CAPUT V.

DE

RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER QUANTITATES $\left(\frac{\partial z}{\partial x}\right)$, $\left(\frac{\partial z}{\partial y}\right)$, ET BINAS TRIUM VARIABILIUM x, y, z QUAECUNQUE DATUR.

Problema 21.

138.

Si posito $\partial z = p\partial x + q\partial y$, debeat esse $px + qy = 0$, functionis z indolem per x et y in genere investigare.

Solutio.

Cum sit $q = -\frac{px}{y}$, erit

$$\partial z = p\partial x - \frac{px\partial y}{y} = px\left(\frac{\partial x}{x} - \frac{\partial y}{y}\right), \text{ seu}$$

$$\partial z = py\left(\frac{\partial x}{y} - \frac{x\partial y}{yy}\right) = py\,\partial\,.\,\frac{x}{y}.$$

Unde patet py esse debere functionem ipsius $\frac{x}{y}$; ac si ponatur

$$py = f' : \frac{x}{y}, \text{ fore } z = f : \frac{x}{y}.$$

Perpetuo scilicet in designandis functionibus hac lege utemur, ut sit

$$\partial\,.\,f : v = \partial v f' : v,$$

sicque porro

$$\partial\,.\,f' : v = \partial v f'' : v \text{ et } \partial\,.\,f'' : v = \partial v f''' : v, \text{ etc.}$$

At $f : \frac{x}{y}$ denotat functionem quamcunque homogeneam ipsarum x et

y nullius dimensionis, ac si z fuerit talis functio quaecunque, et differentiando prodeat $\partial z = p\partial x + q\partial y$, semper erit

$$p x + q y = 0.$$

Corollarium 1.

139. Quodsi ergo z fuerit functio homogenea nullius dimensionis ipsarum x et y, ob

$$p = \left(\frac{\partial z}{\partial x}\right) \text{ et } q = \left(\frac{\partial z}{\partial y}\right), \text{ erit}$$

$$x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = 0,$$

quam veritatem quidem jam supra elicuimus.

Corollarium 2.

140. Tum vero cum sit

$$p = \frac{1}{y} \, f' : \frac{x}{y} \quad \text{et} \quad q = \frac{-x}{yy} \, f' : \frac{x}{y},$$

erit p functio homogenea ipsarum x et y numeri dimensionem $= -1$, et si sit $q = \frac{-px}{y}$, ipsa functio z reperitur ex integratione $z = \int py\partial \cdot \frac{x}{y}$.

Scholion.

141. Simili modo solvitur problema, si posito

$$\partial z = p\,\partial x + q\,\partial y,$$

fieri debeat $mpx + nqy = a$. Tum enim ob $q = \frac{a}{ny} - \frac{mpx}{ny}$, erit

$$\partial z = \frac{a\partial y}{ny} + p\partial x - \frac{mpx\partial y}{ny}, \text{ seu}$$

$$\partial z = \frac{a\partial y}{ny} + \frac{px}{n}\left(\frac{n\partial x}{x} - \frac{m\partial y}{y}\right) = \frac{a\partial y}{ny} + \frac{py^m}{nx^{n-1}}\,\partial \cdot \frac{x^n}{y^m};$$

unde solutio praebet

$$\frac{py^m}{nx^{n-1}} = f' : \frac{x^n}{y^m} \text{ et } z = \frac{a}{n}\,ly + f : \frac{x^n}{y^m}.$$

Quin etiam hoc generalius problema resolvi potest, quo esse debet $pX + qY = A$, existente X functione ipsius x, et Y ipsius y. Cum enim inde fiat $q = \frac{A}{Y} - \frac{pX}{Y}$, erit

$$\partial z = \frac{A\partial y}{Y} + p\partial x - \frac{pX\partial y}{Y} = \frac{A\partial y}{Y} + pX\left(\frac{\partial x}{X} - \frac{\partial y}{Y}\right).$$

Statui ergo debet

$$pX = f' : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right).$$

indeque fit

$$z = A\int\frac{\partial y}{Y} + f : \left(\int\frac{\partial x}{X} - \int\frac{\partial y}{Y}\right).$$

Problema 22.

142. Si posito $\partial z = p\partial x + q\partial y$, debet esse $\frac{q}{p}$ aequale functioni datae cuicunque ipsarum x et y, indolem functionis z in genere investigare.

Solutio.

Sit V ista functio data ipsarum x et y, ut sit $q = pV$, et habebitur $\partial z = p(\partial x + V\partial y)$. Dabitur jam multiplicator M itidem functio ipsarum x et y, ut $M(\partial x + V\partial y)$ fiat integrabile. Ponatur ergo $M(\partial x + V\partial y) = \partial S$, ac dabitur etiam S functio ipsarum x et y. Cum ergo sit $\partial z = \frac{p\partial S}{M}$, perspicuum est, quantitatem $\frac{p}{M}$ aequari debere functioni ipsius S, quare si ponamus $\frac{p}{M} = f' : S$, fiet $z = f : S$, indeque erit

$$p = Mf' : S \text{ et } q = MVf' : S.$$

Corollarium 1.

143. Hoc ergo casu functio quaesita z statim invenitur per x et y expressa, quoniam S per x et y datur. Fieri autem potest, ut S prodeat quantitas transcendens; quin etiam ut per methodos adhuc cognitas multiplicator M ne inveniri quidem possit.

Corollarium 2.

144. Si U sit functio nullius dimensionis ipsarum x et y, erit $M = \frac{1}{x + Vy}$. Seu posito $x = vy$, fiet V functio ipsius v, et

$$\partial S = M\,(y\partial v + v\partial y + V\partial y).$$

Capiatur $M = \frac{1}{y\,(v + V)}$, eritque

$$\partial S = \frac{\partial y}{y} + \frac{\partial v}{v + V}; \text{ unde reperitur}$$

$$z = f : \left(ly + \int \frac{\partial v}{v + V}\right).$$

Scholion.

145. Ob permutabilitatem ipsarum p et x item q et y, simili modo sequentia problemata resolvi possunt

I. Si debeat esse $q = xV$, existente V functione quacunque ipsarum p et y, consideretur forma

$$z = px + \int (q\partial y - x\partial p) = px + \int x\,(V\partial y - \partial p).$$

Quaeratur multiplicator M, ut sit

$$M\,(V\partial y - \partial p) = \partial S,$$

erit S functio ipsarum p et y, atque

$$z = px + \int \frac{x\partial S}{M};$$

ex quo colligitur haec solutio

$$\frac{x}{M} = f' : S \text{ et } z = pMf' : S + f : S.$$

II. Si debeat esse $y = pV$, existente V functione quacunque ipsarum x et q. Consideretur forma

$$z = qy + \int (p\partial x - y\partial q) = qy + \int p\,(\partial x - V\partial q).$$

Quaeratur multiplicator M, ut sit

$$M\,(\partial x - V\partial q) = \partial S,$$

erit S functio ipsarum x et q, et

$$z = qy + \int \frac{p\partial S}{M}.$$

Quare fit

$$\frac{p}{M} = f' : S \text{ et } z = qy + f : S,$$

seu ob $p = \frac{y}{V}$, erit

$$y = MVf' : S \text{ et } z = qMVf' : S + f : S.$$

III. Si debeat esse $y = xV$, existente V functione quacunque ipsarum p et q, consideretur haec forma

$$z = px + qy - \int (x\partial p + xV\partial q).$$

Quaeratur multiplicator M, ut fiat

$$M (\partial p + V\partial q) = \partial S,$$

erit S functio ipsarum p et q, et

$$z = px + qy - \int \frac{x\partial S}{M};$$

unde haec solutio nascitur

$$\frac{x}{M} = f' : S \text{ et } z = px + qy - f : S.$$

Omnes hi casus huc redeunt, ut quaternarum quantitatum p, x, q, y, vel $\frac{q}{p}$, vel $\frac{q}{x}$, vel $\frac{y}{p}$, vel $\frac{y}{x}$, aequetur functioni cuicunque binarum reliquarum.

Problema 23.

146. Si posito $\partial z = p\partial x + q\partial y$, requiratur ut sit $q = pV + U$, existente tam V quam U functione quacunque binarum variabilium x et y, indolem functionis z in genere investigare.

Solutio.

Cum ob $q = pV + U$ sit

$$\partial z = p(\partial x + V\partial y) + U\partial y,$$

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quaeratur primo multiplicator M formulam $\partial x + V \partial y$ reddens integrabilem, sitque

$$M\,(\partial x + V\partial y) = \partial S,$$

erunt M et S functiones ipsarum x et y, fietque

$$\partial z = \frac{p\partial S}{M} + U\partial y.$$

Cum jam sit S functio ipsarum x et y, inde x per y et S definiri potest, quo valore introducto fient U et M functiones ipsarum y et S. Nunc sumto S constante, integretur formula $U\partial y$, sitque

$$\int U\partial y = T + f : S,$$

ac posito

$$\partial T = U\partial y + W\partial S, \text{ fiet}$$

$$\frac{p}{M} = W + f' : S \text{ et } z = T + f : S,$$

sicque omnia per binas variabiles y et S exprimentur.

Corollarium 1.

147. Datis ergo binarum variabilium x et y functionibus V et U, ut sit $q = pV + U$, solutio problematis primo postulat, ut multiplicator M investigetur formulam $\partial x + V\partial y$ integrabilem reddens, quo invento habetur functio S earundem variabilium x et y, ut sit

$$S = \int M\,(\partial x + V\partial y).$$

Corollarium 2.

148. In hunc finem considerari conveniet aequationem differentialem $\partial x + V\partial y = 0$, haec enim si integrari poterit, simul inde colligi potest multiplicator M, ut formula $M\,(\partial y + V\partial y)$ fiat verum differentiale cujusdam functionis S, quae propterea hinc invenietur.

Corollarium 3.

149. Inventa porro hac functione S, quantitas x per y et S exprimi debet, ita ut x aequetur functioni ipsarum y et S, quo valore in quantitate U substituto, quaeratur integrale $\int U\partial y = T$, spectata S ut constante, sicque obtinebitur T functio ipsarum y et S.

Corollarium 4.

150. Denique inventa hac functione T, sit $W = \left(\frac{\partial T}{\partial S}\right)$ unde tandem colligitur solutio problematis his duabus formulis contenta

$$\frac{p}{M} = W + f' : S, \quad \text{et} \quad z = T + f : S;$$

ubi cum S sit functio ipsarum x et y, pro z statim reperitur functio ipsarum x et y.

Corollarium 5.

151. Si U sit functio ipsius y tantum, non opus est illa expressione ipsius x per y et S, sed $T = \int U\partial y$ erit quoque functio ipsius y tantum, hinc $W = \left(\frac{\partial T}{\partial S}\right) = 0$. Hic autem casus manifesto reducitur ad praecedentem ponendo z loco $z - \int U\partial y$.

Exemplum 1.

152. *Si posito $\partial z = p\partial x + q\partial y$, debeat esse $q = \frac{px}{y} + \frac{y}{x}$, indolem functionis z investigare.*

Hic ergo est

$$V = \frac{x}{y} \quad \text{et} \quad U = \frac{y}{x};$$

unde ob

$$\partial x + V\partial y = \partial x + \frac{x\partial y}{y},$$

erit multiplicator $M = y$, et $\partial S = y\partial x + x\partial y$, hinc $S = xy$, sicque habebitur

$$x = \frac{S}{y} \text{ et } U = \frac{yy}{S}.$$

Jam erit

$$T = \int U\partial y = \int \frac{yy\partial y}{S} = \frac{y^3}{3S}, \text{ et } W = \frac{-y^3}{3SS}.$$

Quare pro solutione hujus exempli habebimus

$$\frac{p}{y} = \frac{-y^3}{3SS} + f' : S, \text{ et } z = \frac{y^3}{3S} + f : S,$$

seu ob $S = xy$ erit

$$z = \frac{yy}{3x} + f : xy.$$

Exemplum 2.

153. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse*

$$px + qy = n\sqrt{(xx + yy)},$$

indolem functionis z investigare.

Cum hic sit $q = \frac{-px}{y} + \frac{n}{y}\sqrt{(xx + yy)}$, erit

$$V = \frac{-x}{y} \text{ et } U = \frac{n}{y}\sqrt{(xx + yy)}.$$

Ergo $\partial S = M\left(\partial x - \frac{x\partial y}{y}\right)$, quare capiatur $M = \frac{1}{y}$, ut fiat

$$\partial S = \frac{\partial x}{y} - \frac{x\partial y}{yy}, \text{ et } S = \frac{x}{y}.$$

Hinc oritur

$$x = Sy, \text{ et } U = n\sqrt{(1 + SS)};$$

ideoque posito S constante erit

$$T = \int U\partial y = ny\sqrt{(1 + SS)}, \text{ et } W = \left(\frac{\partial T}{\partial S}\right) = \frac{nyS}{\sqrt{(1 + SS)}};$$

ita ut solutio nostrae quaestionis sit

$$py = \frac{nyS}{\sqrt{(1 + SS)}} + f' : S, \text{ et } z = ny\sqrt{(1 + SS)} + f : S.$$

Cum igitur sit $S = \frac{x}{y}$, erit

$$z = n\sqrt{(xx + yy)} + \mathrm{f} : \frac{x}{y};$$

ubi $\mathrm{f} : \frac{x}{y}$ denotat functionem quamcunque nullius dimensionis ipsarum x et y.

Exemplum 3.

154. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse*
$$pxx + qyy = nxy,$$
functionis z indolem investigare.

Cum sit $q = \frac{-pxx}{yy} + \frac{nx}{y}$, erit
$$V = \frac{-xx}{yy} \text{ et } U = \frac{nx}{y}.$$
Quare ob $\partial S = M\left(\partial x - \frac{xx\partial y}{yy}\right)$, capiatur $M = \frac{1}{xx}$, ut fiat $S = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$. Hinc erit
$$\frac{1}{x} = \frac{1}{y} - S, \text{ et } x = \frac{y}{1 - Sy};$$
ideoque $U = \frac{n}{1 - Sy}$. Sumto igitur S constante habebimus
$$T = \int \frac{n\partial y}{1 - Sy} = -\frac{n}{S}\, l\,(1 - Sy), \text{ et}$$
$$W = +\frac{n}{SS}\, l\,(1 - Sy) + \frac{ny}{S(1 - Sy)}.$$
Consequenter ob
$$S = \frac{x-y}{xy} \text{ et } 1 - Sy = \frac{y}{x},$$
solutio praebet
$$z = \frac{-nxy}{x-y}\, l\,\frac{y}{x} + \mathrm{f} : \frac{x-y}{xy}.$$

Scholion.

155. Ex solutione hujus problematis etiam haec quaestio latius patens resolvi potest. Sint P, Q, item V, U functiones quaecunque datae ipsarum x et y, et quaeri oporteat functionem z, ut sit
$$\partial z = P\partial x + Q\partial y + L\,(V\partial x + U\partial y),$$

seu quod eodem redit, functio L investigari debet, ut ista formula differentialis integrationem admittat. Ad hoc praestandum quaeratur primo multiplicator M formulam $V\partial x + U\partial y$ integrabilem efficiens, ponaturque $\partial S = M(V\partial x + U\partial y)$, unde functio S reperitur per x et y expressa. Ex ea quaeratur valor ipsius x per y et S expressus; et cum sit

$$\partial z = P\partial x + Q\partial y + \frac{L\partial S}{M},$$

hic ubique loco x valor ille substituatur; sit autem inde $\partial x = E\partial y + F\partial S$, unde etiam E et F innotescent, eritque

$$\partial z = EP\partial y + Q\partial y + FP\partial S + \frac{L\partial S}{M}.$$

Sumatur quantitas S pro constante, sitque

$$T = \int (EP + Q)\,\partial y, \text{ erit}$$
$$z = T + f : S,$$

quod quidem ad solutionem sufficit; sed ad L inveniendum, differentietur haec expressio

$$\partial z = (EP + Q)\,\partial y + \partial S \cdot \left(\frac{\partial T}{\partial S}\right) + \partial S f' : S,$$

ac necesse est fiat

$$FP + \frac{L}{M} = \left(\frac{\partial T}{\partial S}\right) + f' : S,$$

ideoque

$$L = -FMP + M\left(\frac{\partial T}{\partial S}\right) + Mf' : S.$$

Caeterum ob permutabilitatem ipsarum p, x et q, y, etiam hinc sequentia problemata resolvi possunt, quae propterea strictim percurram.

Problema 24.

156. Si posito $\partial z = p\partial x + q\partial y$ requiratur, ut sit $q = Vx + U$; existente tam V quam U functione quacunque data ipsarum p et y, investigare indolem functionis quaesitae z.

Solutio.

Utamur formula

$$z = px + \int (q\partial y - x\partial p),$$

et cum loco q valore substituto sit

$$\int (q\partial y - x\partial p) = \int (\mathrm{V}x\partial y - x\partial p + \mathrm{U}\partial y),$$

hanc formulam integrabilem reddi oportet. Sit ea brevitatis gratia $\mathfrak{P}$, et cum sit

$$\partial\mathfrak{P} = x\,(\mathrm{V}\partial y - \partial p) + \mathrm{U}\partial y,$$

quaeratur primo multiplicator M formulam $\mathrm{V}\partial y - \partial p$ integrabilem reddens, ponaturque

$$\mathrm{M}\,(\mathrm{V}\partial y - \partial p) = \partial \mathrm{S},$$

sicque S dabitur per y et p; unde p eliciatur per y et S expressum, quo valore ibi substituto erit

$$\partial\mathfrak{P} = \frac{x\partial \mathrm{S}}{\mathrm{M}} + \mathrm{U}\partial y.$$

Jam sumto S constante sumatur integrale

$$\int \mathrm{U}\partial y = \mathrm{T} + \mathrm{f} : \mathrm{S}, \text{ eritque}$$

$$\frac{x}{\mathrm{M}} = \left(\frac{\partial \mathrm{T}}{\partial \mathrm{S}}\right) + \mathrm{f}' : \mathrm{S}, \text{ et } \mathfrak{P} = \mathrm{T} + \mathrm{f} : \mathrm{S}.$$

Solutio igitur per binas variabiles y et S ita se habebit

$$x = \mathrm{M}\left(\frac{\partial \mathrm{T}}{\partial \mathrm{S}}\right) + \mathrm{M}\mathrm{f}' : \mathrm{S}, \text{ et } z = px + \mathrm{T} + \mathrm{f} : \mathrm{S},$$

ubi nunc quidem S per p et y datur.

Problema 25.

157. Si posito $\partial z = p\,\partial x + q\,\partial y$ requiratur, ut sit $p = \mathrm{V}y + \mathrm{U}$, existentibus V et U functionibus datis ipsarum x et q, indolem functionis z investigare.

Solutio.

Utamur jam forma

$$z = qy + \int (p\partial x - y\partial q),$$

ponaturque formula ad integrationem perducenda

$$\int (p\partial x - y\partial q) = \text{Þ}.$$

Hinc pro p valorem assumtum substituendo erit

$$\partial \text{Þ} = Vy\partial x + U\partial x - y\partial q = y(V\partial x - \partial q) + U\partial x.$$

Quaeramus multiplicatorem M, ut fiat

$$M(V\partial x - \partial q) = \partial S,$$

ac tam M quam S erunt functiones ipsarum x et q, ex quarum posteriori valor ipsius q per x et S expressus eliciatur, in sequenti operatione pro q substituendus. Scilicet cum nunc sit

$$\partial \text{Þ} = \frac{y\partial S}{M} + U\partial x,$$

sumto S constante quaeratur $T = \int U\partial x$, sitque

$$\text{Þ} = T + f : S,$$

unde colligitur

$$\frac{y}{M} = \left(\frac{\partial T}{\partial S}\right) + f' : S, \text{ et } z = qy + T + f : S.$$

ac nunc quidem pro S valorem in x et q restituere licet.

Problema 26.

158. Si posito $\partial z = p\partial x + q\partial y$ requiratur, ut sit $y = Vx + U$, existentibus V et U functionibus quibuscunque datis ipsarum p et q, indolem functionis z in genere investigare.

Solutio.

Hic utendum est formula

$$z = px + qy - \int (x\partial p + y\partial q).$$

Statuatur $\int (x\partial p + y\partial q) = \text{Þ}$, eritque pro y valorem praescriptum substituendo

$$\partial \text{Þ} = x\partial p + Vx\partial q + U\partial q.$$

Quaeratur jam multiplicator M, formulam $\partial p + V\partial q$ integrabilem reddens, sitque

$$M\,(\partial p + V\partial q) = \partial S,$$

ubi M et S per p et q dabuntur; et ex posteriori eliciatur valor ipsius p per q et S expressus, quo deinceps uti oportet. Scilicet cum sit

$$\partial\mathfrak{P} = \frac{x\partial S}{M} + U\partial q,$$

sumto S constante integretur formula $U\partial q$, sitque $T = \int U\partial q$, erit $\mathfrak{P} = T + f : S$ hincque

$$\frac{x}{M} = \left(\frac{\partial T}{\partial S}\right) + f' : S, \quad \text{et } z = px + qy - T - f : S.$$

Omnia ergo per p et q, unde M, S et T cum $\left(\frac{\partial T}{\partial S}\right)$ dantur, ita determinabuntur ut sit

$$x = M\left(\frac{\partial T}{\partial S}\right) + Mf' : S, \quad y = Vx + U, \quad \text{et}$$
$$z = px + qy - T - f : S.$$

Exemplum.

159. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $px + qy = apq$, indolem functionis z investigare.*

Cum ergo sit

$$y = \frac{-px}{q} + ap, \quad \text{erit}$$
$$V = \frac{-p}{q}, \quad U = ap.$$

Quia nunc esse debet

$$M\left(\partial p - \frac{p\partial q}{q}\right) = \partial S,$$

capiatur $M = \frac{1}{q}$ fitque

$$S = \frac{p}{q} \quad \text{et} \quad p = Sq.$$

Hinc $U = aSq$, et sumto S constante

$$T = \int U\partial q = \tfrac{1}{2}aSqq,$$

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ideoque $\left(\frac{\partial T}{\partial s}\right) = \frac{1}{2}aqq$. Quocirca pro solutione habebimus

$$x = \frac{1}{2}aq + \frac{1}{q}f' : \frac{p}{q}, \quad y = \frac{1}{2}ap - \frac{p}{qq}f' : \frac{p}{q}, \quad \text{et}$$

$$z = px + qy - \frac{1}{2}apq - f : \frac{p}{q} = \frac{1}{2}apq + f : \frac{p}{q}.$$

Per reductionem autem supra traditam habebimus

$$y = (aq - x)F' : (qx - \frac{1}{2}aqq), \quad \text{et}$$

$$z = qy + F : (qx - \frac{1}{2}aqq).$$

Scholion.

160. Quatuor problemata haec conjunctim considerata admodum late patent, atque pro formula $\partial z = p\partial x + q\partial y$ omnes relationes inter p, q, x et y complectuntur, in quibus vel x et y, vel p et y, vel x et q, vel p et q, nusquam unam dimensionem superant. Ex quo saepe fieri potest, ut eadem quaestio per duo plurave horum quatuor problematum resolvi possit; veluti evenit in exemplo hoc postremo, in quo cum non solum x et y, sed etiam x et q, itemque p et y, nusquam plus una dimensione occupant, id ad tria praecedentia problemata referri queat, haecque conditio primo tantum problemati adversatur. Quod si autem inter p, q, x et y haec relatio praescribatur, ut esse debeat

$$\alpha px + \beta qy + ap + bq + mx + ny + c = 0,$$

resolutio per omnia quatuor problemata aeque institui potest. Verum etiam resolutiones inde ortae, etiamsi forma discrepent, tamen per reductionem ante expositam ad consensum revocari possunt. At sequens casus latissime patens resolutionem quoque admittit, quem propterea evolvi conveniet.

Problema 27.

161. Si posito $\partial z = p\partial x + q\partial y$, inter p, q et x, y ejusmodi relatio detur, ut functio quaedam ipsarum p et x aequetur

functioni cuipiam ipsarum q et y, functionis z indolem in genere investigare.

Solutio.

Sit P functio illa ipsarum p et x, et Q functio illa ipsarum q et y, quae inter se aequales esse debent. Cum igitur sit $P = Q$, ponatur utraque $= v$, ut sit $P = v$ et $Q = v$. Ex priori ergo p definire licebit per x et v, ex posteriori vero q per y et v; quo facto in formula $\partial z = p\partial x + q\partial y$, cum p sit functio ipsarum x et v, integretur pars $p\partial x$ sumto v constante, sitque $\int p\partial x = R$, simili modo cum q sit functio ipsarum y et v, integretur quoque altera pars $q\partial y$ sumto v constante, sitque $\int q\partial y = S$; erit ergo $R =$ functioni ipsarum x et v, et $S =$ functioni ipsarum y et v. At sumto etiam v variabili sit

$$\partial R = p\partial x + V\partial v, \quad \text{et} \quad \partial S = q\partial y + U\partial v,$$

unde colligitur

$$\partial z = \partial R + \partial S - \partial v\,(V + U),$$

quae forma quia integrabilis esse debet, oportet sit $V + U = f' : v$. Quare solutio problematis his duabus aequationibus continebitur

$$V + U = f' : v \quad \text{et} \quad z = R + S - f : v.$$

Scilicet cum p, R et V dentur per x et v; atque q, S et U per y et v, per aequationem priorem definitur v ex x et y, qui valor in altera substitutus determinabit functionem quaesitam z per x et y.

Corollarium 1.

162. Quoties ergo q ejusmodi functioni ipsarum p, x, y aequari debet, ut inde aequatio formari possit, ex cujus altera

parte tantum binae litterae x et p, ex altera tantum binae reliquae y et q reperiantur, problema resolvi poterit.

Corollarium 2.

163. Si functio illa binarum litterarum p et x, quam posui P, ita sit comparata, ut posita ea $= v$ inde facilius x per p et v definiri possit, tum uti conveniet formula

$$z = px + \int (q\partial y - x\partial p),$$

et evolutio perinde se habebit atque ante.

Corollarium 3.

164. Simili modo si ex functione altera $Q = v$, quantitas y facilius per q et v definiatur, resolutio ex forma

$$z = qy + \int (p\partial x - y\partial q)$$

erit petenda. Sin autem utrumque eveniat, ut tam x per p et v, quam y per q et v definiatur, utendum erit formula

$$z = px + qy - \int (x\partial p + y\partial q).$$

Scholion.

165. Problema hoc innumerabiles complectitur casus in praecedentibus non comprehensos, atque etiam ejus solutio diverso nititur fundamento. Interim tamen longissime adhuc distamus a solutione problematis generalis, cui hoc caput est destinatum et quo in genere solutio desideratur, si inter quaternas quantitates p, q, x, y aequatio quaecunque proponatur; quae autem ob defectum Analyseos ne sperari quidem posse videtur. Contentos ergo nos esse oportet, si quam plurimos casus resolvere docuerimus. Quo autem vis hujus problematis magis perspiciatur aliquot exempla adjungamus.

Exemplum 1.

166. *Si posito* $\partial z = p\partial x + q\partial y$, *esse debeat* $q = \frac{xxyy}{a^4 p}$, *indolem functionis* z *investigare.*

Quia hic p, x, et q, y separare licet, cum sit $\frac{aaq}{yy} = \frac{xx}{aap}$, ponatur $\frac{xx}{aap} = v = \frac{aaq}{yy}$, unde p per x et v, et q per y et v ita definitur, ut sit

$$p = \frac{xx}{aav} \text{ et } q = \frac{vyy}{aa},$$

ideoque

$$\partial z = \frac{xx\partial x}{aav} + \frac{vyy\partial y}{aa}.$$

Hinc colligimus

$$z = \frac{x^3}{3aav} + \frac{vy^3}{3aa} + \frac{1}{3aa}\int\left(\frac{x^3\partial v}{vv} - y^3\partial v\right),$$

sicque $\frac{x^3}{vv} - y^3$ debet esse functio ipsius v. Ac posito

$$\frac{x^3}{vv} - y^3 = f' : v, \text{ seu } y^3 = \frac{x^3}{vv} - f' : v, \text{ erit}$$

$$z = \frac{1}{3aa}\left(\frac{x^3}{v} + vy^3 + f : v\right).$$

Corollarium.

167. Hinc facillime v eliminatur, si ponatur

$$f' : v = \frac{b^3}{vv} - c^3, \text{ hincque } f : v = \frac{-b^3}{v} - c^3 v.$$

Jam prior aequatio dat $y^3 - c^3 = \frac{x^3 - b^3}{vv}$, unde $vv = \frac{x^3 - b^3}{y^3 - c^3}$, et ob

$$3aaz = \frac{x^3 + vvy^3 - b^3 - c^3 vv}{v} = 2v(y^3 - c^3), \text{ erit}$$

$$z = \frac{2}{3aa}\sqrt{(x^3 - b^3)(y^3 - c^3)}.$$

Exemplum 2.

168. *Si posito* $\partial z = p\partial x + q\partial y$, *debeat esse*

$$q = \frac{1}{b}\sqrt{(xx + yy - aapp)},$$

investigare indolem functionis z.

Conditio praescripta redit ad

$$bbqq - yy = xx - aapp = v,$$

unde elicimus

$$q = \frac{1}{b} \sqrt{(yy + v)}, \quad \text{et} \quad p = \frac{1}{a} \sqrt{(xx - v)}.$$

Nunc vero est

$$\int p\partial x = \frac{1}{a} \int \partial x \sqrt{(xx - v)} = \frac{1}{2a} x \sqrt{(xx - v)} - \frac{v}{2a} \int \frac{\partial x}{\sqrt{(xx - v)}}$$

seu $\int p\partial x = \frac{x}{2a} \sqrt{(xx - v)} - \frac{v}{2a} l [x + \sqrt{(xx - v)}] = \mathrm{R}$;

simili modo est

$$\int q\partial y = \frac{y}{2b} \sqrt{(yy + v)} + \frac{v}{2b} l [y + \sqrt{(yy + v)}] = \mathrm{S}.$$

Quare cum sit

$$\mathrm{V} = \left(\frac{\partial \mathrm{R}}{\partial v}\right) = \frac{-x}{4a\sqrt{(xx - v)}} - \frac{1}{2a} l [x + \sqrt{(xx - v)}]$$
$$+ \frac{v}{4a [x + \sqrt{(xx - v)}] \sqrt{(xx - v)}},$$

quae reducitur ad

$$\mathrm{V} = -\frac{1}{4a} - \frac{1}{2a} l [x + \sqrt{(xx - v)}],$$

similique modo

$$\mathrm{U} = \left(\frac{\partial \mathrm{S}}{\partial v}\right) = +\frac{1}{4b} + \frac{1}{2b} l [y + \sqrt{(yy + v)}],$$

ubi cum $\mathrm{V} + \mathrm{U} = \mathrm{f}' : v$, erit

$$\frac{a - b}{4\,a\,b} + l \cdot \frac{[y + \sqrt{(yy + v)}]^{\frac{1}{2b}}}{[x + \sqrt{(xx - v)}]^{\frac{1}{2a}}} = \mathrm{f}' : v;$$

unde valor ipsius v per x et y determinatur. Ex quo tandem colligitur

$$z = \frac{x}{2a} \sqrt{(xx - v)} + \frac{y}{2b} \sqrt{(yy + v)} + v\, l \cdot \frac{[y + \sqrt{(yy + v)}]^{\frac{1}{2b}}}{[x + \sqrt{(xx - v)}]^{\frac{1}{2a}}} - \mathrm{f} : v,$$

seu

$$z = \frac{x}{2a}\sqrt{(xx - v)} + \frac{y}{2b}\sqrt{(yy + v)} - \frac{(a-b)v}{4ab} + v\,\mathrm{f}' : v - \mathrm{f} : v.$$

Scholion.

169. Haec solutio a formulis logarithmicis liberari potest hoc modo. Ponatur

$$\mathrm{f}' : v = lt + \frac{a-b}{4ab},$$

ut sit

$$t^{2ab} = \frac{[y + \sqrt{(yy + v)}]^a}{[x + \sqrt{(xx - v)}]^b},$$

unde v datur per t. Tum vero sit $v = t\mathrm{F}' : t$, et ob

$$\partial v\, \mathrm{f}'' : v = \frac{\partial t}{t} \text{ erit}$$

$$\int v \partial v\, \mathrm{f}'' : v = v\,\mathrm{f}' : v - \mathrm{f} : v = \int \frac{v \partial t}{t} = \mathrm{F} : t,$$

sicque erit

$$z = \frac{x}{2a}\sqrt{(xx - v)} + \frac{y}{2b}\sqrt{(yy + v)} - \frac{(a-b)v}{4ab} + \mathrm{F} : t,$$

ubi est

$$v = t\mathrm{F}' : t, \text{ et } t^{2ab} = \frac{[y + \sqrt{(yy + v)}]^a}{[x + \sqrt{(xx - v)}]^b},$$

unde t et v per x et y definiri potest. Hinc statim patet si capiatur $\mathrm{F}' : t = 0$, fore $v = 0$, $\mathrm{F} : t = 0$ et $z = \frac{xx}{2a} + \frac{yy}{2b}$; hincque $p = \frac{x}{a}$ et $q = \frac{y}{b}$, quo pacto utique conditioni praescriptae satisfit. Caeterum haec ratio quantitates logarithmicas elidendi maxime est notatu digna et in aliis casibus usum amplissimum habere potest.

Exemplum 3.

170. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $x^m y^n = Ap^\mu q^\nu$, indolem functionis z investigare.*

Statuatur ergo

$$\frac{x^m}{p^\mu} = \frac{Aq^\nu}{y^n} = v^{\mu\nu},$$

et hinc deducitur

$$p = \frac{x^{\frac{m}{\mu}}}{v^\nu} \quad \text{et} \quad q = \frac{1}{a}\, y^{\frac{n}{\nu}}\, v^\mu,$$

posito $A = a^\nu$. Unde habebimus

$$\int p\partial x = \frac{\mu\, x^{\frac{m+\mu}{\mu}}}{(m+\mu)\, v^\nu} + \frac{\mu\nu}{m+\mu}\int \frac{x^{\frac{m+\mu}{\mu}}\partial v}{v^{\nu+1}}, \quad \text{et}$$

$$\int q\partial y = \frac{\nu\, y^{\frac{n+\nu}{\nu}}\, v^\mu}{(n+\nu)\, a} - \frac{\mu\nu}{(n+\nu)\, a}\int y^{\frac{n+\nu}{\nu}}\, v^{\mu-1}\,\partial v.$$

Quocirca erit

$$z = \frac{\mu\, x^{\frac{m+\mu}{\mu}}}{(m+\mu)\, v^\nu} + \frac{\nu y^{\frac{n+\nu}{\nu}}\, v^\mu}{(n+\nu)\, a} + \frac{\mu\nu}{(m+\mu)(n+\nu)a} \times$$

$$\int \partial v \left(\frac{(n+\nu)\, a x^{\frac{m+\mu}{\mu}}}{v^{\nu+1}} - (m+\mu)\, y^{\frac{n+\nu}{\nu}}\, v^{\mu-1} \right),$$

ita ut si statuamus

$$\frac{x^{\frac{m+\mu}{\mu}}}{(m+\mu)\, v^{\nu+1}} - \frac{y^{\frac{n+\nu}{\nu}}\, v^{\mu-1}}{(n+\nu)\, a} = f' : v,$$

futurum sit

$$z = \frac{\mu\, x^{\frac{m+\mu}{\mu}}}{(m+\mu)\, v^\nu} + \frac{\nu y^{\frac{n+\nu}{\nu}}\, v^\mu}{(n+\nu)\, a} + \mu\nu f : v.$$

Pro casu simplicissimo ponamus $f' : v = 0$ et $f : v = 0$, eritque

$$y^{\frac{n+\nu}{\nu}} v^{\mu+\nu} = \frac{(n+\nu)\, a}{m+\mu} x^{\frac{m+\mu}{\mu}} \text{ et } v = \left(\frac{(n+\nu)\, a x^{\frac{m+\mu}{\mu}}}{(m+\mu)\, y^{\frac{n+\nu}{\nu}}}\right)^{\frac{1}{\mu+\nu}},$$

tum vero

$$z = \frac{1}{v^{\nu}}\left(\frac{\mu}{m+\mu} x^{\frac{m+\mu}{\mu}} + \frac{\nu}{(n+\nu)\, a} y^{\frac{n+\nu}{\nu}} v^{\mu+\nu}\right), \text{ seu}$$

$$z = \frac{(\mu+\nu)}{(m+\mu)\, v^{\nu}} x^{\frac{m+\mu}{\mu}} = (\mu+\nu)\left(\frac{x^{m+\mu} y^{n+\nu}}{(m+\mu)^{\mu} (n+\nu)^{\nu} A}\right)^{\frac{1}{\mu+\nu}}.$$

Problema 28.

171. Si posito $\partial z = p\partial x + q\partial y$, inter p, q et x, y ejusmodi detur relatio, ut p et q aequentur functionibus quibusdam ipsarum x, y et novae variabilis v, explorare casus, quibus indolem functionis z investigare licet.

Solutio.

Cum sit p functio ipsarum x, y et v, spectatis y et v ut constantibus, quaeratur integrale $\int p\partial x = P$, sitque sumtis omnibus variabilibus

$$\partial P = p\partial x + R\partial y + M\partial v,$$

unde si pro $p\partial x$ valor substituatur, erit

$$\partial z = \partial P + (q - R)\,\partial y - M\partial v.$$

Quodsi jam eveniat, ut $q - R$ sit tantum functio ipsarum y et v, exclusa x, sumto v constante quaeratur $\int (q - R)\,\partial y = T$, sitque deinceps

$$\partial T = (q - R)\,\partial y + V\partial v.$$

Hinc valor ipsius $(q - R)\,\partial y$ ibi substitutus dabit

$$\partial z = \partial P + \partial T - (M + V)\,\partial v,$$

quae forma quia integrabilis esse debet, statuatur

$$M + V = f' : v, \text{ eritque } z = P + T - f : v.$$

Ex operationibus autem susceptis dantur P, R, M, per V, x, y et v, at T et V per y et v tantum; ac resolutio succedit, si modo in forma $q - R$ non amplius x continetur. Pari ratione solutio succedet, si M tantum per y et v detur; tum enim ex y constante quaeratur $\int M\partial v = L$, sitque

$$\partial L = M\partial v + N\partial y, \text{ erit}$$

$$\partial z = \partial P + (q - R + N)\, \partial y - \partial L,$$

ponique conveniet

$$q - R + N = f' : y,$$

ut fiat

$$z = P - L + f : y.$$

Simili modo ab altera parte $\int q\, \partial y$ calculum incipere et prosequi licet.

Introducendo autem functionem ipsarum x, y et v indefinitam K, negotium generalius confici poterit. Sit enim

$$\partial K = F\partial x + G\partial y + H\partial v,$$

ac consideretur haec forma

$$\partial z + \partial K = (p + F)\, \partial x + (q + G)\, \partial y + H\partial v.$$

Nunc sumtis y et v constantibus, quaeratur

$$\int (p + F)\, \partial x = P,$$

sitque

$$\partial P = (p + F)\, \partial x + R\partial y + M\partial v,$$

unde habetur

$$\partial z + \partial K = \partial P + (q + G - R)\, \partial y + (H - M)\, \partial v.$$

Quod si jam eveniat, ut vel $q + G - R$ vel $H - M$ tantum binas variabiles y et v exclusa x contineat, resolutio ut ante est ostensum, absolvi poterit.

Problema 29.

172. Si posito $\partial z = p\partial x + q\partial y$, relatio detur inter binas formulas differentiales p, q et binas variabiles x et z, vel y et z, solutionem problematis quatenus fieri potest, perficere.

Solutio.

Ponamus relationem dari inter p, q et x, z, atque hunc casum facile ad praecedentem revocare licet. Consideretur enim haec formula

$$\partial y = \frac{\partial z - p\partial x}{q},$$

ex principali derivata; voceturque

$$\frac{1}{q} = m \quad \text{et} \quad \frac{p}{q} = -n,$$

ut habeatur

$$\partial y = m\partial z + n\partial x,$$

et ob $q = \frac{1}{m}$ et $p = -\frac{n}{m}$, relatio proposita versabitur inter quaternas quantitates m, n, z et x, ideoque quaestio omnino similis est earum, quas antea tractavimus, hoc tantum discrimine, quod hic quantitas y definiatur, cum ante esset z investigata. Quoniam autem ista determinatio per aequationes absolvitur, perinde est utrum tandem inde z, an y elicere velimus. Quodsi ergo hac reductione facta quaestio in casus ante pertractatos incidat, methodis quoque expositis resolvi poterit.

Exemplum.

173. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $qxz = aap$, indolem functionis z investigare.*

Consideretur formula $\partial y = \frac{\partial z}{q} - \frac{p\partial x}{q}$. Jam quia $\frac{p}{q} = \frac{xz}{aa}$ erit

$$\partial y = \frac{\partial z}{q} - \frac{xz\partial x}{aa} \text{ et } y = \int\left(\frac{\partial z}{q} - \frac{xz\partial x}{aa}\right), \text{ at est}$$

$$\int\frac{xz\partial x}{aa} = \frac{xxz}{2aa} - \int\frac{xx\partial z}{2aa}, \text{ ergo}$$

$$y = \int\partial z\left(\frac{1}{q} + \frac{xx}{2aa}\right) - \frac{xxz}{2aa}.$$

Ponatur ergo

$$\frac{1}{q} + \frac{xx}{2aa} = f' : z, \text{ erit } y = \frac{-xxz}{2aa} + f : z,$$

ex qua aequatione utique z per x et y definitur. Si pro casu simpliciori sumamus $f : z = b + \alpha z$, erit

$$y - b = \left(\alpha - \frac{xx}{2aa}\right)z, \text{ et } z = \frac{2aa(y-b)}{2\alpha aa - xx},$$

et sumtis $\alpha = 0$ et $b = 0$ pro casu simplicissimo erit $z = \frac{-2aay}{xx}$. Hinc autem fit

$$p = \frac{+4aay}{x^3} \text{ et } q = \frac{-2aa}{xx}. \text{ Ergo}$$

$$\frac{p}{q} = -\frac{2y}{x} \text{ et } \frac{xz}{aa} = \frac{-2y}{x}.$$