# CALCULI INTEGRALIS
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## PARS PRIMA,

SEU

INVESTIGATIO FUNCTIONUM DUARUM VARIABILIUM EX DATA DIFFERENTIALIUM CUJUSVIS GRADUS RELATIONE.

## SECTIO SECUNDA,

INVESTIGATIO DUARUM VARIABILIUM FUNCTIONUM EX DATA DIFFERENTIALIUM SECUNDI GRADUS RELATIONE.

# CAPUT I.

## DE FORMULIS DIFFERENTIALIBUS SECUNDI GRADUS IN GENERE.

### Problema 38.

220.

Si $z$ sit functio quaecunque binarum variabilium $x$ et $y$, ejus formulas differentiales secundi gradus exhibere.

### Solutio.

Cum $z$ sit functio binarum variabilium $x$ et $y$, ejus differentiale hujusmodi habebit formam

$$\partial z = p \partial x + q \partial y,$$

ex qua $p$ et $q$ sunt formulae differentiales primi gradus, quas ita denotare solemus

$$p = \left(\frac{\partial z}{\partial x}\right) \text{ et } q = \left(\frac{\partial z}{\partial y}\right).$$

Cum nunc sint quoque $p$ et $q$ functiones ipsarum $x$ et $y$, formulae differentiales inde natae erunt formulae differentiales secundi gradus ipsius $z$, unde intelligitur quatuor hujusmodi formulas nasci

$$\left(\frac{\partial p}{\partial x}\right), \left(\frac{\partial p}{\partial y}\right), \left(\frac{\partial q}{\partial x}\right), \left(\frac{\partial q}{\partial y}\right),$$

quarum autem secundam ac tertiam inter se congruere in calculo differentiali est demonstratum. Sed cum sit $p = \left(\frac{\partial z}{\partial x}\right)$, simili scribendi ratione erit $\left(\frac{\partial p}{\partial x}\right) = \left(\frac{\partial \partial z}{\partial x^2}\right)$, cujus scripturae significatus hinc sponte patet. Deinde eodem modo erit $\left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial \partial z}{\partial x \partial y}\right)$, atque ob $q = \left(\frac{\partial z}{\partial y}\right)$ habebimus

$$\left(\frac{\partial q}{\partial x}\right) = \left(\frac{\partial \partial z}{\partial y \partial x}\right) \text{ et } \left(\frac{\partial q}{\partial y}\right) = \left(\frac{\partial \partial z}{\partial y^2}\right).$$

Quia ergo est $\left(\frac{\partial \partial z}{\partial y \partial x}\right) = \left(\frac{\partial \partial z}{\partial x \partial y}\right)$, functioni $z$ convenient tres formulae differentiales secundi gradus, quae sunt

$$\left(\frac{\partial \partial z}{\partial x^2}\right), \left(\frac{\partial \partial z}{\partial x \partial y}\right) \text{ et } \left(\frac{\partial \partial z}{\partial y^2}\right).$$

### Corollarium 1.

221. Ut ergo functio $z$ duarum variabilium $x$ et $y$ duas habet formulas differentiales primi gradus

$$\left(\frac{\partial z}{\partial x}\right) \text{ et } \left(\frac{\partial z}{\partial y}\right),$$

ita habet tres formulas differentiales secundi gradus

$$\left(\frac{\partial \partial z}{\partial x^2}\right), \left(\frac{\partial \partial z}{\partial x \partial y}\right) \text{ et } \left(\frac{\partial \partial z}{\partial y^2}\right).$$

### Corollarium 2.

222. Hae ergo formulae per duplicem differentiationem nascuntur, unicam tantum quantitatem pro variabili accipienda. In prima scilicet bis eadem $x$ variabilis sumitur, in secunda vero in altera differentiatione $x$, in altera autem $y$ variabilis accipitur; in tertia autem bis $y$.

### Corollarium 3.

223. Simili modo patet, ejusdem functionis $z$ quatuor dari formulas differentiales tertii gradus, scilicet

$$\left(\frac{\partial^3 z}{\partial x^3}\right), \left(\frac{\partial^3 z}{\partial x^2 \partial y}\right), \left(\frac{\partial^3 z}{\partial x \partial y^2}\right), \left(\frac{\partial^3 z}{\partial y^3}\right),$$

quarti autem gradus quinque; quinti, sex, etc.

### Scholion.

224. Formulae hae differentiales secundi gradus ope substitutionis saltem ad formam primi gradus revocari possunt. Veluti

formula $\left(\frac{\partial\partial z}{\partial x^2}\right)$, si ponatur $\left(\frac{\partial z}{\partial x}\right) = p$, transformabitur in $\left(\frac{\partial p}{\partial x}\right)$; formula autem $\left(\frac{\partial\partial z}{\partial x\,\partial y}\right)$ eadem substitutione in hanc $\left(\frac{\partial p}{\partial y}\right)$. At posito $\left(\frac{\partial z}{\partial y}\right) = q$, formula $\left(\frac{\partial\partial z}{\partial x\,\partial y}\right)$ transmutatur in hanc $\left(\frac{\partial q}{\partial x}\right)$; formula autem $\left(\frac{\partial\partial z}{\partial y^2}\right)$ in hanc $\left(\frac{\partial q}{\partial y}\right)$. Vicissim autem uti ex aequalitate $p = \left(\frac{\partial z}{\partial x}\right)$ deduximus

$$\left(\frac{\partial p}{\partial x}\right) = \left(\frac{\partial\partial z}{\partial x^2}\right) \text{ et } \left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial\partial z}{\partial x\,\partial y}\right),$$

ita ex his ulterius progrediendo colligemus

$$\left(\frac{\partial\partial p}{\partial x^2}\right) = \left(\frac{\partial^3 z}{\partial x^3}\right),\ \left(\frac{\partial\partial p}{\partial x\,\partial y}\right) = \left(\frac{\partial^3 z}{\partial x^2\,\partial y}\right),\ \left(\frac{\partial\partial p}{\partial y^2}\right),\ \left(\frac{\partial^3 z}{\partial x\,\partial y^2}\right).$$

Tum vero etiam si ponamus $\left(\frac{\partial q}{\partial x}\right) = \left(\frac{\partial\partial z}{\partial x\,\partial y}\right)$, hinc sequentur istae aequalitates

$$\left(\frac{\partial\partial q}{\partial x^2}\right) = \left(\frac{\partial^3 z}{\partial x^2\,\partial y}\right) \text{ et } \left(\frac{\partial\partial q}{\partial x\,\partial y}\right) = \left(\frac{\partial^3 z}{\partial x\,\partial y^2}\right).$$

Hicque est quasi novus algorithmus, cujus principia per se ita sunt manifesta, ut majore illustratione non indigeant.

### Exemplum 1.

225. *Si sit $z = xy$, ejus formulas differentiales secundi gradus exhibere.*

Cum sit $\left(\frac{\partial z}{\partial x}\right) = y$ et $\left(\frac{\partial z}{\partial y}\right) = x$, erit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = 0,\ \left(\frac{\partial\partial z}{\partial x\,\partial y}\right) = 1 \text{ et } \left(\frac{\partial\partial z}{\partial y^2}\right) = 0.$$

### Exemplum 2.

226. *Si sit $z = x^m y^n$, ejus formulas differentiales secundi gradus exhibere.*

Cum sit $\left(\frac{\partial z}{\partial x}\right) = m x^{m-1} y^n$ et $\left(\frac{\partial z}{\partial y}\right) = n x^m y^{n-1}$, erit

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = m(m-1)x^{m-2}y^n,\ \left(\frac{\partial\partial z}{\partial x\,\partial y}\right) = mn\,x^{m-1}y^{n-1},$$
$$\left(\frac{\partial\partial z}{\partial y^2}\right) = n(n-1)x^m y^{n-2}.$$

Exemplum 3.

227. *Si sit $z = \sqrt{(xx + yy)}$, ejus formulas differentiales secundi gradus exhibere.*

Cum sit
$$\left(\frac{\partial z}{\partial x}\right) = \frac{x}{\sqrt{(xx+yy)}} \text{ et } \left(\frac{\partial z}{\partial y}\right) = \frac{y}{\sqrt{(xx+yy)}}, \text{ erit}$$
$$\left(\frac{\partial \partial z}{\partial x^2}\right) = \frac{yy}{(xx+yy)^{\frac{3}{2}}}, \quad \left(\frac{\partial \partial z}{\partial x \partial y}\right) = \frac{-xy}{(xx+yy)^{\frac{3}{2}}};$$
$$\left(\frac{\partial \partial z}{\partial y^2}\right) = \frac{xx}{(xx+yy)^{\frac{3}{2}}}.$$

Scholion.

228. Quemadmodum binae formulae differentiales primi gradus cujusque functionis $z$ ita sunt comparatae, ut sit
$$\partial z = \partial x \left(\frac{\partial z}{\partial x}\right) + \partial y \left(\frac{\partial z}{\partial y}\right),$$
et integrando
$$z = \int [\partial x \left(\frac{\partial z}{\partial x}\right) + \partial y \left(\frac{\partial z}{\partial y}\right)],$$
ita quoque in formulis secundi gradus erit
$$\left(\frac{\partial z}{\partial x}\right) = \int [\partial x \left(\frac{\partial \partial z}{\partial x^2}\right) + \partial y \left(\frac{\partial \partial z}{\partial x \partial y}\right)] \text{ et}$$
$$\left(\frac{\partial z}{\partial y}\right) = \int [\partial x \left(\frac{\partial \partial z}{\partial x \partial y}\right) + \partial y \left(\frac{\partial \partial z}{\partial y^2}\right)].$$
Tres igitur formulae secundi gradus semper ita sunt comparatae, ut geminam integrationem praebeant, si scilicet cum differentialibus $\partial x$ et $\partial y$ rite combinentur, haecque proprietas quae probe notetur, in sequentibus insigne adjumentum afferet.

Problema 39.

229. Si $z$ sit functio binarum variabilium $x$ et $y$, loco $x$ et $y$ introducantur binae novae variabiles $t$ et $u$, ita ut tam $x$

quam $y$ aequetur certae functioni ipsarum $t$ et $u$, formulas differentiales secundi gradus ipsius $z$ respectu harum novarum variabilium definire.

## Solutio.

Quatenus $z$ per $x$ et $y$ datur, datae sunt ejus formulae differentiales tam primi gradus $\left(\frac{\partial z}{\partial x}\right)$, $\left(\frac{\partial z}{\partial y}\right)$, quam secundi gradus $\left(\frac{\partial \partial z}{\partial x^2}\right)$, $\left(\frac{\partial \partial z}{\partial x \partial y}\right)$, $\left(\frac{\partial \partial z}{\partial y^2}\right)$, ex quibus quomodo formulae differentiales respectu novarum variabilium $t$ et $u$ determinentur definiri oportet. Pro primo gradu autem cum sit

$$\partial z = \partial x \left(\frac{\partial z}{\partial x}\right) + \partial y \left(\frac{\partial z}{\partial y}\right),$$

quia tam $x$ quam $y$ datur per $t$ et $u$ erit

$$\partial x = \partial t \left(\frac{\partial x}{\partial t}\right) + \partial u \left(\frac{\partial x}{\partial u}\right) \text{ et } \partial y = \partial t \left(\frac{\partial y}{\partial t}\right) + \partial u \left(\frac{\partial y}{\partial u}\right),$$

quibus valoribus substitutis habebitur ipsius $z$ differentiale plenum ex variatione utriusque $t$ et $u$ ortum

$$\partial z = \partial t \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial z}{\partial x}\right) + \partial u \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial z}{\partial x}\right) + \partial t \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial z}{\partial y}\right) + \partial u \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial z}{\partial y}\right).$$

Quodsi jam vel sola $t$ variabilis sumatur, vel sola $u$, prodibunt formulae differentiales primi gradus

$$\left(\frac{\partial z}{\partial t}\right) = \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial z}{\partial y}\right), \quad \left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial z}{\partial y}\right).$$

Simili modo ulterius progrediendo, differentiemus formulas

$$\left(\frac{\partial z}{\partial x}\right) = p \text{ et } \left(\frac{\partial z}{\partial y}\right) = q$$

primo generaliter, tum vero loco $x$ et $y$ etiam $t$ et $u$ introducamus; hincque nanciscemur

$$\left(\frac{\partial p}{\partial t}\right) = \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial p}{\partial x}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial p}{\partial y}\right), \quad \left(\frac{\partial p}{\partial u}\right) = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial p}{\partial x}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial p}{\partial y}\right),$$

$$\left(\frac{\partial q}{\partial t}\right) = \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial q}{\partial x}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial q}{\partial y}\right), \quad \left(\frac{\partial q}{\partial u}\right) = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial q}{\partial x}\right) + \left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial q}{\partial y}\right),$$

unde poterimus formulas $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$ pro variabilitate tam solius $t$ quam solius $u$ assignare; scilicet cum sit

$$\left(\frac{\partial z}{\partial t}\right) = p\left(\frac{\partial x}{\partial t}\right) + q\left(\frac{\partial y}{\partial t}\right) \text{ et } \left(\frac{\partial z}{\partial u}\right) = p\left(\frac{\partial x}{\partial u}\right) + q\left(\frac{\partial y}{\partial u}\right),$$

inveniemus

$$\left(\frac{\partial\partial z}{\partial t^2}\right) = \left(\frac{\partial\partial x}{\partial t^2}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial\partial y}{\partial t^2}\right)\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial x}{\partial t}\right)^2\left(\frac{\partial\partial z}{\partial x^2}\right)$$
$$+ 2\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \left(\frac{\partial y}{\partial t}\right)^2\left(\frac{\partial\partial z}{\partial y^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial t\partial u}\right) = \left(\frac{\partial\partial x}{\partial t\partial u}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial\partial y}{\partial t\partial u}\right)\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial\partial z}{\partial x^2}\right)$$
$$+ \left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \left(\frac{\partial y}{\partial t}\right)\left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial\partial z}{\partial y^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial u^2}\right) = \left(\frac{\partial\partial x}{\partial u^2}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial\partial y}{\partial u^2}\right)\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial x}{\partial u}\right)^2\left(\frac{\partial\partial z}{\partial x^2}\right)$$
$$+ 2\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial\partial z}{\partial x\partial y}\right) + \left(\frac{\partial y}{\partial u}\right)^2\left(\frac{\partial\partial z}{\partial y^2}\right).$$

## Corollarium 1.

230. Proposita ergo conditione quadam inter formulas differentiales functionis $z$, quatenus per variabiles $t$ et $u$ definitur, eadem conditio pro eadem functione $z$ transfertur ad alias binas variabiles $x$ et $y$, ab illis utcunque pendentes.

## Corollarium 2.

231. Formulae quidem

$$\left(\frac{\partial x}{\partial t}\right),\ \left(\frac{\partial y}{\partial t}\right),\ \left(\frac{\partial x}{\partial u}\right),\ \left(\frac{\partial y}{\partial u}\right), \text{ etc.}$$

per $t$ et $u$ exprimuntur, ex relatione, quae inter $x$, $y$ et $t$, $u$ assumitur, verum indidem eaedem formulae ad variabiles $x$ et $y$ revocari possunt.

## Scholion.

232. Quemadmodum hic variabilitas quantitatum $t$ et $u$ per formulas differentiales ex variabilibus $x$ et $y$ natas est expressa, ita vicissim si variabiles $t$ et $u$ proponantur, ex quibus certo modo alterae $x$ et $y$ determinentur, sequentes reductiones habebuntur, facta tantum variabilium permutatione. Primo scilicet pro formulis primi gradus

$$\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right)+\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial z}{\partial u}\right),\quad \left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial z}{\partial t}\right)+\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial z}{\partial u}\right).$$

Pro formulis autem differentialibus secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right)=\left(\frac{\partial\partial t}{\partial x^2}\right)\left(\frac{\partial z}{\partial t}\right)+\left(\frac{\partial\partial u}{\partial x^2}\right)\left(\frac{\partial z}{\partial u}\right)+\left(\frac{\partial t}{\partial x}\right)^2\left(\frac{\partial\partial z}{\partial t^2}\right)$$
$$+2\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\,\partial y}\right)=\left(\frac{\partial\partial t}{\partial x\,\partial y}\right)\left(\frac{\partial z}{\partial t}\right)+\left(\frac{\partial\partial u}{\partial x\,\partial y}\right)\left(\frac{\partial z}{\partial u}\right)+\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t^2}\right)$$
$$+\left(\frac{\partial t}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right)=\left(\frac{\partial\partial t}{\partial y^2}\right)\left(\frac{\partial z}{\partial t}\right)+\left(\frac{\partial\partial u}{\partial y^2}\right)\left(\frac{\partial z}{\partial u}\right)+\left(\frac{\partial t}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial t^2}\right)$$
$$+2\left(\frac{\partial t}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial\partial z}{\partial u^2}\right),$$

ubi determinatio litterarum $t$ et $u$ per alteras $x$ et $y$ considerari debet. Quoniam scilicet in conditionibus praescriptis binis variabilibus $x$ et $y$ uti solemus, earum loco alias quascunque $t$ et $u$ introducendo, loco illarum formularum differentialium has novas formas ad variabiles $t$ et $u$ relatas adhibere poterimus, ubi deinceps relatio inter variabiles $x$, $y$ et $t$, $u$ ita est constituenda, ut quaestio solutu facilior evadat. Pro variis igitur hujusmodi relationibus exempla evoluamus.

### Exemplum 1.

233. *Si inter variabiles $x$, $y$ et $t$, $u$ haec relatio constituatur, ut sit*

$$t=\alpha x+\beta y \quad et \quad u=\gamma x+\delta y,$$

*reductionem formularum differentialium exhibere.*

Cum sit

$$\left(\frac{\partial t}{\partial x}\right)=\alpha,\ \left(\frac{\partial t}{\partial y}\right)=\beta,\ \left(\frac{\partial u}{\partial x}\right)=\gamma,\ \left(\frac{\partial u}{\partial y}\right)=\delta,$$

hincque formulae pro secundo gradu evanescant, habebimus pro formulis primi gradus

$$\left(\frac{\partial z}{\partial x}\right)=\alpha\left(\frac{\partial z}{\partial t}\right)+\gamma\left(\frac{\partial z}{\partial u}\right),\quad \left(\frac{\partial z}{\partial y}\right)=\beta\left(\frac{\partial z}{\partial t}\right)+\delta\left(\frac{\partial z}{\partial u}\right).$$

pro formulis autem secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \alpha\alpha\left(\frac{\partial\partial z}{\partial t^2}\right) + 2\alpha\gamma\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \gamma\gamma\left(\frac{\partial\partial z}{\partial u^2}\right),$$
$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \alpha\beta\left(\frac{\partial\partial z}{\partial t^2}\right) + (\alpha\delta + \beta\gamma)\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \gamma\delta\left(\frac{\partial\partial z}{\partial u^2}\right),$$
$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \beta\beta\left(\frac{\partial\partial z}{\partial t^2}\right) + 2\beta\delta\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \delta\delta\left(\frac{\partial\partial z}{\partial u^2}\right).$$

### Corollarium 1.

234. Si sumatur $t = x$ et $u = x + y$, erit $\alpha = 1$, $\beta = 0$, $\gamma = 1$ et $\delta = 1$, ergo $\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial z}{\partial t}\right) + \left(\frac{\partial z}{\partial u}\right)$, $\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial z}{\partial u}\right)$, atque

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial z}{\partial t^2}\right) + 2\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \left(\frac{\partial\partial z}{\partial u^2}\right),$$
$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \left(\frac{\partial\partial z}{\partial t\partial u}\right) + \left(\frac{\partial\partial z}{\partial u^2}\right),$$
$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial z}{\partial u^2}\right).$$

### Corollarium 2.

235. Etsi ergo hic est $t = x$, tamen non est $\left(\frac{\partial z}{\partial t}\right) = \left(\frac{\partial z}{\partial x}\right)$, cujus rei ratio est, quod in forma $\left(\frac{\partial z}{\partial x}\right)$ quantitas $y$ sumitur constans, in $\left(\frac{\partial z}{\partial t}\right)$ vero quantitas $u = x + y$, id quod in genere notasse iuvat, ne ex aequalitate $t = x$ ad aequalitatem formularum $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial t}\right)$ concludamus.

### Exemplum 2.

236. *Si inter variabiles $t$, $u$ et $x$, $y$ haec relatio constituatur, ut sit $t = \alpha x^m$ et $u = \beta y^n$, reductionem exhibere.*

Hic ergo erit

$$\left(\frac{\partial t}{\partial x}\right) = m\alpha x^{m-1},\quad \left(\frac{\partial t}{\partial y}\right) = 0,\quad \left(\frac{\partial\partial t}{\partial x^2}\right) = m(m-1)\alpha x^{m-2},$$
$$\left(\frac{\partial u}{\partial x}\right) = 0,\quad \left(\frac{\partial u}{\partial y}\right) = n\beta y^{n-1},\quad \left(\frac{\partial\partial u}{\partial y^2}\right) = n(n-1)\beta y^{n-2},$$

unde obtinemus pro formulis primi gradus

$$\left(\frac{\partial z}{\partial x}\right) = m\,\alpha\,x^{m-1}\left(\frac{\partial z}{\partial t}\right),\quad \left(\frac{\partial z}{\partial y}\right) = n\,\beta\,y^{n-1}\left(\frac{\partial z}{\partial u}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = m(m-1)\,\alpha\,x^{m-2}\left(\frac{\partial z}{\partial t}\right) + m\,m\,\alpha\,\alpha\,x^{2m-2}\left(\frac{\partial\partial z}{\partial t^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = m\,n\,\alpha\,\beta\,x^{m-1}\,y^{n-1}\left(\frac{\partial\partial z}{\partial t\partial u}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = n(n-1)\,\beta\,y^{n-2}\left(\frac{\partial z}{\partial u}\right) + n\,n\,\beta\,\beta\,y^{2n-2}\left(\frac{\partial\partial z}{\partial u^2}\right),$$

in quibus formulis jam loco $x$ et $y$ earum valores per $t$ et $u$ induci debent.

## Exemplum 3.

237. *Si inter variabiles $t$, $u$ et $x$, $y$ haec relatio constituatur, ut sit $x = t$ et $\frac{x}{y} = u$, formularum differentialium reductionem exhibere.*

Cum sit $t = x$ et $u = \frac{x}{y}$, erit

$$\left(\frac{\partial t}{\partial x}\right) = 1,\quad \left(\frac{\partial t}{\partial y}\right) = 0,$$

hincque formulae involuentes $\partial\partial t$ evanescunt. Porro

$$\left(\frac{\partial u}{\partial x}\right) = \frac{1}{y} = \frac{u}{t},\quad \left(\frac{\partial u}{\partial y}\right) = \frac{-x}{yy} = \frac{-uu}{t},$$

$$\left(\frac{\partial\partial u}{\partial x^2}\right) = 0,\quad \left(\frac{\partial\partial u}{\partial x\partial y}\right) = \frac{-1}{yy} = \frac{-uu}{tt},\quad \left(\frac{\partial\partial u}{\partial y^2}\right) = \frac{2x}{y^3} = \frac{2u^3}{tt},$$

unde pro formulis primi gradus habebimus

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial z}{\partial t}\right) + \frac{u}{t}\left(\frac{\partial z}{\partial u}\right),\quad \left(\frac{\partial z}{\partial y}\right) = \frac{-uu}{t}\left(\frac{\partial z}{\partial u}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial z}{\partial t^2}\right) + \frac{2u}{t}\left(\frac{\partial\partial z}{\partial t\partial u}\right) + \frac{uu}{tt}\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \frac{-uu}{tt}\left(\frac{\partial z}{\partial u}\right) - \frac{uu}{t}\left(\frac{\partial\partial z}{\partial t\partial u}\right) - \frac{u^3}{tt}\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{2u^3}{tt}\left(\frac{\partial z}{\partial u}\right) + \frac{u^4}{tt}\left(\frac{\partial\partial z}{\partial u^2}\right).$$

## Exemplum 4.

238. *Si inter variabiles* $t$, $u$ *et* $x$, $y$ *haec relatio constituatur, ut sit* $t = e^x$ *et* $u = e^x y$, *seu* $x = l\,t$ *et* $y = \frac{u}{t}$, *reductionem formularum differentialium exhibere.*

Hic ergo est

$$(\frac{\partial t}{\partial x}) = e^x = t,\ (\frac{\partial t}{\partial y}) = 0,\ (\frac{\partial \partial t}{\partial x^2}) = e^x = t,\ (\frac{\partial \partial t}{\partial x \partial y}) = 0.$$

Deinde

$$(\frac{\partial u}{\partial x}) = e^x y = u,\ (\frac{\partial u}{\partial y}) = e^x = t,$$

tum vero

$$(\frac{\partial \partial u}{\partial x^2}) = e^x y = u,\ (\frac{\partial \partial u}{\partial x \partial y}) = e^x = t,\ (\frac{\partial \partial u}{\partial y^2}) = 0.$$

Quare pro formulis primi gradus habebimus

$$(\frac{\partial z}{\partial x}) = t\,(\frac{\partial z}{\partial t}) + u\,(\frac{\partial z}{\partial u}),\ (\frac{\partial z}{\partial y}) = t\,(\frac{\partial z}{\partial u}).$$

Pro formulis autem secundi gradus

$$(\frac{\partial \partial z}{\partial x^2}) = t\,(\frac{\partial z}{\partial t}) + u\,(\frac{\partial z}{\partial u}) + t\,t\,(\frac{\partial \partial z}{\partial t^2}) + 2\,t\,u\,(\frac{\partial \partial z}{\partial t \partial u}) + u\,u\,(\frac{\partial \partial z}{\partial u^2}),$$

$$(\frac{\partial \partial z}{\partial x \partial y}) = t\,(\frac{\partial z}{\partial u}) + t\,t\,(\frac{\partial \partial z}{\partial t \partial u}) + t\,u\,(\frac{\partial \partial z}{\partial u^2}),$$

$$(\frac{\partial \partial z}{\partial y^2}) = t\,t\,(\frac{\partial \partial z}{\partial u^2}).$$

## Scholion.

239. In formulis generalibus §. 232. datis assumsimus valores variabilium $t$ et $u$ per $x$ et $y$ expressos dari, et universa evolutione facta tum demum pro $x$ et $y$ variabiles $t$ et $u$ restitui. Commodius ergo videatur, si statim variabilium $x$ et $y$ valores per $t$ et $u$ expressi habeantur; verum inde valores formularum $(\frac{\partial t}{\partial x})$, $(\frac{\partial t}{\partial y})$, etc. nimis complicate exprimerentur, quam ut eas in calculum introducere liceret. Scilicet si $x$ et $y$ per $t$ et $u$ dentur, inde fit

$$\left(\frac{\partial t}{\partial x}\right) = \frac{\left(\frac{\partial y}{\partial u}\right)}{\left(\frac{\partial x}{\partial t}\right)\left(\frac{\partial y}{\partial u}\right) - \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial t}\right)},$$

ac formulae secundi gradus multo magis proditurae sunt perplexae. Quovis ergo casu, quo hujusmodi reductione utendum videtur, conjectura potius quam certa ratione idoneam variabilium immutationem colligi conveniet. Alia vero etiam datur reductio saepe insignem utilitatem afferens, dum ipsius functionis $z$ quaesitae forma mutatur; veluti si ponatur $z = Vv$, denotante V functionem datam ipsarum $x$ et $y$, ita ut jam $v$ sit functio quaesita; quin etiam haec nova quaesita $v$ alio modo cum datis implicari potest.

## Problema 40.

240. Proposita functione $z$ binarum variabilium $x$ et $y$, ac posita $z = Pv$, ita ut P sit data quaedam functio ipsarum $x$ et $y$, formulas differentiales novae functionis $v$ exprimere.

## Solutio.

Cum sit $z = Pv$, ex regulis differentiandi traditis habebimus primo formulas differentiales primi gradus

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial P}{\partial x}\right) v + P\left(\frac{\partial v}{\partial x}\right) \text{ et } \left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial P}{\partial y}\right) v + P\left(\frac{\partial v}{\partial y}\right).$$

Atque hinc deinceps formulae differentiales secundi ordinis ita prodibunt expressae

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = \left(\frac{\partial\partial P}{\partial y^2}\right) v + 2\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + P\left(\frac{\partial\partial v}{\partial x^2}\right),$$

$$\left(\frac{\partial\partial z}{\partial x\partial y}\right) = \left(\frac{\partial\partial P}{\partial x\partial y}\right) v + \left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right) + P\left(\frac{\partial\partial v}{\partial x\partial y}\right),$$

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \left(\frac{\partial\partial P}{\partial y^2}\right) v + 2\left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right) + P\left(\frac{\partial\partial v}{\partial y^2}\right),$$

ubi cum P sit functio data ipsarum $x$ et $y$, ejus formulae differentiales simul habentur.

### Corollarium 1.

241. Si P esset functio ipsius $x$ tantum, puta X, tum posito $z = Xv$ erit

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial X}{\partial x}\right) . v + X \left(\frac{\partial v}{\partial x}\right) \text{ et } \left(\frac{\partial z}{\partial y}\right) = X \left(\frac{\partial v}{\partial y}\right), \text{ tum}$$

$$\left(\frac{\partial \partial z}{\partial x^2}\right) = \left(\frac{\partial \partial X}{\partial x^2}\right) . v + \frac{2 \partial X}{\partial x} \left(\frac{\partial v}{\partial x}\right) + X \left(\frac{\partial \partial v}{\partial x^2}\right),$$

$$\left(\frac{\partial \partial z}{\partial x \partial y}\right) = \frac{\partial X}{\partial x} \left(\frac{\partial v}{\partial y}\right) + X \left(\frac{\partial \partial v}{\partial x \partial y}\right),$$

$$\left(\frac{\partial \partial z}{\partial y^2}\right) = X \left(\frac{\partial \partial v}{\partial y^2}\right).$$

### Corollarium 2.

242. Transformatio haec easdem variabiles $x$ et $y$ servat, et tantum loco functionis $z$ alia $v$ introducitur; cum ante manente eadem functione $z$, binae variabiles $x$ et $y$ ad alias $t$ et $u$ sint reductae. Ex quo hae duae transformationes genere sunt diversae.

### Scholion 1.

243. Casus simplicior fuisset, si per additionem posuissemus $z = P + v$, ut esset P functio quaedam data ipsarum $x$ et $y$; verum tum transformatio ita fit obvia, ut investigatione non indigeat; est enim manifesto

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial v}{\partial x}\right), \quad \left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial P}{\partial y}\right) + \left(\frac{\partial v}{\partial y}\right),$$

$$\left(\frac{\partial \partial z}{\partial x^2}\right) = \left(\frac{\partial \partial P}{\partial x^2}\right) + \left(\frac{\partial \partial v}{\partial x^2}\right),$$

$$\left(\frac{\partial \partial z}{\partial x \partial y}\right) = \left(\frac{\partial \partial P}{\partial x \partial y}\right) + \left(\frac{\partial \partial v}{\partial x \partial y}\right),$$

$$\left(\frac{\partial \partial z}{\partial y^2}\right) = \left(\frac{\partial \partial P}{\partial y^2}\right) + \left(\frac{\partial \partial v}{\partial y^2}\right).$$

Neque vero etiam formas magis compositas evolvi necesse est, veluti, si ponamus $z = \sqrt{(PP + vv)}$, quandoquidem talis forma vix unquam usum foret habitura.

## Scholion 2.

244. Praemissis his principiis et transformationibus, negotium aggrediamur, et methodos aperiamus, ex data relatione inter formulas differentiales secundi gradus, et primi gradus, itemque ipsas quantitates principales, harum ipsarum relationem investigandi. Hic scilicet praeter ipsas quantitates $x$, $y$ et $z$, earumque formulas differentiales primi gradus $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$ considerandae veniunt tres formulae differentiales secundi gradus $\left(\frac{\partial\partial z}{\partial x^2}\right)$, $\left(\frac{\partial\partial z}{\partial x\partial y}\right)$ et $\left(\frac{\partial\partial z}{\partial y^2}\right)$; quarum vel una, vel binae, vel omnes tres in relationem propositam ingredi possunt, ubi insuper ingens discrimen formulae primi gradus, sive in relationem ingrediantur, sive secus, constituunt. Non solum autem nimis longum foret omnes combinationes, uti in praecedente sectione fecimus, prosequi, sed etiam defectus idonearum methodorum impedit, quo minus singula quaestionum huc pertinentium genera percurramus. Capita igitur pertractanda ita instituamus, prout methodus solvendi patietur, ea, ubi nihil praestare licet, penitus praetermissuri.