CAPUT I.

DE
NATURA AEQUATIONUM DIFFERENTIALIUM QUIBUS FUNCTIONES DUARUM VARIABILIUM DETERMINANTUR IN GENERE.

Problema 1.

1.

Si z sit functio quaecunque duarum variabilium x et y, definire indolem aequationis differentialis, qua relatio differentialium ∂x, ∂y et ∂z exprimitur.

Solutio.

Sit $P\partial x + Q\partial y + R\partial z = 0$ aequatio relationem differentialium ∂x, ∂y et ∂z exprimens, in qua P, Q et R sint functiones quaecunque ipsarum x, y et z. Ac primo quidem necesse est, ut haec aequatio nata sit ex differentiatione aequationis cujuspiam finitae, postquam differentiale per quampiam quantitatem fuerit divisum. Dabitur ergo quidam multiplicator puta M, per quem formula $P\partial x + Q\partial y + R\partial z$ multiplicata fiat integrabilis; nisi enim talis multiplicator existeret, aequatio differentialis proposita foret absurda, nihilque omnino declararet. Totum ergo negotium huc redit, ut character assignetur, cujus ope hujusmodi aequationes differentiales absurdae nihilque significantes a realibus dignosci queant. Hunc in finem contemplemur aequationem propositam $P\partial x + Q\partial y + R\partial z = 0$ tanquam realem. Sit M multiplicator eam reddens integrabilem, ita ut haec formula

$$MP\partial x + MQ\partial y + MR\partial z$$

sit verum differentiale cujuspiam functionis trium variabilium x, y et z; quae functio si ponatur $= V$, haec aequatio $V = \text{Const.}$ futura sit integrale completum aequationis propositae. Sive igitur x, sive y, sive z accipiatur constans, singulas has formulas

$$MQ\partial y + MR\partial z, \; MR\partial z + MP\partial x, \; MP\partial x + MQ\partial y,$$

seorsim integrabiles esse oportet; unde ex natura differentialium erit

$$\left(\frac{\partial \,.\, MQ}{\partial z}\right) - \left(\frac{\partial \,.\, MR}{\partial y}\right) = 0, \quad \left(\frac{\partial \,.\, MR}{\partial x}\right) - \left(\frac{\partial \,.\, MP}{\partial z}\right) = 0,$$
$$\left(\frac{\partial \,.\, MP}{\partial y}\right) - \left(\frac{\partial \,.\, MQ}{\partial x}\right) = 0,$$

unde per evolutionem hae tres oriuntur aequationes

$$\text{I.}\quad M\left(\frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial M}{\partial z}\right) - M\left(\frac{\partial R}{\partial y}\right) - R\left(\frac{\partial M}{\partial y}\right) = 0$$

$$\text{II.}\quad M\left(\frac{\partial R}{\partial x}\right) + R\left(\frac{\partial M}{\partial x}\right) - M\left(\frac{\partial P}{\partial z}\right) - P\left(\frac{\partial M}{\partial z}\right) = 0$$

$$\text{III.}\quad M\left(\frac{\partial P}{\partial y}\right) + P\left(\frac{\partial M}{\partial y}\right) - M\left(\frac{\partial Q}{\partial x}\right) - Q\left(\frac{\partial M}{\partial x}\right) = 0$$

quarum si prima per P, secunda per Q et tertia per R multiplicetur, in summa omnia differentialia ipsius M se tollent, et reliqua aequatio per M divisa erit

$$P\left(\frac{\partial Q}{\partial z}\right) - P\left(\frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x}\right) - Q\left(\frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y}\right) - R\left(\frac{\partial Q}{\partial x}\right) = 0$$

quae continet characterem, aequationes differentiales reales ab absurdis discernentem, et quoties inter quantitates P, Q et R haec conditio locum habet, toties aequatio differentialis proposita

$$P\partial x + Q\partial y + R\partial z = 0$$

est realis. Caeterum hic meminisse oportet, hujusmodi formulam uncinulis inclusam $\left(\frac{\partial Q}{\partial z}\right)$ significare valorem $\frac{\partial Q}{\partial z}$, si in differentiatione ipsius Q sola quantitas z ut variabilis tractetur; quod idem de caeteris est tenendum, quae ergo semper ad functiones finitas reducuntur.

Corollarium 1.

2. Proposita ergo aequatione differentiali inter tres variabiles

$$P\partial x + Q\partial y + R\partial z = 0,$$

ante omnia dispiciendum est, utrum character inventus locum habeat, nec ne? priori casu aequatio erit realis, posteriori vero absurda et nihil plane significans, neque unquam ad talem aequationem ullius problematis solutio perducere valet.

Corollarium 2.

3. Character inventus etiam hoc modo exprimi potest

$$\left(\frac{P\partial Q - Q\partial P}{\partial z}\right) + \left(\frac{Q\partial R - R\partial Q}{\partial x}\right) + \left(\frac{R\partial P - P\partial R}{\partial y}\right) = 0,$$

quandoquidem uncinulae non quantitates finitas afficiunt, sed solam differentiationem ad certam variabilem restringunt.

Corollarium 3.

4. Simili modo si aequatio haec characterem continens per PQR dividatur, ea hanc formam induet

$$\left(\frac{\partial\,.\,l\frac{Q}{P}}{R\partial z}\right) + \left(\frac{\partial\,.\,l\frac{R}{Q}}{P\partial x}\right) + \left(\frac{\partial\,.\,l\frac{P}{R}}{Q\partial y}\right) = 0$$

quae etiam ita exprimi potest

$$\left(\frac{\frac{\partial Q}{Q} - \frac{\partial P}{P}}{R\partial z}\right) + \left(\frac{\frac{\partial R}{R} - \frac{\partial Q}{Q}}{P\partial x}\right) + \left(\frac{\frac{\partial P}{P} - \frac{\partial R}{R}}{Q\partial y}\right) = 0.$$

Scholion 1.

5. Quemadmodum omnes aequationes differentiales inter binas variabiles semper sunt reales, semperque per eas relatio certa inter ipsas variabiles definitur, ita hinc discimus, rem secus se habere in aequationibus differentialibus, quae tres variabiles involvant, atque hujusmodi aequationes

$$P \partial x + Q \partial y + R \partial z = 0$$

non certam relationem inter ipsas quantitates finitas x, y et z declarare, nisi quantitates P, Q, R ita fuerint comparatae, ut character inventus locum habeat. Ex quo intelligitur infinitas hujusmodi aequationes differentiales inter ternas variabiles proponi posse, quibus nulla prorsus relatio finita conveniat, et quae propterea nihil plane definiant. Pro arbitrio scilicet hujusmodi aequationes formari possunt, nullo scopo proposito ad quem sint accommodatae; statim enim ac certum quoddam problema ad aequationem differentialem inter ternas variabiles perducit, semper necesse est characterem assignatum ei convenire, cum alioquin nihil omnino significaret. Talis aequatio nihil significans est exempli gratia

$$z \partial x + x \partial y + y \partial z = 0,$$

neque pro z ulla quidem functio ipsarum x et y cogitari potest quae isti aequationi satisfaciat; quin etiam character noster pro hoc exemplo dat $-x-y-z$, quae quantitas cum non evanescat, absurditatem illius aequationis declarat.

Scholion 2.

6. Quo character inventus facilius ad quosvis casus oblatos accommodari queat, ex aequatione

$$P \partial x + Q \partial y + R \partial z = 0$$

primo evolvantur sequentes valores

$$\left(\frac{\partial Q}{\partial z}\right) - \left(\frac{\partial R}{\partial y}\right) = L, \quad \left(\frac{\partial R}{\partial x}\right) - \left(\frac{\partial P}{\partial z}\right) = M, \quad \left(\frac{\partial P}{\partial y}\right) - \left(\frac{\partial Q}{\partial x}\right) = N,$$

et character noster hac continebitur expressione

$$LP + MQ + NR,$$

quae si evanescat, aequatio proposita erit realis, et aequationem quandam finitam agnoscet; sin autem ea ad nihilum non redigatur, aequatio proposita erit absurda, atque de ejus integratione ne cogitandum quidem erit. Ita in exemplo supra posito erit

$$P = z, \quad Q = x, \quad R = y,$$

hinc

$$L = -1, \quad M = -1, \quad N = -1,$$

unde character $-x - y - z$ absurditatem indicat. Proferamus vero etiam exemplum aequationis realis

$$\partial x (yy + nyz + zz) - x (y + nz) \partial y - xz\partial z = 0,$$

in qua ob

$$P = yy + nyz + zz, \quad Q = -xy - nxz \quad \text{et} \quad R = -xz,$$

erit

$$L = -nx, \quad M = -3z - ny \quad \text{et} \quad N = 3y + 2nz,$$

unde

$$LP + MQ + NR$$

$$= -nx (yy + nyz + zz) + x (y + nz) (3z + ny) - xz (3y + 2nz)$$

$$= x[-nyy - nnyz - nzz + 3yz + 3nzz + nyy + nnyz - 3yz - 2nzz] = 0,$$

quare cum hic character evanescat, aequatio haec differentialis pro reali est habenda. Simili modo proposita hac aequatione

$$2\partial x (y + z) + \partial y (x + 3y + 2z) + \partial z (x + y) = 0, \text{ ob}$$

$$P = 2y + 2z, \quad Q = x + 3y + 2z, \quad R = x + y, \text{ fit}$$

$$L = 2 - 1 = 1, \quad M = 1 - 2 = -1, \quad \text{et} \quad N = 2 - 1 = 1,$$

hincque

$$LP + MQ + NR = 2yz - 2zx - 3yz - 2zx + xy = 0,$$

unde ista aequatio differentialis erit realis.

Problema 2.

7. Proposita aequatione differentiali inter ternas variabiles x, y, z, quae sit realis, ejus integrale investigare, unde pateat, qualis functio una earum sit binarum reliquarum.

Solutio.

Sit aequatio differentialis proposita

$$P\partial x + Q\partial y + R\partial z = 0,$$

in qua P, Q, R, ejusmodi sint functiones ipsarum x, y et z, ut characteri realitatis ante inventus satisfaciat. Nisi enim ista aequatio esset realis, ridiculum foret, ejus integrationem tentare. Sumamus ergo hanc aequationem esse realem, atque dabitur relatio inter ipsas quantitates x, y et z, aequationi propositae satisfaciens, ad quam inveniendam perpendatur; si in aequatione integrali una variabilium, puta z, constans spectetur, ex ejus differentiali nihilo aequali posito nasci debere aequationem

$$P\partial x + Q\partial y = 0.$$

Vicissim ergo una variabili puta z ut constante tractata, integratio aequationis differentialis

$$P\partial x + Q\partial y = 0,$$

quae duas tantum variabiles continet, perducet ad aequationem integralem quaesitam, si modo in quantitatem constantem per integrationem ingressam illa quantitas z rite involvatur. Ex quo hanc regulam pro integratione aequationis propositae colligimus. Consideretur una variabilium puta z ut constans, ut habeatur haec aequatio $P\partial x + Q\partial y = 0$, duas tantum variabiles x et y implicans; tum ejus investigetur aequatio integralis completa, quae ergo constantem arbitrariam C complectetur. Deinde haec constans C consideretur ut functio quaecunque ipsius z, atque hac z nunc etiam pro variabili habita, aequatio integralis inventa denuo differentietur; ut omnes tres x, y et z tanquam variabiles tractentur, et aequatio differentialis resultans comparetur cum proposita

$$P \partial x + Q \partial y + R \partial z = 0,$$

vbi quidem functiones P et Q sponte prodibunt, at functio R cum ea quantitate, qua elementum ∂z afficitur, collata determinabit rationem, qua quantitas z in illam litteram C ingreditur, sicque obtinebitur aequatio integralis quaesita, quae simul erit completa, cum semper in illa litterae C pars quaedam constans vere arbitraria relinquatur, cum haec determinatio ex differentiali ipsius C sit petenda.

Corollarium 1.

8. Reducitur ergo integratio hujusmodi aequationum differentialium tres variabiles continentium ad integrationem aequationum differentialium inter duas tantum variabiles, quae ergo quoties licet per methodos in superiori libro traditas, est instituenda.

Corollarium 2.

9. Haec ergo integratio tribus modis institui potest, prout primo vel z, vel y, vel x tanquam constans spectatur. Semper autem necesse est, ut eadem aequatio integralis resultet, siquidem aequatio differentialis fuerit realis.

Corollarium 3.

10. Quodsi haec methodus tentetur in aequatione differentiali impossibili, determinatio illius constantis C non ita succedet,

ut eam variabilem, quae pro constante est habita, solam involvat; atque etiam ex hoc criterium realitatis peti poterit.

Scholion.

11. Quo haec operatio facilius intelligatur, periculum faciamus primo in aequatione impossibili hac

$$z\partial x + x\partial y + y\partial z = 0,$$

hic sumta z pro constante erit

$$z\partial x + x\partial y = 0, \text{ seu } \frac{z\partial x}{x} + \partial y = 0,$$

cujus integrale est $z\,lx + y = C$, existente C functione ipsius z. Differentietur ergo haec aequatio sumendo etiam z variabile, positoque $\partial C = D\partial z$, ut D sit etiam functio ipsius z tantum, erit

$$\frac{z\partial x}{x} + \partial y + \partial z\, lx = D\partial z, \text{ seu}$$
$$z\partial x + x\partial y + \partial z\,(x\,lx - Dx) = 0:$$

deberet ergo esse $x\,lx - Dx = y$, seu $D = lx - \frac{y}{x}$, quod est absurdum.

Deinde in aequatione reali

$$2\partial x\,(y + z) + \partial y\,(x + 3y + 2z) + \partial z\,(x + y) = 0$$

operatio exposita ita instituatur. Sumatur y constans, ut sit

$$2\partial x\,(y + z) + \partial z\,(x + y) = 0, \text{ seu } \frac{2\partial x}{x + y} + \frac{\partial z}{y + z} = 0,$$

cujus integrale est

$$2\,l\,(x + y) + l\,(y + z) = C,$$

ubi C etiam y involvat. Sit ergo $\partial C = D\partial y$, et sumto etiam y variabili, differentiatio praebet

$$\frac{2\partial x + 2\partial y}{x + y} + \frac{\partial y + \partial z}{y + z} = D\partial y, \text{ seu}$$

$$2\partial x(y+z)+2\partial y(y+z)+\partial y(x+y)+\partial z(x+y)$$
$$=D\partial y(x+y)(y+z),$$

quae expressio cum forma proposita collata praebet $D=0$, ideoque $\partial C=0$, et C fit constans vera; ita ut integrale sit

$$(x+y)^2(y+z)=\text{Const.}$$

Hujusmodi igitur exempla aliquot evolvamus.

Exemplum 1.

12. *Hujus aequationis realis*

$$\partial x(y+z)+\partial y(x+z)+\partial z(x+y)=0$$

integrale investigare.

Primo quidem patet hanc aequationem esse realem, cum sit

$$P=y+z,\quad L=1-1=0,$$
$$Q=x+z,\quad M=1-1=0,$$
$$R=x+y,\quad N=1-1=0,$$

sumatur igitur z constans, et aequatio prodibit

$$\partial x(y+z)+\partial y(x+z)=0,\ \text{seu}\ \frac{\partial x}{x+z}+\frac{\partial y}{y+z}=0,$$

cujus integrale est

$$l(x+z)+l(y+z)=f:z.$$

Statuatur ergo

$$(x+z)(y+z)=Z,$$

ubi natura functionis Z ex differentiatione debet erui. Fit autem

$$\partial x(y+z)+\partial y(x+z)+\partial z(x+y+2z)=\partial Z,$$

a qua si proposita auferatur, relinquitur $2z\partial z=\partial Z$; hinc $Z=zz+C$, ita ut aequatio integralis completa sit

**

$$(x + z)(y + z) = zz + C, \text{ seu } xy + xz + yz = C;$$

quae quidem ex ipsa proposita

$$y\partial x + z\partial x + x\partial y + z\partial y + x\partial z + y\partial z = 0,$$

facile elicitur, cum bina membra juncta sit integrabilia.

Exemplum 2.

13. *Hujus differentialis aequationis realis*

$$\partial x\,(ay - bz) + \partial y\,(cz - ax) + \partial z\,(bx - cy) = 0$$

aequationem integralem completam invenire.

Realitas hujus aequationis ita ostenditur. Cum sit

$$\begin{aligned} P &= ay - bz, & \text{erit } L &= 2c, \\ Q &= cz - ax, & M &= 2b, \\ R &= bx - cy, & N &= 2a, \end{aligned}$$

hincque manifesto $LP + MQ + NR = 0$. Jam sumatur z constans; ut habeatur

$$\frac{\partial x}{cz - ax} + \frac{\partial y}{ay - bz} = 0, \text{ ergo } \frac{1}{a}\, l\frac{ay - bz}{cz - ax} = f : z,$$

statuatur ergo $\frac{ay - bz}{cz - ax} = Z$, et differentiatio praebet

$$\frac{a\partial x\,(ay - bz) + a\partial y\,(cz - ax) + a\partial z\,(bx - cy)}{(cz - ax)^2} = \partial Z,$$

ex cujus comparatione cum proposita fit $\partial Z = 0$ et $Z = C$, ita ut aequatio integralis completa sit

$$\frac{ay - bz}{cz - ax} = n, \text{ seu } ay + nax = (b + nc)\,z.$$

Quodsi aequatio integralis ponatur

$$Ax + By + Cz = 0;$$

hae constantes ita debent esse comparatae, ut sit

$$Ac + Bb + Ca = 0,$$

sicque constans arbitraria concinnius inducitur.

Corollarium.

Haec ergo aequatio integrabilis redditur, si dividatur per $(cz - ax)^2$, atque ob eandem rationem etiam hi divisores

$(ay - bz)^2$ et $(bx - cy)^2$

idem praestant. Vi enim integralis hi divisores constantem inter se tenent rationem. Namque si $\frac{ay - bz}{cz - ax} = n$, erit

$$\frac{bx - cy}{cz - ax} = \frac{-b - nc}{a}, \text{ et } \frac{bx - cy}{ay - bz} = \frac{-b - nc}{na}.$$

Exemplum 3.

15. *Hujus aequationis differentialis realis*

$$\partial x(yy + yz + zz) + \partial y(zz + xz + xx) + \partial y(xx + xy + yy) = 0$$

aequationem integralem investigare.

Realitas hujus aequationis inde patet, quod sit

$P = yy + yz + zz$, hincque $L = 2z + x - x - 2y = 2(z - y)$,

$Q = zz + xz + xx$, $\quad M = 2x + y - y - 2z = 2(x - z)$,

$R = xx + xy + yy$, $\quad N = 2y + z - z - 2x = 2(y - x)$,

unde fit

$$LP + MQ + NR = 2(z^3 - y^3) + 2(x^3 - z^3) + 2(y^3 - x^3) = 0.$$

Ad integrale ergo investigandum sumatur z constans, eritque

$$\frac{\partial x}{xx + xz + zz} + \frac{\partial y}{yy + yz + zz} = 0,$$

cujus integrale est

$$\frac{2}{z\sqrt{3}} \text{ Ang. tang. } \frac{x\sqrt{3}}{2z + x} + \frac{2}{z\sqrt{3}} \text{ Ang. tang. } \frac{y\sqrt{3}}{2z + y} = f:z,$$

quae per collectionem horum angulorum abit in

$$\frac{2}{z\sqrt{3}} \text{ Ang. tang. } \frac{(xz + yz + xy)\sqrt{3}}{2zz + xz + yz - xy} = f:z.$$

Statuatur ergo $\frac{xz + yz + xy}{2zz + xz + yz + xy} = Z$, haecque aequatio differentietur sumtis omnibus tribus x, y et z variabilibus, ac prodibit

$$\frac{2z\partial x(yy+yz+zz)+2z\partial y(zz+xz+xx)-2x\partial z(zz+yz+yy)-2y\partial z(zz+xz+xx)}{(2zz+xz+yz-xy)^2}=\partial Z,$$

cum igitur ex aequatione proposita sit

$$\partial x(yy+yz+zz)+\partial y(zz+xz+xx)=-\partial z(xx+xy+yy),$$

erit facta substitutione

$$\frac{-2z\partial z(xx+xy+yy)-2x\partial z(zz+yz+yy)-2y\partial z(zz+xz+xx)}{(2zz+xz+yz-xy)^2}=\partial Z,$$

seu

$$\frac{-2\partial z(xxz+xzz+yyz+yzz+xxy+xyy+3xyz)}{(2zz+xz+yz-xy)^2}=\partial Z,$$

quae in hanc formam reducitur

$$\frac{-2\partial z(x+y+z)(xy+xz+yz)}{(2zz+xz+yz-xy)^2}=\partial Z,$$

At ob $Z=\frac{xy+xz+yz}{2zz+xz+yz-xy}$, erit

$$\frac{-2ZZ\partial z(x+y+z)}{xy+xz+yz}=\partial Z, \text{ seu } \frac{-\partial Z}{ZZ}=\frac{2\partial z(x+y+z)}{xy+xz+yz}.$$

Necesse ergo est ut etiam $\frac{xy+xz+yz}{x+y+z}$ sit functio ipsius z tantum, quae vocetur Σ, ut sit $-\frac{\partial Z}{ZZ}=\frac{2\partial z}{\Sigma}$. Verum ex sola forma functionis Z negotium confici oportet; quod ita expediri potest. Cum sit

$$Z=\frac{xy+xz+yz}{2zz+xz+yz-xy}, \text{ erit } 1+Z=\frac{2zz+2xz+2yz}{2zz+xz+yz-xy},$$

hinc $\frac{1+Z}{Z}=\frac{2z(x+y+z)}{xy+xz+yz}$, cujus valoris ope quantitates x et y ex aequatione differentiali eliduntur, fitque

$$-\frac{\partial Z}{ZZ}=\partial z\cdot\frac{2(x+y+z)}{xy+xz+yz}=\partial z\cdot\frac{1+Z}{Zz}; \text{ unde}$$

$$\frac{-\partial Z}{Z(1+Z)}=\frac{\partial z}{z}=\frac{-\partial Z}{Z}+\frac{\partial Z}{1+Z},$$

et integrando $lz=l\frac{1+Z}{Z}+la$.

Ergo $\frac{1+Z}{Z}=\frac{z}{a}$ et $Z=\frac{a}{z-a}$,

ita ut aequatio integralis quaesita sit

[illegible] $\frac{xy+xz+yz}{2zz+xz+yz-xy}$, seu $xy+xz+yz = a(x+y+z)$
quae simplicissima forma statim colligitur ex aequatione

$$\frac{2z(x+y+z)}{xy+xz+yz} = \frac{1+Z}{Z} = \frac{z}{a}.$$

Corollarium.

16. Cum aequationis propositae integrale completum sit

$$xy+xz+yz = (x+y+z) \text{ seu } \frac{xy+xz+yz}{x+y+z} = \text{Const.}$$

ex hujus differentiatione etiam ipsa aequatio proposita resultare deprehenditur. Unde patet aequationem propositam integrabilem reddi, si dividatur per

$$(x+y+z)^2, \text{ vel etiam per } (xy+xz+yz)^2.$$

Scholion.

17. Ex hoc exemplo intelligitur, determinationem functionis per integrationem illatae interdum haud exiguis difficultatibus esse obnoxiam; siquidem hic functionem Z non sine ambagibus elicui-mus. Verum et hic ista investigatio multo facilius institui potuisset; statim enim atque invenimus

$$\frac{xy+xz+yz}{2zz+xz+yz-xy} = Z = f:z,$$

hanc ipsam expressionem concinniorem reddere licuisset. Nempe cum sit

$$\frac{1}{Z} = \frac{2zz+xz+yz-xy}{xy+xz+yz}, \text{ erit}$$

$$1+\frac{1}{Z} = \frac{2z(x+y+z)}{xy+xz+yz}, \text{ ideoque}$$

$$\frac{xy+xz+yz}{x+y+z} = \frac{2Zz}{1+Z} = f:z.$$

Relicta ergo functione Z statim ponatur

$$\frac{xy+xz+yz}{x+y+z} = \Sigma = f:z,$$

et sumtis differentialibus per se liquebit, fieri $\partial\Sigma = 0$, ideoque

$\Sigma = \text{Const.}$ Adhuc facilius hoc problema resolvitur, si etiam sumto y constante ejus integrale quaeratur, tum enim simili modo pervenitur ad hujusmodi aequationem

$$\frac{xy + xz + yz}{x + y + z} = \mathrm{Y} = f : y;$$

quare cum haec expressio aeque esse debeat functio ipsius z atque ipsius y, necesse est, ut ea sit constans; eritque propterea aequatio integralis completa

$$xy + xz + yz = a\,(x + y + z).$$

Exemplum 4.

18. *Hujus aequationis differentialis realis*

$$\partial x(xx - yy + zz) - zz\partial y + z\partial z(y - x) + \frac{x\partial z}{z}(yy - xx) = 0$$

aequationem integralem completam investigare.

Realitas hujus aequationis ita ostenditur.

Ob $\mathrm{P} = xx - yy + zz$, erit $\mathrm{L} = -3z - \frac{2xy}{z}$

$\mathrm{Q} = -zz$, $\mathrm{M} = -3z + \frac{yy}{z} - \frac{3xx}{z}$

$\mathrm{R} = z(y - x) + \frac{x}{z}(yy - xx)$, $\mathrm{N} = -2y$;

unde calculo subducto formula $\mathrm{LP} + \mathrm{MQ} + \mathrm{NR}$ evanescit.

Sumamus jam z constans, et habebimus hanc aequationem

$$\partial x\,(xx - yy + zz) - zz\partial y = 0,$$

cujus quidem integratio non constaret, nisi perspiceremus ei satisfacere particulariter $y = x$. Hinc autem ponendo $y = x + \frac{zz}{v}$, integrale completum eruere poterimus; fit enim

$$\partial x\left(zz - \frac{2xzz}{v} - \frac{z^4}{vv}\right) - zz\partial x + \frac{z^4\partial v}{vv} = 0$$

hincque $\partial v - \frac{2xv\partial x}{zz} = \partial x$, quae per $e^{\frac{-xx}{zz}}$ multiplicata praebet integrale

$$e^{\frac{-xx}{zz}}v = \int e^{\frac{-xx}{zz}}\partial x + f:z;$$

ubi quidem notandum est in integratione formulae $\int e^{\frac{-xx}{zz}}\partial x$ quantitatem z ut constantem tractari, esseque $v = \frac{zz}{y-x}$: ita ut sit

$$\int e^{\frac{-xx}{zz}}\partial x = \frac{e^{\frac{-xx}{zz}}zz}{y-x} + Z.$$

Quodsi jam hanc aequationem differentiare velimus sumta etiam z variabili, difficultas hic occurrit, quomodo quantitas $\int e^{\frac{-xx}{zz}}\partial x$ differentiale ex variabilitate ipsius z oriundum definiri debeat. Hic ex principiis repeti debet, si fuerit $\partial V = S\partial x + T\partial z$, fore $\left(\frac{\partial T}{\partial x}\right) = \left(\frac{\partial S}{\partial z}\right)$, ideoque si z constans sumatur, $T = \int \partial x \left(\frac{\partial S}{\partial z}\right)$. Jam nostro casu est

$$S = e^{\frac{-xx}{zz}} \text{ et } V = \int e^{\frac{-xx}{zz}}\partial x, \text{ sumta } z \text{ constante;}$$

quare cum sit

$$\left(\frac{\partial S}{\partial z}\right) = e^{\frac{-xx}{zz}}\cdot\frac{2xx}{z^3}, \text{ erit } T = \frac{2}{z^3}\int e^{\frac{-xx}{zz}}xx\partial x.$$

Quocirca quantitatis $\int e^{\frac{-xx}{zz}}\partial x$ differentiale plenum ex variabilitate utriusque x et z oriundum est

$$e^{\frac{-xx}{zz}}\partial x + \frac{2\partial z}{z^3}\int e^{\frac{-xx}{zz}}xx\partial x,$$

cui aequari debet alterius partis $\frac{e^{\frac{-xx}{zz}}zz}{y-x} + Z$ differentiale, quod est

$$e^{\frac{-xx}{zz}}\left(\frac{2z\partial z}{y-x} - \frac{zz\partial y + zz\partial x}{(y-x)^2} + \frac{2xx\partial z - 2xz\partial x}{z(y-x)}\right) + \partial Z.$$

Turbat vero adhuc formula integralis $\int e^{\frac{-xx}{zz}} xx\partial x$, in qua z pro constante habetur: reduci autem potest ad priorem $\int e^{\frac{-xx}{zz}}\partial x$, si ponatur

$$\int e^{\frac{-xx}{zz}} xx\partial x = A e^{\frac{-xx}{zz}} x + B\int e^{\frac{-xx}{zz}}\partial x,$$

prodit enim sola x pro variabili habita, differentiando

$$xx\partial x = A\partial x - \frac{2Axx\partial x}{zz} + B\partial x;\ \text{ergo}$$

$$A = -\tfrac{1}{2}zz,\ \text{et}\ B = -A = \tfrac{1}{2}zz,$$

ita ut sit

$$\int e^{\frac{-xx}{zz}} xx\partial x = -\tfrac{1}{2}e^{\frac{-xx}{zz}} xzz + \tfrac{1}{2}zz\int e^{\frac{-xx}{zz}}\partial x:$$

quare cum sit:

$$\int e^{\frac{-xx}{zz}}\partial x = \frac{e^{\frac{-xx}{zz}} zz}{y-x} + Z,\ \text{erit}$$

$$\int e^{\frac{-xx}{zz}} xx\partial x = -\tfrac{1}{2}e^{\frac{-xx}{zz}} xzz + \frac{e^{\frac{-xx}{zz}} z^4}{2(y-x)} + \tfrac{1}{2}Zzz.$$

Facta ergo substitutione haec orietur aequatio differentialis

$$e^{\frac{-xx}{zz}}\left(\partial x - \frac{x\partial z}{z} + \frac{z\partial z}{y-x}\right) + \frac{Z\partial z}{z} =$$

$$e^{\frac{-xx}{zz}}\left(\frac{2z\partial z}{y-x} - \frac{zz\partial y}{(y-x)^2} + \frac{zz\partial x}{(y-x)^2} - \frac{2x\partial x}{y-x} + \frac{2xx\partial z}{z(y-x)}\right) + \partial Z,$$

quae transit in hanc formam.

$$e^{\frac{-xx}{zz}}\left(\frac{\partial x\,(y+x)}{y-x} - \frac{zz\partial x}{(y-x)^2} + \frac{zz\partial y}{(y-x)^2} - \frac{z\partial z}{y-x} - \frac{x(y+x)\partial z}{z(y-x)}\right) = \frac{z\partial Z - Z\partial z}{z}$$

seu

$$\frac{e^{\frac{-xx}{zz}}}{(y-x)^2}\left[\partial x\,(yy-xx-zz) + zz\partial y - z\partial z(y-x) - \frac{x\partial z}{z}(yy-xx)\right] = \frac{z\partial Z - Z\partial z}{z}.$$

qua cum proposita collata evidens est, esse debere

seu $z\partial Z - Z\partial z = 0$ seu $Z = nz$;

ita ut aequationis propositae integrale completum sit

$$\int e^{\frac{-xx}{zz}}\partial x = \frac{e^{\frac{-xx}{zz}}\,zz}{y-x} + nz,$$

siquidem in integrali $\int e^{\frac{-xx}{zz}}\partial x$ quantitas z pro constante habeatur.

Corollarium.

19. Aequatio ergo proposita integrabilis redditur, si multiplicetur per $\frac{1}{(y-x)^2}e^{\frac{-xx}{zz}}$; ac tum integrale est ipsa aequatio, quam invenimus.

Scholion 1.

20. Exemplum hoc imprimis est notatu dignum, quod in ejus solutione quaedam artificia sunt in subsidium vocata, quibus in praecedentibus non erat opus. Per formulam autem $\int e^{\frac{-xx}{zz}}\partial x$ integrale non satis determinatum videtur. Cum enim in ea z constans ponatur, constans per integrationem introducenda per nz non definitur, siquidem lex non praescribitur, secundum quam integrale $\int e^{\frac{-xx}{zz}}\partial x$ capi oporteat, utrum ita ut evanescat facto $x = 0$, an alio quocunque modo? Dubium autem hoc diluetur, si aequationem inventam per z dividamus, ut formula integralis sit $\int e^{\frac{-xx}{zz}}\frac{\partial x}{z}$; ubi cum $\frac{\partial x}{z}$ sit $\partial\cdot\frac{x}{z}$, evidens est ea exprimi functionem quandam ipsius $\frac{x}{z}$; ac si ponatur $\frac{x}{z} = p$, fore aequationem nostram integralem

$$\int e^{-pp}\partial p + \text{Const.} = e^{-pp}\frac{z}{y-x}$$

neque hic amplius conditio illa, qua in formula integrali quantitas z pro constante sit habenda, locum habet, sed integrale perinde determinatur, ac si aequatio duas tantum variabiles contineret. Hanc circumstantiam si perpendissemus, plenum differentiale formulae $\int e^{\frac{-xx}{zz}} \partial x$, ex variabilitate utriusque x et z nullam difficultatem peperisset. Postquam enim pervenimus ad aequationem

$$\int e^{\frac{-xx}{zz}} \partial x = e^{\frac{-xx}{zz}} \cdot \frac{zz}{y-z} + \mathrm{f} : z,$$

eam ita repraesentemus

$$\int e^{\frac{-xx}{zz}} \cdot \frac{\partial x}{z} = \int e^{\frac{-xx}{zz}} \partial \cdot \frac{x}{z} = e^{\frac{-xx}{zz}} \cdot \frac{z}{y-z} + Z,$$

ubi cum in formulam integralem etiam variabilitas ipsius z sit inducta, si ea differentietur sumtis omnibus x, y et z variabilibus orietur

$$e^{\frac{-xx}{zz}} \left(\frac{\partial x}{z} - \frac{x \partial z}{zz}\right) = e^{\frac{-xx}{zz}} \left(\frac{\partial z}{y-z} + \frac{z\partial x - z\partial y}{(y-x)^2} - \frac{2x\partial x}{z(y-x)} + \frac{2xx\partial z}{zz(y-x)}\right) + \partial Z$$

seu

$$e^{\frac{-xx}{zz}} \left(\frac{\partial x(y+x)}{z(y-x)} - \frac{z\partial x}{(y-x)^2} + \frac{z\partial y}{(y-x)^2} - \frac{x\partial z(y+x)}{zz(y-x)} - \frac{\partial z}{y-x}\right) = \partial Z$$

quae reducitur ad hanc formam

$$\frac{e^{\frac{-xx}{zz}}}{z(y-x)^2} \left[\partial x(yy-xx-zz) + zz\partial y - z\partial z(y-x) - \frac{x\partial z}{z}(yy-xx)\right] = \partial Z;$$

unde patet esse debere $\partial Z = 0$ et $Z = \mathrm{Const.}$ sicque elicitur aequatio integralis ante inventa.

Scholion 2.

21. Idem integrale prodiisset, si loco z altera reliquarum x vel y pro constante fuisset assumta; ubi in genere notari convenit, si hujusmodi aequationem

$$P\partial x + Q\partial y + R\partial z = 0$$

sumta z constante tractare licuerit, etiam resolutionem, quaecunque trium variabilium pro constante assumatur, succedere debere, etiamsi id quandoque minus perspiciatur. Ita in aequatione proposita si y pro constante habeatur, resolvenda erit haec aequatio

$$\partial x\,(xx + zz - yy) - z\partial z\,(x - y) - \frac{x\partial z}{z}(xx - yy) = 0,$$

quae per z multiplicata cum in hanc formam abeat

$$(z\partial x - x\partial z)(xx + zz - yy) + yzz\partial z = 0,$$

facile patet, eam simpliciorem reddi ponendo $x = pz$, tum enim ob

$$z\partial x - x\partial z = zz\partial p$$

prodibit

$$\partial p\,(ppzz + zz - yy) + y\partial z = 0;$$

sit porro $z = qy$, fietque

$$\partial p\,(ppqq + qq - 1) + \partial q = 0,$$

cui cum satisfaciat $q = \frac{1}{p}$, statuatur $q = \frac{1}{p} + \frac{1}{r}$, habebiturque

$$\partial p\left(\frac{2p}{r} + \frac{pp}{rr} + \frac{1}{pp} + \frac{2}{pr} + \frac{1}{rr}\right) - \frac{\partial p}{pp} - \frac{\partial r}{rr} = 0, \quad \text{seu}$$

$$\partial p\,(2ppr + p^3 + 2r + p) - p\partial r = 0, \quad \text{vel}$$

$$\partial r - \frac{2r\partial p\,(pp+1)}{p} = \partial p\,(pp + 1),$$

quae multiplicata per $\frac{1}{pp}e^{-pp}$ et integrata dat

$$e^{-pp}\frac{r}{pp} = \int e^{-pp}\,.\,\frac{\partial p\,(1 + pp)}{pp}.$$

$$\text{At}\ \int e^{-pp}\frac{\partial p}{pp} = -\,e^{-pp}\frac{1}{p} - 2\int e^{-pp}\,\partial p,$$

$$\text{unde}\ e^{-pp}\left(\frac{r}{pp} + \frac{1}{p}\right) = -\int e^{-pp}\,\partial p.$$

Cum nunc sit

$$p = \frac{x}{z} \quad \text{et} \quad \frac{1}{r} = \frac{z}{y} - \frac{z}{x} = \frac{z\,(x - y)}{xy}, \quad \text{erit}$$

$$r = \frac{xy}{z(x-y)},\ \frac{r}{pp} = \frac{yz}{x(x-y)} \text{ et } \frac{r}{pp} + \frac{1}{p} = \frac{z}{x-y}.$$

Unde aequatio nostra integralis erit

$$\int e^{\frac{-xx}{zz}}\, \partial . \frac{x}{z} = e^{\frac{-xx}{zz}} . \frac{z}{y-x} + \mathrm{f} : y,$$

cujus differentiale, si etiam y pro variabili habeatur, cum aequatione proposita comparatum, dabit ut ante $\mathrm{f} : y = \text{Const.}$

Caeterum cum in his exemplis variabiles x, y, z ubique eundem dimensionem numerum impleant, methodum generalem hujusmodi aequationes tractandi exponam.

Problema 3.

22. Si in aequatione differentiali

$$P\,\partial x + Q\,\partial y + R\,\partial z = 0$$

functiones P, Q, R fuerint homogeneae ipsarum x, y et z ejusdem numeri dimensionum; ejus integrationem, si quidem fuerit realis, investigare.

Solutio.

Sit n numerus dimensionum, quas ternae varibiles x, y et z in functionibus P, Q, R constituunt; ac posito $x = pz$ et $y = qz$, fiet

$$P = z^n S,\ Q = z^n T \text{ et } R = z^n V,$$

ita ut jam S, T, V, futurae sint functiones binarum tantum variabilium p et q. Cum jam sit

$$\partial x = p\,\partial z + z\,\partial p \text{ et } \partial y = q\,\partial z + z\,\partial q,$$

aequatio nostra hanc induet formam

$$\partial z(pS + qT + V) + Sz\,\partial p + Tz\,\partial q = 0, \text{ seu}$$

$$\frac{\partial z}{z} + \frac{S\,\partial p + T\,\partial q}{pS + qT + V} = 0$$

quae aequatio realis esse nequit, nisi formula differentialis binas variabiles p et q involvens $\frac{S\partial p + T\partial q}{pS + qT + V}$ per se fuerit integrabilis; quod eveniet si fuerit

$$(qT + V)\left(\frac{\partial S}{\partial q}\right) + pT\left(\frac{\partial S}{\partial p}\right) - (pS + V)\left(\frac{\partial T}{\partial p}\right) - qS\left(\frac{\partial T}{\partial q}\right) - S\left(\frac{\partial V}{\partial q}\right) + T\left(\frac{\partial V}{\partial p}\right) = 0.$$

Quoties ergo hic character locum habet, nostra aequatio erit realis, ejusque integrale erit

$$lz + \int \frac{S\partial p + T\partial q}{pS + qT + V} = \text{Const:}$$

ubi tantum opus est, ut loco litterarum p et q valores assumti $\frac{x}{z}$ et $\frac{y}{z}$ restituantur.

Corollarium 1.

23. Ita in nostro primo exemplo (§. 12.) cum sit

$P = y + z$, $Q = x + z$, $R = x + y$, erit

$S = q + 1$, $T = p + 1$, $V = p + q$ et

$$\frac{\partial z}{z} + \frac{(q+1)\partial p + (p+1)\partial q}{2pq + 2p + 2q} = 0;$$

cujus integrale est

$$lz + \tfrac{1}{2}l(pq + p + q) = \tfrac{1}{2}l(xy + xz + yz) = C, \text{ seu}$$
$$xy + xz + yz = C.$$

Corollarium 2.

24. In secundo exemplo (§. 13.) est

$P = ay - bz$, $Q = cz - ax$, $R = bx - cy$, hinc

$S = aq - b$, $T = c - ap$, $V = bp - cq$;

Ergo $\frac{\partial z}{z} + \frac{(aq - b)\partial p + (c - ap)\partial q}{0} = 0$;

hincque

$$(aq - b)\partial p + (c - ap)\partial q = 0,$$

et integrando

$$l\,\frac{aq-b}{c-ap} = l\,\frac{ay-bz}{cz-ax} = \text{C}.$$

Corollarium 3.

25. In tertio exemplo (§. 14.) fit

$$\text{S} = qq+q+1,\ \text{T} = pp+p+1,\ \text{et}\ \text{V} = pp+pq+qq,$$

hincque

$$\frac{\partial z}{z} + \frac{\partial p\,(qq+q+1) + \partial q\,(pp+p+1)}{ppq+pqq+pp+3pq+qq+p+q} = 0,$$

qui denominator est $= (p+q+1)(pq+p+q)$, unde haec fractio resolvitur in has duas

$$\frac{-\partial p - \partial q}{p+q+1} + \frac{\partial p\,(q+1) + \partial q\,(p+1)}{pq+p+q}:$$

ex quo integrale a logarithmis ad numeros perductum oritur

$$\frac{z\,(pq+p+q)}{p+q+1} = \frac{xy+xz+yz}{x+y+z} = \text{C}.$$

Corollarium 4.

26. In exemplo quarto (§. 18.) fit

$$\text{S} = pp-qq+1,\ \text{T} = -1,\ \text{V} = q-p+p\,(qq-pp),$$

hincque

$$\frac{\partial z}{z} + \frac{\partial p\,(pp-qq+1) - \partial q}{0} = 0,$$

ideoque

$$\partial q = \partial p\,(pp-qq+1).$$

Cum ergo satisfaciat $q = p$, ponatur $q = p + \frac{1}{r}$, fiet

$$\partial r - 2pr\partial p = \partial p;\ \text{et integrando}$$

$$e^{-pp}\,r = \int e^{-pp}\,\partial p = e^{-pp} \cdot \frac{1}{q-p},$$

ita ut integrale sit

$$e^{\frac{-xx}{zz}} \cdot \frac{z}{y-x} = \int e^{\frac{-xx}{zz}}\,\partial\,.\,\frac{x}{z} + \text{Const.}$$

Scholion.

27. Cum igitur aequationes differentiales tres variabiles involventes nullam habeant difficultatem sibi propriam, quoniam earum resolutio, siquidem fuerint reales, semper ad aequationes differentiales duarum variabilium reduci potest; hoc argumentum fusius non prosequor. Quod enim ad ejusmodi aequationes differentiales trium variabilium attinet, in quibus ipsa differentialia ad plures dimensiones ascendunt, veluti est

$$P\partial x^2 + Q\partial y^2 + R\partial z^2 + 2S\partial x\partial y + 2T\partial x\partial z + 2V\partial y\partial z = 0,$$

de iis generatim tenendum est, nisi per radicis extractionem ad formam

$$P\partial x + Q\partial y + R\partial z = 0,$$

reduci queant, eas semper esse absurdas. Quomodocunque enim aequatio integralis esset comparata, ex ea valor ipsius z ita definiri posset, ut z aequetur functioni binarum variabilium x et y, unde foret $\partial z = p\partial x + q\partial y$; neque hae variabiles x et y ullo modo a se penderent. Hic ergo valor $p\partial x + q\partial y$ loco ∂z in aequatione differentiali substitutus, ita satisfacere deberet, ut omnes termini se mutuo destruerent, quod autem fieri non posset, si ex aequationis resolutione ∂z ita definiretur, ut differentialia ∂x et ∂y signis radicalibus essent involuta. Hinc aequatio illa exempli loco allata, cum per resolutionem det

$$\partial z = \frac{-T\partial x - V\partial y \pm \sqrt{[(TT-PR)\partial x^2 + (TV-RS)\partial x\partial y + (VV-QR)\partial y^2]}}{R},$$

realis esse nequit, nisi radix extrahi queat, hoc est nisi ipsa aequatio in factores formae

$$P\partial x + Q\partial y + R\partial z,$$

resolvi possit. Atque etiamsi hoc eveniat, et hi factores nihilo aequales statuantur, tamen aequatio non erit realis, nisi criterium supra traditum locum habeat. Ex his perspicuum est, ne ejusmodi quidem aequationes, quae quatuor pluresve variabiles involvant, plus difficultatis habere.

Problema 4.

28. Si V sit functio quaecunque binarum variabilium x et y, in formula autem integrali $\int V\partial x$ quantitas y pro constante sit habita, definire hujus formae $\int V\partial x$ differentiale, si praeter x etiam y variabilis assumatur.

Solutio.

Ponatur ista formula integralis $\int V\partial x = Z$, eritque Z utique functio ambarum variabilium x et y, etiamsi in ipsa integratione y pro constante habeatur. Evidens autem est, si vicissim in differentiatione y constans sumatur, fore $\partial Z = V\partial x$. Quare si etiam y variabilis statuatur, differentiale ipsius $Z = \int V\partial x$ hujusmodi habebit formam

$$\partial Z = V\partial x + Q\partial y,$$

et quaestio huc redit, ut ista quantitas Q determinetur. Quia autem forma $V\partial x + Q\partial y$ est verum differentiale, necesse est sit $\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right)$; hincque $\partial x\left(\frac{\partial Q}{\partial x}\right) = \partial x\left(\frac{\partial V}{\partial y}\right)$. At $\partial x\left(\frac{\partial Q}{\partial x}\right)$ est differentiale ipsius Q, si y pro constante habeatur; unde Q reperietur si formula $\partial x\left(\frac{\partial V}{\partial y}\right)$ ita integretur, ut y tanquam constans tractetur, seu erit $Q = \int \partial x\left(\frac{\partial V}{\partial y}\right)$. Quocirca formulae $Z = \int V\partial x$ differentiale ex variabilitate utriusque x et y oriundum erit

$$\partial Z = V\partial x + \partial y \cdot \int \partial x\left(\frac{\partial V}{\partial y}\right).$$

Corollarium 1.

29. Quoniam V est functio ipsarum x et y, si ponatur $\partial V = R\partial x + S\partial y$, erit $S = \left(\frac{\partial V}{\partial y}\right)$; unde fit

$$\partial Z = \partial \cdot \int V\partial x = V\partial x + \partial y \int S\partial x,$$

scilicet in formulae $\int S\partial x$ integratione, perinde ac formulae $\int V\partial x$ sola quantitas x pro variabili est habenda.

Corollarium 2.

30. Si V fuerit functio homogenea ipsarum x et y existe numero dimensionum $= n$, posito $\partial V = R\partial x + S\partial y$, erit $Rx + Sy = nV$, ideoque $S = \frac{nV}{y} - \frac{Rx}{y}$, hinc

$$\int S\partial x = \frac{n}{y}\int V\partial x - \frac{1}{y}\int Rx\partial x.$$

At ob y constans est $R\partial x = \partial V$, hinc

$$\int Rx\partial x = \int x\partial V = Vx - \int V\partial x, \text{ ideoque}$$

$$\int S\partial x = \frac{n+1}{y}\int V\partial x - \frac{Vx}{y}, \text{ et}$$

$$\partial Z = \partial . \int V\partial x = V\partial x - \frac{Vx\partial y}{y} + \frac{(n+1)\partial y}{y}\int V\partial x.$$

Corollarium 3.

31. Idem facilius invenitur ex consideratione quod functio $Z = \int V\partial x$ futura sit homogenea $n+1$ dimensionum, quare posito $\partial Z = V\partial x + Q\partial y$, erit $Vx + Qy = (n+1)Z$; ideoque $Q = \frac{(n+1)Z}{y} - \frac{Vx}{y}$, ut ante.

Scholion.

32. Problemate jam ante, et in praecedente quidem libro, sum usus, neque tamen abs re fore putavi, si id data opera hic tractarem, quandoquidem hic liber in functionibus binarum pluriumve variabilium occupatur. Praecipuum autem negotium non in ejusmodi aequationibus differentialibus, quales in hoc capite integrare docui, versatur, quod quidem brevi esset absolum, sed cum differentiatio functionis binarum variabilium x et y duplices formulas $\left(\frac{\partial V}{\partial x}\right)$ et $\left(\frac{\partial V}{\partial y}\right)$ suppeditet, existente V hujusmodi functione, hoc loco ejusmodi quaestiones potissimum contemplabimur, quibus talis functio

binarum variabilium versantur, sint x et y binae variabiles, et z earum functio et data quadam differentialium relatione definienda, ita ut aequatio finita inter x, y et z requiratur. Ponamus autem $\partial z = p\partial x + q\partial y$, ita ut sit recepto signandi modo $p = \left(\frac{\partial z}{\partial x}\right)$ et $q = \left(\frac{\partial z}{\partial y}\right)$, atque ideo p et q sint formulae differentiales, quae in relationem propositam ingrediantur. In genere ergo relatio ista erit aequatio quaecunque inter quantitates p, q, x, y et z proposita, atque haec sectio perfecte absolveretur, si methodus constaret, ex data aequatione quacunque inter has quantitates p, q, x, y et z eruendi aequationem inter x, y et z; quod autem cum in genere ne pro functionibus quidem unicae variabilis praestari possit, multo minus hic est expectandum, ex quo eos casus tantum evolvi conveniet, qui resolutionem admittant. Primo autem resolutio succedit, si in aequatione proposita altera formularum differentialium p vel q plane desit, ita ut aequatio vel inter p, x, y et z vel inter q, x, y et z proponatur. Deinde aequationes, quae solas binas formulas differentiales p et q continent, ita ut altera debeat esse functio quaecunque alterius, commode resolvere licet. Tum igitur sequentur aequationes, quae praeter p et q unicam quantitatem finitarum x vel y vel z complectantur, ex quo genere cujusmodi casus resolvi queant videamus. Ordo porro postulat, ut ad aequationes, quae praeter binas formulas differentiales p et q insuper binas quantitatum finitarum, vel x et y, vel x et z, vel y et z, involvunt, progrediamur; ac denique de resolutione aequationum omnes litteras p, q, x, y et z implicantium, agemus, subsidia transformationis deinceps exposituri.

V ex data quacunque relatione harum duarum formularum $\left(\frac{\partial V}{\partial x}\right)$ et $\left(\frac{\partial V}{\partial y}\right)$ est difinienda. Relatio autem haec per aequationem inter istas formulas et binas variabiles x et y, quam etiam ipsa functio quaesita V ingredi potest, exprimitur, ex cujus aequationis indole divisio tractationis erit petenda. Problema scilicet generale, in quo solvendo ista sectio est occupata, ita se habet, ut ea binarum variabilium x et y functio V inveniatur, quae satisfaciat aequationi cuicunque inter quantitates x, y, V, $\left(\frac{\partial V}{\partial x}\right)$ et $\left(\frac{\partial V}{\partial y}\right)$ propositae. Quodsi in hanc aequationem altera tantum binarum formularum differentialium $\left(\frac{\partial V}{\partial x}\right)$ vel $\left(\frac{\partial V}{\partial y}\right)$ ingrediatur, resolutio non est difficilis, atque ad casum aequationum differentialium duas tantum variabiles involventium reducitur; quando autem ambae istae formulae in aequatione proposita insunt, quaestio multo magis est ardua ac saepenumero ne resolvi quidem potest, etiamsi resolutio aequationum differentialium duas tantum variabiles complectentium admittatur: in hoc enim negotio, quoties resolutionem ad integrationem aequationum differentialium inter duas variabiles reducere licet, problema pro resoluto erit habendum. Cum igitur ex aequatione proposita formula $\left(\frac{\partial V}{\partial y}\right)$ aequetur functioni utcunque ex quantitatibus x, y, V et $\left(\frac{\partial V}{\partial x}\right)$ conflatae, ex indole hujus functionis, prout fuerit simplicior, et vel solam formulam $\left(\frac{\partial V}{\partial x}\right)$, vel praeter eam unicam ex reliquis, vel etiam binas, vel adeo omnes comprehendat, tractationem sequentem distribuemus. Hoc enim ordine servato facillime apparebit, quantum adhuc praestare liceat, et quantum adhuc desideretur. Praeterea vero nonnulla subsidia circa transformationem binarum formularum differentialium ad alias variabiles exponenda occurrent.

Divisio hujus Sectionis.

Quo partes, quas in hac sectione pertractari conveniet, clarius conspectui exponantur, quoniam hae quaestiones circa functiones versantur, sint x et y binae variabiles, et z binarum variabilium earum functio et data quadam differentialium relatione definienda, ita ut aequatio finita inter x, y et z requiratur. Ponamus autem $\partial z = p\partial x + q\partial y$, ita ut sit recepto signandi modo $p = \left(\frac{\partial z}{\partial x}\right)$ et $q = \left(\frac{\partial z}{\partial y}\right)$, atque ideo p et q sint formulae differentiales, quae in relationem propositam ingrediantur. In genere ergo relatio ista erit aequatio quaecunque inter quantitates p, q, x, y et z proposita, atque haec sectio perfecte absolveretur, si methodus constaret, ex data aequatione quacunque inter has quantitates p, q, x, y et z eruendi aequationem inter x, y et z; quod autem cum in genere ne pro functionibus quidem unicae variabilis praestari possit, multo minus hic est expectandum, ex quo eos casus tantum evolvi conveniet, qui resolutionem admittant. Primo autem resolutio succedit, si in aequatione proposita altera formularum differentialium p vel q plane desit, ita ut aequatio vel inter p, x, y et z vel inter q, x, y et z proponatur. Deinde aequationes, quae solas binas formulas differentiales p et q continent, ita ut altera debeat esse functio quaecunque alterius, commode resolvere licet. Tum igitur sequentur aequationes, quae praeter p et q unicam quantitatem finitarum x vel y vel z complectantur, ex quo genere cujusmodi casus resolvi queant videamus. Ordo porro postulat, ut ad aequationes, quae praeter binas formulas differentiales p et q insuper binas quantitatum finitarum, vel x et y, vel x et z, vel y et z, involvunt, progrediamur; ac denique de resolutione aequationum omnes litteras p, q, x, y et z implicantium, agemus, subsidia transformationis deinceps expondituri.