# CAPUT IV.

## DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER BINAS FORMULAS DIFFERENTIALES ET UNICAM TRIUM QUANTITATUM VARIABILIUM PROPONITUR.

### Problema 15.

97.

Si $z$ ejusmodi esse debeat functio binarum variabilium $x$ et $y$, ut posito $\partial z = p\,\partial x + q\,\partial y$ sit $q = \frac{px}{a}$, indolem hujus functionis in genere investigare.

### Solutio.

Cum sit

$$\partial z = p\,\partial x + \frac{px\,\partial y}{a} = px\left(\frac{\partial x}{x} + \frac{\partial y}{a}\right),$$

haecque formula esse debeat integrabilis, neeesse est ut $px$ ac proinde etiam $z$, sit functio quantitatis $lx + \frac{y}{a}$. Quare solutio nostri problematis in genere ita se habebit, ut sit

$$z = \mathrm{f}:\left(lx + \tfrac{y}{a}\right) \text{ et } px = \mathrm{f}':\left(lx + \tfrac{y}{a}\right),$$

sumendo scilicet perpetuo $\partial \,.\, \mathrm{f}:u = \partial u\,\mathrm{f}':u$. Hinc autem erit

$$p = \tfrac{1}{x}\,\mathrm{f}':\left(lx + \tfrac{y}{a}\right) \text{ et } q = \tfrac{1}{a}\,\mathrm{f}':\left(lx + \tfrac{y}{a}\right),$$

sicque $q = \frac{px}{a}$ omnino uti requiritur.

### Corollarium 1.

98. Cum sit

$$z = p x - \int x \partial p + \int \frac{p x \partial y}{a} = p x + \int p x \left(\frac{\partial y}{a} - \frac{\partial p}{p}\right);$$

hinc alia solutio deduci potest. Si enim ponamus

$$\int p x \left(\frac{\partial y}{a} - \frac{\partial p}{p}\right) = \mathrm{f}:\left(\frac{y}{a} - l p\right), \text{ erit } p x = \mathrm{f}':\left(\frac{y}{a} - l p\right),$$

indeque

$$z = \mathrm{f}':\left(\frac{y}{a} - l p\right) + \mathrm{f}:\left(\frac{y}{a} - l p\right).$$

### Corollarium 2.

99. Hac ergo solutione nova introducitur variabilis $p$, ex qua cum $y$ conjuncta definitur primo

$$x = \frac{1}{p} \mathrm{f}':\left(\frac{y}{a} - l p\right),$$

tum vero ipsa functio quaesita

$$z = p x + \mathrm{f}:\left(\frac{y}{a} - l p\right).$$

Huic autem solutioni praecedens sine dubio antecellit, cum illa quantitatem $z$ immediate per $x$ et $y$ exprimat.

### Scholion.

100. Quo has duas solutiones inter se comparare queamus, quoniam functio arbitraria in utraque diversae est indolis, etiam charactere diverso utamur. Cum igitur prima praebeat

$$z = \mathrm{f}:\left(\frac{y}{a} + l x\right) \text{ et } p x = \mathrm{f}':\left(\frac{y}{a} + l x\right),$$

altera vero

$$z = \mathrm{F}:\left(\frac{y}{a} - l p\right) + \mathrm{F}':\left(\frac{y}{a} - l p\right) \text{ et } p x = \mathrm{F}':\left(\frac{y}{a} - l p\right),$$

patet fore

$$\mathrm{f}':\left(\frac{y}{a} + l x\right) = \mathrm{F}':\left(\frac{y}{a} - l p\right) \text{ et}$$

$$f' : \left(\frac{y}{a} + l\,x\right) = \mathrm{F} : \left(\frac{y}{a} - l\,p\right) + \mathrm{F}' : \left(\frac{y}{a} - l\,p\right),$$

unde non solum relatio inter utriusque functionis $f$ et F indolem definitur, sed etiam inde sequi debet, fore

$$p\,x = f' : \left(\frac{y}{a} + l\,x\right);$$

id quod non parum videtur absconditum. Verum ob hoc ipsum istud problema eo magis est notatu dignum, quod solutio altera, qua nova variabilis $p$ introducitur, congruit cum priore, ubi $z$ per $x$ et $y$ immediate definitur, neque tamen consensus harum solutionum perspicue monstrari potest. Quamobrem quando ad ejusmodi solutiones pervenimus, uti in problematibus posterioribus capitis praecedentis usu venit, in quibus nova variabilis introducitur, non omnem statim spem ejus eliminandae abjicere debemus, cum isto casu altera solutio ad priorem certe sit reductibilis, etiamsi methodus reducendi non perspiciatur, quam tamen infra §. 119. exhibebimus.

## Problema 16.

101. Si $z$ ejusmodi esse debeat functio binarum variabilium $x$ et $y$, ut posito $\partial z = p\,\partial x + q\,\partial y$, sit $q = p\,\mathrm{X} + \mathrm{T}$, existentibus X et T functionibus quibuscunque ipsius $x$, indolem istius functionis $z$ in genere investigare.

## Solutio.

Cum ergo sit $\partial z = p\,\partial x + p\,\mathrm{X}\,\partial y + \mathrm{T}\,\partial y$, statuatur $p = r - \frac{\mathrm{T}}{\mathrm{X}}$ ut prodeat

$$\partial z = r\,\partial x - \frac{\mathrm{T}\,\partial x}{\mathrm{X}} + r\,\mathrm{X}\,\partial y = \frac{-\mathrm{T}\,\partial x}{\mathrm{X}} + r\,\mathrm{X}\left(\frac{\partial x}{\mathrm{X}} + \partial y\right),$$

qua reductione facta perspicuum est, tam $r\,\mathrm{X}$ quam $\int r\,\mathrm{X}\left(\frac{\partial x}{\mathrm{X}} + \partial y\right)$ fore functionem quantitatis $y + \int\frac{\partial x}{\mathrm{X}}$. Quare si ponamus

$$\int r X \left(\frac{\partial x}{X} + \partial y\right) = f : \left(y + \int \frac{\partial x}{X}\right), \text{ erit}$$

$$r X = f' : \left(y + \int \frac{\partial x}{X}\right),$$

ac tum functio quaesita erit

$$z = -\int \frac{T \partial x}{X} + f : \left(y + \int \frac{\partial x}{X}\right),$$

quae ob functionem indefinitam f: est completa. Tum vero erit

$$p = \frac{-T}{X} + \frac{1}{X} f' : \left(y + \int \frac{\partial x}{X}\right) \text{ et}$$

$$q = f' : \left(y + \int \frac{\partial x}{X}\right):$$

unde patet fore utique $q = p X + T$. Quoniam vero X et T sunt functiones datae ipsius $x$, formulae integrales $\int \frac{\partial x}{X}$ et $\int \frac{T \partial x}{X}$ solutionem non turbant.

### Corollarium 1.

102. Solutio aliquanto facilior redditur sumendo ex conditione praescripta $p = \frac{q}{X} - \frac{T}{X}$, unde fit

$$\partial z = -\frac{T \partial x}{X} + \frac{q \partial x}{X} + q \partial y \text{ et}$$

$$z = -\int \frac{T \partial x}{X} + \int q \left(\partial y + \frac{\partial x}{X}\right).$$

Jam manifesto est

$$\int q \left(\partial y + \frac{\partial x}{X}\right) = f : \left(y + \int \frac{\partial x}{X}\right),$$

sicque ipsa solutio praecedens resultat.

### Corollarium 2.

103. Eodem modo resolvitur problema, si proponatur conditio $q = p Y + V$, existentibus Y et V functionibus datis ipsius $y$. Tum enim erit

$$\partial z = p \partial x + p Y \partial y + V \partial y, \text{ et } z = \int V \partial y + \int p (\partial x + Y \partial y).$$

Hic ergo fit

$$\int p\,(\partial x + Y\partial y) = f:(x + \int Y\partial y),$$

et solutio erit

$$z = \int V\partial y + f:(x + \int Y\partial y);$$

unde fit

$$p = f':(x + \int Y\partial y) \text{ et } q = V + Y\,f':(x + \int Y\partial y).$$

## Scholion.

104. Ex forma solutionis hic inventae discere poterimus, quomodo problema comparatum esse debeat, ut ejus solutio hac ratione perfici, et functio $z$ per binas variabiles $x$ et $y$ exhiberi queat. Sint enim K et V functiones quaecunque ipsarum $x$ et $y$, indeque differentiando

$$\partial K = L\partial x + M\partial y \text{ et } \partial V = P\partial x + Q\partial y.$$

Jam a solutione incipiamus, ponamusque

$$z = K + f:V,$$

eritque differentiando

$$\partial z = L\partial x + M\partial y + (P\partial x + Q\partial y)\,f':V.$$

Cum jam hanc formam cum assumta

$$\partial z = p\partial x + q\partial y$$

comparando, fit

$$p = L + P\,f':V \text{ et } q = M + Q\,f':V, \text{ erit}$$

$$Qp - Pq = LQ - MP.$$

Quare si hoc problema proponatur, ut posito

$$\partial z = p\partial x + q\partial y,$$

fieri debeat

$$q = \tfrac{Q}{P}p + M - \tfrac{LQ}{P},$$

solutio erit $z = K + f:V$; dummodo M et L itemque P et Q ita sint comparatae, ut sit

$$L\partial x + M\partial y = \partial K \text{ et } P\partial x + Q\partial y = \partial V,$$

verùm hi casus ad sequens caput sunt referendi.

## Problema 17.

105. Si $z$ ejusmodi esse debeat functio binarum variabilium $x$ et $y$, ut posito $\partial z = p\partial x + q\partial y$, sit $q = Px + \Pi$ existentibus P et $\Pi$ functionibus datis ipsius $p$; indolem istius functionis $z$ in genere investigare.

## Solutio.

Cum igitur sit

$$\partial z = p\partial x + Px\partial y + \Pi\partial y, \text{ erit}$$
$$z = px + \int (Px\partial y + \Pi\partial y - x\partial p).$$

Sumatur $Px + \Pi = v$, ut sit $x = \frac{v - \Pi}{P}$, fietque

$$z = px + \int (v\partial y - \frac{v\partial p}{P} + \frac{\Pi\partial p}{P}).$$

Quare cum P et $\Pi$ sint functiones ipsius $p$, ideoque formula $\int \frac{\Pi\partial p}{P}$ data, habebitur

$$z = px + \int \frac{\Pi\partial p}{P} + \int v\,(\partial y - \frac{\partial p}{P}),$$

unde patet tam $v$ quam $\int v\,(\partial y - \frac{\partial p}{P})$ functionem esse debere formulae $y - \int \frac{\partial p}{P}$. Ponamus ergo

$$\int v\,(\partial y - \frac{\partial p}{P}) = f:(y - \int \frac{\partial p}{P}), \text{ eritque}$$
$$v = Px + \Pi = f':(y - \int \frac{\partial p}{P}).$$

et hinc

$$x = \frac{-\Pi}{P} + \frac{1}{P} f':(y - \int \frac{\partial p}{P}),$$

tum vero

$$z = \int \frac{\Pi\partial p}{P} - \frac{\Pi p}{P} + \frac{p}{P} f':(y - \int \frac{\partial p}{P}) + f:(y - \int \frac{\partial p}{P}).$$

### Corollarium 1.

106. In solutione hujus problematis iterum nova variabilis $p$ introducitur, ex qua cum $y$ conjunctim primo variabilis $x$, tum vero ipsa functio quaesita $z$ determinatur.

### Corollarium 2.

107. Neque vero hinc istam novam variabilem $p$ ex calculo elidere licet, uti ante usu venit; propterea quod hic P et $\Pi$ functiones ipsius $p$ denotant, quarum indoles jam in ipsum problema ingreditur.

### Corollarium 3.

108. Simili modo problema resolvetur, si permutandis $x$ et $y$, quantitas $p$ ita per $y$ et $q$ detur, ut sit $p = Qy + \Xi$, denotantibus Q et $\Xi$ functiones datas ipsius $q$.

### Scholion.

109. In hoc capite constituimus ejusmodi problemata tractare, quorum conditio aequatione inter binas formulas differentiales $\left(\frac{\partial z}{\partial x}\right) = p$, $\left(\frac{\partial z}{\partial y}\right) = q$ et unam ex tribus variabilibus $x$, $y$ et $z$ utcunque exprimitur. Problemata autem bina evoluta ex hoc genere certos casus complectuntur, quorum solutio peculiari methodo expediri potest, simulque ad formulas simpliciores perducitur. In posteriori quidem relationem inter $p$, $q$ et $x$ ita assumsimus, ut sit $q = Px + \Pi$, seu ut in valore ipsius $q$, per $p$ et $x$ expresso, quantitas $x$ unam dimensionem non excedat; in priori vero ita ut sit $q = pX + T$, seu ut in valore ipsius $q$, per $p$ et $x$ expresso, quantitas $p$ unicam obtineat dimensionem. In genere autem notasse juvabit, tam quantitates $p$ et $x$ quam $q$ et $y$ inter se esse permutabiles. Cum enim sit

$$\int p\,\partial x = px - \int x\,\partial p,$$

loco

$$z = \int(p\,\partial x + q\,\partial y),$$

erit

$$z = px + \int(q\,\partial y - x\,\partial p).$$

Simili modo est

$$z = qy + \int(p\,\partial x - y\,\partial q),$$

tum vero etiam

$$z = px + qy - \int(x\,\partial p + y\,\partial q).$$

Quibus ergo casibus una harum quatuor formularum integralium redditur integrabilis, iisdem ternae reliquae etiam integrationem admittent. Cum igitur in superiori capite primam formulam resolverimus, si $p$ vel $q$ quomodocunque detur per $x$ et $y$; ita eodem modo resolvetur formula secunda, si $q$ per $p$ et $y$, tertia autem si $p$ per $x$ et $q$, at quarta si vel $x$ per $p$ et $q$ vel $y$ per $p$ et $q$ utcunque datur; quae quaestiones cum generaliter expediri queant, eas in sequenti problemate evolvamus.

### Problema 18.

110. Posito $\partial z = p\,\partial x + q\,\partial y$, si relatio inter $p$, $q$ et $x$ aequatione quacunque definiatur, indolem functionis $z$, quemadmodum ex binis variabilibus $x$ et $y$ determinetur, in genere investigare.

### Solutio.

Ex aequatione inter $p$, $q$ et $x$ proposita quaeratur valor ipsius $x$ qui functioni cuipiam ipsarum $p$ et $q$ aequabitur. Cum jam sit

$$z = px + qy - \int(x\,\partial p + y\,\partial q),$$

quoniam $x$ est functio data ipsarum $p$ et $q$, formula $x\partial p$ integretur sumta quantitate $q$ constante, sitque

$$\int x\,\partial p = \mathrm{V} + \mathrm{f} : q,$$

et erit V functio cognita ipsarum $p$ et $q$, qua differentiata prodeat

$$\partial \mathrm{V} = x\,\partial p + \mathrm{S}\,\partial q,$$

ubi S quoque erit functio data ipsarum $p$ et $q$. Quia jam forma $\int(x\partial p + y\partial q)$ integrationem admittere debet, aequabitur formae $\mathrm{V} + \mathrm{f} : q$, unde differentiando concluditur

$$x\partial p + y\partial q = x\partial p + \mathrm{S}\partial q + \partial q\,\mathrm{f}' : q,$$

ideoque

$$y = \mathrm{S} + \mathrm{f}' : q \text{ et } z = px + qy - \mathrm{V} - \mathrm{f} : q, \text{ seu}$$
$$z = px + \mathrm{S}q + q\,\mathrm{f}' : q - \mathrm{f} : q - \mathrm{V}.$$

Solutio ergo ita se habet: primo ex conditione praescripta datur $x$ per $p$ et $q$, tum sumta $q$ constante sit $\mathrm{V} = \int x\,\partial p$, et vicissim $\partial \mathrm{V} = x\partial p + \mathrm{S}\,\partial q$; inventis autem V et S per $p$ et $q$, reliquae quantitates $y$ et $z$ ita per easdem exprimentur ut sit

$$y = \mathrm{S} + \mathrm{f}' : q \text{ et } z = px + \mathrm{S}q + q\,\mathrm{f}' : q - \mathrm{f} : q - \mathrm{V},$$

quae solutio, quia $\mathrm{f} : q$ functionem quamcunque ipsius $q$ sive continuam sive discontinuam denotat, utique pro completa latissimeque patente est habenda.

## Aliter.

111. Vel ex aequatione inter $p$, $q$ et $x$ data, quaeratur valor ipsius $p$ per $x$ et $q$ expressus, ita ut $p$ aequetur functioni cuipiam datae binarum variabilium $x$ et $q$, per quas etiam reliquas quantitates $y$ et $z$ definire conemur. Ad hoc utamur formula

$$z = qy + \int(p\,\partial x - y\,\partial q),$$

et quia $p$ est functio ipsarum $x$ et $q$, dabitur earundem ejusmodi functio V ut sit

$$\partial V = p\partial x + R\partial q.$$

Statuatur ergo

$$\int (p\partial x - y\partial q) = V + f:q,$$

eritque

$$y = - R - f':q \text{ et } z = qy + V + f:q.$$

### Corollarium 1.

112. Utraque solutio aeque commode adhiberi potest, si ex relatione inter $p$, $q$ et $x$ proposita, tam quantitatem $x$ quam $p$ aeque commode definire liceat. Sin autem earum altera commodius definiri queat, ea solutione, quae ad istum casum est accommodata, erit utendum.

### Corollarium 2.

113. Sin autem neque $p$ neque $x$ commode elici queat, tum nihilo minus hic resolutio aequationum cujusque ordinis, quin etiam transcendentium tanquam concessa assumitur. Caeterum etiamsi $q$ facile per $p$ et $x$ definiatur, hinc calculus nihil juvatur.

### Corollarium 3.

114. Ex hoc problemate utpote latissime patente etiam bina praecedentia resolvi possunt; solutio autem hinc inventa a praecedente discrepabit, cum illa ex methodo particulari sit deducta: operae autem pretium erit, has duplices solutiones inter se comparare.

### Exemplum 1.

115. *Si fuerit $q = pX + T$, existentibus $X$ et $T$ functionibus ipsius $x$, indolem functionis $z$ investigare.*

Hic solutione utendum est posteriori, pro qua est $p = \frac{q - T}{X}$; nunc posita $q$ constante prodit

$$V = \int p \partial x = q \int \frac{\partial x}{X} - \int \frac{T \partial x}{X},$$

hincque

$$R = \left(\frac{\partial V}{\partial q}\right) = \int \frac{\partial x}{X};$$

unde solutio his formulis continetur

$$q = pX + T, \ y = -\int \frac{\partial x}{X} - f' : q, \ z = -\int \frac{T \partial x}{X} - q f' : q + f : q,$$

solutio autem superior ita se habebat

$$q = pX + T, \ q = f' : \left(y + \int \frac{\partial x}{X}\right) \text{ et } z = -\int \frac{T \partial x}{X} + f : \left(y + \int \frac{\partial x}{X}\right).$$

## Scholion.

116. Consensus harum duarum solutionum ita ostendi potest, ut ex ea, quam hic invenimus, antecedens per legitimam consequentiam formetur. Cum enim sit

$$f' : q = -y - \int \frac{\partial x}{X},$$

statuatur brevitatis gratia $y + \int \frac{\partial x}{X} = v$, ut sit $f' : q = -v$, erit ergo vicissim $q$ aequalis functioni cuidam ipsius $v$, quae ponatur $q = F' : v$, unde fit $\partial q = \partial v F'' : v$, ergo

$$\partial q f' : q = - v \partial v F'' : v = - v \partial . F' : v,$$

ergo integrando

$$f : q = -\int v \partial . F' : v = -v F' : v + \int \partial v . F' : v = -v F' : v + F : v.$$

Quare cum sit

$$z = -\int \frac{T \partial t}{X} - q f' : q + f' : q, \text{ erit}$$

$$z = -\int \frac{T \partial x}{X} + v F' : v - v F' : v + F : v, \text{ seu}$$

$$z = -\int \frac{T \partial x}{X} + F : \left(y + \int \frac{\partial x}{X}\right),$$

quae est ipsa solutio praecedens.

Exemplum 2.

117. *Si fuerit $q = Px + \Pi$, existentibus $P$ et $\Pi$ functionibus datis ipsius $p$, indolem functionis $z$; ut sit*

$$\partial z = p\partial x + q\partial y,$$

*investigare.*

Hic solutione priori utendum, cum sit $x = \frac{q - \Pi}{P}$. Sumto ergo $q$ constante quaeratur

$$V = \int x \partial p = q \int \frac{\partial p}{P} - \int \frac{\Pi \partial p}{P},$$

unde fit

$$S = \left(\frac{\partial V}{\partial q}\right) = \int \frac{\partial p}{P}.$$

Solutio ergo praebet

$$y = \int \frac{\partial p}{P} + f' : q \quad \text{et}$$

$$z = \frac{pq}{P} - \frac{p\Pi}{P} + q\int \frac{\partial p}{P} + qf' : q - f : q - q\int \frac{\partial p}{P} + \int \frac{\Pi \partial p}{P},$$

sive

$$z = \frac{p(q - \Pi)}{P} + \int \frac{\Pi \partial p}{P} + q f' : q - f : q.$$

Solutio autem ejusdem casus supra (105.) inventa erat

$$x = \frac{-\Pi}{P} + \frac{1}{P} f' : \left(y - \int \frac{\partial p}{P}\right) \text{ et } q = Px + \Pi,$$

atque

$$z = \frac{-p\Pi}{P} + \int \frac{\Pi \partial p}{P} + \frac{p}{P} f' : \left(y - \int \frac{\partial p}{P}\right) + f : \left(y - \int \frac{\partial p}{P}\right).$$

Scholion 1.

118. Videamus quomodo solutio hic inventa ad superiorem reduci queat. Cum ibi invenerimus

$$y - \int \frac{\partial p}{P} = f' : q,$$

vicissim $q$ aequabitur functioni quantitatis $y - \int \frac{\partial p}{P}$, ponatur ergo

$$q = F' : \left(y - \int \frac{\partial p}{P}\right),$$

eritque statim

$$x = \frac{-\Pi}{P} + \frac{1}{P} F' : \left(y - \int \frac{\partial p}{P}\right);$$

sit brevitatis gratia $y - \int \frac{\partial p}{P} = v$, ut fiat

$$q = F' : v \quad \text{et} \quad v = f' : q, \quad \text{erit}$$

$$F : v = \int q \partial v = qv - \int v \partial q = qv - \int \partial q f' : q.$$

Ergo $F : v = qv - f : q$, ita ut sit

$$f : q = q\left(y - \int \frac{\partial p}{P}\right) - F : \left(y - \int \frac{\partial p}{P}\right), \quad \text{seu}$$

$$f : q = \left(y - \int \frac{\partial p}{P}\right) F' : \left(y - \int \frac{\partial p}{P}\right) - F : \left(y - \int \frac{\partial p}{P}\right).$$

Quibus valoribus substitutis habebimus

$$x = \frac{-\Pi}{P} + \frac{1}{P} F' : \left(y - \int \frac{\partial p}{P}\right) \quad \text{et}$$

$$z = \frac{-p\Pi}{P} + \frac{p}{P} F' : \left(y - \int \frac{\partial p}{P}\right) + \int \frac{\Pi \partial p}{P} + \left(y - \int \frac{\partial p}{P}\right) F' : \left(y - \int \frac{\partial p}{P}\right)$$
$$- \left(y - \int \frac{\partial p}{P}\right) F' : \left(y - \int \frac{\partial p}{P}\right) + F : \left(y - \int \frac{\partial p}{P}\right), \quad \text{seu}$$

$$z = \frac{-p\Pi}{P} + \frac{p}{P} F' : \left(y - \int \frac{\partial p}{P}\right) + \int \frac{\Pi \partial p}{P} + F : \left(y - \int \frac{\partial p}{P}\right),$$

quae est ipsa solutio ante inventa.

## Scholion 2.

119. Hoc consensu ostenso etiam consensum supra observatum §. 100. demonstrare poterimus, qui multo magis absconditus videtur. Altera autem solutio ibi inventa erat

$$px = F' : \left(\frac{y}{a} - lp\right) \quad \text{et} \quad z = px + F : \left(\frac{y}{a} - lp\right),$$

ex quarum formula priori patet, fore vicissim $\frac{y}{a} - lp$ functionem ipsius $px$; hinc etiam $\frac{y}{a} - lp + lpx$ seu $\frac{y}{a} + lx$ aequabitur functioni ipsius $px$. Denuo ergo vicissim $px$ aequabitur functioni cuipiam ipsius $\frac{y}{a} + lx$.

$$\partial . F : \left(\frac{y}{a} - lp\right) = \left(\frac{\partial y}{a} - \frac{\partial p}{p}\right) F' : \left(\frac{y}{a} - lp\right), \quad \text{erit}$$

$$x\,\partial u + B\,\partial x + u u\,\partial x + (A-1)\,u\,\partial x = 0,$$

hincque $\frac{\partial x}{x} = \frac{-\partial u}{uu+(A-1)u+B}$; cui particulariter satisfit ponendo $uu + (A-1)\,u + B = 0$.

Sit primo $uu + (A-1)\,u + B = (u+f)\,(u+g)$, erit particulariter $l\,y = -f\,l\,x$ et $y = x^{-f}$, similique modo $y = x^{-g}$, unde integrale completum erit.

$$y = \alpha\,x^{-f} + \beta\,x^{-g}.$$

Si sit $g = f$, statuatur $g = f - \omega$ evanescente $\omega$, erit

$$x^{-g} = x^{-f}.\,x^{\omega} = x^{-f}\,(1 + \omega\,l\,x),$$

ergo hoc casu fit

$$y = x^{-f}\,(\alpha + \beta\,l\,x).$$

Sit denique

$$uu + (A-1)\,u + B = uu + 2\,f\,u\cos.\zeta + ff,\ \text{erit}$$

$$u = -f\,(\cos.\zeta \pm \sqrt{-1}.\sin.\zeta),$$

ergo particulariter

$$y = x^{-f\cos.\zeta}.\,x^{\pm f\sqrt{-1}.\sin.\zeta} = x^{-f\cos.\zeta}\,[\cos.(f\sin.\zeta.\,lx) \pm \sqrt{-1}.\sin.(f\sin.\zeta.\,lx)],$$

quare integrale completum erit

$$y = C\,x^{-f\cos.\zeta}\sin.(f\sin.\zeta.\,l\,x + \gamma).$$

### Corollarium 1.

848. Hujus ergo aequationis

$$\partial\partial y + (f+g+1)\,\frac{\partial y\,\partial x}{x} + \frac{f\,g\,y\,\partial x^2}{x\,x} = 0,$$

integrale completum est

$$y = \alpha\,x^{-f} + \beta\,x^{-g}.$$

Hujus autem

$$\partial\partial y + (2\,f+1)\,\frac{\partial y\,\partial x}{x} + \frac{f\,f\,y\,\partial x^2}{x\,x} = 0,$$

integrale completum est

$$y = x^{-f}(\alpha + \beta\, l\, x).$$

Corollarium 2.

849. At si aequatio proposita hujusmodi formam habuerit

$$\partial\partial y + (1 + 2f \cos.\zeta)\frac{\partial y \partial x}{x} + \frac{ffy\partial x}{xx} = 0,$$

tum ejus integrale completum erit

$$y = C\, x^{-f\cos.\zeta} \sin.(f \sin.\zeta\, l\, x + \gamma).$$

Scholion.

850. Similem resolutionem quoque admittit haec aequatio differentio-differentialis

$$\partial\partial y - \frac{n\partial y \partial x}{x} + A x^n \partial y \partial x + B x^{2n} y \partial x^2 = 0.$$

Ponatur enim $\partial y = x^n y u \partial x$, et cum sit

$$\partial\partial y = x^n y \partial x \partial u + n x^{n-1} y u \partial x^2 + x^{2n} y u^2 \partial x^2,$$

erit per $y$ dividendo

$$x^n \partial x \partial u + n x^{n-1} u \partial x^2 + x^{2n} u u \partial x^2 - n x^{n-1} u \partial x^2 + A x^{2n} u \partial x^2 + B x^{2n} \partial x^2 = 0,$$

hinc

$$\partial u + x^n u u \partial x + A x^n u \partial x + B x^n \partial x = 0,$$

ideoque

$$x^n \partial x = \frac{-\partial u}{uu + Au + B},$$

cui particulariter satisfit ponendo $uu + Au + B = 0$, unde $u$ duplicem consequitur valorem constantem quorum alter sit $u = -f$ alter $u = -g$. Quocirca integralia particularia erunt

$$y = e^{\frac{-f x^{n+1}}{n+1}} \text{ et } y = e^{\frac{-g x^{n+1}}{n+1}}.$$

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quantitatis $y+\frac{m}{m+1}x^{\frac{m+1}{m}}$, quae quantitas si ponatur $=v$ et $q=\mathrm{F}':z$, ut sit $v=\mathrm{f}':q$, erit

$$\mathrm{f}:q=\int\partial q\,\mathrm{f}':q=\int v\partial v\mathrm{F}'':v,\quad \text{ob } \partial q=\partial v\mathrm{F}'':v;$$

unde concluditur

$$\mathrm{f}:q=v\mathrm{F}':v-\mathrm{F}:v \text{ et } z=\mathrm{F}:v=\mathrm{F}:\left(y+\frac{m}{m+1}x^{\frac{m+1}{m}}\right).$$

## Exemplum 4.

123. *Duarum variabilium $x$ et $y$ ejusmodi functionem $z$ investigare, ut posito*

$$\partial z=p\partial x+q\partial y \text{ fiat } p^3+x^3=3pqx.$$

Consideretur forma

$$z=qy+\int(p\partial x-y\partial q),$$

ubi jam formulam $p\partial x-y\partial q$ integrabilem reddi oportet. Statuatur $p=ux$, et conditio praescripta dat

$$x(1+u^3)=3qu;$$

unde fit

$$x=\frac{3qu}{1+u^3} \text{ et } p=\frac{3quu}{1+u^3};$$

tum vero

$$\partial x=\frac{3q\partial u(1-2u^3)}{(1+u^3)^2}+\frac{3u\partial q}{1+u^3};$$

sicque habebitur

$$z=qy+\int\left(\frac{9qquu\partial u(1-2u^3)}{(1+u^3)^3}+\frac{9qu^3\partial q}{(1+u^3)^2}-y\partial q\right), \text{ at}$$

$$\int\frac{9qq\partial u(1-2u^3)}{(1+u^3)^3}=\frac{3qq(1+4u^3)}{2(1+u^3)^2}-\int\frac{3q(1+4u^3)\partial q}{(1+u^3)^2}.$$

Ergo

$$z=qy+\frac{3qq(1+4u^3)}{(1+u^3)^2}-\int\partial q\left(y+\frac{3q}{1+u^3}\right).$$

Quare necesse est esse $\frac{3q}{1+u^3}$ functionem ipsius $q$ tantum, quae sit $= - f':q$, unde fit

$$y = -\frac{3q}{1+u^3} - f':q \quad \text{et} \quad z = qy + \frac{3qq(1+4u^3)}{2(1+u^3)^2} + f:q,$$

seu $z = \frac{3qq(2u^3-1)}{2(1+u^3)^2} - qf':q + f:q$, existente $x = \frac{3qu}{1+u^3}$. Ex quibus tribus aequationibus si eliminentur binae quantitates $q$ et $u$, orietur aequatio inter $z$ et $x$, $y$, quae quaeritur.

## Corollarium 1.

124. Ex aequatione pro $y$ inventa colligitur $\frac{3}{2+u^3} = \frac{-y-f':q}{q}$, aequatio autem pro $z$ inventa abit in hanc

$$z = \frac{3qq}{1+u^3} - \frac{9qq}{2(1+u^3)^2} - qf':q + f:q,$$

quae eliso $u$ transmutatur in hanc

$$z = -qy - 2qf':q - \tfrac{1}{2}(y+f':q)^2 + f:q;$$

tum vero est

$$x = -u(y+f':q),$$

unde reperitur $u = \frac{-x}{y+f':q}$, hincque

$$x^3 = 3q(y+f':q)^2 + (y+f':q)^3.$$

## Corollarium 2.

125. Si sumamus $f':q = a$, erit $f:q = aq + b$, et postrema aequatio praebet $q = \frac{x^3-(y+a)^3}{3(y+a)^2}$. Cum deinde pro hoc casu fiat

$$z = -qy - aq - \tfrac{1}{2}(y+a)^2 + b,$$

proveniet loco $q$ valorem inventum substituendo

$$z = \frac{6b(y+a) - (y+a)^3 - 2x^3}{6(y+a)}.$$

## Corollarium 3.

126. Cum in genere sit

$$x^3 = (y+f':q)^2(y+3q+f':q),$$

ponamus $f':q = a - 3q$, ideoque $f:q = b + aq - \tfrac{3}{2}qq$, ut fiat $(y + a - 3q)^2 = \frac{x^3}{y+a}$, eritque

$$y + a - 3q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} \text{ et } q = \tfrac{1}{3}(y+a) - \frac{x\sqrt{x}}{3\sqrt{(y+a)}}.$$

Hinc ergo prodit

$$f:q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} - y \text{ et}$$

$$f:q = b + \frac{a(y+a)}{3} - \frac{ax\sqrt{x}}{3\sqrt{(y+a)}} - \tfrac{1}{6}(y+a)^2 + \tfrac{1}{3}x\sqrt{x}\,(y+a) - \frac{x^3}{6(y+a)}$$

$$\text{seu } f:q = b + \frac{aa - yy}{6} + \frac{xy\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{6(y+a)},$$

atque

$$z = -\tfrac{1}{3}y(y+a) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - 2aq + 6qq - \frac{x^3}{2(y+a)} + b + aq - \tfrac{3}{2}qq,$$

$$\text{seu } z = b - \tfrac{1}{3}y(y+a) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{2(y+a)} - aq + \tfrac{9}{2}qq,$$

et facta reductione

$$z = b + \tfrac{1}{6}(y+a)^2 - \tfrac{2}{3}x\sqrt{x}\,(y+a).$$

## Corollarium 4.

127. Quodsi hic sumatur $a = 0$ et $b = 0$, erit per expressionem satis simplicem

$$z = \tfrac{1}{6}yy - \tfrac{2}{3}x\sqrt{xy};$$

quae quomodo conditioni praescriptae satisfaciat, ita apparet. Per differentiationem colligitur

$$p = \left(\frac{\partial z}{\partial x}\right) = -\sqrt{xy} \text{ et } q = \left(\frac{\partial z}{\partial y}\right) = \tfrac{1}{3}y - \frac{x\sqrt{x}}{3\sqrt{y}},$$

hincque

$$p^3 + x^3 = -xy\sqrt{xy} + x^3; \text{ at}$$

$$3pq = xx - y\sqrt{xy}, \text{ ideoque}$$

$$3pqx = x^3 - xy\sqrt{xy}, \text{ ergo}$$

$$p^3 + x^3 = 3pqx.$$

### Scholion.

128. Successit ergo solutio, quando aequatio quaecunque inter $p$, $q$ et $x$ proponitur, etiamsi casibus, quibus inde neque $x$ neque $p$ elici potest, difficultas quaedam restat, quae autem resolutionem aequationum finitarum potissimum afficit, quam hic merito concedi postulamus. Interim ex postremo exemplo perspicitur, quomodo operatio sit instituenda, si ope substitutionis idoneae aequatio proposita ad resolutionem accommodari queat, cui autem negotio hic amplius non immoror. Neque etiam eos casus, quibus inter $p$, $q$ et $y$ relatio quaedam praescribitur, hic seorsim evolvam, cum ob permutabilitatem ipsarum $x$ et $y$, qua etiam $p$ et $q$ permutantur, hi casus ad praecedentes sponte revocentur. Superest igitur casus, quo aequatio inter $p$, $q$ et $z$ proponitur, ubi quidem statim manifestum est, in aequatione $\partial z = p\partial x + q\partial y$ quantitates $p$ et $q$ non uti functiones ipsarum $x$ et $y$ spectari posse, quoniam etiam a $z$ pendent, neque ergo earum indoles inde determinari poterit, ut formula $p\partial x + q\partial y$ integrabilis evadat. Verum sine discrimine conditio ea est definienda, ut aequatio differentialis

$$\partial z - p\partial x - q\partial y = 0$$

fiat possibilis; ad quod ex principiis supra stabilitis §. 6. requiritur, ut posito

$$\left(\frac{\partial q}{\partial z}\right) = \mathrm{L}, \; -\left(\frac{\partial p}{\partial z}\right) = \mathrm{M}, \text{ et } \left(\frac{\partial p}{\partial y}\right) - \left(\frac{\partial q}{\partial x}\right) = \mathrm{N}, \text{ sit}$$

$$\mathrm{L}p + \mathrm{M}q - \mathrm{N} = 0, \text{ seu } p\left(\frac{\partial q}{\partial z}\right) - q\left(\frac{\partial p}{\partial z}\right) + \left(\frac{\partial q}{\partial x}\right) - \left(\frac{\partial p}{\partial y}\right) = 0.$$

Quare proposita aequatione quacunque inter $p$, $q$ et $z$, eas conditiones in genere investigare oportet, ut huic requisito satisfiat.

## Problema 19.

129. Si posito $\partial z = p\partial x + q\partial y$ debeat esse $p + q = \frac{z}{a}$, relationem functionis $z$ ad variabiles $x$ et $y$ in genere investigare.

### Solutio.

Cum sit $q = \frac{z}{a} - p$, aequatio nostra hanc induet formam

$$\partial z = p\partial x - p\partial y + \frac{z\partial y}{a}, \text{ seu}$$

$$p(\partial x - \partial y) = \frac{a\partial z - z\partial y}{a} = z\left(\frac{\partial z}{z} - \frac{\partial y}{a}\right).$$

Quoniam igitur ambae formulae

$$\partial x - \partial y \text{ et } \frac{\partial z}{z} - \frac{\partial y}{a}$$

per se sunt integrabiles, ob

$$\frac{\partial z}{z} - \frac{\partial y}{a} = \frac{p}{z}(\partial x - \partial y),$$

necesse est ut $\frac{p}{z}$ sit functio quantitatis $x - y$, ponatur ergo

$$\frac{p}{z} = f' : (x - y), \text{ ut fiat } lz - \frac{y}{a} = f : (x - y).$$

Definiri ergo potest $z$ per $x$ et $y$, et cum sit $e^{f:(x-y)}$ etiam functio ipsius $x - y$, si ea ponatur $= F : (x - y)$, erit

$$z = e^{\frac{y}{a}} F : (x - y), \text{ unde fit}$$

$$\left(\frac{\partial z}{\partial y}\right) = p = e^{\frac{y}{a}} F' : (x - y) \text{ et}$$

$$\left(\frac{\partial z}{\partial y}\right) = q = -e^{\frac{y}{a}} F' : (x - y) + \frac{1}{a} e^{\frac{y}{a}} F : (x - y);$$

ideoque

$$p + q = \frac{1}{a} e^{\frac{y}{a}} F : (x - y) = \frac{z}{a},$$

uti requiritur.

Corollarium 1.

130. Ex hoc exemplo intelligitur, quomodo certa functio ipsarum $p$ et $q$ quantitati $z$ aequari possit, etiamsi $p$ et $q$ sint functiones ipsarum $x$ et $y$. Simul scilicet ratio integralis formulae

$$\partial z = p\,\partial x + q\,\partial y$$

introducitur in calculum.

Corollarium 2.

131. Forma $e^{\frac{y}{a}}\,\mathrm{F}:(x-y)$ pro valore ipsius $z$ inventa per functionem quamvis ipsius $x-y$ multiplicari potest. Si ergo multiplicetur per

$$e^{\frac{x-y}{a}},\ \text{fit}\ z = e^{\frac{x}{a}}\,\mathrm{F}:(x-y).$$

Sin autem multiplicetur per

$$e^{\frac{x-y}{2a}},\ \text{fit}\ z = e^{\frac{x+y}{2a}}\ \mathrm{F}:(x-y),$$

quae formae problemati aeque satisfaciunt.

Problema 20.

132. Si posito $\partial z = p\partial x + q\partial y$, quantitas $z$ aequari debeat functioni datae ipsarum $p$ et $q$, indolem, qua $z$ per $x$ et $y$ definitur, in genere investigare.

Solutio.

Ex formula proposita habemus $\partial y = \frac{\partial z}{q} - \frac{p\partial x}{q}$; statuatur $p = qr$, ut sit $z$ aequalis functioni ipsarum $q$ et $r$, et ex $\partial y = \frac{\partial z}{q} - r\partial x$ elicitur

$$y = \frac{z}{q} - rx + \int \left(\frac{z\partial q}{qq} + x\partial r\right),$$

quam formulam integrabilem reddi oportet. Cum igitur $z$ sit functio data ipsarum $q$ et $r$, posito $r$ constante quaeratur integrale formulae $\frac{z\partial q}{qq}$, sitque

$$\int \frac{z\partial q}{qq} = \mathrm{V} + \mathrm{f} : r,$$

unde differentiando prodeat

$$\partial \mathrm{V} = \frac{z\partial q}{qq} + \mathrm{R}\partial r,$$

ac jam patet esse debere $x = \mathrm{R} + \mathrm{f}' : r$, indeque obtineri

$$y = \frac{z}{q} - \mathrm{R}r - r\mathrm{f}'r + \mathrm{V} + \mathrm{f} : r,$$

quibus duabus aequationibus relatio inter quantitates propositas determinatur. Primo igitur posito $p = qr$, datur $z$ per $q$ et $r$. Deinde sumto $r$ constante integretur formula $\frac{z\partial q}{qq}$, sitque integrale resultans $\mathrm{V} = \int \frac{z\partial q}{qq}$, quod etiam per $q$ et $r$ datur; unde sumto $q$ constante colligitur $\mathrm{R} = \left(\frac{\partial \mathrm{V}}{\partial r}\right)$. Quibus inventis erit

$$x = \mathrm{R} + \mathrm{f}' : r \text{ et } y = \frac{z}{q} - rx + \mathrm{V} + \mathrm{f} : r,$$

sicque omnes quantitates per binas variabiles $q$ et $r$ determinantur.

### Corollarium 1.

133. Quia permutatis $x$ et $y$ litterae $p$ et $q$ permutantur, simili modo nostram investigationem incipere potuissemus ab aequatione

$$\partial x = \frac{\partial z}{p} - \frac{q\partial y}{p},$$

similisque solutio prodiisset, quae quidem forma diversa at re congruens esset.

### Corollarium 2.

134. Jam scilicet posito $q = ps$, ut sit

$$\partial x = \frac{\partial z}{p} - s\partial y, \text{ erit}$$

$$x = \frac{z}{p} - sy + \int\left(\frac{z\partial p}{pp} + y\partial s\right).$$

Jam sumto $s$ constante ponatur $\int\frac{z\partial p}{pp} = U$, quae quantitas per $p$ et $s$ determinatur, ex ea vero prodeat $\left(\frac{\partial U}{\partial s}\right) = S$, erit

$$y = S + f' : s \text{ et } x = \frac{z}{p} - sy + U + f : s.$$

## Exemplum 1.

135. *Si esse debeat $p + q = \frac{z}{a}$, solutionem pro hoc casu exhibere.*

Posito $p = qr$, erit $z = aq(1 + r)$, nunc sumto $r$ constante erit

$$V = \int\frac{z\partial q}{qq} = a(1 + r)\, lq \text{ et } R = \left(\frac{\partial V}{\partial r}\right) = a lq.$$

Hinc reperitur

$$x = alq + f' : r, \text{ et } y = \frac{z}{q} - arlq - rf' : r + a(1 + r)\, lq + f : r, \text{ seu}$$

$$y = a(1 + r) + a lq - rf' : r + f : r.$$

Si hinc $q$ elidere velimus, ob $q = \frac{z}{a(1 + r)}$ solutio his duabus aequationibus continetur

$$x = a l \frac{z}{a(1 + rr)} + f' : r, \text{ et}$$

$$y = a l \frac{z}{a(1 + r)} + a(1 + r) - rf' : r + f : r.$$

Unde sequenti modo praecedens solutio elici potest, ex forma priori est

$$\frac{x}{a} - l\frac{z}{a} = - l(1 + r) + \frac{1}{a} f' : r = \text{funct. } r,$$

ex ambabus vero

$$y - x = a(1 + r) - (1 + r) f' : r + f : r = \text{funct. } r.$$

Cum ergo tam $\frac{x}{a} - l\frac{z}{a}$, seu $ze^{-\frac{x}{a}}$, quam $y - x$ sit functio ipsius $r$, altera forma aequabitur functioni alterius; unde statui potest

$$ze^{-\frac{x}{a}} = F:(y-x), \quad \text{seu} \quad z = e^{\frac{x}{a}} F:(y-x),$$

quae est solutio ante inventa.

### Exemplum 2.

136. *Si posito $\partial z = p\partial x + q\partial y$ debeat esse $z = apq$, relationem inter $x$, $y$ et $z$ investigare.*

Posito $p = qr$ erit $z = aqqr$, et sumto $r$ constante sit $V = \int \frac{z\partial q}{qq} = aqr$, hincque $R = \left(\frac{\partial V}{\partial r}\right) = aq$. Quocirca habebimus

$$x = aq + f':r \quad \text{et} \quad y = aqr - rf':r + f:r,$$

seu ob $r = \frac{z}{aqq}$, erit

$$x = aq + f':\frac{z}{aqq} \quad \text{et} \quad y = \frac{z}{q} - \frac{z}{aqq} f':\frac{z}{aqq} + f:\frac{z}{aqq}.$$

Hic in genere notemus si sit $f':r = v$, ponamusque $r = F':v$, ob $\partial r = \partial v F'':v$, fore

$$f:r = \int \partial r f':r = \int v\partial v F'':v = vF':v - F:v, \quad \text{seu}$$

$$f:r = vF':v - F:v, \quad \text{hincque}$$

$$f:r - rf':r = -F:v.$$

Quare cum sit $f':r = x - aq$, si ponamus $r = F':(x-aq)$, erit

$$f:r - rf':r = -F:(x-aq) \quad \text{et}$$

$$y = aqF':(x-aq) - F:(x-aq), \quad \text{atque}$$

$$z = aqqF':(x-aq).$$

### Scholion.

137. Hae postremae formulae ita statim ex conditione quaestionis elici possunt. Nam ob $p = \frac{z}{aq}$ erit

$$\partial z = \frac{z\partial x}{aq} + q\partial y, \text{ et } \partial y = \frac{\partial z}{q} - \frac{z\partial x}{aqq},$$

hincque

$$y = \frac{z}{q} + \int\left(\frac{z\partial q}{qq} - \frac{z\partial x}{aqq}\right) = \frac{z}{q} + \int\frac{z}{qq}\left(\partial q - \frac{\partial x}{a}\right);$$

ubi manifestum est esse $\frac{z}{qq}$ functionem quantitatis $q - \frac{x}{a}$. Quare posito

$$\frac{z}{qq} = \mathrm{F}' : \left(q - \frac{x}{a}\right), \text{ erit}$$
$$y = \frac{z}{q} + \mathrm{F} : \left(q - \frac{x}{a}\right).$$

Quin etiam indidem alia solutio deduci potest ponendo

$$\partial x = \frac{aq}{z}(\partial z - q\partial y),$$

quae posito $z = qv$ abit in

$$\partial x = \frac{a}{v}(v\partial q + q\partial v - q\partial y), \text{ unde}$$
$$x = aq + \int\frac{aq}{v}(\partial v - \partial y).$$

Quare ponatur

$$\frac{aq}{v} = \mathrm{f}' : (v - y), \text{ eritque}$$
$$x = aq + \mathrm{f} : (v - y).$$

Jam restituto valore $v = \frac{z}{q}$ habebitur

$$\frac{aqq}{z} = \mathrm{f}' : \left(\frac{z}{q} - y\right) \text{ et } x - aq = \mathrm{f} : \left(\frac{z}{q} - y\right).$$

Prima autem solutio ad eliminanda $q$ et $r$ est aptissima in exemplis. Si enim ponatur

$$\mathrm{f}' : r = \frac{b}{\sqrt{r}} + c, \text{ erit } \mathrm{f} : r = 2b\sqrt{r} + cr + d;$$

hinc

$$z = aqqr \text{ et } x = aq + \frac{b}{\sqrt{r}} + c,$$

atque

$$y = aqr + b\sqrt{r} + d.$$

Jam ob $r = \frac{z}{aqq}$ fit

$$x = aq + bq\sqrt{\frac{a}{z}} + c \text{ et } y = \frac{z}{q} + \frac{b}{q}\sqrt{\frac{z}{a}} + d.$$

Hinc

$$x - c = q\left(a + \frac{b\sqrt{a}}{\sqrt{z}}\right) \text{ et } y - d = \frac{z}{aq}\left(a + \frac{b\sqrt{a}}{\sqrt{z}}\right),$$

et multiplicando iliditur $q$, fitque

$$(x - c)(y - d) = \frac{z}{a}\left(a + \frac{b\sqrt{a}}{\sqrt{z}}\right)^2 = (b + \sqrt{az})^2,$$

ita ut sit

$$b + \sqrt{az} = \sqrt{(x - c)(y - d)},$$

et proinde

$$z = \frac{(x-c)(y-d) - 2b\sqrt{(x-c)(y-d)} + bb}{a},$$

quae si $b = c = d = 0$ dat casum simplicissimum $z = \frac{xy}{a}$.