

332. Hae aequationes etiam sequenti forma magis perspicua repraesentari possunt, prior quidem

$$\frac{1}{(\Phi' : y)^2} \left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{1}{(\pi' : x)^2} \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{\Phi'' : y}{(\Phi' : y)^3} \left(\frac{\partial z}{\partial y}\right)$$
$$+ \left(\frac{\pi'' : y}{(\pi' : x)^3} - \frac{2m}{\pi : x . \pi' : x}\right) \left(\frac{\partial z}{\partial x}\right) - \frac{n}{(\pi : x)^2} z = 0,$$

et posterior

$$\frac{1}{(\Phi' : y)^2} \left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{1}{(\pi' : x)^2} \left(\frac{\partial\partial z}{\partial x^2}\right) + \left(\frac{2m}{\Phi : y . \Phi' : y} - \frac{\Phi'' : y}{(\Phi' : y)^3}\right) \left(\frac{\partial z}{\partial y}\right)$$
$$+ \frac{\pi'' : x}{(\pi' : x)^3} \left(\frac{\partial z}{\partial x}\right) + \frac{n}{(\Phi : y)^2} z = 0.$$

Casus 1.

333. Ponamus $\pi' : x = a$, et $\Phi' : y = b$, erit $\pi : x = ax$ et $\Phi : y = by$ tum vero $\pi'' : x = 0$ et $\Phi'' : y = 0$; unde forma prior prodibit

$$\frac{1}{bb} \left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{1}{aa} \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2m}{aax} \left(\frac{\partial z}{\partial x}\right) - \frac{n}{aaxx} z = 0,$$

quae reducitur ad formam supra resolutam ponendo

$$t = ax + by \text{ et } u = ax - by.$$

Posterior vero forma est

$$\frac{1}{bb} \left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{1}{aa} \left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{2m}{bby} \left(\frac{\partial z}{\partial y}\right) + \frac{n}{bbyy} z = 0,$$

quae reducitur ad formam supra resolutam ponendo

$$t = ax + by \text{ et } u = by - ax,$$

utraque autem est integrabilis casu

$$n = (m + i)(m - i - 1);$$

Reductione enim ad variabiles t et u facta oritur haec aequatio

$$\left(\frac{\partial\partial z}{\partial t \partial u}\right) + \frac{m}{t+u} \left(\frac{\partial z}{\partial t}\right) + \frac{m}{t+u} \left(\frac{\partial z}{\partial u}\right) + \frac{n}{(t+u)^2} z = 0.$$

Corollarium 1.

334. Si sumatur $n = 0$, hae ambae aequationes

$$\frac{aa}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right) - \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2m}{x}\left(\frac{\partial z}{\partial x}\right) = 0, \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{bb}{aa}\left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{2m}{y}\left(\frac{\partial z}{\partial y}\right) = 0,$$

sunt integrabiles, quoties m fuerit numerus integer, ideoque $2m$ numerus par.

Corollarium 2.

335. En ergo aequationes ob simplicitatem notatu dignas, ex tribus tantum terminis constantes, quae infinitis casibus integrationem admittunt. Integrale autem quovis casu facile exhibetur ex §. 328, si modo ibi loco x et y scribatur t et u.

Casus 2.

336. Sit $\pi' : x = ax^{\mu}$ et $\Phi' : y = b$, erit

$$\pi : x = \frac{1}{\mu+1} ax^{\mu+1} \text{ et } \Phi : y = by,$$

tum vero

$$\pi'' : x = \mu a x^{\mu-1} \text{ et } \Phi'' : y = 0.$$

Unde forma prior provenit

$$\frac{1}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{1}{aax^{2\mu}}\left(\frac{\partial\partial x}{\partial x^2}\right) + \frac{\mu - 2m\mu - 2m}{aax^{2\mu+1}}\left(\frac{\partial z}{\partial x}\right) - \frac{n(\mu+1)^2}{aax^{2\mu+2}} z = 0,$$

quae reducitur ad formam supra resolutam ponendo

$$t = \frac{1}{\mu+1} ax^{\mu+1} + by \text{ et } u = \frac{1}{\mu+1} ax^{\mu+1} - by.$$

Posterior vero forma fit

$$\frac{1}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{1}{aax^{2\mu}}\left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{2m}{bby}\left(\frac{\partial z}{\partial y}\right) + \frac{\mu}{aax^{2\mu+1}}\left(\frac{\partial z}{\partial x}\right) + \frac{n}{bbyy} z = 0,$$

cujus reductio absolvitur ponendo

**

$$t = \frac{1}{\mu+1} a x^{\mu+1} + by \text{ et } u = by - \frac{1}{\mu+1} a x^{\mu+1}.$$

Haeque ambae aequationes integrationem admittunt, quoties fuerit $n = (m+i)\ (m-i-1)$.

Corollarium 1.

337. Ex priori forma casus maxime notabilis existit, si capiatur $m = \frac{\mu}{2\mu+2}$, et $n = 0$, tum enim erit

$$\frac{aa}{bb} x^{2\mu} \left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right),$$

quae est integrabilis, quoties $\frac{\mu}{2\mu+2}$ fuerit numerus integer m sive positivus sive negativus.

Corollarium 2.

338. Vel cum sit $\mu = \frac{-2m}{2m-1}$, haec aequatio

$$\frac{aa}{bb} x^{\frac{-4m}{2m-1}} \left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right), \text{ seu } \left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{4m}{2m-1}} \left(\frac{\partial\partial z}{\partial x^2}\right),$$

erit integrabilis, quoties m fuerit numerus integer sive positivus sive negativus, reductio autem fit ponendo

$$t = -(2m-1)\, a x^{\frac{-1}{2m-1}} + by \text{ et}$$

$$u = -(2m-1)\, a x^{\frac{-1}{2m-1}} - by.$$

Casus 3.

339. Sit $\pi' : x = a x^{\mu}$ et $\Phi' : y = b y^{\nu}$, erit

$$\pi : x = \frac{1}{\mu-1} a x^{\mu+1} \text{ et } \Phi : y = \frac{1}{\nu+1} b y^{\nu+1},$$

tum vero

$$\pi'' : x = \mu a x^{\mu-1} \text{ et } \Phi'' : y = \nu b y^{\nu-1}.$$

Hinc prior forma resultat

$$\frac{1}{bby^{2\nu}}\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{\partial\partial z}{\partial x^2}\right)-\frac{\nu}{bby^{2\nu+1}}\left(\frac{\partial z}{\partial y}\right)$$
$$+\frac{\mu-2m\mu-2m}{aax^{2\mu+1}}\left(\frac{\partial z}{\partial x}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+2}}z=0,$$

quae reducitur ponendo

$$t=\tfrac{1}{\mu+1}ax^{\mu+1}+\tfrac{1}{\nu+1}by^{\nu+1}\text{ et}$$
$$u=\tfrac{1}{\mu+1}ax^{\mu+1}-\tfrac{1}{\nu+1}by^{\nu+1}.$$

Posterior vero forma evadit

$$\frac{1}{bby^{2\nu}}\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{\partial\partial z}{\partial x^2}\right)+\frac{2m\nu+2m-\nu}{bby^{2\nu+1}}\left(\frac{\partial z}{\partial y}\right)$$
$$+\frac{\mu}{aax^{2\mu+1}}\left(\frac{\partial z}{\partial x}\right)+\frac{n(\nu+1)^2}{bby^{2\nu+2}}z=0.$$

cujus reductio fit hac substitutione

$$t=\tfrac{1}{\mu+1}ax^{\mu+1}+\tfrac{1}{\nu+1}by^{\nu+1}\text{ et}$$
$$u=\tfrac{-1}{\mu+1}ax^{\mu+1}+\tfrac{1}{\nu+1}by^{\nu+1}.$$

Vel cum hic tantum ratio inter a et b in computum ingrediatur, pro priori poni poterit

$$t=\tfrac{1}{2}x^{\mu+1}+\tfrac{(\mu+1)b}{2(\nu+1)a}y^{\nu+1}\text{ et}$$
$$u=\tfrac{1}{2}x^{\mu+1}-\tfrac{(\mu+1)b}{2(\nu+1)a}y^{\nu+1},$$

ut fiat $t+u=x^{\mu+1}$, quo expressio integralis fiat simplicior.

Corollarium 1.

340. Si ponatur in forma priori $\mu=\frac{-2m}{2m-1}$ minuetur ea uno termino, fietque

$$\frac{1}{bby^{2\nu}}\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{1}{aa}x^{\frac{4m}{2m-1}}\left(\frac{\partial\partial z}{\partial x^2}\right)-\frac{\nu}{bby^{2\nu+1}}\left(\frac{\partial z}{\partial y}\right)$$
$$-\frac{n}{(2m-1)^2 aa}x^{\frac{2}{2m-1}}z=0.$$

Statuatur $a=b$, et capiatur quoque $\nu=\frac{-2m}{2m-1}$, ut prodeat

$$y^{\frac{4m}{2m-1}}\left(\frac{\partial\partial z}{\partial y^2}\right)-x^{\frac{4m}{2m-1}}\left(\frac{\partial\partial z}{\partial x^2}\right)+\frac{2m}{2m-1}y^{\frac{2m+1}{2m-1}}\left(\frac{\partial z}{\partial y}\right)$$
$$-\frac{n}{(2m-1)^2}x^{\frac{2}{2m-1}}z=0.$$

Corollarium 2.

341. Sumatur porro in priori forma $\nu=\mu$, at fiat $\mu-2m\mu-2m=-\mu$, seu $m=\frac{\mu}{\mu+1}$, ut prodeat

$$\frac{1}{bby^{2\mu}}\left(\frac{\partial\partial z}{\partial y^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{\partial\partial z}{\partial x^2}\right)-\frac{\mu}{bby^{2\mu+1}}\left(\frac{\partial z}{\partial y}\right)$$
$$-\frac{\mu}{aax^{2\mu+1}}\left(\frac{\partial z}{\partial x}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+1}}z=0,$$

quae integrabilis existit, quoties fuerit

$$n=-\frac{[\mu+(\mu+1)i]\,[(\mu+1)i+1]}{(\mu+1)^2},\ \text{seu}$$
$$n=-\left(i+\frac{\mu}{\mu+1}\right)\left(i+\frac{1}{\mu+1}\right).$$

Scholion.

342. Largissima ergo hinc nobis suppeditatur copia aequationum satis concinnarum, quas ope methodi hic traditae integrare licet. Atque hic imprimis duo casus conspiciuntur, quorum alter

$$\left(\frac{\partial\partial z}{\partial y^2}\right)=\frac{bb}{aa}x^{\frac{4m}{2m-1}}\left(\frac{\partial\partial z}{\partial x^2}\right)$$

pro motu cordarum inaequali crassitie praeditarum determinando est inventus, alter autem hac aequatione

$$\frac{aa}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right) - \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2m}{x}\left(\frac{\partial z}{\partial x}\right) = 0$$

contentus ideo est memorabilis, quod in analysi pro soni propagatione instituta ad talem formam pervenitur. Hae igitur binae aequationes prae caeteris merentur, ut pro casibus integrabilitatis integralia exhibeamus.

Problema 54.

343. Proposita aequatione differentiali

$$\frac{aa}{bb}\left(\frac{\partial\partial z}{\partial y^2}\right) - \left(\frac{\partial\partial z}{\partial x^2}\right) - \frac{2m}{x}\left(\frac{\partial z}{\partial x}\right) = 0,$$

casibus quibus m est numerus integer sive positivus sive negativus, ejus integrale completum exhibere.

Solutio.

Facta substitutione $t = \frac{1}{2}x + \frac{b}{2a}y$ et $u = \frac{1}{2}x - \frac{b}{2a}y$, aequatio nostra hanc induit formam

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial t}\right) + \frac{m}{t+u}\left(\frac{\partial z}{\partial u}\right) = 0.$$

Cum igitur sit $t + u = x$, si ponamus

$$\mathfrak{A} = f:\frac{ax+by}{2a} + F:\frac{ax-by}{2a},$$

$$\mathfrak{B} = x\left(f':\frac{ax+by}{2a} + F':\frac{ax-by}{2a}\right),$$

$$\mathfrak{C} = x^2\left(f'':\frac{ax+by}{2a} + F'':\frac{ax-by}{2a}\right),$$

$$\mathfrak{D} = x^3\left(f''':\frac{ax+by}{2a} + F''':\frac{ax-by}{2a}\right),$$

$$\mathfrak{E} = x^4\left(f^{IV}:\frac{ax+by}{2a} + F^{IV}:\frac{ax-by}{2a}\right),$$

$$\mathfrak{F} = x^5\left(f^{V}:\frac{ax+by}{2a} + F^{V}:\frac{ax-by}{2a}\right), \text{ etc.}$$

casus integrabiles ita se habebunt, primo negativi

si $m = 0$; $z = \mathfrak{A}$,

si $m = -1$; $z = \mathfrak{A} - \frac{1}{2}\mathfrak{B}$,

si $m = -2$; $z = \mathfrak{A} - \frac{2}{4}\mathfrak{B} + \frac{1}{4.3}\mathfrak{C}$,

si $m = -3$; $z = \mathfrak{A} - \frac{3}{6}\mathfrak{B} + \frac{1}{6.5}\mathfrak{C} - \frac{1}{6.5.4}\mathfrak{D}$,

si $m = -4$; $z = \mathfrak{A} - \frac{4}{8}\mathfrak{B} + \frac{6}{8.7}\mathfrak{C} - \frac{4}{8.7.6}\mathfrak{D} + \frac{1}{8.7.6.5}\mathfrak{E}$,

si $m = -5$; $z = \mathfrak{A} - \frac{5}{10}\mathfrak{B} + \frac{10}{10.9}\mathfrak{C} - \frac{10}{10.9.8}\mathfrak{D} + \frac{5}{10.9.8.7}\mathfrak{E} - \frac{1}{10.9.8.7.6}\mathfrak{F}$, etc.

Tum vero pro valoribus positivis ipsius m

si $m = 1$; $xz = \mathfrak{A}$,

si $m = 2$; $x^3 z = \mathfrak{A} - \frac{1}{2}\mathfrak{B}$,

si $m = 3$; $x^5 z = \mathfrak{A} - \frac{2}{4}\mathfrak{B} + \frac{1}{4.3}\mathfrak{C}$,

si $m = 4$; $x^7 z = \mathfrak{A} - \frac{3}{6}\mathfrak{B} + \frac{3}{6.5}\mathfrak{C} - \frac{1}{6.5.4}\mathfrak{D}$,

si $m = 5$; $x^9 z = \mathfrak{A} - \frac{4}{8}\mathfrak{B} + \frac{6}{8.7}\mathfrak{C} - \frac{1}{8.7.6}\mathfrak{D} + \frac{1}{8.7.6.5}\mathfrak{E}$,

si $m = 6$; $x^{11} z = \mathfrak{A} - \frac{5}{10}\mathfrak{B} + \frac{10}{10.9}\mathfrak{C} - \frac{10}{10.9.8}\mathfrak{D} + \frac{5}{10.9.8.7}\mathfrak{E} - \frac{1}{10.9.8.7.6}\mathfrak{F}$, etc.

Cui ergo expressioni casu $m = -i$ aequatur valor z, eidem aequatur casu $m = i + 1$ valor ipsius $x^{2i+1} z$.

Scholion.

344. Valores ipsarum t et u ita hic assumsi, ut fieret $t + u = x$, atque eosdem valores quoque in functionibus adhiberi oportet. Etsi enim $f : \frac{ax+by}{2a}$ etiam est functio ipsius $ax + by$, tamen functiones per differentiationem inde derivatae discrepant. Namque si ponamus

$$f : \frac{ax+by}{2a} = \phi : (ax + by),$$

erit differentiando

$$\frac{(a\partial x + b\partial y)}{2a} f' : (\frac{ax+by}{2a}) = (a\partial x + b\partial y)\, \Phi' : (ax + by),$$

unde erit

$$f' : \frac{ax+by}{2a} = 2a\Phi' : (ax + by),$$

neque ergo hae functiones differentiales sunt aequales etiamsi principales assumtae sint aequales, simili modo erit

$$f'' : \frac{ax+by}{2a} = 4aa\Phi'' : (ax + by), \text{ et}$$

$$f''' : \frac{ax+by}{2a} = 8a^3\Phi''' : (ax + by), \text{ etc.}$$

et ita porro.

Problema 55.

345. Proposita aequatione differentiali

$$(\frac{\partial\partial z}{\partial y^2}) = \frac{bb}{aa}\, x^{\frac{-4m}{2m-1}}\, (\frac{\partial\partial z}{\partial x^2}),$$

casibus quibus m est numerus integer sive positivus sive negativus, integrale completum exhibere.

Solutio.

Introductis novis variabilibus t et u, ita ut sit

$$t = \tfrac{1}{2}\, x^{\frac{-1}{2m-1}} - \frac{b}{2(2m-1)a}\, y \text{ et } u = \tfrac{1}{2}\, x^{\frac{-1}{2m-1}} + \frac{b}{2(2m-1)a}\, y,$$

aequatio nostra hanc induit formam

$$(\frac{\partial\partial z}{\partial t\partial u}) + \frac{m}{t+u}\,(\frac{\partial z}{\partial t}) + \frac{m}{t+u}\,(\frac{\partial z}{\partial u}) = 0,$$

ubi est

$$t + u = x^{\frac{-1}{2m-1}}.$$

Posito igitur

$$\mathfrak{A} = f : t + F : u,\ \mathfrak{B} = x^{\frac{-1}{2m-1}}(f' : t + F' : u),$$

$$\mathfrak{C} = x^{\frac{-2}{2m-1}} (f'':t + F'':u), \quad \mathfrak{D} = x^{\frac{-3}{2m-1}} (f''':t + F''':u),$$
$$\mathfrak{E} = x^{\frac{-4}{2m-1}} (f^{IV}:t + F^{IV}:u), \quad \mathfrak{F} = x^{\frac{-5}{2m-1}} (f^{V}:t + F^{V}:u), \text{ etc.}$$

percurramus primo casus, quibus m a cyphra per numeros negativos decrescit.

I. Si $m = 0$, aequationis
$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} \left(\frac{\partial\partial z}{\partial x^2}\right)$ integrale
$z = f:(\frac{1}{2}x + \frac{b}{2a}y) + F:(\frac{1}{2}x - \frac{b}{2a}y)$.

II. Si $m = -1$, ob

$$t = \tfrac{1}{2}x^{\frac{1}{3}} + \frac{b}{6a}y \text{ et } u = \tfrac{1}{2}x^{\frac{1}{3}} - \frac{b}{6a}y,$$

erit aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{4}{3}} \left(\frac{\partial\partial z}{\partial x^2}\right) \text{ integrale}$$

$$z = f:t + F:u - \tfrac{1}{2}x^{\frac{1}{3}} (f':t + F':u).$$

III. Si $m = -2$, ob

$$t = \tfrac{1}{2}x^{\frac{1}{5}} + \frac{b}{10a}y \text{ et } u = \tfrac{1}{2}x^{\frac{1}{5}} - \frac{b}{10a}y,$$

erit aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{8}{5}} \left(\frac{\partial\partial z}{\partial x^2}\right) \text{ integrale}$$

$$z = f:t + F:u - \tfrac{2}{4}x^{\frac{1}{5}} (f':t + F':u) + \frac{1}{4.3} x^{\frac{2}{5}} (f'':t + F'':u).$$

IV. Si $m = -3$, ob

$$t = \tfrac{1}{2}x^{\frac{1}{7}} + \frac{b}{14a}y \text{ et } u = \tfrac{1}{2}x^{\frac{1}{7}} - \frac{b}{14a}y,$$

erit aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa}\, x^{\frac{12}{7}} \left(\frac{\partial\partial z}{\partial x^2}\right) \text{ integrale}$$

$$z = \mathrm{f}:t + \mathrm{F}:u - \tfrac{3}{6}\, x^{\frac{1}{7}} (\mathrm{f}':t + \mathrm{F}':u) + \tfrac{3}{6.5}\, x^{\frac{2}{7}} (\mathrm{f}'':t + \mathrm{F}'':u)$$

$$- \tfrac{1}{6.5.4}\, x^{\frac{3}{7}} (\mathrm{f}''':t + \mathrm{F}''':u).$$

V. Si $m = -4$, ob

$$t = \tfrac{1}{2}\, x^{\frac{1}{9}} + \tfrac{b}{18a}\, y \text{ et } u = \tfrac{1}{2}\, x^{\frac{1}{9}} - \tfrac{b}{18a}\, y,$$

erit aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa}\, x^{\frac{16}{9}} \left(\frac{\partial\partial z}{\partial x^2}\right) \text{ integrale}$$

$$z = \mathrm{f}:t + \mathrm{F}:u - \tfrac{4}{8}\, x^{\frac{1}{9}} (\mathrm{f}':t + \mathrm{F}':u) + \tfrac{6}{8.7}\, x^{\frac{2}{9}} (\mathrm{f}'':t + \mathrm{F}'':u)$$

$$- \tfrac{4}{8.7.6}\, x^{\frac{3}{9}} (\mathrm{f}''':t + \mathrm{F}''':u) + \tfrac{1}{8.7.6.5}\, x^{\frac{4}{9}} (\mathrm{f}^{\mathrm{IV}}:t + \mathrm{F}^{\mathrm{IV}}:u),$$

et ita porro.

Pro altero vero casu ubi m habet valores positivos, integralia sequenti modo exprimentur

I. Si sit $m = 1$, seu $\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa}\, x^4 \left(\frac{\partial\partial z}{\partial x^2}\right)$, ob $t = \frac{1}{2}\, x^{-1} - \frac{b}{2a}\, y$ et $u = \frac{1}{2}\, x^{-1} + \frac{b}{2a}\, y$,
erit integrale

$$x^{-1} z = \mathrm{f}:t + \mathrm{F}:u, \text{ seu } z = x(\mathrm{f}:t + \mathrm{F}:u).$$

II. Si sit $m = 2$, seu $\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa}\, x^{\frac{8}{3}} \left(\frac{\partial\partial z}{\partial x^2}\right)$,
ob $t = \frac{1}{2}\, x^{-\frac{1}{3}} - \frac{b}{6a}\, y$ et $u = \frac{1}{2}\, x^{-\frac{1}{3}} + \frac{b}{6a}\, y$,
erit integrale

$$z = x(\mathrm{f}:t + \mathrm{F}:u) - \tfrac{1}{2}\, x^{\frac{2}{3}} (\mathrm{f}':t + \mathrm{F}':u).$$

III. Si sit $m = 3$, seu $\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{12}{5}} \left(\frac{\partial\partial z}{\partial x^2}\right)$,

ob $t = \frac{1}{2} x^{-\frac{1}{5}} - \frac{b}{10a} y$ et $u = \frac{1}{2} x^{-\frac{1}{5}} + \frac{b}{10a} y$,

erit integrale

$$z = x(\mathrm{f}:t + \mathrm{F}:u) - \frac{2}{4} x^{\frac{4}{5}} (\mathrm{f}':t + \mathrm{F}':u) + \frac{1}{4.3} x^{\frac{3}{5}} (\mathrm{f}'':t + \mathrm{F}'':u).$$

IV. Si sit $m = 4$, seu $\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{16}{7}} \left(\frac{\partial\partial z}{\partial x^2}\right)$,

ob $t = \frac{1}{2} x^{-\frac{1}{7}} - \frac{b}{14a} y$ et $u = \frac{1}{2} x^{-\frac{1}{7}} + \frac{b}{14a} y$,

erit integrale

$$z = x(\mathrm{f}:t + \mathrm{F}:u) - \frac{3}{6} x^{\frac{6}{7}} (\mathrm{f}':t + \mathrm{F}':u) + \frac{3}{6.5} x^{\frac{5}{7}} (\mathrm{f}'':t + \mathrm{F}'':u)$$
$$- \frac{1}{6.5.4} x^{\frac{4}{7}} (\mathrm{f}''':t + \mathrm{F}''':u).$$

V. Si sit $m = 5$, seu $\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa} x^{\frac{20}{9}} \left(\frac{\partial\partial z}{\partial x^2}\right)$,

ob $t = \frac{1}{2} x^{-\frac{1}{9}} - \frac{b}{18a} y$ et $u = \frac{1}{2} x^{-\frac{1}{9}} + \frac{b}{18a} y$,

erit integrale

$$z = x(\mathrm{f}:t + \mathrm{F}:u) - \frac{4}{8} x^{\frac{8}{9}} (\mathrm{f}':t + \mathrm{F}':u) + \frac{6}{8.7} x^{\frac{7}{9}} (\mathrm{f}'':t + \mathrm{F}'':u)$$
$$- \frac{4}{8.7.6} x^{\frac{6}{9}} (\mathrm{f}''':t + \mathrm{F}''':u) + \frac{1}{8.7.6.5} x^{\frac{5}{9}} (\mathrm{f}^{\mathrm{IV}}:t + \mathrm{F}^{\mathrm{IV}}:u). \text{ etc.}$$

unde lex, qua has expressiones ulterius continuare licet, per se est manifesta.

Scholion 1.

346. Casus isti integrabilitatis congruunt cum iis, qui in aequatione *Riccatiana* dicta deprehenduntur, novimus scilicet aequa-

tionem hanc

$$\partial y + yy\partial x = ax^{\frac{-4m}{2m-1}}\partial x$$

integrari posse quoties m est numerus integer sive positivus sive negativus. Haec autem aequatio haud levi vinculo cum nostra forma est connexa, quod ita ostendi potest. Proposita forma generali

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = X\left(\frac{\partial\partial z}{\partial x^2}\right),$$

pro integralibus particularibus inveniendis statuatur $z = e^{\alpha y} v$, ut v sit functio ipsius x tantum, erit

$$\left(\frac{\partial z}{\partial x}\right) = e^{\alpha y}\cdot\frac{\partial v}{\partial x} \text{ et } \left(\frac{\partial\partial z}{\partial x^2}\right) = e^{\alpha y}\cdot\frac{\partial\partial v}{\partial x^2};$$

tum vero $\left(\frac{\partial\partial z}{\partial y^2}\right) = \alpha\alpha e^{\alpha y} v$; unde prodit haec aequatio $\alpha\alpha v = \frac{X\partial\partial v}{\partial x^2}$; in qua si porro statuatur $v = e^{\int p\partial x}$, oritur

$$\frac{\alpha\alpha\partial x}{X} = \partial p + pp\partial x,$$

ac si $X = Ax^{\frac{4m}{2m-1}}$, ut in nostro casu, haec aequatio sit

$$\partial p + pp\partial x = ax^{\frac{-4m}{2m-1}}\partial x.$$

Haud temere igitur evenire putandum est, quod utraque aequatio iisdem casibus integrationem admittat. Interim tamen notatu dignum occurit, quod casus $m = \infty$, qui in forma Riccatiana fit facillimus, idem in nostra aequatione neutiquam integrationem admittat. Habetur quippe haec aequatio

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{bb}{aa}\, xx \left(\frac{\partial\partial z}{\partial x^2}\right),$$

cujus reductio modo supra §. 330. adhibito non succedit. Nam ob

$$Q = \frac{bx}{a},\ R = 0,\ S = 0 \text{ et } T = 0,$$

pro novis variabilibus ponitur

$$t = \int p\left(\partial x + \frac{bx\partial y}{a}\right) \text{ et } u = \int q\left(\partial x - \frac{bx\partial y}{a}\right);$$

unde ob $M = \frac{bbx}{aa} = N$, oritur haec aequatio

$$\left(\frac{\partial\partial z}{\partial t \partial u}\right) - \frac{1}{4qx}\left(\frac{\partial z}{\partial t}\right) - \frac{1}{4px}\left(\frac{\partial z}{\partial u}\right) = 0,$$

quae sumendo

$$p = \frac{1}{x} \text{ et } q = \frac{1}{x},$$

ut sit

$$t = lx + \frac{by}{a} \text{ et } u = lx - \frac{by}{a},$$

transit in

$$\left(\frac{\partial\partial z}{\partial t \partial u}\right) - \frac{1}{4}\left(\frac{\partial z}{\partial t}\right) - \frac{1}{4}\left(\frac{\partial z}{\partial u}\right) = 0,$$

cujus integratio haud perspicitur.

Scholion 2.

347. Aequationis autem $\left(\frac{\partial\partial z}{\partial y^2}\right) = xx\left(\frac{\partial\partial z}{\partial x^2}\right)$ integralia particularia infinita exhibere licet, in hac forma $z = Ax^\lambda e^{\mu y}$ contenta. Cum enim hinc sit

$$\left(\frac{\partial z}{\partial y}\right) = \mu A x^\lambda e^{\mu y} \text{ et } \left(\frac{\partial z}{\partial x}\right) = \lambda A x^{\lambda-1} e^{\mu y}, \text{ erit}$$

$$\mu\mu A x^\lambda e^{\mu y} = \lambda(\lambda-1) A x^\lambda e^{\mu y}, \text{ ideoque}$$

$\mu = \sqrt{\lambda(\lambda-1)}$, unde ex quovis numero pro λ assumto bini valores pro μ oriuntur, ita ut habeatur

$$z = Ax^\lambda e^{y\sqrt{\lambda(\lambda-1)}} + Bx^\lambda e^{-y\sqrt{\lambda(\lambda-1)}},$$

et hujusmodi membrorum numerus variando λ in infinitum multiplicari potest. Interim tamen singula haec membra adhuc generaliora reddi possunt. Posito enim $z = x^\lambda e^{\mu y} v$, videamus an v necessario constans esse debeat: hinc autem fit

$$\left(\frac{\partial z}{\partial y}\right) = \mu x^\lambda e^{\mu y} v + x^\lambda e^{\mu y}\left(\frac{\partial v}{\partial y}\right) \text{ et}$$

$$\left(\frac{\partial z}{\partial x}\right) = \lambda x^{\lambda-1} e^{\mu y} v + x^\lambda e^{\mu y}\left(\frac{\partial v}{\partial x}\right),$$

ideoque nostra aequatio praebet per $x^\lambda e^{\mu y}$ divisa

$$\mu\mu v + 2\mu \left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial\partial v}{\partial y^2}\right)$$
$$= \lambda(\lambda - 1) v + 2\lambda x \left(\frac{\partial v}{\partial x}\right) + xx \left(\frac{\partial\partial v}{\partial x^2}\right).$$

Statuatur ut ante $\mu\mu = \lambda(\lambda - 1)$, sitque $v = \alpha lx + \beta y$, erit

$$2\beta\mu = 2\alpha\lambda - \alpha, \text{ seu } \frac{\alpha}{\beta} = \frac{2\mu}{2\lambda - 1} = \frac{2\sqrt{\lambda(\lambda-1)}}{2\lambda - 1};$$

unde cujusque membri ex numero λ nati forma erit

$$z = x^\lambda \left\{ \begin{array}{l} e^{y\sqrt{\lambda(\lambda-1)}} \left(\mathfrak{A} + \frac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{A}} lx + \frac{2\lambda - 1}{\mathfrak{A}} y\right) \\ + e^{-y\sqrt{\lambda(\lambda-1)}} \left(\mathrm{B} - \frac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{B}} lx + \frac{2\lambda-1}{\mathfrak{B}} y\right) \end{array} \right\}.$$

Quomodocunque igitur non solum exponens λ sed etiam quantitates A, $\mathfrak{A}$, B, $\mathfrak{B}$ varientur, infinita hujusmodi membra formari possunt, quae omnia junctim sumta valorem completum functionis z praebere sunt censenda. Quin etiam pro λ imaginaria assumi possunt, posito enim

$$\lambda = a + b\sqrt{-1} \text{ fit } \mu = p + q\sqrt{-1},$$

existente

$$pp - qq = aa - a - bb \text{ et}$$
$$pp + qq = \sqrt{(aa + bb)(aa - 2a + 1 + bb)},$$

tum vero est

$$x^\lambda = a^a (\cos.\, blx + \sqrt{-1}\,.\,\sin.\, blx) \text{ et}$$
$$e^{\mu y} = e^{py} (\cos.\, qy + \sqrt{-1}\,.\,\sin.\, qy),$$

unde colligitur forma realis

$$z = x^a e^{py} \left\{ \begin{array}{l} \mathrm{A} \cos.(blx + qy) + \mathrm{B}\,[2plx + (2a-1)y] \cos.(blx + qy) \\ \qquad - \mathrm{B}(2qlx + 2by) \sin.(blx + qy) \\ \mathfrak{A} \sin.(blx + qy) + \mathfrak{B}\,[2plx + (2a-1)y] \sin.(blx + qy) \\ \qquad + \mathfrak{B}(2qlx + 2by) \cos.(blx + qy) \end{array} \right\},$$

ubi quantitates a et b pro lubitu assumere licet, unde simul p et q

definiuntur. Quodsi hic litteras b et q ut datas spectemus, binae reliquae a et p ex iis ita determinentur, ut sit

$$2a - 1 = q\surd\left(\frac{1}{qq - bb} - 4\right) \text{ et } p = \frac{b}{2}\surd\left(\frac{1}{qq - bb} - 4\right),$$

hic ergo necesse est sit $qq > bb$ et $qq < bb + \frac{1}{4}$, seu qq inter hos arctos limites bb et $bb + \frac{1}{4}$ contineri debet; statuatur $q = c$ et $\surd\left(\frac{1}{qq - bb} - 4\right) = 2f$, ut sit

$$\frac{1}{qq - bb} = 4(1 + ff), \text{ seu } cc - bb = \frac{1}{4(1 + ff)},$$

atque $2a - 1 = 2cf$ et $p = bf$,

ex quo forma integralium particularium erit

$$z = x^{cf + \frac{1}{2}} e^{bfy} \left\{ \begin{array}{l} \text{A}\cos(blx + cy) + 2\text{B}f(blx + cy)\cos.(blx + cy) \\ \qquad - 2\text{B}(clx + by)\sin.(blx + cy) \\ \mathfrak{A}\sin.(blx + cy) + 2\mathfrak{B}f(blx + cy)\sin.(blx + cy) \\ \qquad + 2\mathfrak{B}(clx + by)\cos.(blx + cy) \end{array} \right\},$$

quae posito brevitatis gratia angulo $blx + cy = \Phi$ transformatur in hanc

$$z = x^{cf + \frac{1}{2}} e^{bfy} \left\{ \begin{array}{l} \text{A}\cos.(\Phi + \alpha) + \text{B}f(blx + cy)\sin.(\Phi + \beta) \\ \qquad + \text{B}(clx + by)\cos.(\Phi + \beta) \end{array} \right\};$$

ubi quantitates b, c, A, B, α, β ab arbitrio nostro pendent.

Scholion 3.

348. Resolutio ergo aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = xx\left(\frac{\partial\partial z}{\partial x^2}\right),$$

ita institui potest, ut fingatur

$$z = x^\lambda e^{\mu y}(mlx + ny),$$

unde fit

$$\left(\frac{\partial z}{\partial x}\right) = \lambda x^{\lambda - 1} e^{\mu y}(mlx + ny) + mx^{\lambda - 1} e^{\mu y} \text{ et}$$

$$\left(\frac{\partial z}{\partial y}\right) = \mu x^\lambda e^{\mu y}(mlx + ny) + nx^\lambda e^{\mu y},$$

hincque ulterius differentiando

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = x^{\lambda-2} e^{\mu y} [m(2\lambda - 1) + \lambda(\lambda - 1) mlx + \lambda(\lambda - 1) ny] \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = x^{\lambda} e^{\mu y} (2\mu n + \mu\mu mlx + \mu\mu ny).$$

Ex quo colligitur primo $\mu = \sqrt{\lambda(\lambda - 1)}$, deinde

$$2n\sqrt{\lambda(\lambda - 1)} = m(2\lambda - 1),$$

ut sit

$$\frac{m}{n} = \frac{2\sqrt{\lambda(\lambda - 1)}}{2\lambda - 1},$$

sicque eadem prodit integratio quam modo ante dedimus.