CAPUT II.

DE
UNA FORMULA DIFFERENTIALI SECUNDI GRADUS PER RELIQUAS QUANTITATES UTCUNQUE DATA.

Problema 41.

245.

Si z debeat esse ejusmodi functio ipsarum x et y, ut formula secundi gradus $\left(\frac{\partial\partial z}{\partial x^2}\right)$ aequetur functioni datae ipsarum x et y; indolem functionis z investigare.

Solutio.

Sit P functio ista data ipsarum x et y, ita ut esse debeat $\left(\frac{\partial\partial z}{\partial x^2}\right) = \mathrm{P}$. Sumatur jam y constans, et cum sit

$$\partial \,.\, \left(\frac{\partial z}{\partial x}\right) = \partial x \left(\frac{\partial\partial z}{\partial x^2}\right), \text{ erit } \partial \,.\, \left(\frac{\partial z}{\partial x}\right) = \mathrm{P}\partial x,$$

unde integrando prodit

$$\left(\frac{\partial z}{\partial x}\right) = \int \mathrm{P}\partial x + \text{Const.}$$

Ubi in integratione $\int \mathrm{P}\partial x$ quantitas y pro constante habetur, et constans adjicienda functionem quamcunque ipsius y denotabit, ita ut haec prima integratio praebeat

$$\left(\frac{\partial z}{\partial x}\right) = \int \mathrm{P}\partial x + \mathrm{f} : y.$$

Nunc iterum quantitate y ut constante spectata, erit

$$\partial z = \partial x \left(\frac{\partial z}{\partial x}\right) \text{ seu } \partial z = \partial x \int \mathrm{P}\partial x + \partial x \,\mathrm{f} : y,$$

ubi cum $\int P\partial x$ sit functio ipsarum x et y, quarum haec y constans assumitur, integratio denuo instituta dabit

$$z = \int\partial x\int P\partial x + x f : y + F : y,$$

quod est integrale completum aequationis differentio - differentialis propositae $\left(\frac{\partial\partial z}{\partial x^2}\right) = P$; propterea quod duas functiones arbitrarias $f : y$ et $F : y$ complectitur, quarum utramque ita pro lubitu accipere licet, ut etiam functiones discontinuae non excludantur.

Corollarium 1.

246. Quodsi ergo proponatur haec conditio $\left(\frac{\partial\partial z}{\partial x^2}\right) = 0$ ejus integratio completa dabit

$$z = x f : y + F : y,$$

ob $P = 0$, cujus veritas ex differentiatione perspicitur, unde fit primo $\left(\frac{\partial z}{\partial x}\right) = f : y$, tum vero $\left(\frac{\partial\partial z}{\partial x^2}\right) = 0$.

Corollarium 2.

247. Eodem modo in genere integrale inventum per differentiationem comprobatur. Cum enim invenerimus

$$z = \int\partial x\int P\partial x + x f : y + F : y,$$

prima differentiatio praebet

$$\left(\frac{\partial z}{\partial x}\right) = \int P\partial x + f : y,$$

repetita vero $\left(\frac{\partial\partial z}{\partial x^2}\right) = P$.

Corollarium 3.

248. Simili modo si haec proponatur conditio $\left(\frac{\partial\partial z}{\partial y^2}\right) = Q$, existente Q functione quacunque ipsarum x et y, integrale completum reperitur

$$z = \int\partial y\int Q\partial y + y f : x + F : x,$$

ubi in geminato integrali $\int \partial y \int Q \partial y$ quantitas x pro constante habetur.

Scholion.

249. Hinc ratio integralium completorum, quae ex formulis differentialibus secundi gradus nascuntur, in genere perspicitur, quae in hoc est sita, ut duae functiones arbitrariae invehantur, ubi iterum notandum est, has functiones tam discontinuas quam continuas esse posse. Nisi ergo per totam hanc sectionem integralia duas hujusmodi functiones arbitrarias involvant, ea pro completis haberi nequeunt. Quotiescunque enim problema ad hujusmodi aequationem $(\frac{\partial\partial z}{\partial x^2}) = P$ perducit, ejus indoles semper ita est comparata, ut tributo ipsi x certo quodam valore $x = a$, tam formula $(\frac{\partial z}{\partial x})$ quam ipsa quantitas z datae cuipiam functioni ipsius y aequari possit. Quare si tam integrale $\int P \partial x$ quam hoc $\int \partial x \int P \partial x$ ita accipiatur, ut posito $x = a$ evanescat, erit pro eodem casu $x = a$, valor

$$(\frac{\partial z}{\partial x}) = f:y \quad \text{et} \quad z = af:y + F:y,$$

unde ex problematis natura utraque functio $f:y$ et $F:y$ definitur. Haec autem applicatio ad omnes casus fieri non posset, nisi integrale completum haberetur; quamobrem in hoc praecipue est incumbendum, ut omnium hujusmodi problematum integralia completa habeantur. Caeterum hic in perpetuum monendum duco, quoties hujusmodi formula integralis $\int P \partial x$ occurrit, semper solam quantitatem x variabilem accipi esse intelligendam; siquidem si etiam y variabilis acciperetur, formula $\int P \partial x$ ne significatum quidem admitteret. Simili modo in formula $\int \partial x \int P \partial x$ intelligi debet, in utraque integratione solam x variabilem assumi. Sin autem talis forma $\int \partial y \int P \partial x$ occurrat, intelligendum est, integrale $\int P \partial x$ ex variabilitate solius x colligi debere, quod si ponatur $= R$, ut habeatur $\int R \partial y$, hic jam sola y pro variabili erit habenda.

Exemplum 1.

250. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{xy}{a}$.

Cum hic sit $P = \frac{xy}{a}$, erit

$$\int P\partial x = \frac{xxy}{2a} \text{ et } \int \partial x \int P\partial x = \frac{x^3 y}{6a},$$

sicque habebitur ex prima integratione

$$\left(\frac{\partial z}{\partial x}\right) = \frac{xxy}{2a} + \mathrm{f}:y,$$

ita ut posito $x = a$, formula $\left(\frac{\partial z}{\partial x}\right)$ functioni cuicunque ipsius y aequari possit, seu applicatae curvae cujuscunque respondenti abscissae y. Tum vero altera integratione instituta, erit

$$z = \frac{x^3 y}{6a} + x\mathrm{f}:y + \mathrm{F}:y,$$

qui valor casu $x = a$ denuo functioni cuicunque ipsius y aequari potest.

Fxemplum 2.

251. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{ax}{\sqrt{(xx+yy)}}$.

Ob $P = \frac{ax}{\sqrt{(xx+yy)}}$, erit

$$\int P\partial x = a\sqrt{(xx+yy)}, \text{ et}$$

$$\int \partial x \int P\partial x = a\int \partial x \sqrt{(xx+yy)} = \tfrac{1}{2} ax\sqrt{(xx+yy)} + \tfrac{1}{2} ayyl\left[x + \sqrt{(xx+yy)}\right];$$

unde prima integratio praebet

$$\left(\frac{\partial z}{\partial x}\right) = a\sqrt{(xx+yy)} + \mathrm{f}:y \text{ altera vero}$$

$$z = \tfrac{1}{2} ax\sqrt{(xx+yy)} + \tfrac{1}{2} ayyl\left[x + \sqrt{(xx+yy)}\right] + x\mathrm{f}:x + \mathrm{F}:y.$$

Exemplum 3.

252. *Quaeratur binarum variabilium x et y ejusmodo functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{1}{\sqrt{(aa - xx - yy)}}$,

Cum sit $P = \frac{1}{\sqrt{(aa - xx - yy)}}$, erit

$$\int P\partial x = \text{Ang. sin. } \frac{x}{\sqrt{(aa - yy)}},$$

tum vero

$$\int\partial x\int P\partial x = x \text{ Ang. sin. } \frac{x}{\sqrt{(aa - yy)}} - \int\frac{x\partial x}{\sqrt{(aa - xx - yy)}}.$$

Quare integratio prima praebet

$$\left(\frac{\partial z}{\partial x}\right) = \text{Ang. sin. } \frac{x}{\sqrt{(aa - yy)}} + f:y,$$

hincque ipsa functio quaesita erit

$$z = x \text{ Ang. sin } \frac{x}{\sqrt{(aa - yy)}} + \sqrt{(aa - xx - yy)} + xf:y + F:y.$$

Exemplum 4.

253. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = x \sin.(x + y)$.

Ob $P = x \sin.(x + y)$, erit

$$\int P\partial x = \int x\partial x \sin.(x + y) = -x\cos.(x + y) + \int\partial x \cos.(x + y)$$

$$\text{seu } \int P\partial x = -x\cos.(x + y) + \sin.(x + y).$$

Tum vero est

$$\int x\partial x \cos.(x + y) = x\sin.(x + y) + \cos.(x + y),$$

ideoque

$$\int\partial x\int P\partial x = -2\cos.(x + y) - x\sin.(x + y).$$

Quocirca ambo nostra integralia erunt

$$\left(\frac{\partial z}{\partial x}\right) = \sin.(x + y) - x\cos.(x + y) + f:y \quad \text{et}$$

$$z = -2\cos.(x + y) - x\sin.(x + y) + xf:y + F:y.$$

Problema 42.

254. Si z debeat esse ejusmodi functio veriabilium x et y, ut sit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = P\left(\frac{\partial z}{\partial x}\right) + Q,$$

existentibus P et Q functionibus quibusvis ipsarum x et y, indolem functionis z in genere investigare.

Solutio.

Ponamus hic $\left(\frac{\partial z}{\partial x}\right) = v$, ut sit $\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial v}{\partial x}\right)$, erit nostra aequatio integranda

$$\left(\frac{\partial v}{\partial x}\right) = Pv + Q.$$

Spectetur ergo sola x ut variabilis, et ob $\partial v = \partial x\left(\frac{\partial v}{\partial x}\right)$, erit

$$\partial v = Pv\partial x + Q\partial x,$$

quae per $e^{-\int P\partial x}$ multiplicata et integrata dat

$$e^{-\int P\partial x} v = \int e^{-\int P\partial x} Q\partial x + f : y,$$

ideoque

$$\left(\frac{\partial z}{\partial x}\right) = e^{\int P\partial x}\int e^{-\int P\partial x} Q\partial x + e^{\int P\partial x} f : y.$$

Retineatur sola x variabilis, spectata y ut constante, et ob

$$\partial z = \partial x\left(\frac{\partial z}{\partial x}\right) \text{ erit}$$

$$z = \int e^{\int P\partial x}\partial x\int e^{-\int P\partial x} Q\partial x + f : y\int e^{\int P\partial x}\partial x + F : y,$$

quod ob binas functiones arbitrarias $f : y$ et $F : y$ est integrale completum.

Corollarium 1.

255. Problema hoc multo latius patet praecedente, cum conditio proposita etiam formulam primi gradus $\left(\frac{\partial z}{\partial x}\right)$ involvat, nihilo vero minus solutio feliciter successit.

Corollarium 2.

256. Hic ergo quadruplici integratione est opus, primo scilicet quaeri debet integrale $\int P\partial x$, quod si ponatur $= lR$, quaeri porro debet integrale

$$\int e^{\int P\partial x}\,\partial x = \int R\partial x,$$

quod si ponamus $= S$, restat integrale

$$\int R\partial x \int \frac{Q\partial x}{R} = \int \partial S \int \frac{Q\partial x}{R},$$

quod abit in

$$S\int \frac{Q\partial x}{R} - \int \frac{QS\partial x}{R},$$

ita ut insuper hae duae formae integrari debeant.

Corollarium 3.

257. Eodem omnino modo resolvitur problema, quo esse debet

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial z}{\partial y}\right) + Q,$$

si P et Q fuerint functiones quaecunque datae ipsarum x et y. Reperitur enim

$$\left(\frac{\partial z}{\partial y}\right) = e^{\int P\partial y}\int e^{-\int P\partial y}\,Q\partial y + e^{\int P\partial y}.\,f:x \text{ et}$$

$$z = \int e^{\int P\partial y}\,\partial y \int e^{-\int P\partial y}\,Q\partial y + f:x.\int e^{\int P\partial y}\,\partial y + F:x.$$

Exemplum 1.

258. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{n}{x}\left(\frac{\partial z}{\partial x}\right)$.*

Posito $\left(\frac{\partial z}{\partial x}\right) = v$, sumtoque solo x variabili, erit $\frac{\partial v}{\partial x} = \frac{nv}{x}$, ideoque $\frac{\partial v}{v} = \frac{n\partial x}{x}$, cujus integrale dat

$$v = \left(\frac{\partial z}{\partial x}\right) = x^n\, f:y.$$

Jam iterum sola x pro variabili habita, erit

$$\partial z = x^n \partial x f : y,$$

cujus integrale completum est

$$z = \frac{1}{n+1} x^{n+1} f : y + F : y.$$

Casu autem $n = -1$, seu $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{-1}{x}\left(\frac{\partial z}{\partial x}\right)$, erit

$$\left(\frac{\partial z}{\partial x}\right) = \frac{1}{x} f : y, \text{ et } z = lx \,.\, f : y + F : y.$$

Exemplum 2.

259. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{n}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{a}{xy}$.

Posito $\left(\frac{\partial z}{\partial x}\right) = v$, sumtoque solo x variabili, erit

$$\partial v = \frac{nv\partial x}{x} + \frac{a\partial x}{xy},$$

quae aequatio per x^n divisa et integrata praebet

$$\frac{v}{x^n} = \frac{a}{y}\int\frac{\partial x}{x^{n+1}} = \frac{-a}{nx^n y} + f : y, \text{ seu}$$

$$v = \left(\frac{\partial z}{\partial x}\right) = \frac{-a}{ny} + x^n f : y.$$

Sit iterum sola x variabilis, ut habeatur

$$\partial z = \frac{-a\partial x}{ny} + x^n \partial x f : y,$$

prodibitque integrale completum

$$z = \frac{-ax}{ny} + \frac{1}{n+1} x^{n+1} f : y + F : y.$$

Exemplum 3.

260. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{2nx}{xx+yy}\left(\frac{\partial z}{\partial x}\right) + \frac{x}{ay}$.

Posito $\left(\frac{\partial z}{\partial x}\right) = v$, erit sumendo y constans

$$\partial v = \frac{2nxv\partial x}{xx+yy} + \frac{x\partial x}{ay},$$

quae aequatio per $(xx+yy)^n$ divisa et integrata dat

$$\frac{v}{(xx+yy)^n} = \frac{1}{ay}\int\frac{x\partial x}{(xx+yy)^n} = -\frac{1}{2(n-1)ay(xx+yy)^{n-1}} + \mathrm{f}:y,$$

seu

$$v = \left(\frac{\partial z}{\partial x}\right) = \frac{-(xx+yy)}{2(n-1)ay} + (xx+yy)^n\,\mathrm{f}:y.$$

Hinc sumto iterum y constante, fit

$$z = \frac{-x(xx+3yy)}{6(n-1)ay} + \mathrm{f}:y.\int(xx+yy)^n\partial x + \mathrm{F}:y.$$

Casu quo $n=1$, seu

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{2x}{xx+yy}\left(\frac{\partial z}{\partial x}\right) + \frac{x}{ay}, \text{ erit}$$

$$\frac{v}{xx+yy} = \frac{1}{ay}\int\frac{x\partial x}{xx+yy} = \frac{1}{2ay}\,l(xx+yy) + \mathrm{f}:y,$$

hinc

$$\left(\frac{\partial z}{\partial x}\right) = \frac{xx+yy}{2ay}\,l(xx+yy) + (xx+yy)\,\mathrm{f}:y, \text{ et}$$

$$z = \frac{x(xx+3yy)}{6ay}\,l(xx+yy) - \frac{1}{9ay}\left(x^3+6xy^2-6y^3\,\text{Ang. tang.}\,\frac{x}{y}\right) + \frac{1}{3}x(xx+3yy)\,\mathrm{f}:y + \mathrm{F}:y.$$

Problema 43.

261. Si z debeat esse ejusmodi functio binarum variabilium x et y, ut sit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \mathrm{P}\left(\frac{\partial z}{\partial x}\right) + \mathrm{Q},$$

existentibus P et Q functionibus quibuscunque datis omnium trium variabilium x, y et z, indolem functionis z investigare.

Solutio.

Posita quantitate y constante, erit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{\partial\partial z}{\partial x^2} \text{ et } \left(\frac{\partial z}{\partial x}\right) = \frac{\partial z}{\partial x};$$

ideoque habebitur aequatio differentialis secundi gradus ad librum praecedentem pertinens

$$\partial\partial z = P\partial x\partial z + Q\partial x^2,$$

quae duas tantum variabiles x et z involvere est censenda, quia y in ea tanquam constans spectatur. Tentetur ergo integratio hujus aequationis per methodos ibi expositas; quae si successerit, loco binarum constantium, quas duplex integratio invehit, scribantur ipsius y functiones indefinitae $f:y$ et $F:y$, quae adeo discontinuae accipi possunt, sicque habebitur aequationis propositae integrale completum.

Corollarium 1.

262. Reducitur ergo solutio hujus problematis ad methodum integrandi in superiori libro traditam, ubi functionem unius variabilis ex data differentialium secundi gradus relatione investigari oportebat.

Corollarium 2.

263. Quodsi ergo resolutionem omnium aequationum differentialium secundi gradus, quae binas tantum variabiles involvunt, hic nobis concedi postulemus, solutio nostri problematis pro confecta est censenda.

Corollarium 3.

264. Me non monente intelligitur, eodem modo aequationem

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = P\left(\frac{\partial z}{\partial y}\right) + Q,$$

tractari oportere, ejusque solutionem tanquam confectam spectari posse, quaecunque fuerint P et Q functiones ipsarum x, y et z.

Scholion 1.

265. Ex solutionis ratione intelligitur, problema multo latius patens simili modo resolvi posse: si enim formula $\left(\frac{\partial\partial z}{\partial x^2}\right)$ quomodo-

cunque per quantitates principales x, y et z ac praeterea formulam $\left(\frac{\partial z}{\partial x}\right)$ determinetur, ita ut etiam hujus formulae $\left(\frac{\partial z}{\partial x}\right)$ potestates aliaeve functiones quaecunque ingrediantur, solutio semper ad librum superiorem revocabitur; quia ponendo y constans fit

$$\left(\frac{\partial z}{\partial x}\right) = \frac{\partial z}{\partial x} \quad \text{et} \quad \left(\frac{\partial \partial z}{\partial x^2}\right) = \frac{\partial \partial x}{\partial x^2},$$

ideoque resultat aequatio differentialis secundi gradus formae consuetae duas tantum variabiles x et z involvens. Hoc tantum teneatur, loco constantium per utramque integrationem ingredientium scribi oportere formas $f:y$ et $F:y$. Satis igitur notabilem partem propositi nostri expedivimus, scilicet cum vel $\left(\frac{\partial \partial z}{\partial x^2}\right)$ utcunque per x, y, z et $\left(\frac{\partial z}{\partial x}\right)$, vel $\left(\frac{\partial \partial z}{\partial y^2}\right)$ utcunque per x, y, z et $\left(\frac{\partial z}{\partial y}\right)$ determinatur, ibi nempe excluditur formula primi gradus $\left(\frac{\partial z}{\partial y}\right)$, hic vero formula $\left(\frac{\partial z}{\partial x}\right)$. Quae si accederet, quaestio hac methodo neutiquam tractari posset; quemadmodum vel ex hoc casu simplicissimo $\left(\frac{\partial \partial z}{\partial x^2}\right) = \left(\frac{\partial z}{\partial y}\right)$ intelligere licet, cujus resolutio maxime ardua est putanda.

Scholion 2.

266. Cum igitur trium formularum differentialium secundi gradus $\left(\frac{\partial \partial z}{\partial x^2}\right)$, $\left(\frac{\partial \partial z}{\partial x \partial y}\right)$, $\left(\frac{\partial \partial z}{\partial y^2}\right)$ primam ac tertiam hactenus sim contemplatus, quatenus earum per reliquas quantitates, determinatio resolutionem admittit methodo quidem hic adhibita: superest ut formulam quoque secundam $\left(\frac{\partial \partial z}{\partial x \partial y}\right)$ consideremus, et quibusnam determinationibus per reliquas quantitates x, y, z, $\left(\frac{\partial z}{\partial x}\right)$, $\left(\frac{\partial z}{\partial y}\right)$ solutio absolvi queat, investigemus, in quo negotio a casibus simplicissimis exordiri conveniet.

Problema 44.

267. Si z ejusmodi debeat esse functio binarum variabilium x et y, ut fiat $\left(\frac{\partial\partial z}{\partial x \partial y}\right) = \mathrm{P}$, existente P functione quacunque data ipsarum x et y, indolem functionis z generaliter determinare.

Solutio.

Ponatur $\left(\frac{\partial z}{\partial x}\right) = v$, eritque $\left(\frac{\partial\partial z}{\partial x \partial y}\right) = \left(\frac{\partial v}{\partial y}\right)$, ideoque habebitur $\left(\frac{\partial x}{\partial y}\right) = \mathrm{P}$. Jam spectetur quantitas x ut constans, ita ut P solam variabilem y contineat, eritque $\partial v = \mathrm{P}\partial y$, unde in hypothesi quantitatis x constantis integrando prodit

$$v = \left(\frac{\partial z}{\partial x}\right) = \int \mathrm{P}\partial y + f' : x,$$

ubi $\int \mathrm{P}\partial y$ erit functio data ipsarum x et y. Nunc porro spectetur x ut variabilis, y vero ut constans, ut adipiscamur hanc aequationem differentialem

$$\partial z = \partial x \int \mathrm{P}\partial y + \partial x f' : x,$$

quae integrata dat

$$z = \int \partial x \int \mathrm{P}\partial y + f : x + \mathrm{F} : y,$$

ubi cum habeantur duae functiones arbitrariae, id indicio est, hoc integrale esse completum.

Corollarium 1.

268. Si ordine inverso primum y tum vero x constans posuissemus, invenissemus

$$\left(\frac{\partial z}{\partial y}\right) = \int \mathrm{P}\partial x + f' : y, \text{ et } z = \int \partial y \int \mathrm{P}\partial x + f : y + \mathrm{F} : x,$$

qui valor aeque satisfacit ac praecedens.

Corollarium 2.

269. Patet ergo vel fore

$$\int \partial x \int \mathrm{P}\partial y = \int \partial y \int \mathrm{P}\partial x,$$

vel differentiam saltem exprimi per aggregatum ex functione ipsius x et functione ipsius y. Quod etiam inde patet quod posito

$$\int \partial x \int P \partial y = \int \partial y \int P \partial x = V,$$

fiat utrinque $P = \left(\frac{\partial \partial V}{\partial x \partial y}\right)$.

Corollarium 3.

270. Si sit $P = 0$, seu debeat esse $\left(\frac{\partial \partial z}{\partial x \partial y}\right) = 0$, reperitur pro indole functionis z haec forma

$$z = f : x + F : y.$$

Scholion.

271. Hic casus in doctrina solidorum frequenter occurrit, si enim natura superficiei exprimatur aequatione inter ternas coordinatas x, y et u, erit soliditas $= \int \partial x \int u \partial y$, quare si soliditas exprimatur per z, erit $\left(\frac{\partial \partial z}{\partial x \partial y}\right) = u$, ordinatae scilicet ad binas x et y normali. Tum vero si ponatur

$$\partial u = p \partial x + q \partial y,$$

superficies hujus solidi erit

$$= \int \partial x \int \partial y \, \sqrt{(1 + pp + qq)},$$

quae superficies si exprimatur littera z, erit

$$\left(\frac{\partial \partial z}{\partial x \partial y}\right) = \sqrt{(1 + pp + qq)}.$$

Quando ergo in nostro problemate ejusmodi functio z ipsarum x et y quaeritur, ut sit $\left(\frac{\partial \partial z}{\partial x \partial y}\right) = P$, idem est ac si quaeratur soliditas respondens superficiei, cujus natura aequatione inter ternas coordinatas x, y et P exprimitur. Exemplis igitur aliquot hunc calculum illustremus.

Exemplum 1.

272. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial \partial z}{\partial x \partial y}\right) = \alpha x + \beta y$.

Cum hic sit $P = \alpha x + \beta y$, erit

$\int P \partial y = \alpha x y + \frac{1}{2} \beta y y$ et

$\int \partial x \int P \partial y = \frac{1}{2} \alpha x x y + \frac{1}{2} \beta x y y = \frac{1}{2} x y (\alpha x + \beta y)$,

unde functio quaesita z ita exprimitur, ut sit

$$z = \frac{1}{2} x y (\alpha x + \beta y) + f : x + F : y.$$

Exemplum 2.

273. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial \partial y}{\partial x \partial y}\right) = \sqrt{(a a - y y)}$.

Hic est $P = \sqrt{(a a - y y)}$, ergo

$\int P \partial x = x \sqrt{(a a - y y)}$,

ubi quia perinde est, a variabilitate ipsius x incipio. Hinc igitur fit

$$\int \partial y \int P \partial x = x \int \partial y \sqrt{(a a - y y)}$$
$$= \frac{1}{2} x y \sqrt{(a a - y y)} + \frac{1}{2} a a x \int \frac{\partial y}{\sqrt{(a a - y y)}},$$

ex quo integrale completum erit

$$z = \frac{1}{2} x y \sqrt{(a a - y y)} + \frac{1}{2} a a x \text{ Ang. sin. } \frac{y}{a} + f : x + F : y,$$

Exemplum 3.

274. *Quaeratur binarum variabilium x et y ejusmodi functio z, ut sit* $\left(\frac{\partial \partial z}{\partial x \partial y}\right) = \frac{a}{\sqrt{(a a - x x - y y)}}$.

Ob $P = \frac{a}{\sqrt{(a a - x x - y y)}}$, erit

$\int P \partial y = a \text{ Ang. sin. } \frac{y}{\sqrt{(a a - x x)}}$, hinc

$\int \partial x \int P \partial y = a \int \partial x \text{ Ang. sin. } \frac{y}{\sqrt{(a a - x x)}}$.

Ponatur brevitatis gratia

$$\text{Ang. sin.} \frac{y}{\sqrt{(aa - xx)}} = \Phi, \text{ erit}$$

$$\int \partial x \int P \partial y = a \int \Phi \partial x = a x \Phi - a \int x \partial x \left(\frac{\partial \Phi}{\partial x}\right),$$

in hac enim integratione y pro constante habetur. Quare ob $\frac{y}{\sqrt{(aa - xx)}} = \sin. \Phi$, erit

$$\frac{y x}{(a a - x x)^{\frac{3}{2}}} = \left(\frac{\partial \Phi}{\partial x}\right) \cos. \Phi.$$

At vero est

$$\cos. \Phi = \frac{\sqrt{(aa - xx - yy)}}{\sqrt{(aa - xx)}}, \text{ hincque}$$

$$\left(\frac{\partial \Phi}{\partial x}\right) = \frac{y x}{(aa - xx)\sqrt{(aa - xx - yy)}}, \text{ et}$$

$$\int x \partial x \left(\frac{\partial \Phi}{\partial x}\right) = y \int \frac{x x \partial x}{(aa - xx)\sqrt{(aa - xx - yy)}};$$

quo integrali invento, erit

$$z = a x \text{ Ang. sin.} \frac{y}{\sqrt{(aa - xx)}} - a y \int \frac{x x \partial x}{(aa - xx)\sqrt{(aa - xx - yy)}} + f : x + F : y,$$

quae forma per integrationem evoluta reducitur ad hanc

$$z = a x \text{ Ang. sin.} \frac{y}{\sqrt{(aa - xx)}} + a y \text{ Ang. sin.} \frac{x}{\sqrt{(aa - yy)}}$$
$$- a a \text{ Ang. sin.} \frac{x y}{\sqrt{(aa - xx)(aa - yy)}} + f : x + F : y.$$

Formulae enim $\int \frac{a a \partial x}{(aa - xx)\sqrt{(aa - xx - yy)}}$ integrale ita facillime elicitur. Ponatur $\frac{x}{\sqrt{(aa - xx - yy)}} = p$, erit $x x = \frac{pp(aa - yy)}{1 + pp}$, et ob y constans per logarithmos differentiando

$$\frac{\partial x}{x} = \frac{\partial p}{p} - \frac{p \partial p}{1 + pp} = \frac{\partial p}{p(1 + pp)},$$

tum per illam formulam multiplicando

$$\frac{\partial x}{\sqrt{(aa - xx - yy)}} = \frac{\partial p}{1 + pp}.$$

Porro est

$$a a - x x = \frac{aa + ppyy}{1 + pp},$$

unde formula integralis fit

$$\int \frac{aa\partial x}{(aa-xx)\sqrt{(aa-xx-yy)}} = \int \frac{aa\partial p}{aa+ppyy} = \frac{aa}{yy}\int \frac{\partial p}{\frac{aa}{yy}+pp}$$

$$= \frac{a}{y}\text{ Ang. tang. }\frac{py}{a} = \frac{a}{y}\text{ Ang. tang. }\frac{xy}{a\sqrt{(aa-xx-yy)}}$$

$$= \frac{a}{y}\text{ Ang. sin. }\frac{xy}{\sqrt{(aa-xx)(aa-yy)}}.$$

Problema 45.

275. Si z ejusmodi esse debeat functio binarum variabilium x et y, ut sit

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = P\left(\frac{\partial z}{\partial x}\right) + Q,$$

existentibus P et Q functionibus quibuscunque ipsarum x et y, investigare indolem functionis z.

Solutio.

Ponatur $\left(\frac{\partial z}{\partial x}\right) = v$, ut oriatur ista aequatio

$$\left(\frac{\partial v}{\partial y}\right) = Pv + Q,$$

quae continet quantitates x, y et v; statuatur ergo x constans, eritque

$$\partial v = Pv\partial y + Q\partial y,$$

quae per $e^{-\int P\partial y}$ multiplicata praebet

$$e^{-\int P\partial y}v = \int e^{-\int P\partial y}Q\partial y + f':x,$$

ideoque

$$v = \left(\frac{\partial z}{\partial x}\right) = e^{\int P\partial y}\int e^{-\int P\partial y}Q\partial y + e^{\int P\partial y}f':x.$$

Nunc cum haec integralia determinate contineant x et y, spectetur y ut constans, et sequens integratio praebet

$$z = \int e^{\int P \partial y} \partial x \int e^{-\int P \partial y} Q \partial y + \int e^{\int P \partial y} \partial x \, f' : x + F : y,$$

quae integralia quovis casu evoluta fiunt manifesta.

Corollarium 1.

276. Ad hoc ergo problema resolvendum, per integrationem primo quaeratur R, ut sit $\int P \partial y = l R$; deinde quaeratur S, ut sit $\int \frac{Q \partial y}{R} = S$. Denique sit $\int R S \partial x = T$; ita ut in illis sola quantitas y, hic vero sola x pro variabili habeatur. Quo facto erit nostrum integrale completum

$$z = T + \int R \partial x \, f' : x + F : y.$$

Corollarium 2.

277. Hic ergo functio arbitraria $f : x$ in formula integrali est involuta, quae tamen si per applicatam curvae cujuscunque respondentem abscissae x exhibeatur, hoc integrale $\int R \partial x \, f' : x$ pro quovis valore ipsius y seorsim construi poterit, siquidem in hac integratione quantitas y ut constans spectatur.

Scholion.

278. Eodem plane modo resolvitur permutandis variabilibus x et y hoc problema, quo functio z quaeritur, ut sit

$$\left(\frac{\partial \partial z}{\partial x \partial y}\right) = P \left(\frac{\partial z}{\partial y}\right) + Q,$$

dummodo P et Q sint functiones ipsarum x et y tantum, ipsam functionem z non implicantes. Solutio enim ita se habebit

$$z = \int e^{\int P \partial x} \partial y \int e^{-\int P \partial x} Q \partial x + \int e^{\int P \partial x} \partial y \, f' : y + F : x.$$

Quin etiam utrinque problema latius extendi potest, ac prius resolutionem admittet, si formula $\left(\frac{\partial \partial z}{\partial x \partial y}\right)$ aequetur functioni cuicunque trium quantitatum x, y et $\left(\frac{\partial z}{\partial x}\right)$, posterius vero si $\left(\frac{\partial \partial z}{\partial x \partial y}\right)$ aequetur functioni cuicunque harum trium quantitatum x, y et $(\frac{\partial z}{\partial x})$, utroque enim casu res reducitur ad aequationem differentialem primi gradus. Neque vero haec solvendi methodus succedit, si utraque formula primi gradus $(\frac{\partial z}{\partial x})$ et $(\frac{\partial z}{\partial y})$ simul ingrediatur, vel si functiones P et Q etiam ipsam quantitatem z complectantur.

Exemplum 1.

279. *Quaeratur binarum variabilium x et y functio z, ut sit* $(\frac{\partial \partial z}{\partial x \partial y}) = \frac{n}{y}(\frac{\partial z}{\partial x}) + \frac{m}{x}$.

Sit $(\frac{\partial z}{\partial x}) = v$, erit

$$(\frac{\partial v}{\partial y}) = \frac{nv}{y} + \frac{m}{x},$$

et spectata x ut constante, erit

$$\partial v = \frac{nv\partial y}{y} + \frac{m\partial y}{x},$$

unde per y^n dividendo prodit

$$\frac{v}{y^n} = \frac{m}{x}\int\frac{\partial y}{y^n} = \frac{-m}{(n-1)\,x\,y^{n-1}} + \mathrm{f}' : x;$$

ita ut sit

$$v = (\frac{\partial z}{\partial x}) = \frac{-my}{(n-1)x} + y^n \mathrm{f}' : x:$$

sumatur jam y constans, et denuo integrando obtinetur

$$z = \frac{-m}{n-1}\, y\, l\, x + y^n \mathrm{f} : x + \mathrm{F} : y.$$

Exemplum 2.

280. *Quaeratur binarum x et y functio z, ut sit*

$$(\frac{\partial \partial z}{\partial x \partial y}) = \frac{y}{xx+yy}(\frac{\partial z}{\partial x}) + \frac{a}{xx+yy}.$$

Posito $(\frac{\partial z}{\partial x}) = v$ et sumto x constante, erit

$$\partial v = \frac{v y \partial y}{xx + yy} + \frac{a \partial y}{xx + yy},$$

quae aequatio per $\sqrt{(xx + yy)}$ divisa dat

$$\frac{v}{\sqrt{(xx + yy)}} = a \int \frac{\partial y}{(xx + yy)^{\frac{3}{2}}} = \frac{a y}{xx \sqrt{(xx + yy)}} + \mathrm{f} : x.$$

Ergo

$$v = \left(\frac{\partial z}{\partial x}\right) = \frac{a y}{xx} + \sqrt{(xx + yy)} \,.\, \mathrm{f} : x,$$

sit jam y constans, reperieturque

$$z = \frac{-a y}{x} + \int \mathrm{f} : x \times \partial x \sqrt{(xx + yy)} + \mathrm{F} : y,$$

ubi quidem integrale

$$\int \mathrm{f} : x \times \partial x \sqrt{(xx + yy)},$$

ob functionem indeterminatam $\mathrm{f} : x$, etsi y constans ponitur, in genere exprimi nequit, ita ut explicate per y et functiones ipsius x exhiberi possit.

Scholion.

281. Formula ergo secundi gradus $\left(\frac{\partial \partial z}{\partial x \partial y}\right)$ non tam largam casuum resolubilium copiam admittit, quam binae reliquae $\left(\frac{\partial \partial z}{\partial x^2}\right)$ et $\left(\frac{\partial \partial z}{\partial y^2}\right)$, cum in his solutio succedat, etiamsi ipsa quantitas z quoque in earum determinationem ingrediatur, quod hic secus evenit, cum methodus non pateat hujusmodi aequationem $\left(\frac{\partial \partial z}{\partial x \partial y}\right) = \mathrm{P}\left(\frac{\partial z}{\partial x}\right) + \mathrm{Q}$, quando litterae P et Q quantitatem z continent resolvendi; neque etiam solutio locum habet, quando praeter formulam primi gradus $\left(\frac{\partial z}{\partial x}\right)$ simul quoque altera $\left(\frac{\partial z}{\partial y}\right)$ adest. Interim tamen dantur casus, quibus solutiones particulares exhiberi possunt, eaeque adeo infinitae, quae junctim sumtae solutioni generali aequivalere videntur,

etiamsi in applicatione ad usum practicum parum subsidii plerumque afferant, formas tamen hujusmodi solutionum notasse juvabit.

Problema 46.

282. Si z ejusmodi debeat esse functio binarum variabilium x et y, ut fiat $(\frac{\partial \partial z}{\partial x \partial y}) = a z$, indolem hujus functionis z particulariter saltem investigare.

Solutio.

Cum quantitas z unam ubique teneat dimensionem evidens est, si statuatur $z = e^p q$, quantitatem exponentialem e^p ex calculo evanescere. Ponamus igitur $z = e^{\alpha x} Y$, ita ut Y functionem ipsius y tantum contineat, eritque

$$(\frac{\partial z}{\partial x}) = \alpha e^{\alpha x} Y \text{ et } (\frac{\partial \partial z}{\partial x \partial y}) = \alpha e^{\alpha x} \frac{\partial Y}{\partial y} = a e^{\alpha x} Y,$$

unde fit

$$\frac{\alpha \partial Y}{Y} = a \partial y \text{ et } Y = e^{\frac{a y}{\alpha}},$$

sicque jam solutionem particularem habemus

$$z = A e^{\alpha x + \frac{a y}{\alpha}};$$

quae autem satis late patet, cum tam A quam α pro lubitu assumi possit. Plures autem valores ipsius x seorsim satisfacientes, etiam junctim sumti satisfaciunt, unde hujusmodi expressionem multo generaliorem deducimus

$$z = A e^{\alpha x + \frac{a}{\alpha} y} + B e^{\beta x + \frac{a}{\beta} y} + C e^{\gamma x + \frac{a}{\gamma} y} + D e^{\delta x + \frac{a}{\delta} y},$$

ubi cum A, B, C, etc. item α, β, γ, etc. omnes valores possibiles recipere queant, haec forma pro maxime universali est habiles recipere queant, haec forma pro maxime universali est habenda, neque si ad amplitudinem spectamus, quicquam cedere videtur superioribus solutionibus, quae binas functiones arbitrarias involvunt, propterea quod hic duplicis generis coëfficientes arbitrarii occurrunt, interim tamen haud liquet, quomodo functiones discontinuae hac relatione repraesentari queant.

Corollarium 1.

283. Pro solutione ergo particulari invenienda, sumantur bini numeri m et n, ut eorum productum sit $mn = a$, eritque $z = \mathrm{A}\, e^{mx+ny}$. Atque etiam ex iisdem numeris permutatis erit $z = \mathrm{A}\, e^{nx+my}$.

Corollarium 2.

284. Ex tali numerorum m et n pari, ut sit $mn = a$, solutiones quoque per sinus et cosinus angulorum exhiberi possunt; erit enim

$$z = \mathrm{B}\,\sin.(mx - ny),\ \text{vel}\ z = \mathrm{B}\,\cos.(mx - ny),$$

vel etiam permutando

$$z = \mathrm{B}\,\sin.(nx - my),\ \text{vel}\ z = \mathrm{B}\,\cos.(nx - my).$$

Corollarium 3.

285. Cum igitur hujusmodi formulae innumerabiles exhiberi queant, singulae per constantes quascunque multiplicatae et in unam summam collectae dabunt solutionem generalem problematis.

Scholion.

286. Neque tamen haec solutio, etsi infinities infinitas determinationes recipit, ita est comparata, ut ejusmodi solutionibus, quae binas functiones arbitrarias involvunt, aequivalens aestimari possit; propterea quod non patet, quomodo singulas litteras assumi opor-

teat, ut pro dato casu, verbi gratia $y = 0$, quantitas z vel $\left(\frac{\partial z}{\partial x}\right)$ seu $\left(\frac{\partial z}{\partial y}\right)$ data functioni ipsius x aequalis evadat, cujuscunque etiam indolis fuerit haec functio. Semper autem solutio generalis duplicis hujusmodi determinationis capax esse debet. Quando autem talem solutionem impetrare non licet, utique ejusmodi solutionibus, uti hic invenimus, contenti esse debemus. Ac tales quidem solutiones simili modo obtinere possumus, si proponatur ejusmodi aequatio:

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + Rz = 0,$$

si modo litterae P, Q, R denotent functiones ipsius x tantum. Posito enim $z = e^{\alpha y} X$, ut X sit functio solius x, ob

$$\left(\frac{\partial z}{\partial x}\right) = e^{\alpha y} \frac{\partial X}{\partial x}, \quad \left(\frac{\partial z}{\partial y}\right) = \alpha e^{\alpha y} X, \text{ et ob } \left(\frac{\partial\partial z}{\partial x \partial y}\right) = \alpha e^{\alpha y} \left(\frac{\partial X}{\partial x}\right),$$

erit

$$\frac{\alpha \partial X}{\partial x} + \frac{P \partial X}{\partial x} + \alpha Q X + R X = 0,$$

unde reperitur

$$\frac{\partial X}{X} = \frac{-\partial x\,(\alpha Q + R)}{\alpha + P};$$

sicque elicitur pro quovis numero α idoneus valor ipsius X. Quare sumendis infinitis numeris α, hoc modo expressio infinities infinitas determinationes recipiens colligitur

$$z = Ae^{\alpha y} X + Be^{\beta y} X' + Ce^{\gamma y} X'' + \text{etc.}$$

Verumtamen dantur etiam casus ejusmodi aequationum, quae solutiones vere completas admittunt, quarum rationem in sequente problemate indagemus.

Problema 47.

287. Proposita aequatione resolvenda

$$\left(\frac{\partial\partial z}{\partial x \partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + Rz + S = 0,$$

investigare cujusmodi functiones ipsarum x et y esse debeant quantitates P, Q, R et S, ut haec aequatio solutionem vere completam admittat.

Solutio.

Sit V functio quaecunque ipsarum x et y, ac ponatur $z = e^V v$, ita ut jam v sit quantitas incognita, cujus valorem investigari oporteat. Cum igitur sit

$$\left(\frac{\partial z}{\partial x}\right) = e^V \left[\left(\frac{\partial v}{\partial x}\right) + v \left(\frac{\partial V}{\partial x}\right)\right], \quad \left(\frac{\partial z}{\partial y}\right) = e^V \left[\left(\frac{\partial v}{\partial y}\right) + v \left(\frac{\partial V}{\partial y}\right)\right],$$

facta substitutione totaque aequatione per e^V divisa prodibit sequens aequatio

$$\left.\begin{array}{l} e^{-V} S + \left(\frac{\partial \partial v}{\partial x \partial y}\right) + \left(\frac{\partial V}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial V}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial V}{\partial x}\right)\left(\frac{\partial V}{\partial y}\right) v \\ \quad + P \left(\frac{\partial v}{\partial x}\right) + Q \left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial \partial V}{\partial x \partial y}\right) v \\ \quad + P \left(\frac{\partial V}{\partial x}\right) v \\ \quad + Q \left(\frac{\partial V}{\partial y}\right) v \\ \quad + R v \end{array}\right\} = 0.$$

Efficiendum jam est, ut haec aequatio resolutionem completam admittat. Cum igitur ante viderimus, talem aequationem

$$\left(\frac{\partial \partial v}{\partial x \partial y}\right) + T \left(\frac{\partial v}{\partial x}\right) + e^{-V} S = 0$$

generaliter resolvi posse, qualescunque etiam functiones ipsarum x et y pro S, T et V accipiantur, ad hanc aequationem illam redigamus. Necesse igitur est statui

$$P + \left(\frac{\partial V}{\partial y}\right) = T, \quad Q + \left(\frac{\partial V}{\partial x}\right) = 0 \quad \text{et}$$

$$R + Q \left(\frac{\partial V}{\partial y}\right) + P \left(\frac{\partial V}{\partial x}\right) + \left(\frac{\partial V}{\partial x}\right)\left(\frac{\partial V}{\partial y}\right) + \left(\frac{\partial \partial V}{\partial x \partial y}\right) = 0,$$

unde obtinemus

$$P = T - \left(\frac{\partial V}{\partial y}\right), \quad Q = -\left(\frac{\partial V}{\partial x}\right) \quad \text{et}$$

$$R = \left(\frac{\partial V}{\partial x}\right)\left(\frac{\partial V}{\partial y}\right) - T \left(\frac{\partial V}{\partial x}\right) - \left(\frac{\partial \partial V}{\partial x \partial y}\right).$$

Cum igitur per §. 275. reperietur

$$v = -\int e^{-\int T \partial y} \partial y \int e^{\int T \partial y - V} S \partial y + \int e^{-\int T \partial y} \partial x \, f : x + F : y,$$

erit aequationis propositae

$$\left(\frac{\partial \partial z}{\partial x \partial y}\right) + P \left(\frac{\partial z}{\partial x}\right) + Q \left(\frac{\partial z}{\partial y}\right) + R z + S = 0,$$

si modo litterae P, Q, R assignatos teneant valores, integrale completum

$$z = -e^{V}\int e^{-\int T\partial y}\partial x \int e^{\int T\partial y - V} S\partial y + e^{V}\int e^{-\int T\partial y}\partial x\, f:x + e^{V}F:y,$$

quandoquidem hic formae $f:x$ et $F:y$ functiones quascunque ipsius x et y denotant.

Corollarium 1.

288. Quaecunque ergo functiones ipsarum x et y pro litteris T et V accipiantur, inde oriuntur valores idonei pro litteris P, Q, R assumendi, ut aequatio resolutionem completam admittat, functio autem S arbitrio nostro relinquitur.

Corollarium 2.

289. Possunt etiam in aequatione proposita functiones P et Q indefinitae relinqui, eritque tum

$$V = -\int Q\partial x \text{ et } \left(\frac{\partial V}{\partial y}\right) = -\int \partial x \left(\frac{\partial Q}{\partial y}\right), \text{ atque}$$
$$\left(\frac{\partial\partial V}{\partial x\partial y}\right) = -\left(\frac{\partial Q}{\partial y}\right);$$

unde tantum quantitas R ita determinari debet, ut sit

$$R - PQ - \left(\frac{\partial Q}{\partial y}\right) = 0, \text{ seu}$$
$$R = PQ + \left(\frac{\partial Q}{\partial y}\right).$$

Corollarium 3.

290. Quia hic pro $\int Q\partial x$ scribi potest $\int Q\partial x + Y$, denotante Y functionem quamcunque ipsius y, ob

$$V = -\int Q\partial x - Y,$$

complete integrabilis erit haec aequatio:

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + \left[PQ + \left(\frac{\partial Q}{\partial y}\right)\right] z + S = 0,$$

cujus integrale est

$$z = e^{-\int Q\partial x - Y} v, \text{ existente}$$

$$\left(\frac{\partial\partial v}{\partial x\partial y}\right) + \left[P - \int\partial x\left(\frac{\partial Q}{\partial y}\right) - \frac{\partial Y}{\partial y}\right]\left(\frac{\partial v}{\partial x}\right) + e^{-V} S = 0,$$

ubi

$$T = P - \int\partial x\left(\frac{\partial Q}{\partial y}\right) - \frac{\partial Y}{\partial y},$$

ac propterea

$$\int T\partial y = \int P\partial y - \int Q\partial x - Y,$$

unde valor ipsius v facile definitur.

Scholion.

291. In hoc calculo, quo differentialia formularum integralium capi oportet, dum alia quantitas variabilis assumitur, atque in integratione supponitur, haec regula est tenenda, quod si fuerit $V = \int Q\partial x$, fore $\left(\frac{\partial V}{\partial y}\right) = \int\partial x\left(\frac{\partial Q}{\partial y}\right)$. Cum enim sit $\left(\frac{\partial V}{\partial x}\right) = Q$, erit $\left(\frac{\partial\partial V}{\partial x\partial y}\right) = \left(\frac{\partial Q}{\partial y}\right)$. Quodsi ergo statuatur $\left(\frac{\partial V}{\partial y}\right) = S$, erit $\left(\frac{\partial S}{\partial x}\right) = \left(\frac{\partial Q}{\partial y}\right)$, et

$$S = \left(\frac{\partial V}{\partial y}\right) = \int\partial x\left(\frac{\partial Q}{\partial y}\right);$$

unde vicissim colligitur, si fuerit $S = \int\partial x\left(\frac{\partial Q}{\partial y}\right)$, fore ob $\int S\partial y = V$, integrando $\int S\partial y = \int Q\partial x$; quod cum ex principiis ante stabilitis per se sit manifestum, non opus esse judico, pro hoc quasi novo algorithmi genere praecepta seorsim tradere. Videamus autem in aliquot exemplis, cujusmodi aequationes ope hujus methodi complete resolvere liceat.

Exemplum 1.

292. *Proposita aequatione differentio-differentiali*

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + a\left(\frac{\partial z}{\partial x}\right) + b\left(\frac{\partial z}{\partial y}\right) + Rz + S = 0,$$

definire indolem functionis R, ut haec aequatio resolutionem admittat, existente S functione quacunque ipsarum x et y.

Cum sit $P = a$ et $Q = b$, erit $R = ab$ et $V = -bx$; tuto enim functio Y omitti potest, quia in sequente integratione jam binae functiones arbitrariae introducuntur, erit ergo $T = a$; unde posito $z = e^{-bx} v$, habebitur haec aequatio:

$$\left(\frac{\partial\partial v}{\partial x \partial y}\right) + a\left(\frac{\partial v}{\partial x}\right) + e^{bx}\, S = 0,$$

ac posito $\left(\frac{\partial v}{\partial x}\right) = u$, fit

$$\left(\frac{\partial u}{\partial y}\right) + au + e^{bx}\, S = u,$$

et sumto x constante

$$e^{ay}\, u = -\int e^{ay+bx} S\partial y + f' : y, \text{ ergo}$$

$$u = \left(\frac{\partial v}{\partial x}\right) = -e^{-ay}\int e^{ay+bx} S\partial y + e^{-ay} f' : x,$$

et sumto jam y constante

$$v = -e^{-ay}\int\partial x\int e^{ay+bx} S\partial y + e^{-ay} f : x + F : y,$$

sumendo

$$\int\partial x f' : x = f : x.$$

Quod si jam pro $e^{-bx} f : x$ scribatur $f : x$, erit

$$z = -e^{-ay-bx}\int\partial x\int e^{ay+bx} S\partial y + e^{-ay} f : x + e^{-bx} F : y.$$

Aliter.

Si sumsissemus $V = -bx - ay$, prodiisset $T = a - a = 0$; ideoque posito $z = e^{-bx-ay} v$, quantitas v ex hac aequatione:

$$\left(\frac{\partial\partial v}{\partial x \partial y}\right) + e^{bx+ay}\, S = 0$$

definiri deberet, quae dat

$$\left(\frac{\partial v}{\partial x}\right) = -\int e^{bx+ay} S\partial y + f' : x, \text{ et}$$

$$v = -\int\partial x\int e^{bx+ay} S\partial y + f : x + F : y, \text{ et}$$

$$z = e^{-bx-ay}\left(-\int\partial x\int e^{bx+ay} S\partial y + f : x + F : y\right),$$

quae forma simplicior est praecedente, etiamsi eodem redeat, estque hoc integrale completum aequationis

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + a\left(\frac{\partial z}{\partial x}\right) + b\left(\frac{\partial z}{\partial y}\right) + abz + S = 0.$$

Exemplum 2.

293. *Proposita aequatione differentio-differentiali*

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \frac{a}{y}\left(\frac{\partial z}{\partial x}\right) + \frac{b}{x}\left(\frac{\partial z}{\partial y}\right) + Rz + S = 0,$$

definiri indolem functionis R, ut haec aequatio resolutionem admittat, existente S functione quacunque ipsarum x et y.

Cum sit $P = \frac{a}{y}$ et $Q = \frac{b}{x}$, erit $V = -blx - Y$, hincque $R = \frac{ab}{xy}$, et aequatio integrabilis erit

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \frac{a}{y}\left(\frac{\partial z}{\partial x}\right) + \frac{b}{x}\left(\frac{\partial z}{\partial y}\right) + \frac{ab}{xy}z + S = 0.$$

Quoniam igitur fit

$$T = P + \left(\frac{\partial V}{\partial y}\right) = \frac{a}{y} - \frac{\partial Y}{\partial y},$$

sumamus $Y = +aly$, ut fiat $T = 0$, ac posito:

$$z = e^{-blx - aly}v = x^{-b}y^{-a}v,$$

quantitas v ex hac aequatione definiri debet

$$\left(\frac{\partial\partial v}{\partial x\partial y}\right) + x^b y^a S = 0,$$

unde fit

$$\left(\frac{\partial v}{\partial x}\right) = -x^b \int y^a S\partial y + f' : x \quad \text{et}$$

$$v = -\int x^b \partial x \int y^a S\partial y + f : x + F : y,$$

ideoque

$$z = \frac{-\int x^b \partial x \int y^a S\partial y + f : x + F : y}{x^b y^a}.$$

Scholion 1.

294. Hinc igitur patet ope istius methodi in genere integrari posse hanc aequationem:

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + \left[PQ + \left(\frac{\partial Q}{\partial y}\right)\right] z + S = 0,$$

quaecunque functiones ipsarum x et y pro P, Q et S accipiantur. Ac resolutio quidem ita se habet, ut posito

$$z = e^{-\int Q\partial x - Y} v,$$

haec quantitas v determinetur hac aequatione:

$$\left(\frac{\partial\partial v}{\partial x\partial y}\right) + \left[P - \int\partial x\left(\frac{\partial Q}{\partial y}\right) - \frac{\partial Y}{\partial y}\right]\left(\frac{\partial v}{\partial x}\right) + e^{\int Q\partial x + Y} S = 0,$$

ubi jam pro Y talis functio ipsius y accipi potest, ut hujus aequationis forma simplicissima evadat; id quod potissimum evenit, si expressio

$$P - \int\partial x\left(\frac{\partial Q}{\partial y}\right) - \frac{\partial Y}{\partial y}$$

ad nihilum redigi queat. In genere autem reperitur

$$v = -\int e^{-\int P\partial y + \int Q\partial x + Y}\partial x \int e^{\int P\partial y} S\partial y$$
$$+ \int e^{-\int P\partial y + \int Q\partial x + Y}\partial x\, f:x + F:y,$$

qui valor ergo per $e^{-\int Q\partial x - Y}$ multiplicatus praebet formam functionis z. Hoc modo autem functio Y ab arbitrio nostro pendens penitus e calculo egreditur, fitque

$$z = -e^{-\int Q\partial x}\int e^{-\int P\partial y + \int Q\partial x}\partial x \int e^{\int P\partial y} S\partial y$$
$$+ e^{-\int Q\partial x}\int e^{-\int P\partial y + \int Q\partial x}\partial x\, f:x + e^{-\int Q\partial x} F:y,$$

quod est integrale completum hujus aequationis:

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + \left[PQ + \left(\frac{\partial Q}{\partial y}\right)\right] z + S = 0.$$

Scholion 2.

295. Permutandis autem variabilibus x et y etiam haec aequatio complete integrari potest:

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) + P\left(\frac{\partial z}{\partial x}\right) + Q\left(\frac{\partial z}{\partial y}\right) + \left[PQ + \left(\frac{\partial P}{\partial x}\right)\right] z + S = 0,$$

cujus integrale erit

$$z = -e^{-\int P\partial y}\int e^{-\int Q\partial x+\int P\partial y}\partial y\int e^{\int Q\partial x}S\partial x$$
$$+e^{-\int P\partial y}\int e^{-\int Q\partial x+\int P\partial y}\partial y\,\mathrm{f}:y+e^{-\int P\partial y}\mathrm{F}:x,$$

ubi praecipue hic casus in utraque forma contentus notari meretur, si fuerit $P = Y$ et $Q = X$, existente X functione ipsius x et Y ipsius y tantum; tum enim hujus aequationis

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right)+Y\left(\frac{\partial z}{\partial x}\right)+X\left(\frac{\partial z}{\partial y}\right)+XYz+S=0,$$

integrale completum erit

$$z = -e^{-\int X\partial x-\int Y\partial y}\int e^{\int X\partial x}\partial x\int e^{\int Y\partial y}S\partial y$$
$$+e^{-\int X\partial x-\int Y\partial y}(\mathrm{f}:x+\mathrm{F}:y),$$

quod etiam ita exhiberi potest:

$$e^{\int X\partial x+\int Y\partial y}z = \mathrm{f}:x+\mathrm{F}:y-\int e^{\int X\partial x}\partial x\int e^{\int Y\partial y}S\partial y,$$

vel etiam hoc modo:

$$e^{\int X\partial x+\int Y\partial y}z = \mathrm{f}:x+\mathrm{F}:y-\int e^{\int Y\partial y}\partial y\int e^{\int X\partial x}S\partial x.$$