CAPUT III.

SI DUAE VEL OMNES FORMULAE SECUNDI GRADUS PER RELIQUAS QUANTITATES DETERMINANTUR.

Problema 43.

296.

Si z ejusmodi debeat esse functio ipsarum x et y, ut fiat

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = aa \left(\frac{\partial\partial z}{\partial x^2}\right),$$

indolem functionis z determinare.

Solutio.

Introducantur binae novae variabiles t et u, ut sit $t = \alpha x + \beta y$ et $u = \gamma x + \delta y$, atque ex §. 231. omnes formulae differentiales sequentes mutationes subibunt:

$$\left(\frac{\partial z}{\partial x}\right) = \alpha \left(\frac{\partial z}{\partial t}\right) = \gamma \left(\frac{\partial z}{\partial u}\right); \quad \left(\frac{\partial z}{\partial y}\right) = \beta \left(\frac{\partial z}{\partial t}\right) + \delta \left(\frac{\partial z}{\partial u}\right),$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \alpha\alpha \left(\frac{\partial\partial z}{\partial t^2}\right) + 2\,\alpha\gamma \left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \gamma\gamma \left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) = \alpha\beta \left(\frac{\partial\partial z}{\partial t^2}\right) + (\alpha\delta + \beta\gamma) \left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \gamma\delta \left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \beta\beta \left(\frac{\partial\partial z}{\partial t^2}\right) + 2\,\beta\delta \left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \delta\delta \left(\frac{\partial\partial z}{\partial u^2}\right),$$

unde nostra aequatio transibit in hanc:

$$(\beta\beta - \alpha\alpha\,aa) \left(\frac{\partial\partial z}{\partial t^2}\right) + 2\,(\beta\delta - \alpha\gamma\,aa) \left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$$
$$+ (\delta\delta - \gamma\gamma\,aa) \left(\frac{\partial\partial z}{\partial u^2}\right) = 0.$$

Ponatur ergo

$$\beta\beta = \alpha\alpha\,aa \text{ et } \delta\delta = \gamma\gamma\,aa, \text{ seu}$$
$$\alpha = 1,\ \gamma = 1,\ \beta = a \text{ et } \delta = -a,$$

ut binae formulae extremae evanescant, quod fit ponendo

$$t = x + ay \text{ et } u = x - ay,$$

eritque

$$-2(aa + aa)\left(\frac{\partial\partial z}{\partial t\partial u}\right) = 0, \text{ seu } \left(\frac{\partial\partial z}{\partial t\partial u}\right) = 0,$$

unde per §. 269. colligitur integrale completum

$$z = f : t + F : u,$$

ac pro t et u restitutis valoribus

$$z = f : (x + ay) + F : (x - ay),$$

quae forma manifesto satisfacit, cum sit

$$\left(\frac{\partial z}{\partial x}\right) = f' : (x + ay) + F' : (x - ay),$$

$$\left(\frac{\partial z}{\partial y}\right) = af' : (x + ay) - aF' : (x - ay),$$

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = f'' : (x + ay) + F'' : (x - ay),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = aaf'' : (x + ay) + aaF'' : (x - ay).$$

Corollarium 1.

297. Valor igitur ipsius z aequatur aggregato duarum functionum arbitrariarum, alterius ipsius $x + ay$, alterius ipsius $x - ay$, atque ambae hae functiones ita ad arbitrium assumi possunt, ut etiam functiones discontinuas earum loco capere liceat.

Corollarium 2.

298. Pro lubitu ergo binae curvae quaecunque etiam libero manus tractu descriptae ad hunc usum adhiberi possunt. Scilicet si in una curva abscissa capiatur $= x + ay$, in altera vero abscissa $= x - ay$, summa applicatarum semper valorem idoneum pro functione z suppeditabit.

Scholion 1.

299. Hoc fere primum est problema, quod in hoc novo calculi genere solvendum occurrit; perduxerat autem solutio generalis problematis de cordis vibrantibus ad hanc ipsam aequationem, quam hic tractavimus. Celeb. *Alembertus*, qui hoc problema primus felici successu est aggressus, methodo singulari aequationem integravit; scilicet cum esse oporteat $\left(\frac{\partial\partial z}{\partial y^2}\right) = a^2 \left(\frac{\partial\partial z}{\partial x^2}\right)$, posito $\partial z = p\partial x + q\partial y$, indeque

$$\partial p = r\partial x + s\partial y \text{ et } \partial q = s\partial x + t\partial y,$$

illa aequatio postulat ut sit $t = aar$. Consideratis porro istis aequationibus:

$$\left.\begin{array}{l}\partial p = r\partial x + s\partial y \\ \partial q = s\partial x + aar\partial y\end{array}\right| \begin{array}{l}\text{elicitur combinando} \\ a\partial p + \partial q = ar(\partial x + a\partial y) + s(a\partial y + \partial x),\end{array}$$

seu $a\partial p + \partial q = (ar + s)(\partial x + a\partial y)$,

unde patet $ar + s$ functioni ipsius $x + ay$ aequari debere, ex quo etiam $ap + q$ tali functioni aequatur. Atque quia a aeque negative ac positive accipi potest, habentur duae hujusmodi aequationes:

$$ap + q = 2af' : (x + ay) \text{ et } q - ap = 2aF' : (x - ay),$$

unde colligitur

$$q = af' : (x + ay) + aF' : (x - ay), \text{ et}$$

$$p = f' : (x + ay) - F' : (x - ay),$$

hincque aequatio $\partial z = p\partial x + q\partial y$ sponte integratur, fitque

$$z = f : (x + ay) - F : (x - ay).$$

Hoc modo sagacissimus Vir integrale completum est adeptus, sed non animadvertit, loco functionum harum introductarum, non solum omnis generis functiones continuas, sed etiam omni continuitatis lege destitutas, accipi licere.

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Scholion 2.

300. Cum plurimum intersit, in hoc novo calculi genere quam plurimas methodos persequi. ab aliis solutio nostrae aequationis ita est tentata, ut ponerent $\left(\frac{\partial z}{\partial y}\right) = k\left(\frac{\partial z}{\partial x}\right)$, unde fit primo $\left(\frac{\partial\partial z}{\partial x\partial y}\right) = k\left(\frac{\partial\partial z}{\partial x^2}\right)$, tum vero $\left(\frac{\partial\partial z}{\partial y^2}\right) = k\left(\frac{\partial\partial z}{\partial x\partial y}\right)$, ex quo colligitur $\left(\frac{\partial\partial z}{\partial y^2}\right) = kk\left(\frac{\partial\partial z}{\partial x^2}\right)$. Evidens ergo est pro nostro casu capi debere $kk = aa$, seu $k = \pm a$. Sit ergo $k = a$, et ob $\left(\frac{\partial z}{\partial y}\right) = a\left(\frac{\partial z}{\partial x}\right)$, fiet

$$\partial z = \partial x\left(\frac{\partial z}{\partial x}\right) + \partial y\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial z}{\partial x}\right)(\partial x + a\partial y),$$

hincque manifestum est fore $z = \mathrm{f}:(x + ay)$, et ob a ambiguum, quoniam bini valores seorsim satisfacientes etiam juncti satisfacient, concluditur ipsa solutio inventa. Hoc etiam modo negotium confici potest: Statuatur

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = aa\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial v}{\partial x\partial y}\right), \text{ eritque}$$

$$\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial v}{\partial x}\right) \text{ et } aa\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial v}{\partial y}\right).$$

Inventis nunc formulis primi gradus $\left(\frac{\partial v}{\partial x}\right)$ et $\left(\frac{\partial v}{\partial y}\right)$, ob

$$\partial v = \partial x\left(\frac{\partial v}{\partial x}\right) + \partial y\left(\frac{\partial v}{\partial y}\right),$$

habebimus has aequationes:

$$\partial z = \partial x\left(\frac{\partial z}{\partial x}\right) + \partial y\left(\frac{\partial z}{\partial y}\right) \text{ et}$$

$$\partial v = \partial x\left(\frac{\partial z}{\partial y}\right) + aa\partial y\left(\frac{\partial z}{\partial x}\right),$$

ex quarum combinatione colligimus

$$\partial v + a\partial z = (\partial x + a\partial y)\left[\left(\frac{\partial z}{\partial y}\right) + a\left(\frac{\partial z}{\partial x}\right)\right],$$

hincque

$$v + az = \mathrm{f}:(x + ay) \text{ et } v - az = \mathrm{F}:(x - ay),$$

sicque pro z eadem forma exsurgit. Methodus vero, quam in solutione sum secutus, ad naturam rei magis videtur accommodata, cum etiam in aliis problematibus magis complicatis insignem utilitatem afferat.

Scholion 3.

301. Nostra autem solutio hoc habet incommodi, quod pro hac aequatione $\left(\frac{\partial\partial z}{\partial y^2}\right) + aa\left(\frac{\partial\partial z}{\partial x^2}\right) = 0$, ad expressionem imaginariam deducit, scilicet

$$z = f:(x + ay\sqrt{-1}) + F:(x - ay\sqrt{-1}).$$

Quoties autem functiones f et F sunt continuae, cujuscunque demum fuerint indolis, semper earum valores ad hanc formam $P \pm Q\sqrt{-1}$ reduci possunt, unde sequens forma, ex illa facile deducenda, semper valorem realem exhibebit:

$$z = \tfrac{1}{2}f:(x + ay\sqrt{-1}) + \tfrac{1}{2}f:(x - ay\sqrt{-1}) + \frac{1}{2\sqrt{-1}}F:(x + ay\sqrt{-1}) - \frac{1}{2\sqrt{-1}}F:(x - ay\sqrt{-1})$$

pro cujus ad realitatem reductione notasse juvabit, posito

$$x = s\cos.\phi \quad \text{et} \quad ay = s\sin.\phi, \quad \text{fore}$$

$$(x \pm ay\sqrt{-1})^n = s^n(\cos.n\phi \pm \sqrt{-1}.\sin.n\phi).$$

Quare quoties functiones propositae per operationes analyticas sunt conflatae, hoc est, continuae, earum valores realiter per cosinus et sinus ipsius ϕ exhiberi possunt. Quando autem functiones illae sunt discontinae, talis reductio neutiquam locum habet, etiamsi certum videatur, etiam tunc formam allatam valorem realem esse adepturam. Quis autem in curva quacunque, libero manus ductu descripta, applicatas abscissis $x + ay\sqrt{-1}$ et $x - ay\sqrt{-1}$ respondentes animo saltem imaginari, ac summam earum realem assignare valuerit, aut differentiam, quae per $\sqrt{-1}$ divisa etiam erit realis? Hic ergo haud exiguus defectus calculi cernitur, quem nullo adhuc modo supplere licet; atque ob hunc ipsum defectum hujusmodi solutiones universales plurimum de sua vi perdunt.

Problema 49.

302. Proposita aequatione $\left(\frac{\partial\partial z}{\partial y^2}\right) = PP\left(\frac{\partial\partial z}{\partial x^2}\right)$, inquirere, quales functiones ipsarum x et y pro P assumere liceat, ut integratio ope reductionis succedat.

Solutio.

Reductionem hanc ita fieri assumo, ut loco x et y binae aliae variabiles t et u introducantur, qua substitutione secundum §. 231. in genere facta prodit haec aequatio:

$$\left.\begin{array}{l} +\left(\frac{\partial\partial t}{\partial y^2}\right)\left(\frac{\partial z}{\partial t}\right)+\left(\frac{\partial\partial u}{\partial y^2}\right)\left(\frac{\partial z}{\partial u}\right)+\left(\frac{\partial t}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial t^2}\right)+2\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t\partial u}\right)+\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial u^2}\right) \\ -PP\left(\frac{\partial\partial t}{\partial x^2}\right)-PP\left(\frac{\partial\partial u}{\partial x^2}\right)-PP\left(\frac{\partial t}{\partial x}\right)^2-2PP\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)-PP\left(\frac{\partial u}{\partial x}\right)^2 \end{array}\right\} = 0.$$

Jam relatio inter binas variabiles t, u et praecedentes x, y ejusmodi statuatur, ut binae formulae $\left(\frac{\partial\partial z}{\partial t^2}\right)$ et $\left(\frac{\partial\partial z}{\partial u^2}\right)$ ex calculo egrediantur, id quod fiet ponendo

$$\left(\frac{\partial t}{\partial y}\right) + P\left(\frac{\partial t}{\partial x}\right) = 0 \text{ et } \left(\frac{\partial u}{\partial y}\right) - P\left(\frac{\partial u}{\partial x}\right) = 0.$$

Tum autem erit

$$\left(\frac{\partial\partial t}{\partial y^2}\right) = -P\left(\frac{\partial\partial t}{\partial x\partial y}\right) - \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial t}{\partial x}\right);$$

at cum sit indidem

$$\left(\frac{\partial\partial t}{\partial x\partial y}\right) = -P\left(\frac{\partial\partial t}{\partial x^2}\right) - \left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial t}{\partial x}\right), \text{ erit}$$

$$\left(\frac{\partial\partial t}{\partial y^2}\right) = PP\left(\frac{\partial\partial t}{\partial x^2}\right) + P\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial t}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial t}{\partial x}\right),$$

similique modo sumendo P negative

$$\left(\frac{\partial\partial u}{\partial y^2}\right) = PP\left(\frac{\partial\partial u}{\partial x^2}\right) + P\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right).$$

His substitutis nostra aequatio hanc induet formam:

$$\left[P\left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right)\right]\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right) + \left[P\left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial P}{\partial y}\right)\right]\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial z}{\partial u}\right)$$
$$- 4PP\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial\partial z}{\partial t\partial u}\right) = 0,$$

quae cum unicam formulam secundi gradus $\left(\frac{\partial\partial z}{\partial t\partial u}\right)$ contineat, integrationem admittit, si vel $(\frac{\partial z}{\partial t})$ vel $(\frac{\partial z}{\partial u})$ e calculo excesserit. Ponamus ergo insuper

$$P(\frac{\partial P}{\partial x}) - (\frac{\partial P}{\partial y}) = 0,$$

qua aequatione indoles quaesitae functionis P definitur; quo facto aequatio integranda, per $2P(\frac{\partial u}{\partial x})$ divisa, erit

$$(\frac{\partial P}{\partial x})(\frac{\partial z}{\partial u}) - 2P(\frac{\partial t}{\partial x})(\frac{\partial\partial z}{\partial t\partial u}) = 0,$$

cujus integrale, posito $(\frac{\partial z}{\partial u}) = v$, fit

$$2lv = \int \frac{\partial t\,(\frac{\partial P}{\partial x})}{P\,(\frac{\partial t}{\partial x})} = l\left(\frac{\partial z}{\partial u}\right).$$

Verum prius ipsam functionem P per x et y definiri oportet. Cum igitur sit $(\frac{\partial P}{\partial y}) = P(\frac{\partial P}{\partial x})$, erit

$$\partial P = \partial x\,(\frac{\partial P}{\partial x}) + P\partial y\,(\frac{\partial P}{\partial x}),$$

hincque ponendo brevitatis ergo $(\frac{\partial P}{\partial x}) = p$, fit

$$\partial x = \frac{\partial P}{p} - P\partial y, \text{ atque}$$

$$x = -Py + \int\partial P\,(y + \frac{1}{p}).$$

Statuatur ergo $y + \frac{1}{p} = f' : P$, ac reperitur

$$x + Py = f : P \quad \text{et} \quad p = (\frac{\partial P}{\partial x}) = \frac{1}{f' : P - y}$$

ac $(\frac{\partial P}{\partial y}) = \frac{P}{f' : P - y}$, unde ratio determinationis quantitatis P per x et y definitur. Pro novis autem variabilibus t et u, ob

$$(\frac{\partial t}{\partial y}) = -P(\frac{\partial t}{\partial x}), \text{ erit}$$

$$\partial t = (\frac{\partial t}{\partial x})(\partial x - P\partial y)$$

et ob $x = -Py + f : P$, fit

$$\partial t = (\frac{\partial t}{\partial x})(\partial P f' : P - 2P\partial y - y\partial P)$$

$$= P^{\frac{1}{2}}(\frac{\partial t}{\partial x})(\frac{\partial P}{\sqrt{P}} f' : P - 2\partial y\sqrt{P} - \frac{y\partial P}{\sqrt{P}}),$$

cujus postremae formulae cum integrale sit

$$\int \frac{\partial P}{\surd P} f' : P - 2y\surd P, \text{ erit}$$
$$t = F : (\int \frac{\partial P}{\surd P} f' : P - 2y\surd P).$$

Deinde ob $(\frac{\partial u}{\partial y}) = P(\frac{\partial u}{\partial x})$, habetur

$$\partial u = (\frac{\partial u}{\partial x})(\partial x + P\partial y) = (\frac{\partial u}{\partial x})(\partial P f' : P - y\partial P),$$

ideoque

$$\partial u = (\frac{\partial u}{\partial x})(f' : P - y)\,\partial P;$$

quare u aequabitur functioni ipsius P. In hoc autem negotio functiones quascunque accipere licet, quia sequente demum integratione universalitas solutionis obtinetur. Quare ponamus

$$t = \int \frac{\partial P}{\surd P} f' : P - 2y\surd P \text{ et } u = P, \text{ existente}$$
$$x + Py = f : P.$$

Denique ad ipsum integrale inveniendum, quia est

$$2l(\frac{\partial z}{\partial u}) = \int \frac{\partial t\,(\frac{\partial P}{\partial x})}{P\,(\frac{\partial t}{\partial x})},$$

in qua integratione u seu P sumitur constans, pur superiora erit

$$\frac{\partial t}{(\frac{\partial t}{\partial x})} = \partial P f' : P - 2P\partial y - y\partial P = -2P\partial y,$$

ob P constans, et $(\frac{\partial P}{\partial x}) = \frac{1}{f' : P - y}$, unde fit

$$2l(\frac{\partial z}{\partial P}) = \int \frac{-2\partial y}{f' : P - y} = 2l(f' : P - y) + 2lF : P, \text{ seu}$$
$$(\frac{\partial z}{\partial P}) = (f' : P - y)\,F : P,$$

hincque porro

$$z = \int \partial P\,(f' : P - y)\,F : P,$$

sumendo hinc t constans. Cum igitur sit

$$y = +\frac{1}{2\surd P}\int \frac{\partial P}{\surd P} f' : P - \frac{t}{2\surd P},$$

ideoque

$$f':P - y = f':P - \frac{1}{2\sqrt{P}}\int\frac{\partial P}{\sqrt{P}}f':P + \frac{t}{2\sqrt{P}},$$

unde conficitur

$$z = \int\partial P\,(f':P - \frac{1}{2\sqrt{P}}\int\frac{\partial P}{\sqrt{P}}f':P)\,F:P$$
$$+ (\tfrac{1}{2}\int\frac{\partial P}{\sqrt{P}}f':P - y\sqrt{P})\int\frac{\partial P}{\sqrt{P}}F:P + \Phi:(\int\frac{\partial P}{\sqrt{P}}f':P - 2y\sqrt{P}),$$

quae expressio duas continet functiones arbitrarias F et Φ.

Corollarium 1.

303. Primum hujus formae membrum ita transformari potest:

$$\int\frac{\partial P}{\sqrt{P}}(\sqrt{P}\,.\,f':P - \tfrac{1}{2}\int\frac{\partial P}{\sqrt{P}}f':P)\,F:P, \text{ at}$$
$$\sqrt{P}\,.\,f':P - \tfrac{1}{2}\int\frac{\partial P}{\sqrt{P}}f':P = \int\partial P\sqrt{P}\,.\,f'':P,$$

unde primum membrum erit

$$\int\frac{\partial P}{\sqrt{P}}F:P\,.\int\partial P\sqrt{P}\,.\,f'':P.$$

Corollarium 2.

304. Cum autem hoc primum membrum sit functio indefinita ipsius P, si ea indicetur per Π : P, erit

$$\frac{\partial P}{\sqrt{P}}F:P = \frac{\partial P\,\Pi':P}{\int\partial P\sqrt{P}\,.\,f'':P},$$

unde forma integralis fit

$$z = \Pi:P + \Phi:(\int\frac{\partial P}{\sqrt{P}}f':P - 2y\sqrt{P})$$
$$+ (\int\frac{\partial P}{\sqrt{P}}f':P - 2y\sqrt{P})\int\frac{\partial P\,\Pi':P}{2\int\partial P\sqrt{P}\,.\,f'':P}.$$

Corollarium 3.

305. Solutio magis particularis nascitur sumendo $\Pi:P = 0$, hincque z aequabitur functioni cuicunque quantitatis $\int\frac{\partial P}{\sqrt{P}}f':P - 2y\sqrt{P}$, quae ob $x + Py = f:P$ per x et y exhiberi censenda est.

Scholion.

306. Quanquam hic eadem methodo sum usus atque in problemate praecedente, tamen, quod mirum videatur, casus praecedentis problematis, quo erat $P = a$, in hac solutione non continetur. Ratio hujus paradoxi in resolutione aequationis $(\frac{\partial P}{\partial y}) = P(\frac{\partial P}{\partial x})$ est sita, cui manifesto satisfacit valor $P = a$, etiamsi in forma inde derivata $x + Py = f:P$ non contineatur. Hic scilicet simile quiddam usu venit, quod jam supra observavimus, saepe aequationi differentiali valorem quendam satisfacere posse, qui in integrali non contineatur. Veluti aequationi $\partial y \sqrt{(a-x)} = \partial x$ satisfacere videmus valorem $x = a$, quem tamen integrale $y = C - 2\sqrt{(a-x)}$ excludit. Quare etiam nostro casu valor $P = a$ peculiarem evolutionem postulat, in priore problemate peractam. De reliquis, ubi pro $f:P$ certa quaedam functio ipsius P assumitur, exempla quaedam evolvamus.

Exemplum 1.

307. *Sumto* $f:P = 0$, *ut sit* $P = -\frac{x}{y}$, *integrale completum hujus aequationis:*

$$(\frac{\partial\partial z}{\partial y^2}) = \frac{xx}{yy}(\frac{\partial\partial z}{\partial x^2}),$$

investigare.

Cum sit $f':P = o$, solutio inventa, ob $\int\frac{\partial P}{\sqrt{P}}f':P = C$, praebet

$$z = \frac{-C}{2}\int\frac{\partial P}{\sqrt{P}}F:P + (\tfrac{1}{2}C - y\sqrt{P})\int\frac{\partial P}{\sqrt{P}}F:P + \Phi:(C - 2y\sqrt{P}).$$

Statuatur $\int\frac{\partial P}{\sqrt{P}}F:P = \Pi:P$, prodibitque

$$z = -y\sqrt{P}\,.\,\Pi:P + \Phi:y\sqrt{P}.$$

Restituatur pro P valor $\frac{-x}{y}$, et ob $y\sqrt{P} = \sqrt{-xy}$, imaginarium $\sqrt{-1}$ in functiones involvendo erit

$$z = \sqrt{xy} \,.\, \Pi : \frac{x}{y} + \Phi : \sqrt{xy},$$

quae forma facile in hanc transfunditur:

$$z = x\Gamma : \frac{x}{y} + \Theta : xy,$$

ubi $x\Gamma : \frac{x}{y}$ denotat functionem quamcunque homogeneam unius dimensionis ipsarum x et y. Resolutio autem instituetur loco x et y has novas variabiles t et u introducendo, ut sit $t = C - 2\sqrt{-xy}$ et $u = -\frac{x}{y}$, vel etiam simplicius $t = 2\sqrt{xy}$ et $u = \frac{x}{y}$, unde fit

$$\left(\frac{\partial t}{\partial x}\right) = \frac{\sqrt{y}}{\sqrt{x}},\ \left(\frac{\partial t}{\partial y}\right) = \frac{\sqrt{x}}{\sqrt{y}},\ \left(\frac{\partial\partial t}{\partial x^2}\right) = \frac{-\sqrt{y}}{2x\sqrt{x}},\ \left(\frac{\partial\partial t}{\partial y^2}\right) = \frac{-\sqrt{x}}{2y\sqrt{y}}.$$

$$\left(\frac{\partial u}{\partial x}\right) = \frac{1}{y},\ \left(\frac{\partial u}{\partial y}\right) = \frac{-x}{yy},\ \left(\frac{\partial\partial u}{\partial x^2}\right) = 0,\ \left(\frac{\partial\partial u}{\partial y^2}\right) = \frac{2x}{y^3},$$

et ob $PP = \frac{xx}{yy}$ aequatio proposita hanc induit formam:

$$0\left(\frac{\partial z}{\partial t}\right) + \frac{2x}{y^3}\left(\frac{\partial z}{\partial u}\right) - \frac{4x\sqrt{x}}{yy\sqrt{y}}\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) = 0.$$

Nunc cum sit

$$ttu = 4xx, \text{ et } x = \tfrac{1}{2}t\sqrt{u},$$

atque $y = \frac{t}{2\sqrt{u}}$, habebimus

$$\frac{8uu}{tt}\left(\frac{\partial z}{\partial u}\right) - \frac{8uu}{t}\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) = 0, \text{ seu } \left(\frac{\partial z}{\partial u}\right) = t\left(\frac{\partial\partial z}{\partial t\,\partial u}\right).$$

Fiat $\left(\frac{\partial z}{\partial u}\right) = v$, ut sit $v = t\left(\frac{\partial v}{\partial t}\right)$, et sumto u constante, $\frac{\partial t}{t} = \frac{\partial v}{v}$ ergo $v = \left(\frac{\partial z}{\partial u}\right) = t\,f' : u$. Sit jam t constans, fietque

$$z = t\,f : u + F : t = 2\sqrt{xy} \,.\, f : \frac{x}{y} + F : \sqrt{xy},$$

ut ante.

Corollarium.

308. Quemadmodum autem expressio inventa

$$z = x\Gamma : \frac{x}{y} + \Theta : xy$$

satisfaciat, differentialibus rite sumtis perspicietur

$$\left(\frac{\partial z}{\partial x}\right) = \Gamma : \frac{x}{y} + \frac{x}{y}F' : \frac{x}{y} + y\,\Theta' : xy,\ \left(\frac{\partial z}{\partial y}\right) = \frac{-xx}{yy}\Gamma' : \frac{x}{y} + x\,\Theta' : xy,$$

unde porro fit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{2}{y}\Gamma' : \frac{x}{y} + \frac{x}{yy}\Gamma'' : \frac{x}{y} + yy\,\Theta'' : xy \text{ et}$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{2xx}{y^3}\Gamma' : \frac{x}{y} + \frac{x^3}{y^4}\Gamma'' : \frac{x}{y} + xx\,\Theta'' : xy.$$

Exemplum 2.

309. *Sumto* $f : P = \frac{PP}{2a}$, *ut sit*

$PP = 2aPy + 2ax$ *et* $P = ay \mp \surd\,(aayy + 2ax)$,

hujus aequationis:

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = [2aayy + 2ax + 2ay\surd\,(aayy + 2ax)]\left(\frac{\partial\partial z}{\partial x^2}\right),$$

integrale completum investigare.

Cum sit $f : P = \frac{PP}{2a}$, erit

$$f' : P = \frac{P}{a}, \text{ et } \int\frac{\partial P}{\surd P} f' : P = \int\frac{1}{a}\partial P\surd P = \frac{2}{3a}P\surd P,$$

unde forma generalis supra inventa abit in

$$z = \int\partial P . \frac{2P}{3a}F : P + \left(\frac{P\surd P}{3a} - y\surd P\right)\int\frac{\partial P}{\surd P}F : P + \Phi : \left(\frac{2}{3a}P\surd P - y\surd P\right).$$

Statuatur $\int\frac{\partial P}{\surd P} . F : P = \Pi : P$, erit

$$\partial P . F : P = \partial P\surd P . \Pi' P,$$

atque

$$z = \frac{2}{3a}\int P^{\frac{3}{2}}\partial P . \Pi' : P + \left(\frac{P\surd P}{3a} - y\surd P\right)\Pi : P + \Phi : \left(\frac{P\surd P}{3a} - y\surd P\right).$$

Est autem

$$\frac{P}{3a} - y = \frac{-2}{3}y + \frac{1}{3}\surd\left(yy + \frac{2x}{a}\right);$$

quarum formularum evolutio deducit ad expressiones nimis perplexas. At substitutiones ad scopum perducentes sunt

$$t = \frac{2}{3a}P\surd P - 2y\surd P \text{ et } u = P.$$

Corollarium.

310. Si pro solutione magis restricta ponatur

$$\Pi : P = P^{n-\frac{1}{2}}, \text{ erit}$$

$$\Pi' : P = (n - \tfrac{1}{2})\, P^{n-\frac{3}{2}},$$

hincque colligitur

$$z = \frac{n}{(n+1)a} P^{n+1} - P^n y + \Phi : \left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right).$$

Sit $n = 1$, et functio Φ evanescat, eritque

$$z = \frac{1}{2a} P P - P y = x;$$

at casus $n = 2$ dat

$$z = \frac{2}{3a} P^3 - P^2 y = \tfrac{2}{3} axy + \tfrac{2}{3} P (2x + ayy), \text{ seu}$$

$$z = aay^3 + 3axy + (ayy + 2x) \sqrt{(aayy + 2ax)}.$$

Scholion.

311. Forma integralis inventa sequenti modo simplicior effici potest: Ponatur

$$\int \frac{\partial P}{\sqrt{P}} F : P = \Pi : P, \text{ fiet}$$

$$F : P = \sqrt{P} \,.\, \Pi' : P,$$

eritque (omittendo postremum membrum

$$\Phi \left(\int \frac{\partial P}{\sqrt{P}} f' : P - 2 y \sqrt{P}\right),$$

quod nulla reductione indiget)

$$z = \int \partial P \left(\sqrt{P} \,.\, f' : P - \tfrac{1}{2} \int \frac{\partial P}{\sqrt{P}} f' : P\right) \Pi' : P$$

$$+ \tfrac{1}{2} \Pi : P \int \frac{\partial P}{\sqrt{P}} f' : P - y\sqrt{P} \,.\, \Pi : P; \text{ at}$$

$$\tfrac{1}{2} \Pi : P \,.\, \int \frac{\partial P}{\sqrt{P}} f' : P = \int \left(\tfrac{1}{2} \partial P\, \Pi' : P \,.\, \int \frac{\partial P}{\sqrt{P}} f' : P + \tfrac{1}{2} \frac{\partial P}{\sqrt{P}} \Pi : P \,.\, f' : P\right),$$

unde fit

$$z = \int \Pi' : P \,.\, \partial P \sqrt{P} f' : P + \tfrac{1}{2} \int \Pi : P \,.\, \frac{\partial P}{\sqrt{P}} f' : P - y\sqrt{P} \,.\, \Pi : P.$$

Porro est

$$\int \partial P . \Pi' : P . \sqrt{P} . f' : P = \Pi : P . \sqrt{P} . f' : P - \int \Pi : P \left(\frac{\partial P}{2\sqrt{P}} f' : P + \partial P \sqrt{P} . f'' : P \right),$$

ideoque

$$z = \Pi : P . \sqrt{P} . f' : P - \int \partial P . \Pi : P . \sqrt{P} . f'' : P - y \sqrt{P} . \Pi : P.$$

Statuatur porro

$$\int \partial P \Pi : P . \sqrt{P} . f'' : P = \Theta : P., \text{ erit}$$

$$\Pi : P = \frac{\Theta' : P}{\sqrt{P} . f'' : P} \text{ et}$$

$$z = \frac{\Theta' : P}{f'' : P} (f' : P - y) - \Theta : P + \Phi \left(\int \frac{\partial P}{\sqrt{P}} f' : P - 2 y \sqrt{P} \right),$$

quae forma sine dubio multo est simplicior quam primo inventa.

Problema 50.

312. Proposita aequatione

$$\left(\frac{\partial \partial z}{\partial y^2}\right) - PP \left(\frac{\partial \partial z}{\partial x^2}\right) + Q \left(\frac{\partial z}{\partial y}\right) + R \left(\frac{\partial z}{\partial x}\right) = 0.$$

invenire valores quantitatum P, Q, R, quibus integratio ope reductionis ante adhibitae succedit.

Solutio.

Introductis binis novis variabilibus t et u, habebimus

$$0 = \left(\frac{\partial \partial t}{\partial y^2}\right) \left(\frac{\partial z}{\partial t}\right) + \left(\frac{\partial \partial u}{\partial y^2}\right) \left(\frac{\partial z}{\partial u}\right) + \left(\frac{\partial t}{\partial y}\right)^2 \left(\frac{\partial \partial z}{\partial t^2}\right) + 2 \left(\frac{\partial t}{\partial y}\right) \left(\frac{\partial u}{\partial y}\right) \left(\frac{\partial \partial z}{\partial t \partial u}\right) + \left(\frac{\partial u}{\partial y}\right)^2 \left(\frac{\partial \partial z}{\partial u^2}\right)$$

$$- PP \left(\frac{\partial \partial t}{\partial x^2}\right) - PP \left(\frac{\partial \partial u}{\partial x^2}\right) - PP \left(\frac{\partial t}{\partial x}\right)^2 - 2 PP \left(\frac{\partial t}{\partial x}\right) \left(\frac{\partial u}{\partial x}\right) - PP \left(\frac{\partial u}{\partial x}\right)^2$$

$$+ Q \left(\frac{\partial t}{\partial y}\right) + Q \left(\frac{\partial u}{\partial y}\right)$$

$$+ R \left(\frac{\partial t}{\partial x}\right) + Q \left(\frac{\partial u}{\partial x}\right).$$

Statuamus ergo ut ante

$$\left(\frac{\partial t}{\partial y}\right) = P \left(\frac{\partial t}{\partial x}\right) \text{ et } \left(\frac{\partial u}{\partial y}\right) = - P \left(\frac{\partial u}{\partial x}\right),$$

unde fit

$$\left(\frac{\partial \partial t}{\partial x \partial y}\right) = P \left(\frac{\partial \partial t}{\partial x^2}\right) + \left(\frac{\partial P}{\partial x}\right) \left(\frac{\partial t}{\partial x}\right), \text{ et}$$

$$\left(\frac{\partial \partial t}{\partial y^2}\right) = PP \left(\frac{\partial \partial t}{\partial x^2}\right) + P \left(\frac{\partial P}{\partial x}\right) \left(\frac{\partial t}{\partial x}\right) + \left(\frac{\partial P}{\partial y}\right) \left(\frac{\partial t}{\partial x}\right),$$

atque

$$\left(\frac{\partial\partial u}{\partial y^2}\right) = PP\left(\frac{\partial\partial u}{\partial x^2}\right) + P\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right),$$

et aequatio resolvenda erit

$$0 = \left[P\left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial P}{\partial y}\right) + PQ + R\right]\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right) - 4\,PP\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial\partial z}{\partial t\partial u}\right)$$
$$+ \left[P\left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right) - PQ + R\right]\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial z}{\partial u}\right).$$

Jam evidens est integrationem institui posse, si alterutra formula $\left(\frac{\partial z}{\partial t}\right)$ vel $\left(\frac{\partial z}{\partial u}\right)$ ex calculo abeat. Ponamus ergo esse

$$P\left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right) - PQ + R = 0, \text{ seu}$$
$$R = PQ + \left(\frac{\partial P}{\partial y}\right) - P\left(\frac{\partial P}{\partial x}\right),$$

et aequatio resultans, per $\left(\frac{\partial t}{\partial x}\right)$ divisa, fit

$$0 = 2\left[PQ + \left(\frac{\partial P}{\partial y}\right)\right]\left(\frac{\partial z}{\partial t}\right) - 4\,PP\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial\partial z}{\partial t\partial u}\right).$$

Fiat $\left(\frac{\partial z}{\partial t}\right) = v$, erit

$$\left[PQ + \left(\frac{\partial P}{\partial y}\right)\right]v - 2\,PP\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial u}\right) = 0.$$

Sumatur t constans, ut fiat

$$\frac{\partial v}{v} = \frac{\left[PQ + \left(\frac{\partial P}{\partial y}\right)\right]\partial u}{2\,PP\left(\frac{\partial u}{\partial x}\right)},$$

ut necesse est, ut quantitates P, Q, $\left(\frac{\partial P}{\partial y}\right)$ et $\left(\frac{\partial u}{\partial x}\right)$ per novas variabiles t et u exprimantur, quas ergo primum definiri convenit. Cum igitur sit

$$\left(\frac{\partial t}{\partial y}\right) = P\left(\frac{\partial t}{\partial x}\right) \text{ et } \left(\frac{\partial u}{\partial y}\right) = -P\left(\frac{\partial u}{\partial x}\right), \text{ erit}$$
$$\partial t = \left(\frac{\partial t}{\partial x}\right)(\partial x + P\partial y) \text{ et } \partial u = \left(\frac{\partial u}{\partial x}\right)(\partial x - P\partial y).$$

Sunt ergo $\left(\frac{\partial t}{\partial x}\right)$ et $\left(\frac{\partial u}{\partial x}\right)$ factores integrabiles reddentes formulas $\partial x + P\partial y$ et $\partial x - P\partial y$: non enim opus est ut hinc valores t et u generalissime definiantur. Sint p et q tales multiplicatores, per x et y dati, eritque

$$t = \int p\,(\partial x + P\partial y) \text{ et } u = \int q\,(\partial x - P\partial y),$$

unde superior integratio fit

$$\frac{\partial v}{v} = \frac{[PQ + (\frac{\partial P}{\partial y})]\,\partial u}{2PPq},$$

in qua integratione quantitas $t = \int p\,(\partial x + P\partial y)$ constans est spectanda. Seu ob $\partial u = q\,(\partial x - P\partial y)$ erit

$$\frac{\partial v}{v} = \frac{[PQ + (\frac{\partial P}{\partial y})]\,(\partial x - P\partial y)}{2PP}.$$

Verum ob $\partial t = 0$ est $\partial x = -P\partial y$, ita ut prodeat

$$\frac{\partial v}{v} = -\frac{\partial y}{P}\,[PQ + (\frac{\partial P}{\partial y})],$$

ubi ob t constans, et datum per x et y, valor ipsius x per y et t expressus substitui potest, ut sola y variabilis insit, et invento integrali

$$-\int \frac{\partial y}{P}\,[PQ + (\frac{\partial P}{\partial y})] = lV,$$

erit $v = V\,f : t = (\frac{\partial z}{\partial t})$.

Nunc ponatur u constans eritque

$$z = \int V\partial t\, f : t + F : u.$$

Conditio autem, sub qua haec integratio locum habet, postulat ut sit

$$R = PQ + (\frac{\partial P}{\partial y}) - P\,(\frac{\partial P}{\partial x}).$$

Corollarium 1.

313. Eodem modo aequatio proposita resolutionem admittet, si fuerit

$$R = -PQ - (\frac{\partial P}{\partial y}) - P\,(\frac{\partial P}{\partial x});$$

manetque ut ante

$$t = \int p\,(\partial x + P\partial y) \text{ et } u = \int q\,(\partial x - P\partial y).$$

Tum vero fit

$$0 = -[PQ + (\tfrac{\partial P}{\partial x})] (\tfrac{\partial z}{\partial u}) - 2PP (\tfrac{\partial t}{\partial x}) (\tfrac{\partial\partial z}{\partial t \partial u}),$$

quae posito $(\frac{\partial z}{\partial u}) = v$, sumtoque u constante dat

$$\frac{\partial v}{v} = \frac{-[PQ + (\frac{\partial P}{\partial y})]\, \partial t}{2PP\, (\frac{\partial t}{\partial x})} = \frac{-[PQ + (\frac{\partial P}{\partial y})]\, (\partial x + P\partial y)}{2PP}$$

Corollarium 2.

314. Si porro habita ratione, quod

$$u = \int q\, (\partial x - P\partial y)$$

sit constans et $\partial x = P\partial y$, ponatur

$$\int - \frac{\partial y\, [PQ + (\frac{\partial P}{\partial y})]}{P} = l V, \text{ erit}$$

$$v = V\, f : u = (\tfrac{\partial z}{\partial u}),$$

unde tandem, sumendo jam

$$t = \int p\, (\partial x + P\partial y),$$

colligitur

$$z \int V \partial u\; f : u + F : t.$$

Exemplum 1.

315. *Si sumatur $P = a$ et $R = aQ$, quaecunque fuerit Q functio ipsarum x et y, integrare aequationem:*

$$(\tfrac{\partial\partial z}{\partial y^2}) - aa\, (\tfrac{\partial\partial z}{\partial x^2}) + Q\, (\tfrac{\partial z}{\partial y}) + aQ\, (\tfrac{\partial z}{\partial x}) = 0.$$

Cum hic sit $P = a$, erit $p = 1$, $q = 1$ et $t = x + ay$ atque $u = x - ay$, unde posito $(\frac{\partial z}{\partial t}) = v$ fit

$$\frac{\partial v}{v} = \frac{aQ\partial u}{2aa} = \frac{Q\partial u}{2a}.$$

Quoniam igitur est

$$x = \frac{t+u}{2} \text{ et } y = \frac{t-u}{2a},$$

his valoribus substitutis fit Q functio ipsarum t et u, ac spectata t ut constante erit

$$lv = \frac{1}{2a}\int Q\partial u + l\,\mathrm{f} : t, \text{ seu}$$

$$\left(\frac{\partial z}{\partial t}\right) = e^{\frac{1}{2a}\int Q\partial u}\,\mathrm{f} : t,$$

et sumta jam u constante

$$z = \int e^{\frac{1}{2a}\int Q\partial u}\partial t\,\mathrm{f} : t + \mathrm{F} : u.$$

Corollarium 1.

316. Si Q sit constans $= 2ab$, aequationis hujus

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - aa\left(\frac{\partial\partial z}{\partial x^2}\right) + 2ab\left(\frac{\partial z}{\partial y}\right) + 2aab\left(\frac{\partial z}{\partial x}\right) = 0,$$

integrale erit

$$z = e^{bu}\mathrm{f} : t + \mathrm{F} : u = e^{b(x-ay)}\mathrm{f} : (x+ay) + \mathrm{F} : (x-ay),$$

sive

$$z = e^{b(x-ay)}[\mathrm{f} : (x+ay) + \mathrm{F} : (x-ay)].$$

Corollarium 2.

317. Si $Q = \frac{a}{x}$, hujus aequationis

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - aa\left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{a}{x}\left(\frac{\partial z}{\partial y}\right) + \frac{aa}{x}\left(\frac{\partial z}{\partial x}\right) = 0$$

integrale ob

$$\int Q\partial u = \int\frac{a\partial u}{x} = \int\frac{2a\partial u}{t+u} = 2al(t+u), \text{ erit}$$

$$z = \int(t+u)\partial t\,\mathrm{f} : t + \mathrm{F} : u = \int t\partial t\,\mathrm{f} : t + u\int\partial t\,\mathrm{f} : t + \mathrm{F} : u.$$

Vel sit $\mathrm{f} : t = \Pi' : t$, erit

$$\int\partial t\,\mathrm{f} : t = \Pi' : t \text{ et}$$

$$\int t\partial t\,\mathrm{f} : t = \int t\partial . \Pi'' : t = t\Pi'' : t - \int\partial t . \Pi'' : t = t\Pi'' : t - \Pi : t,$$

ergo

$$z = (t+u)\,\Pi' : t - \Pi : t + F : u, \text{ seu}$$
$$z = 2x\Pi' : (x+ay) - \Pi : (x+ay) + F : (x-ay).$$

Exemplum 2.

318. *Sit* $P = \frac{x}{y}$, *et* $R = \frac{-x}{y}Q + \frac{x}{yy} - \frac{x}{yy} = \frac{-x}{y}Q$, *sumaturque* $Q = \frac{1}{x}$, *ut sit* $R = \frac{-1}{y}$, *et haec aequatio integrari debeat*
$$\left(\frac{\partial\partial z}{\partial y^2}\right) - \frac{xx}{yy}\left(\frac{\partial\partial z}{\partial x^2}\right) + \frac{1}{x}\left(\frac{\partial z}{\partial y}\right) - \frac{1}{y}\left(\frac{\partial z}{\partial x}\right) = 0.$$

Cum ergo sit

$$t = \int p\ \left(\partial x + \frac{x\partial y}{y}\right) \text{ et } u = \int q\ \left(\partial x - \frac{x\partial y}{y}\right),$$

sumatur $p = y$ et $q = \frac{1}{y}$, ut fiat $t = xy$ et $u = \frac{x}{y}$.

Posito nunc $\left(\frac{\partial z}{\partial u}\right) = v$ sumtoque u constante, ex Corollario 1. fit:

$$\frac{\partial v}{v} = \frac{-\left(\frac{1}{y} - \frac{x}{yy}\right)\partial t}{\frac{2xx}{yy}\cdot y} = \frac{-(y-x)\partial t}{2xxy}.$$

Est vero $tu = xx$, hincque $x = \sqrt{tu}$ et $y = \sqrt{\frac{t}{u}}$, atque

$$2xxy = 2t\sqrt{tu};$$

unde fit

$$\frac{\partial v}{v} = \frac{\left(\sqrt{tu} - \sqrt{\frac{t}{u}}\right)\partial t}{2t\sqrt{tu}} = \frac{\partial t}{2t} - \frac{\partial t}{2tu},$$

et ob u constans

$$lv = \tfrac{1}{2}lt - \tfrac{1}{2u}lt,$$

$$\left(\frac{\partial z}{\partial u}\right) = t^{\frac{1}{2}}\int t^{-\frac{1}{2u}}\,f : u.$$

Quare sumto jam t constante erit

$$z = t^{\frac{1}{2}}\int t^{-\frac{1}{2u}}\,\partial u\, f : u + F : t.$$

**

Vel ponatur $-\frac{1}{2u} = s$, ut sit $s = -\frac{y}{2x}$ eritque

$$z = t^{\frac{1}{2}} \int t^{s} \partial s \; \mathrm{f} : s + \mathrm{F} : t.$$

In hac integratione $\int t^{s} \partial s \; \mathrm{f} : s$ sola s est variabilis, ac demum integrali sumto restitui debet $t = xy$ et $s = \frac{-y}{2x}$. Caeterum patet functionem quamcunque ipsius xy particulariter satisfacere.

Problema 51.

319. Proposita aequatione generali

$$\left(\frac{\partial\partial z}{\partial y^2}\right) - 2\mathrm{P}\left(\frac{\partial\partial z}{\partial x \partial y}\right) + (\mathrm{PP} - \mathrm{QQ})\left(\frac{\partial\partial z}{\partial x^2}\right) + \mathrm{R}\left(\frac{\partial z}{\partial y}\right) + \mathrm{S}\left(\frac{\partial z}{\partial x}\right)$$
$$+ \mathrm{T}z + \mathrm{V} = 0,$$

invenire conditiones quantitatum P, Q, R, S, T, ut integratio ope reductionis adhibitae succedat.

Solutio.

Facta eadem substitutione introducendis binis novis variabilibus t et u, aequatio nostra sequentem induet formam

$$\left.\begin{array}{lllll}
\mathrm{V} + \mathrm{T}z + \left(\frac{\partial\partial t}{\partial y^2}\right)\left(\frac{\partial z}{\partial t}\right) & + \left(\frac{\partial\partial u}{\partial y^2}\right)\left(\frac{\partial z}{\partial u}\right) & + \left(\frac{\partial t}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial t^2}\right) & + 2\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t \partial u}\right) & + \left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial u^2}\right) \\
- 2\mathrm{P}\left(\frac{\partial\partial t}{\partial x \partial y}\right) & - 2\mathrm{P}\left(\frac{\partial\partial u}{\partial x \partial y}\right) & - 2\mathrm{P}\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial t}{\partial y}\right) & - 2\mathrm{P}\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) & - 2\mathrm{P}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) \\
+ (\mathrm{PP} - \mathrm{QQ})\left(\frac{\partial\partial t}{\partial x^2}\right) & + (\mathrm{PP} - \mathrm{QQ})\left(\frac{\partial\partial u}{\partial x^2}\right) & + (\mathrm{PP} - \mathrm{QQ})\left(\frac{\partial t}{\partial x}\right)^2 & - 2\mathrm{P}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial t}{\partial y}\right) & + (\mathrm{PP} - \mathrm{QQ})\left(\frac{\partial u}{\partial x}\right)^2 \\
+ \mathrm{R}\left(\frac{\partial t}{\partial y}\right) & + \mathrm{R}\left(\frac{\partial u}{\partial y}\right) & & + 2(\mathrm{PP} - \mathrm{QQ})\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) & \\
+ \mathrm{S}\left(\frac{\partial t}{\partial x}\right) & + \mathrm{S}\left(\frac{\partial u}{\partial x}\right). & & &
\end{array}\right\} = 0.$$

Determinentur jam hae duae novae variabiles t et u ita per x et y, ut formulae $\left(\frac{\partial\partial z}{\partial t^2}\right)$ et $\left(\frac{\partial\partial z}{\partial u^2}\right)$ evanescant: debebitque esse

$$\left(\frac{\partial t}{\partial y}\right) = (\mathrm{P} + \mathrm{Q})\left(\frac{\partial t}{\partial x}\right) \text{ et } \left(\frac{\partial u}{\partial y}\right) = (\mathrm{P} - \mathrm{Q})\left(\frac{\partial u}{\partial x}\right),$$

unde patet has variabiles sequenti modo determinari

$$t = \int p\,[\partial x + (\mathrm{P} + \mathrm{Q})\,\partial y] \text{ et } u = \int q\,[\partial x + (\mathrm{P} - \mathrm{Q})\,\partial y],$$

sumendo p et q ita ut hae formulae integrationem admittant.

Cum nunc sit

$$\left(\frac{\partial\partial t}{\partial x\partial y}\right) = (P+Q)\left(\frac{\partial\partial t}{\partial x^2}\right) + \left[\left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial Q}{\partial x}\right)\right]\left(\frac{\partial t}{\partial x}\right),$$

$$\left(\frac{\partial\partial t}{\partial y^2}\right) = (P+Q)^2\left(\frac{\partial\partial t}{\partial x^2}\right) + (P+Q)\left[\left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial Q}{\partial x}\right)\right]\left(\frac{\partial t}{\partial x}\right)$$
$$+ \left[\left(\frac{\partial P}{\partial y}\right) + \left(\frac{\partial Q}{\partial y}\right)\right]\left(\frac{\partial t}{\partial x}\right),$$

$$\left(\frac{\partial\partial u}{\partial x\partial y}\right) = (P-Q)\left(\frac{\partial\partial u}{\partial x^2}\right) + \left[\left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial Q}{\partial x}\right)\right]\left(\frac{\partial u}{\partial x}\right),$$

$$\left(\frac{\partial\partial u}{\partial y^2}\right) = (P-Q)^2\left(\frac{\partial\partial u}{\partial x^2}\right) + (P-Q)\left[\left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial Q}{\partial x}\right)\right]\left(\frac{\partial u}{\partial x}\right)$$
$$+ \left[\left(\frac{\partial P}{\partial y}\right) - \left(\frac{\partial Q}{\partial y}\right)\right]\left(\frac{\partial u}{\partial x}\right).$$

Hinc reperitur formulae $2\left(\frac{\partial\partial z}{\partial t\partial u}\right)$ coëfficiens $= -2QQ\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)$,
termini $\left(\frac{\partial z}{\partial t}\right)$ coëfficiens $=$

$$\left[-(P-Q)\left(\frac{\partial P+\partial Q}{\partial x}\right) + \left(\frac{\partial P+\partial Q}{\partial y}\right) + R(P+Q) + S\right]\left(\frac{\partial t}{\partial x}\right),$$

termini vero $\left(\frac{\partial z}{\partial u}\right)$ coëfficiens $=$

$$\left[-(P+Q)\left(\frac{\partial P-\partial Q}{\partial x}\right) + \left(\frac{\partial P-\partial Q}{\partial y}\right) + R(P-Q) + S\right]\left(\frac{\partial u}{\partial x}\right).$$

Est vero $\left(\frac{\partial t}{\partial x}\right) = p$ et $\left(\frac{\partial u}{\partial x}\right) = q$, unde si brevitatis gratia vocetur

$$S + R(P+Q) + \left(\frac{\partial P+\partial Q}{\partial y}\right) - (P-Q)\left(\frac{\partial P+\partial Q}{\partial x}\right) = M \text{ et}$$
$$S + R(P-Q) + \left(\frac{\partial P-\partial Q}{\partial y}\right) - (P+Q)\left(\frac{\partial P-\partial Q}{\partial x}\right) = N,$$

aequatio nostra resolvenda erit

$$0 = V + Tz + Mp\left(\frac{\partial z}{\partial t}\right) + Nq\left(\frac{\partial z}{\partial u}\right) - 4QQpq\left(\frac{\partial\partial z}{\partial t\partial u}\right);$$

seu ut cum formis supra §§ 294 et 295. exhibitis comparari queat

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right) - \frac{M}{4QQq}\left(\frac{\partial z}{\partial t}\right) - \frac{N}{4QQp}\left(\frac{\partial z}{\partial u}\right) - \frac{T}{4QQpq}z - \frac{V}{4QQpq} = 0,$$

quae si porro brevitatis gratia ponatur

$$\frac{M}{4QQq} = K \text{ et } \frac{N}{4QQp} = L,$$

duplici casu integrationem admittit: altero si fuerit

$$-\frac{T}{4QQpq} = +KL - \left(\frac{\partial L}{\partial u}\right), \text{ seu } T = 4QQpq\left(\frac{\partial L}{\partial u}\right) - \frac{MN}{4QQ},$$

altero vero si fuerit

$$-\frac{T}{4QQpq} = KL - \left(\frac{\partial K}{\partial t}\right), \text{ seu } T = 4QQpq\left(\frac{\partial K}{\partial t}\right) - \frac{MN}{4QQ}.$$

Quoniam vero K et L per x et y dantur, formulae illae $\left(\frac{\partial K}{\partial t}\right)$ et $\left(\frac{\partial L}{\partial u}\right)$ ita reduci possunt ut sit

$$\left(\frac{\partial K}{\partial t}\right) = \frac{Q-P}{2Qp}\left(\frac{\partial K}{\partial x}\right) + \frac{1}{2Qp}\left(\frac{\partial K}{\partial y}\right) \text{ et}$$
$$\left(\frac{\partial L}{\partial u}\right) = \frac{P+Q}{2Qp}\left(\frac{\partial L}{\partial x}\right) - \frac{1}{2Qq}\left(\frac{\partial L}{\partial y}\right).$$

Quemadmodum autem ipsa integralia his casibus inveniri debeant, id quidem supra est declaratum; unde superfluum foret calculos illos taediosos hic repetere: quovis enim casu oblato solutio inde peti poterit.

Scholion. 1.

320. Quod ad hanc reductionem formularum attinet, ea sequenti modo instituitur. Cum sit in genere

$$\partial z = \partial x \left(\frac{\partial z}{\partial x}\right) + \partial y\left(\frac{\partial z}{\partial y}\right),$$

ex formulis

$$\partial t = p\partial x + p(P+Q)\partial y \text{ et } \partial u = q\partial x + q(P-Q)\partial y \text{ erit}$$
$$q\partial t - p\partial u = 2pqQ\partial y, \text{ seu } \partial y = \frac{q\partial t - p\partial u}{2Qpq} \text{ et}$$
$$q(P-Q)\partial t - p(P+Q)\partial u = -2Qpq\partial x, \text{ seu}$$
$$\partial x = \frac{p(P+Q)\partial u - q(P-Q)\partial t}{2Qpq}.$$

Quibus valoribus substitutis obtinebitur

$$\partial z = \left[\frac{(P+Q)\partial u}{2Qq} - \frac{(P-Q)\partial t}{2Qp}\right]\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial t}{2Qp} - \frac{\partial u}{2Qq}\right)\left(\frac{\partial z}{\partial y}\right),$$

ita ut ∂z per differentialia ∂t et ∂u exprimatur. Posito ergo u constante et $\partial u = 0$, erit

$$\left(\frac{\partial z}{\partial t}\right) = \frac{Q-P}{2Qp}\left(\frac{\partial z}{\partial x}\right) + \frac{1}{2Qp}\left(\frac{\partial z}{\partial y}\right);$$

at posito t constante et $\partial t = 0$, erit

$$\left(\frac{\partial z}{\partial u}\right) = \frac{P+Q}{2Qq}\left(\frac{\partial z}{\partial x}\right) - \frac{1}{2Qq}\left(\frac{\partial z}{\partial y}\right).$$

Scholion 2.

321. Methodus igitur hoc capite tradita in hoc consistit, ut hujusmodi aequationes ope introductionis binarum novarum variabilium t et u ad hanc formam reducantur

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right)+P\left(\frac{\partial z}{\partial t}\right)+Q\left(\frac{\partial z}{\partial u}\right)+Rz+S=0,$$

de qua in praecedente capite vidimus, quibusnam casibus ea integrari queat. Iisdem igitur quoque casibus omnes aequationes, quae ad talem formam se reduci patiuntur, integrationem admittent. Est vero ejusdem formae casus quidam maxime singularis, cujus integratio absolvi potest, unde denuo infinita multitudo aliarum aequationum, quae quidem eo reduci queant, oritur integrationem pariter admittentium. Quem propterea casum sequenti capite diligentius evoluamus.