CAPUT VI.

DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER BINAS FORMULAS DIFFERENTIALES $\left(\frac{\partial z}{\partial x}\right)$, $\left(\frac{\partial z}{\partial y}\right)$, ET OMNES TRES VARIABILES x, y, z QUAECUNQUE DATUR.

Problema 30.

174.

Si posito $\partial z = p\partial x + q\partial y$, debeat esse $nz = px + qy$, indolem functionis z in genere investigare.

Solutio.

Ope relationis datae elidatur vel p vel q. Scilicet cum sit $q = \frac{nz}{y} - \frac{px}{y}$, erit

$$\partial z = p\partial x + \frac{nz\partial y}{y} - \frac{px\partial y}{y},$$

quae aequatio in hanc formam transfundatur

$$\partial z - \frac{nz\partial y}{y} = p\left(\partial x - \frac{x\partial y}{y}\right) = py\partial \,.\, \frac{x}{y}.$$

Ut prius membrum $\partial z - \frac{nz\partial y}{y}$ integrabile reddatur, multiplicetur aequatio per $\frac{1}{z}$. funct. $\frac{z}{y^n}$, seu particulariter per $\frac{1}{y^n}$, eritque

$$\partial \,.\, \frac{z}{y^n} = py^{1-n}\, \partial \,.\, \frac{x}{y}.$$

Quo facto evidens est poni debere $py^{1-n} = f' : \frac{x}{y}$, ut fiat

$\frac{z}{y^n} = \mathrm{f} : \frac{x}{y}$, seu $z = y^n \mathrm{f} : \frac{x}{y}$. Unde patet fore z functionem homogeneam ipsarum x et y, dimensionum numero existente $= n$.

Si in genere aequatio multiplicetur per $\frac{1}{z}$. funct. $\frac{z}{y^n}$, erit partis prioris integrale $\mathrm{F} : \frac{z}{y^n}$, pro parte autem altera si ponatur $\frac{py}{z}$ funct. $\frac{z}{y^n} = \mathrm{f}' : \frac{x}{y}$, erit $\mathrm{F} : \frac{z}{y^n} = \mathrm{f} : \frac{x}{y}$, atque ut ante $\frac{z}{y^n}$ aequabitur functioni cuicunque ipsius $\frac{x}{y}$.

Corollarium 1.

175. Cum z aequetur functioni homogeneae n dimensionum ipsarum x et y, erunt p et q functiones $n-1$ dimensionum. Scilicet cum sit $z = y^n \mathrm{f} : \frac{x}{y}$, erit

$$p = y^{n-1} \mathrm{f}' : \frac{x}{y}, \text{ et } q = ny^{n-1} \mathrm{f} : \frac{x}{y} - xy^{n-2} \mathrm{f}' : \frac{x}{y},$$

unde fit manifesto $nz = px + qy$.

Corollarium 2.

176. Si p et q fuerint functiones $n-1$ dimensionum ipsarum x et y, ac formula $p\partial x + q\partial y$ sit integrabilis seu $\left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial q}{\partial x}\right)$, tum integrale certo erit $\frac{px + qy}{n}$, quae proprietas nonnunquam insignem usum habere potest.

Scholion.

177. Fundamentum hujus solutionis in hoc consistit, quod aequatio integranda in duas partes resolvatur, quarum utraque ope certi multiplicatoris integrabilis reddi queat, unde deinceps una

quantitas variabilis, cujus differentiale in aequatione non occurrit determinetur. Hinc aequatio nostra

$$\partial z - \frac{nz\partial y}{y} = p\ (\partial x - \frac{x\partial y}{y})$$

etiam ita repraesentari potest

$$\frac{\partial x}{y} - \frac{x\partial y}{yy} = \frac{1}{py}\left(\partial z - \frac{nz\partial y}{y}\right) = \frac{y^{n-1}}{p}\left(\frac{\partial z}{y^n} - \frac{nz\partial y}{y^{n+1}}\right), \text{ seu}$$

$$\partial \cdot \frac{x}{y} = \frac{y^{n-1}}{p}\,\partial \cdot \frac{z}{y^n}.$$

Sit ergo

$$\frac{y^{n-1}}{p} = F' : \frac{z}{y^n}, \text{ eritque}$$

$$\frac{x}{y} = F : \frac{z}{y^n}, \text{ ac vicissim } \frac{z}{y^n} = f : \frac{x}{y}, \text{ ut ante.}$$

Possumus etiam statim z ex calculo elidere; cum enim sit

$$nz = px + qy, \text{ erit}$$

$$n\partial z = p\partial x + q\partial y + x\partial p + y\partial q.$$

At est

$$n\partial z = np\partial x + nq\partial y,$$

$$(n-1)\,p\partial x - x\partial p + (n-1)\,q\partial y - y\partial q = 0, \text{ seu}$$

$$x^n\left(\frac{(n-1)\,p\partial x}{x^n} - \frac{\partial p}{x^{n-1}}\right) + y^n\left(\frac{(n-1)\,q\partial y}{y^n} - \frac{\partial q}{y^{n-1}}\right) = 0,$$

quae reducitur ad hanc formam

$$-\,x^n\,\partial \cdot \frac{p}{x^{n-1}} - y^n\,\partial \cdot \frac{q}{y^{n-1}} = 0, \text{ seu}$$

$$\partial \cdot \frac{q}{y^{n-1}} = -\frac{x^n}{y^n}\,\partial \cdot \frac{p}{x^{n-1}}.$$

Statuatur

$$\frac{x^n}{y^n} = -\,f' : \frac{p}{x^{n-1}}, \text{ erit } \frac{q}{y^{n-1}} = f : \frac{p}{x^{n-1}}.$$

Vel posito $\frac{x}{y} = v$, si ob $v^n = -\,f' : \frac{p}{x^{n-1}}$ reciproce ponatur

$$\frac{p}{x^{n-1}} = u = \frac{1}{v^{n-1}}\,F' : v,$$

ut sit

$$f' : u = -\,v^n,$$

reperietur

$$\int \partial u f' : u = f : u = nF : v - vF' : v.$$

Hinc

$$p = \frac{x^{n-1}}{v^{n-1}}\,F' : v = y^{n-1}\,F' : \frac{x}{y}; \text{ et}$$

$$q = y^{n-1}\,f : u = ny^{n-1}\,F : \frac{x}{y} - xy^{n-2}\,F' : \frac{x}{y};$$

ideoque

$$nz = px + qy = ny^n\,F : \frac{x}{y}, \text{ seu } z = y^n\,F : \frac{x}{y},$$

ut ante.

Problema 31.

178. Si posito $\partial z = p\partial x + q\partial y$, debeat esse

$$\alpha px + \beta qy = nz,$$

indolem functionis z investigare.

Solutio.

Ex conditione praescripta eliciatur ut ante

$$q = \frac{nz}{\beta y} - \frac{\alpha px}{\beta y}, \text{ eritque}$$

$$\partial z - \frac{nz\partial y}{\beta y} = p\partial x - \frac{\alpha px\partial y}{\beta y},$$

quae aequatio per $y^{\frac{\beta}{n}}$ divisa dat

$$\partial \cdot \frac{z}{y^{n:\beta}} = \frac{p}{y^{n:\beta}}\left(\partial x - \frac{\alpha x \partial y}{\beta y}\right) = \frac{p y^{\alpha:\beta}}{y^{n:\beta}} \partial \cdot \frac{x}{y^{\alpha:\beta}}.$$

Quod si ergo ponamus

$$p y^{(\alpha - n):\beta} = f' : \frac{x}{y^{\alpha:\beta}},$$

habebimus solutionem

$$z = y^{n:\beta}\ f : \frac{x}{y^{\alpha:\beta}}.$$

At functio ipsius $\frac{x}{y^{\alpha:\beta}}$ reducitur ad functionem ipsius $\frac{x^{\beta}}{y^{\alpha}}$, unde z etiam ita per x et y determinatur, ut sit

$$z = y^{n:\beta}\ f : \frac{x^{\beta}}{y^{\alpha}},$$

vel etiam

$$z^{\frac{1}{n}} = y^{\frac{1}{\beta}}\ f : \frac{x^{\frac{1}{\alpha}}}{y^{\frac{1}{\beta}}}.$$

Quodsi ergo quantitates $x^{\frac{1}{\alpha}}$ et $y^{\frac{1}{\beta}}$ unam dimensionem constituere censeantur, $z^{\frac{1}{n}}$ aequabitur earundem functioni unius dimensionis, ipsa autem quantitas z earundem functioni n dimensionum. Vel sumta pro z functione quacunque homogenea n dimensionum binarum variabilium t et u, scribatur deinde $t = x^{\frac{1}{\alpha}}$ et $u = y^{\frac{1}{\beta}}$, ac prodibit functio conveniens pro z.

Problema 32.

179. Si posito $\partial z = p\partial x + q\partial y$ debeat esse

$$Z = pX + qY,$$

denotante Z functionem ipsius z, X ipsius x, et Y ipsius y, indolem functionis z in genere investigare.

Solutio.

Ex conditione praescripta elicitur $q = \frac{Z}{Y} - \frac{pX}{Y}$, qui valor substitutus praebet

$$\partial z - \frac{Z\partial y}{Y} = p\left(\partial x - \frac{X\partial y}{Y}\right), \text{ hincque}$$

$$\frac{\partial z}{Z} - \frac{\partial y}{Y} = \frac{p}{Z}\left(\partial x - \frac{X\partial y}{Y}\right) = \frac{pX}{Z}\left(\frac{\partial x}{X} - \frac{\partial y}{Y}\right),$$

ubi jam resolutio est manifesta. Statuatur scilicet

$$\frac{pX}{Z} = f' : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right), \text{ eritque}$$

$$\int\frac{\partial z}{Z} - \int\frac{\partial y}{Y} = f : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right),$$

unde valor ipsius z per x et y definitur.

Corollarium 1.

180. Hic ergo z ita per x et y definiri debet, ut si X, Y et Z datae sint functiones sigillatim ipsarum x, y et z, fiat

$$X\left(\frac{\partial z}{\partial x}\right) + Y\left(\frac{\partial z}{\partial y}\right) = Z;$$

cujus ergo aequationis resolutionem hic invenimus hac aequatione finita contentam

$$\int\frac{\partial z}{Z} = \int\frac{\partial y}{Y} + f : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right).$$

Corollarium 2.

181. Quemadmodum autem hic valor conditioni problematis satisfaciat, ex ejus differentiatione statim patet. Cum enim sit

$$\frac{\partial z}{Z} = \frac{\partial y}{Y} + \left(\frac{\partial x}{X} - \frac{\partial y}{Y}\right) f' : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right), \text{ erit}$$

$$\left(\frac{\partial z}{\partial x}\right) = \frac{Z}{X} f' : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right), \text{ et}$$

$$\left(\frac{\partial z}{\partial y}\right) = \frac{Z}{Y} - \frac{Z}{Y} f' : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right),$$

unde fit

$$X\left(\frac{\partial z}{\partial x}\right) + Y\left(\frac{\partial z}{\partial y}\right) = Z.$$

Scholion.

182. Solutio ergo, eodem modo ut fecimus, sine introductione novarum litterarum p et q absolvi potest, retinendo earum loco valores differentiales $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$; facilius autem singulae litterae scribuntur, calculusque fit brevior. Caeterum ex hoc problematum genere, ubi omnes tres variabiles x, y et z praeter binos valores differentiales p et q in determinationem ingrediuntur, paucissima resolvere licet; ac praeter hoc, quod tractavimus vix unum aut alterum insuper adjungere poterimus. Unde hic insignia adhuc calculi incrementa desiderantur. Quo autem hujus problematis vis penitius inspiciatur, nonnulla exempla subjungamus.

Exemplum 1.

183. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse*

$$zz = pxx + qyy,$$

indolem functionis z in genere investigare.

Hic ergo est $Z = zz$, $X = xx$, et $Y = yy$; unde habemus

$$\int\frac{\partial x}{X} = -\frac{1}{x}, \quad \int\frac{\partial y}{Y} = -\frac{1}{y}, \quad \text{et} \int\frac{\partial z}{Z} = -\frac{1}{z},$$

quibus valoribus substitutis pro solutione adipiscimur

$$-\frac{1}{z} = -\frac{1}{y} + f : \left(\frac{1}{y} - \frac{1}{x}\right), \text{ seu}$$

$$z = \frac{y}{1 - y\,\mathrm{f} : \left(\frac{1}{y} - \frac{1}{x}\right)}.$$

Sumatur ergo functio quaecunque quantitatis

$$\frac{1}{y} - \frac{1}{x} = \frac{x - y}{xy},$$

quae si ponatur V, erit $z = \frac{y}{1 - Vy}$.

Veluti si ponamus $V = \frac{n}{y} - \frac{n}{x}$, erit

$$\frac{1}{z} = \frac{1}{y} - \frac{n}{y} + \frac{n}{x} = \frac{ny - (n-1)x}{xy},$$

hincque $z = \frac{xy}{ny - (n-1)x}$, unde

$$p = \left(\frac{\partial z}{\partial x}\right) = \frac{nyy}{[ny - (n-1)x]^2}, \text{ et } q = \left(\frac{\partial z}{\partial y}\right) = \frac{-(n-1)xx}{[ny - (n-1)x]^2},$$

sicque

$$pxx + qyy = \frac{xxyy}{[ny - (n-1)x]^2} = zz.$$

Exemplum 2.

184. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $\frac{n}{z} = \frac{p}{x} + \frac{q}{y}$, indolem functionis z investigare.*

Cum hic sit

$$X = \frac{1}{x},\quad Y = \frac{1}{y} \text{ et } Z = \frac{n}{z}, \text{ erit}$$

$$\int \frac{\partial x}{X} = \tfrac{1}{2}xx,\quad \int \frac{\partial y}{Y} = \tfrac{1}{2}yy \text{ et } \int \frac{\partial z}{Z} = \frac{1}{2n}zz;$$

unde solutio ita erit comparata

$$\frac{1}{2n}zz = \tfrac{1}{2}yy + \mathrm{f} : (xx - yy), \text{ sive}$$

$$zz = nyy + \mathrm{f} : (xx - yy),$$

non enim est necesse functionem per $2n$ multiplicari, cum ea omnes operationes jam per se involvat.

Si pro hac functione sumatur $\alpha(xx - yy)$, habebitur solutio particularis

$$zz = \alpha xx + (n-\alpha)yy \text{ et } z = \sqrt{[\alpha xx + (n-\alpha)yy]},$$

hincque

$$p = \left(\frac{\partial z}{\partial x}\right) = \frac{\alpha x}{\sqrt{[\alpha xx + (n-\alpha)yy]}}, \text{ et}$$

$$q = \left(\frac{\partial z}{\partial y}\right) = \frac{(n-\alpha)y}{\sqrt{[\alpha xx + (n-\alpha)yy]}},$$

seu $\frac{p}{x} = \frac{\alpha}{z}$ et $\frac{q}{y} = \frac{n-\alpha}{z}$, ideoque $\frac{p}{x} + \frac{q}{y} = \frac{n}{z}$.

Problema 33.

185. Si posito $\partial z = p\partial x + q\partial y$ debeat esse $q = p\mathrm{T} + \mathrm{V}$, existente T functione quacunque ipsarum x et y, ac V functione ipsarum y et z, investigare indolem functionis z.

Solutio.

Substituto loco q valore praescripto, huic aequationi inducatur forma

$$\partial z - \mathrm{V}\partial y = p(\partial x + \mathrm{T}\partial y).$$

Cum jam V tantum binas variabiles y et z involvat, dabitur multiplicator M prius membrum $\partial z - \mathrm{V}\partial y$ integrabile reddens; ponatur ergo

$$\mathrm{M}(\partial z - \mathrm{V}\partial y) = \partial \mathrm{S}.$$

Simili modo quia T tantum x et y continet, dabitur multiplicator L membrum quoque posterius $\partial x + \mathrm{T}\partial y$ integrabile efficiens; sit igitur

$$\mathrm{L}(\partial x + \mathrm{T}\partial y) = \partial \mathrm{R},$$

ita ut nunc sint R et S functiones cognitae, illa ipsarum x et y, haec vero ipsarum y et z. Hinc nostra aequatio induet hanc formam

$$\frac{\partial \mathrm{S}}{\mathrm{M}} = \frac{p\,\partial \mathrm{R}}{\mathrm{L}} \text{ seu } \partial \mathrm{S} = \frac{p\mathrm{M}\,\partial \mathrm{R}}{\mathrm{L}},$$

cujus integrabilitas necessario postulat ut sit $\frac{p\mathrm{M}}{\mathrm{L}}$ functio ipsius R. Ponamus ergo

$$\frac{pM}{L} = f' : R, \text{ eritque } S = f : R$$

qua aequatione relatio inter z et x, y definitur.

Corollarium 1.

186. In hoc problemate praecedens tanquam casus particularis continetur: cum enim ibi esset $Z = pX + qY$, erit $q = -\frac{X}{Y}p + \frac{Z}{Y}$, ideoque hujus problematis applicatione facta fit

$$T = \frac{-X}{Y} \text{ et } V = \frac{Z}{Y}.$$

Corollarium 2.

187. Quanquam autem hoc problema infinite latius patet quam praecedens, arctissimis tamen adhuc limitibus continetur, neque ejus ope vel hunc casum simplicissimum $z = py + qx$ resolvere licet.

Scholion.

188. Omnino est haec forma $z = py + qx$ digna notatu, quod nulla ratione hactenus cognita resolvi posse videtur. Sive enim inde eliciatur $q = \frac{z - py}{x}$, unde fit

$$\partial z - \frac{z\partial y}{x} = p\left(\partial x - \frac{y\partial y}{x'}\right),$$

sive simili modo p, nulla via ad solutionem patet; cujus difficultatis causa in hoc manifesto est posita, quod formula $\partial z - \frac{z\partial y}{x}$ nullo multiplicatore integrabilis reddi potest; seu quod haec aequatio $\partial z - \frac{z\partial y}{x} = 0$ plane est impossibilis, cum x perinde sit variabilis atque y et z. Supra scilicet jam notavi non omnes aequationes differentiales inter ternas variabiles esse possibiles, simulque characterem possibilitatis exhibui, qui pro tali forma

$$\partial x + P\partial x + Q\partial y = 0,$$

huc reducitur, ut sit

$$P\left(\frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial P}{\partial x}\right) = \left(\frac{\partial Q}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right),$$

nostro jam casu est $P = 0$ et $Q = \frac{-z}{x}$, unde hic character dat $0 = \frac{z}{xx}$, quod cum sit falsum, etiam aequatio illa $\partial z - \frac{z \partial y}{x} = 0$ est impossibilis, quod quidem per se est manifestum. Verum tamen pro hoc casu $z = py + qx$ solutio particularis est obvia scilicet $z = n(x + y)$, unde fit $p = q = n$. Deinceps autem methodum dabimus ex hujusmodi solutione particulari generalem eruendi.

Exemplum 1.

189. *Si posito $\partial z = p \partial x + q \partial y$ debeat esse*

$$py + qx = \frac{nxz}{y},$$

indolem functionis z investigare.

Cum hinc sit $q = -\frac{py}{x} + \frac{nz}{y}$, erit

$$T = \frac{-y}{x} \text{ et } V = \frac{nz}{y},$$

unde fit

$$\partial S = M\left(\partial z - \frac{nz\partial y}{y}\right) \text{ et } \partial R = L\left(\partial x - \frac{y\partial y}{x}\right).$$

Sumatur ergo $M = \frac{1}{y^n}$, ut fiat $S = \frac{z}{y^n}$, et $L = 2x$, ut fiat $R = xx - yy$. Quocirca hanc adipiscimur solutionem

$$\frac{z}{y^n} = f:(xx - yy), \text{ seu } z = y^n f:(xx - yy).$$

Exemplum 2.

190. *Si posito $\partial z = p \partial x + q \partial y$ debeat esse*

$$pxx + qyy = nyz,$$

definire indolem functionis z.

Cum ergo sit $q = \frac{-pxx}{yy} + \frac{nz}{y}$, erit

$$T = \frac{-xx}{yy} \text{ et } V = \frac{nz}{y},$$

sicque hic casus in nostro problemate continetur. Unde colligi oportet

$$\partial R = L\left(\partial x - \frac{xx\partial y}{yy}\right) \text{ et } \partial S = M\left(\partial z - \frac{nz\partial y}{y}\right).$$

Quare sumto $L = \frac{1}{xx}$ fit $R = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$; et sumto $M = \frac{1}{y^n}$, fit $S = \frac{z}{y^n}$, ideoque solutio prodit ista

$$\frac{z}{y^n} = f : \frac{x-y}{xy} \text{ et } z = y^n f : \frac{x-y}{xy}.$$

Problema 34.

191. Si posito $\partial z = p\partial x + q\partial y$, debeat esse $p = qT + V$, existente T functione ipsarum x et y, at V functione ipsarum x et z, indolem functionis z investigare.

Solutio.

Simili modo ut ante si loco p valor praescriptus substituatur, obtinebitur

$$\partial z - V\partial x = q(\partial y + T\partial x).$$

Jam ob indolem functionum V et T sequentes integrationes instituere licebit

$$M(\partial z - V\partial x) = \partial S, \; N(\partial y + T\partial x) = \partial R,$$

unde fit

$$\frac{\partial S}{M} = \frac{q\partial R}{N}, \text{ seu } \partial S = \frac{Mq}{N}\partial R.$$

Atque hinc facillime colligitur haec solutio

$$\frac{Mq}{N} = f' : R, \text{ et } S = f : R.$$

Problema 35.

192. Si posito $\partial z = p\partial x + q\partial y$ debeat esse $z = Mp + Nq$, existentibus M et N functionibus quibusvis binarum variabilium x et y; ex quadam solutione particulari, qua constat esse $z = V$, indolem functionis z in genere determinare.

Solutio.

Valor iste particularis V, qui est functio ipsarum x et y differentietur, sitque

$$\partial V = P\partial x + Q\partial y,$$

qui valor quia loco z substitutus satisfacit, ubi fit $p = P$ et $q = Q$, erit per hypothesin

$$V = MP + NQ.$$

Jam generatim ponatur $z = V f : T$, sitque

$$\partial T = R\partial x + S\partial y,$$

et nunc quaeri oportet hanc functionem T. Ex differentiatione autem eruimus

$$p = \left(\frac{\partial z}{\partial x}\right) = P f : T + V R f' : T, \text{ et}$$

$$q = \left(\frac{\partial z}{\partial y}\right) = Q f : T + V S f' : T.$$

Quare cum sit

$$z = Mp + Nq = V f : T, \text{ erit}$$

$$V f : T = (MP + NQ) f : T + V(MR + NS) f' : T,$$

et ob $V = MP + NQ$ per hypothesin habebitur

$$MR + NS = 0, \text{ hinc}$$

$$\partial T = R\left(\partial x - \frac{M\partial y}{N}\right).$$

Jam nosse non oportet R, sed sufficit considerari formulam $N\partial x - M\partial y$, quae ope multiplicatoris cujusdam integrabilis reddi potest. Solutio ergo facillime huc redit, ut ex conditione praescripta $z = Mp + Nq$ formetur aequatio realis

$$\partial T = R(N\partial x - M\partial y),$$

invento enim multiplicatore idoneo R, per integrationem reperitur quantitas T, qua inventa erit $z = V \cdot f : T$.

Aliter.

Facilius valor generalis hoc modo invenitur; ob valorem ipsius z cognitum V, statuatur $z = Vv$, sitque

$$\partial v = r\,\partial x + s\,\partial y; \text{ erit}$$

$$p = Pv + Vr \text{ et } q = Qv + Vs,$$

ideoque

$$z = Mp + Nq = (MP + NQ)v + V(Mr + Ns) = Vv.$$

At est $V = MP + NQ$; ergo

$$Mr + Ns = 0, \text{ seu } s = -\frac{Mr}{N}.$$

Unde fit

$$\partial v = r\left(\partial x - \frac{M\partial y}{N}\right) = \frac{r}{N}(N\,\partial x - M\,\partial y).$$

Statuatur ergo, idoneum multiplicatorem investigando,

$$R(N\,\partial x - M\,\partial y) = \partial T, \text{ erit } \partial v = \frac{r}{NR} \cdot \partial T,$$

ex quo colligitur

$$\frac{r}{NR} = f' : T \text{ et } v = f : T,$$

ita ut in genere sit ut ante $z = Vv$.

Corollarium 1.

193. Proposita ergo conditione $z = Mp + Nq$, ut sit $\partial z = p\,\partial x + q\,\partial y$, statim consideretur aequatio differentialis $R(N\,\partial x - M\,\partial y) = \partial T$, unde tam multiplicator R quam inde integrale T reperitur; haecque operatio non pendet a valore particulari cognito V.

Corollarium 2.

194. Inventa autem quantitate T, si undecunque innotuerit solutio particulariter satisfaciens $z = V$, erit solutio generalis $z = V\,f:T$. Probe autem notetur ex solutione particulari generalem elici non posse, nisi conditio praescripta sit hujusmodi $z = Mp + Nq$.

Exemplum 1.

195. *Si posito* $\partial z = p\,\partial x + q\,\partial y$ *debeat esse* $z = py + qx$, *ex valore particulari* $z = x + y$ *generalem definire.*

Cum hic sit $M = y$ et $N = x$, habebimus hanc aequationem

$$R\,(x\,\partial x - y\,\partial y) = \partial T, \text{ hincque}$$

$$T = f:(xx - yy);$$

ergo solutio generalis erit

$$z = (x + y)\,f:(xx - yy).$$

Exemplum 2.

196. *Si posito* $\partial z = p\,\partial x + q\,\partial y$ *debeat esse*

$$z = p\,(x + y) + q\,(y - x),$$

ex valore particulari $z = \sqrt{(xx + yy)}$ *generalem invenire.*

Ob $M = x + y$ et $N = y - x$ formula $N\,\partial x - M\,\partial y$ deducit ad hanc aequationem

$$R\,(y\,\partial x - x\,\partial x - x\,\partial y - y\,\partial y) = \partial T.$$

Sumatur $R = \frac{1}{xx + yy}$, ut sit

$$\partial T = \frac{y\,\partial x - x\,\partial y}{xx + yy} - \frac{x\,\partial x - y\,\partial y}{xx + yy}, \text{ erit}$$

$$T = \text{Ang. tang.}\,\frac{x}{y} - \tfrac{1}{2}\,l\,(xx + yy).$$

Atque ex valore hoc dupliciter transcendente erit

$$z = \sqrt{(xx + yy)}\, f : T,$$

simulque patet nullum alium dari valorem particularem, qui sit algebraicus, praeter datum $z = \sqrt{(xx + yy)}$.

Exemplum 3.

197. *Si posito $\partial z = p \partial x + q \partial y$ debeat esse*

$$z = p(\alpha x + \beta y) + q(\gamma x + \delta y),$$

ex invento valore particulari $z = V$, indolem functionis z in genere definire.

Hic est $M = \alpha x + \beta y$ et $N = \gamma x + \delta y$, unde deducimur ad hanc aequationem

$$R[(\gamma x + \delta y)\partial x - (\alpha x + \beta y)\partial y] = \partial T,$$

ubi ob formam homogeneam debet esse

$$R = \frac{1}{\gamma xx + (\delta - \alpha) xy - \beta yy},$$

ut sit

$$\partial T = \frac{(\gamma x + \delta y)\partial x - (\alpha x + \beta y)\partial y}{\gamma xx + (\delta - \alpha) xy - \beta yy},$$

ad quod integrale inveniendum ponatur $y = ux$, ac prodibit

$$\partial T = \frac{\partial x}{x} - \frac{(\alpha + \beta u)\partial u}{\gamma + (\delta - \alpha) u - \beta uu}, \text{ sit}$$

$$\int \frac{(\alpha + \beta u)\partial u}{\gamma + (\delta - \alpha) u - \beta uu} = lU, \text{ erit } T = lx - lU,$$

et cum functio ipsius T sit etiam functio ipsius $\frac{x}{U}$, erit in genere $z = V f : \frac{x}{U}$. Patet autem, cum U sit functio ipsius $u = \frac{y}{x}$, fore U functionem homogeneam nullius dimensionis ipsarum x et y, ideoque $\frac{x}{U}$ functionem unius dimensionis.

Scholion.

198. Hoc ergo exemplo difficultas restat, quomodo solutio particularis $z = V$ obtineri queat; nisi enim una saltem hujusmod

solutio particularis constet, solutio generalis ne absolvi quidem potest. Pro hoc autem casu solutionem particularem sequenti modo elicere licet, qui cum aliquid singulare habeat, nullum est dubium, quin ejus ope hoc calculi genus haud parum adjumenti sit consecuturum.

Problema 86.

199. Si posito $\partial z = p\,\partial x + q\,\partial y$ debeat esse

$$z = p(\alpha x + \beta y) + q(\gamma x + \delta y),$$

valorem particularem investigare, qui loco z substitutus huic conditioni satisfaciat.

Solutio.

Negotium hoc succedet, si pro z ejusmodi valorem quaeramus, qui sit functio nullius dimensionis ipsarum x et y, seu posito $y = u x$, qui sit functio ipsius u tantum. Ponamus ergo

$$z = f:u = f:\frac{y}{x}, \text{ eritque } f':u = \frac{\partial z}{\partial u};$$

at ob $\partial u = \frac{\partial y}{x} - \frac{y\partial x}{xx}$, erit

$$\partial z = \left(\frac{\partial y}{x} - \frac{u\partial x}{x}\right) f':u, \text{ hinc}$$

$$p = -\frac{u}{x} f':u = -\frac{u\partial z}{x\partial u} \text{ et } q = \frac{1}{x} f':u = \frac{\partial z}{x\partial u}.$$

Quibus valoribus pro p et q substitutis, conditio praescripta praebet

$$z = x(\alpha + \beta u)p + x(\gamma + \delta u)q = \frac{-u\partial z(\alpha + \beta u) + \partial z(\gamma + \delta u)}{\partial u},$$

unde fit

$$\frac{\partial z}{z} = \frac{\partial u}{\gamma + (\delta - \alpha)u - \beta uu}.$$

Ponamus

$$\int \frac{\partial u}{\gamma + (\delta - \alpha)u - \beta uu} = l\,V,$$

ut fiat $z = V$, eritque V valor particularis pro z satisfaciens.

Corollarium 1.

200. Invento hoc valore V, praecedentis exempli ope solutio generalis facile invenitur. Erit scilicet $z = V\,f : \frac{x}{U}$ existente

$$\frac{\partial U}{U} = \frac{(\alpha+\beta u)\,\partial u}{\gamma+(\delta-\alpha)u-\beta uu};$$

unde patet quantitatem U ex ipso valore particulari V inveniri posse.

Corollarium 2.

201. Erit enim

$$l\,U = -l\sqrt{[\gamma+(\delta-\alpha)\,u-\beta\,u\,u]} + \int \frac{\frac{1}{2}(\delta+\alpha)\,\partial u}{\gamma+(\delta-\alpha)\,u-\beta\,u\,u},$$

ideoque

$$l\,U = -l\sqrt{[\gamma+(\delta-\alpha)u-\beta uu]} + \tfrac{1}{2}(\alpha+\delta)\,l\,V,$$

sive

$$U = \frac{V^{\frac{1}{2}(\alpha+\delta)}}{\sqrt{[\gamma+(\delta-\alpha)\,u-\beta uu]}};\ \text{hinc}$$

$$\frac{x}{U} = \frac{\sqrt{[\gamma xx+(\delta-\alpha)\,xy-\beta yy]}}{V^{\frac{1}{2}(\alpha+\delta)}}.$$

Corollarium 3.

202. Quocirca invento valore particulari $z = V$, ut sit

$$\frac{\partial V}{V} = \frac{\partial u}{\gamma+(\delta-\alpha)u-\beta uu},\ \text{existente}\ u = \frac{y}{x},$$

erit valor generaliter satisfaciens

$$z = V\,f : \frac{\gamma xx+(\delta-\alpha)\,xy-\beta yy}{V^{\alpha+\delta}} = V\,f : \frac{x(\gamma x+\delta y)-y(\alpha x+\beta y)}{V^{\alpha+\delta}}$$

Corollarium 4.

203. Hinc colligitur alius valor particularis, qui semper est algebraicus, erit is scilicet

$$z = [x(\gamma x + \delta y) - y(\alpha x + \beta y)]^{\frac{1}{\alpha+\delta}},$$

vel ejus multiplum quodcunque. Nisi autem V sit quantitas algebraica, omnes reliqui valores erunt transcendentes, et in hac forma contenti

$$z = [x(\gamma x + \delta y) - y(\alpha x + \beta y)]^{\frac{1}{\alpha+\delta}} f : \frac{x(\gamma x + \delta y) - y(\alpha x + \beta y)}{V^{\alpha+\delta}}.$$

Scholion.

204. Unicus casus, quo $\delta = -\alpha$ et conditio proposita

$$z = p(\alpha x + \beta y) + q(\gamma x - \alpha y),$$

peculiarem evolutionem postulat. Primo autem posito $u = \frac{y}{x}$, pro valore particulari $z = V$ erit

$$l V = \int \frac{\partial u}{\gamma - 2\alpha u - \beta uu}.$$

Tum vero ob

$$\frac{\partial U}{U} = \frac{(\alpha + \beta u)\partial u}{\gamma - 2\alpha u - \beta uu}, \text{ erit}$$

$$U = \frac{1}{\sqrt{(\gamma - 2\alpha u - \beta uu)}} \text{ et } \frac{x}{U} = \sqrt{(\gamma xx - 2\alpha xy - \beta yy)},$$

ita ut jam valor generalis sit

$$z = V f : (\gamma xx - 2\alpha xy - \beta yy).$$

Per se enim manifestum est, formam $f : \sqrt{T}$ exprimi posse per $f : T$. Nisi ergo V sit functio algebraica, hoc casu nulla solutio particularis algebraica locum habet.

Exemplum 1.

205. *Si posito $\partial z = p\partial x + q\partial y$ esse debeat $nz = py - qy$, indolem functionis z investigare.*

Comparatione cum forma nostra generali instituta fit

$$\alpha = 0,\ \beta = \tfrac{1}{n},\ \gamma = -\tfrac{1}{n},\ \delta = 0.$$

Hic ergo casus ob $\delta = -\alpha$ pertinet ad §. praecedentem, unde fit

$$l\,V = \int \frac{n\,\partial u}{-1 - uu} = -n \text{ Ang. tang. } u.$$

Cum igitur sit $u = \frac{y}{x}$, forma generalis est

$$z = e^{-n \text{ Ang. tang. } \frac{y}{x}}\, f:(xx + yy).$$

Exemplum 2.

206. *Si posito $\partial z = p\,\partial x + q\,\partial y$ debeat esse*

$$z = p\,(x + y) - q\,(x + y),$$

indolem functionis z investigare.

Comparatione facta fit

$$\alpha = 1,\ \beta = 1,\ \gamma = -1,\ \delta = -1,$$

hincque

$$l\,V = \int \frac{\partial u}{-1 - 2u - uu} = \frac{1}{1 + u}, \text{ et } V = e^{\frac{1}{1+u}},$$

et solutio generalis est

$$z = e^{\frac{x}{x+y}}\, f:(x + y).$$

Exemplum 3.

207. *Si posito $\partial z = p\,\partial x + q\,\partial y$ debeat esse*

$$z = p\,(x - 2y) + q\,(2x - 3y),$$

indolem functionis z investigare.

Cum ergo hic sit

$$\alpha = 1,\ \beta = -2,\ \gamma = 2, \text{ et } \delta = -3,$$

erit primo

$$lV = \int \frac{\partial u}{2 - 4u + 2uu} = \frac{+1}{2(1-u)} = \frac{x}{2(x-y)},$$

et quia non est $\delta = -\alpha$, solutio generalis statim prodit

$$z = (2xx - 4xy + 2yy)^{\frac{-1}{2}} f : \frac{(2xx - 4xy + 2yy)}{V^{-2}},$$

et ob

$$V = e^{\frac{x}{2(x-y)}}, \text{ erit}$$

$$z = \frac{1}{x-y} f : (x-y)^2 e^{\frac{x}{x-y}}.$$

Unde solutio simplicissima est $z = \frac{1}{x-y}$.

Scholion.

208. Hic merito quaerimus, quo pacto haec solutio generalis statim sine adjumento solutionis specialis inveniri potuisset? sequenti autem modo ista investigatio instituenda videtur. Cum sit

$$p(\alpha x + \beta y) = z - q(\gamma x + \delta y) \text{ et}$$
$$q(\gamma x + \delta y) = z - p(\alpha x + \beta y),$$

si uterque valor seorsim in forma

$$\partial z = p\,\partial x + q\,\partial y$$

substituatur, prodibunt binae sequentes aequationes

$$(\alpha x + \beta y)\partial z = z\partial x - q(\gamma x + \delta y)\partial x + q(\alpha x + \beta y)\partial y,$$
$$(\gamma x + \delta y)\partial z = z\partial y + p(\gamma x + \delta y)\partial x - p(\alpha x + \beta y)\partial y.$$

Multiplicetur prior indefinite per M posterior per N, et productorum summa dabit

$$\partial z[\mathrm{M}(\alpha x + \beta y) + \mathrm{N}(\gamma x + \delta y)] - z(\mathrm{M}\partial x + \mathrm{N}\partial y)$$
$$= (\mathrm{N}p - \mathrm{M}q)[(\gamma x + \delta y)\partial x - (\alpha x + \beta y)\partial y],$$

ubi jam M et N ita capi debent, ut prius membrum integrationem admittat, tum enim ejus integrale aequabitur functioni cuicunque quantitatis

$$\int \frac{(\gamma x + \delta y)\partial x - (\alpha x + \beta y)\partial y}{\gamma x x + (\delta - \alpha) x y - \beta y y},$$

quam supra (§. 197.) definire docuimus: unde patet illud integrale fieri $= f : \frac{x}{U}$. Manifestum autem est, M et N ejusmodi functiones esse oportere ut haec aequatio fiat possibilis

$$\frac{\partial z}{z} = \frac{M \partial x + N \partial y}{M(\alpha x + \beta y) + N(\gamma x + \delta y)},$$

seu ut membrum posterius integrationem admittat; quod si enim ejus integrale sit $= l V$, erit $\frac{z}{V} = f : \frac{z}{U}$. Pro hac integrabilitate ponamus $y = u x$, et M et N functionis ipsius u, erit

$$\frac{\partial z}{z} = \frac{(M + N u)\partial x + N x \partial u}{M x(\alpha + \beta u) + N x(\gamma + \delta u)}.$$

Ubi integratio succedit sumendo $M = - N u$, ut sit

$$\frac{\partial z}{z} = \frac{+ \partial u}{\gamma + (\delta - \alpha) u - \beta u u}, \text{ seu}$$

$$l V = \int \frac{\partial u}{\gamma + (\delta - \alpha) u - \beta u u},$$

prorsus ut ante.

Problema 36.

209. Si posito $\partial z = p \partial x + q \partial y$ debeat esse $Z = p P + q Q$, existente Z functione ipsius z tantum, P et Q autem functionibus ipsarum x et y quibusvis datis, indolem functionis z investigare.

Solutio.

Formentur sequentes aequationes ex propositis

$$L \partial z = L p \partial x + L q \partial y, \quad M Z \partial x = M p P \partial x + M q Q \partial x,$$
$$N Z \partial y = N p P \partial y + N q Q \partial y,$$

quae in unam summam collectae dabunt

$$L \partial z + Z (M \partial x + N \partial y) = p [(L + M P) \partial x + N P \partial y]$$
$$+ q [(L + N Q) \partial y + M Q \partial x],$$

Ut jam pars posterior habeat factorem a litteris p et q liberum, fiat

$$L + M P : N P = M Q : L + N Q,$$

unde fit

$$L L + L N Q + L M P = 0, \text{ seu } L = - M P - N Q,$$

quo valore inducto erit

$$- \partial z (M P + N Q) + Z (M \partial x + N \partial y) = (M q - N p) (Q \partial x - P \partial y).$$

Cum nunc P et Q sint functiones datae ipsarum x et y, dabitur multiplicator R, ut fiat

$$R (Q \partial x - P \partial y) = \partial U, \text{ ideoque}$$

$$- \partial z (M P + N Q) + Z (M \partial x + N \partial y) = \frac{M q - N p}{R} . \partial U.$$

Pro parte priori capiantur functiones indefinitae M et N ita ut formula $\frac{M \partial x + N \partial y}{M P + N Q}$ integrabilis evadat, id quod semper fieri licet, sitque

$$\frac{M \partial x + N \partial y}{M P + N Q} = \partial V,$$

et ob

$$M \partial x + N \partial y = (M P + N Q) \partial V,$$

aequatio nostra hanc induet formam

$$(M P + N Q) (- \partial z + Z \partial V) = \frac{M q - N p}{R} . \partial U, \text{ seu}$$

$$\frac{\partial z}{Z} - \partial V = \frac{N p - M q}{R Z (M P + N Q)} . \partial U.$$

Statuatur jam

$$\frac{N p - M q}{R Z (M P + N Q)} = f' : U,$$

atque habebitur

$$\int \frac{\partial z}{Z} - V = f : U, \text{ seu } \int \frac{\partial z}{Z} = V + f : U,$$

unde z determinatur per x et y.

Corollarium 1.

210. Pro solutione ergo problematis quaeratur primo ad formulam $Q \partial x - P \partial y$ multiplicator R eam reddens integrabilem, statuaturque

$$R(Q\,\partial x - P\,\partial y) = \partial U,$$

unde colligitur quantitas U per binas variabiles x et y expressa.

Corollarium 2.

211. Deinde quantitates M et N ita capiantur, ut formula $\frac{M\,\partial x + N\,\partial y}{MP + NQ}$ fiat integrabilis, cujus integrale si statuatur $= V$ statim habetur solutio generalis problematis, quae dat

$$\int \frac{\partial z}{Z} = V + f:U.$$

Exemplum.

212. *Si P et Q sint functiones homogeneae ipsarum x et y utraque dimensionum numeri $= n$, solutionem problematis perficere.*

Ponatur $y = ux$, et tam P quam Q fiet productum ex potestate x^n in functionem quandam ipsius u. Sit ergo

$$P = x^n S \text{ et } Q = x^n T,$$

eruntque S et T functiones datae ipsius u. Tum vero ob

$$\partial y = u\,\partial x + x\,\partial u,$$

formula

$$Q\,\partial x - P\,\partial y$$

abit in

$$x^n T\,\partial x - x^n S u\,\partial x - x^{n+1} S\,\partial u = x^n[(T - Su)\,\partial x - Sx\,\partial u].$$

Sumatur ergo

$$R = \frac{1}{x^{n+1}(T - Su)}, \text{ fietque}$$

$$\partial U = \frac{\partial x}{x} - \frac{S\,\partial u}{T - Su}, \text{ unde colligitur U.}$$

Deinde pro altera quantitate V habebimus hanc aequationem

$$\partial V = \frac{(M + Nu)\,\partial x + Nx\,\partial u}{x^n(MS + NT)},$$

ubi jam facile est, pro M et N ejusmodi functiones ipsius u assumere, ut haec formula integrationem admittat. Integrale scilicet erit

$$V = \frac{-M - Nu}{(n-1)x^{n-1}(MS + NT)},$$

at M et N seu $\frac{M}{N} = K$ ita accipi debet, ut fiat

$$\frac{1}{(n-1)x^{n-1}} \partial \cdot \frac{K+u}{KS+T} = \frac{1}{x^{n-1}} \cdot \frac{\partial u}{KS+T}, \text{ seu}$$

$$-KK\partial S + KS\partial u - uK\partial S - uS\partial K + T\partial K - K\partial T$$
$$+ T\partial u - u\partial T + (n-1)\partial u(KS+T) = 0,$$

quae ad hanc formam reducitur

$$(T - Su)\partial K + K(nS\partial u - u\partial S - \partial T) - KK\partial S + nT\partial u - u\partial T = 0.$$

Ex qua, concessa aequationum resolutione, cognoscitur quantitas K, qua inventa erit

$$V = \frac{-K - u}{(n-1)x^{n-1}(KS+T)}$$

Cum autem illa aequatio solutu difficilis videatur, ponatur statim $\frac{K+u}{KS+T} = v$, eritque

$$K = \frac{Tv - u}{1 - Sv} \text{ et } KS + T = \frac{T - Su}{1 - Sv},$$

unde fit

$$\partial v + \frac{(n-1)\partial u(1 - Sv)}{T - Su} = 0,$$

qua resoluta erit $V = \frac{-v}{(n-1)x^{n-1}}$.

Corollarium.

213. Casus autem quo $n = 1$ singulari evolutione eget. Facile autem patet tum sumi debere $M = -Nu$, ut fiat $\partial V = \frac{\partial u}{T - Su}$, unde postquam quantitas V fuerit inventa, erit semper

$$\int \frac{\partial z}{Z} = V + f : U.$$

Scholion.

214. Cum ternae variabiles x, y, z, sint inter se permutabiles patet hoc problema multo latius extendi posse. Scilicet si conditio proposita hac continetur aequatione $p\,P + q\,Q + R = 0$, non solum solvendi methodus adhibita succedit, si R sit functio ipsius z, et P cum Q functiones ipsarum x et y, sed etiam si fuerit P functio ipsius x et Q et R functiones ipsarum y et z; tum vero etiam si Q functio ipsius y, at P et R functiones binarum reliquarum x et z. Haec vero conditio cum ante tractatis eo redit, ut binae formulae differentiales p et q sint a se invicem separatae, neque plus una dimensione occupent, etiamsi et his casibus ingens restrictio accedat. Quodsi autem conditio magis sit complicata, solutio vix unquam sperari posse videtur, interim tamen casum ejusmodi proferam, quo solutionem expedire licet.

Problema 37.

215. *Si posito* $\partial z = p\,\partial x + q\,\partial y$, *debeat esse*

$$q = A\,p^n\,x^\lambda\,y^\mu\,z^\nu,$$

indolem functionis z *in genere investigare.*

Solutio.

Posito hoc valore loco q, habebimus

$$\partial z = p\,\partial x + A\,p^n\,x^\lambda\,y^\mu\,z^\nu\,\partial y, \text{ unde fit}$$

$$A\,y^\mu\,\partial y = p^{-n}\,x^{-\lambda}\,z^{-\nu}\,(\partial z - p\,\partial x).$$

Ponatur $p^{-n}\,x^{-\lambda}\,z^{-\nu} = t$, ut sit

$$p = t^{-\frac{1}{n}}\,x^{-\frac{\lambda}{n}}\,z^{-\frac{\nu}{n}}, \text{ eritque}$$

$$A\,y^\mu\,\partial y = t\,\partial z - t^{\frac{n-1}{n}}\,x^{-\frac{\lambda}{n}}\,z^{-\frac{\nu}{n}}\,\partial x.$$

Statuatur porro

$$t^{n-1} z^{-\nu} = u^n, \text{ seu } t = z^{\frac{\nu}{n-1}} u^{\frac{n}{n-1}}, \text{ erit}$$

$$A y^\mu \partial y = u^{\frac{n}{n-1}} z^{\frac{\nu}{n-1}} \partial z - u x^{-\frac{\lambda}{n}} \partial x.$$

Jam partibus quoad fieri licet integratis adipiscimur

$$\frac{A}{\mu+1} y^{\mu+1} = \frac{n-1}{n+\nu-1} u^{\frac{n}{n-1}} z^{\frac{n+\nu-1}{n-1}} + \frac{nu}{n-\lambda} x^{\frac{n-\lambda}{n}} - \int \partial u \left(\frac{n}{n+\nu-1} u^{\frac{1}{n-1}} z^{\frac{n+\nu-1}{n-1}} + \frac{n}{n-\lambda} x^{\frac{n-\lambda}{n}}\right),$$

ac nunc solutionem per praecepta supra data expedire licet; scilicet statuatur

$$\frac{1}{n+\nu-1} u^{\frac{1}{n-1}} z^{\frac{n+\nu-1}{n-1}} + \frac{1}{n-\lambda} x^{\frac{n-\lambda}{n}} = f' : u,$$

eritque

$$\frac{A}{\mu+1} y^{\mu+1} = \frac{n-1}{n+\nu-1} u^{\frac{n}{n-1}} z^{\frac{n+\nu-1}{n-1}} + \frac{n}{n-\lambda} u x^{\frac{n-\lambda}{n}} - n f : u,$$

atque ex his binis aequationibus si elidatur u, dabitur utique z per x et y.

Corollarium 1.

216. Casus $n = 1$ peculiarem postulat tractationem, cum enim posito $p = \frac{1}{t} x^{-\lambda} z^{-\nu}$ sit

$$A y^\mu \partial y = t \partial z - x^{-\lambda} z^{-\nu} \partial x, \text{ erit}$$

$$\frac{A}{\mu+1} y^{\mu+1} = \frac{1}{\lambda-1} x^{1-\lambda} z^{-\nu} + \int \partial z \left(t - \frac{\nu}{\lambda-1} x^{1-\lambda} z^{-\nu-1}\right),$$

atque hinc statim concluditur

$$\frac{A}{\mu+1} y^{\mu+1} = \frac{1}{\lambda-1} x^{1-\lambda} z^{-\nu} + f : z, \text{ existente}$$

$$t - \frac{\nu}{\lambda-1} x^{1-\lambda} z^{-\nu-1} = f' : z.$$

Corollarium 2.

217. Casus autem $n + \nu - 1 = 0$ et $n - \lambda = 0$ nullam facessunt molestiam, cum sit priori casu

$$\frac{n-1}{n+\nu-1} z^{\frac{n+\nu-1}{n-1}} = l z,$$

posteriori autem

$$\frac{n}{n-\lambda}x^{\frac{n-\lambda}{n}} = lx$$

quos valores in solutionem introduci oportet.

Exemplum.

218. *Si posito* $\partial z = p\partial x + q\partial y$ *debeat esse* $pqxy = az$, *seu* $q = \frac{az}{pxy}$, *functionem z investigare.*

Erit ergo

$$\partial z = p\,\partial x + \frac{az\partial y}{pxy}, \text{ seu } \frac{a\partial y}{y} = \frac{px}{z}(\partial z - p\,\partial x).$$

Ponatur $\frac{px}{z} = t$, seu $p = \frac{tz}{x}$, erit $\frac{a\partial y}{y} = t\partial z - \frac{ttz\partial x}{x}$.

Statuatur porro $ttz = uu$, seu $t = \frac{u}{\sqrt{z}}$, ut sit

$$\frac{a\partial y}{y} = \frac{u\partial z}{\sqrt{z}} - \frac{uu\partial x}{x},$$

et quoad fieri potest integrando

$$aly = 2u\sqrt{z} - uulx - \int\partial u(2\sqrt{z} - 2ulx),$$

ita ut jam post signum integrale unicum differentiale ∂u reperiatur. Posito ergo

$$\sqrt{z} - ulx = f':u, \text{ erit}$$

$$aly = 2u\sqrt{z} - uulx - 2f:u = uulx + 2uf':u - 2f:u.$$

Pro casu simplicissimo sumatur $f':u = 0$ et $f:u = 0$, erit $u = \frac{\sqrt{z}}{lx}$, ideoque

$$aly = \frac{2z}{lx} - \frac{z}{lx} = \frac{z}{lx},$$

ita ut pro casu simplicissimo sit $z = alx.ly$. Si ponatur

$$f':u = ulc \text{ et } f:u = \tfrac{1}{2}uulc, \text{ erit}$$

$$u = \frac{\sqrt{z}}{lx + lc} = \frac{\sqrt{z}}{lcx} \text{ et}$$

$$aly = \frac{2z}{lcx} - \frac{zlx}{(lcx)^2} - \frac{zlc}{(lcx)^2} = \frac{lz}{lcx},$$

ita ut sit

$$z = aly(lc + lx),$$

magis generaliter autem erit

$$z = a\,(l\,b + l\,y)\,(l\,c + l\,x).$$

Scholion.

219. Methodi hactenus traditae haud mediocriter amplificabuntur, si loco binarum variabilium x et y, quarum functio esse debet z, binae aliae variabiles t et u introducantur, quarum relatio ad illas detur. Ita si z sit functio binarum variabilium x et y, ut inde prodeat

$$\partial z = p\,\partial x + q\,\partial y;$$

ac loco x et y aliae novae variabiles t et u introducantur, ut jam differentiatione instituta prodeat

$$\partial z = r\,\partial t + s\,\partial u;$$

quaeritur quomodo r et s per p et q determinentur, pro relatione inter pristinas variabiles x, y et novas t et u stabilita. Hinc ergo tam x quam y certae cuidam functioni ipsarum t et u aequabitur, quae cum detur sit

$$\partial x = \mathrm{P}\,\partial t + \mathrm{Q}\,\partial u \text{ et } \partial y = \mathrm{R}\,\partial t + \mathrm{S}\,\partial u,$$

ita ut facta hac substitutione z jam sit functio ipsarum t et u. Cum igitur esset

$$\partial z = p\,\partial x + q\,\partial y,$$

erit nunc

$$\partial z = (\mathrm{P}\,p + \mathrm{R}\,q)\,\partial t + (\mathrm{Q}\,p + \mathrm{S}\,q)\,\partial u.$$

Est vero per hypothesin

$$\partial z = r\,\partial t + s\,\partial u,$$

unde habebitur

$$r = \mathrm{P}\,p + \mathrm{R}\,q \text{ et } s = \mathrm{Q}\,p + \mathrm{S}\,q.$$

Quare facta hac substitutione valores differentiales novi ex praecedentibus ita determinabuntur, ut sit

$$\left(\tfrac{\partial z}{\partial t}\right) = \mathrm{P}\left(\tfrac{\partial z}{\partial x}\right) + \mathrm{R}\left(\tfrac{\partial z}{\partial y}\right) \text{ et } \left(\tfrac{\partial z}{\partial u}\right) = \mathrm{Q}\left(\tfrac{\partial z}{\partial x}\right) + \mathrm{S}\left(\tfrac{\partial z}{\partial y}\right).$$

Unde etiam cum sit vicissim

$$Q r - P s = (Q R - P S) q \text{ et } S r - R s = (P S - Q R) p,$$

concludimus fore

$$\left(\frac{\partial z}{\partial x}\right) = \frac{S}{PS - QR} \left(\frac{\partial z}{\partial t}\right) - \frac{R}{PS - QR} \left(\frac{\partial z}{\partial u}\right) \text{ et}$$

$$\left(\frac{\partial z}{\partial y}\right) = \frac{-Q}{PS - QR} \left(\frac{\partial z}{\partial t}\right) + \frac{P}{PS - QR} \left(\frac{\partial z}{\partial u}\right).$$

Vel cum x et y perinde ac z sint functiones ipsarum t et u haec relatio ita exprimi potest, ut sit

$$\left(\frac{\partial z}{\partial t}\right) = \left(\frac{\partial x}{\partial t}\right) \left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial y}{\partial t}\right) \left(\frac{\partial z}{\partial y}\right) \text{ et}$$

$$\left(\frac{\partial z}{\partial u}\right) = \left(\frac{\partial x}{\partial u}\right) \left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial y}{\partial u}\right) \left(\frac{\partial z}{\partial y}\right).$$

Hinc efficitur, ut quae problemata pro data quadam relatione inter p, q, x, y, z resolvi possunt, ea quoque pro relatione inde resultante inter r, s, t, u et z resolvi queant; unde saepe problemata nascuntur, quae solutu vehementer difficilia videantur, ex quo non contemnenda subsidia in hanc Analyseos partem inferri possent; sed quia usus praecipue in formulis differentialibus secundi gradus spectatur, his non fusius immorans ad eas evoluendas progredior.