CAPUT VII.

DE VARIATIONE FORMULARUM INTEGRALIUM TRES VARIABILES INVOLVENTIUM, QUARUM UNA UT FUNCTIO BINARUM RELIQUARUM SPECTATUR.

Problema 19.

159.

Formularum integralium huc pertinentium naturam evolvere, ac rationem qua earum variationes investigari conveniat, explicare.

Solutio.

Cum tres habeantur variabiles x, y et z, quarum una z ut functio binarum reliquarum x et y est spectanda, etiamsi in ipsa variationis investigatione ratio hujus functionis pro incognita haberi debet, formulae integrales quae in hoc calculi genere occurrunt, plurimum discrepant ab iis, quae circa binas tantum variabiles proponi solent. Quemadmodum enim talis forma integralis $\int V\partial x$, ubi V duas variabiles x et y implicare censetur, quarum y ab x pendere concipitur, quasi summa omnium valorum elementarium $V\partial x$ per omnes valores ipsius x collectorum considerari potest; ita quando tres variabiles x, y et z habentur, quarum haec z a binis x et y simul pendere concipitur, integralia huc pertinentia collectionem omnium elementorum ad omnes valores tam ipsius x, quam ipsius y relatorum involvunt, ideoque duplicem integrationem alteram per omnes valores ipsius x, alteram vero ipsius y elementa congregantem requirunt. Ex quo hujusmodi integralia tali forma $\int\int V\partial x\partial y$ contineri debent, qua scilicet duplex integratio innuatur; cujus evolutio ita institui solet, ut primo altera variabilis y ut constans spectetur, et formulae $\int V\partial x$ valor per terminos integrationis extensus quaeratur; in quo cum jam x obtineat valorem vel datum vel ab y pendentem, hoc integrale $\int V\partial x$ in functionem ipsius y tantum abibit, qua in ∂y ducta superest ut integrale $\int\partial y\int V\partial x$ investigetur, quae ergo forma $\int\partial y\int\partial V\delta x$ hoc modo tractata illi $\int\int V\partial x\partial y$ aequivalere est censenda. Ac si ordine inverso primo quantitas x constans accipiatur, et integrale $\int V\partial y$ per terminos praescriptos extendatur, id deinceps ut functio ipsius x spectari et integrale quaesitum $\int\partial x\int V\partial y$ inveniri poterit. Perinde autem est utro modo valorem integralis formulae duplicatae $\int\int V\partial x\partial y$ utamur.

Cum igitur in hoc genere aliae formulae integrales nisi hujusmodi $\int\int V\partial x\partial y$ occurrere nequeant, totum negotium huc redit, ut quemadmodum hujusmodi formae variationem inveniri oporteat, ostendamus. Quoniam autem quantitates x et y variationis expertes assumimus, ex iis quae initio sunt demonstrata facile colligitur fore

$$\delta\int\int V\partial x\partial y = \int\int\delta V\partial x\partial y,$$

ubi δV variationem ipsius V denotat; hicque integratione pariter duplici est opus, prorsus ut modo ante innuimus.

Corollarium 1.

160. Si ponamus integrale $\int\int V\partial x\partial y = W$, cum sit $\int\partial\int x V\partial y = W$, erit per solam x differentiando

$$\int V\partial y = \left(\frac{\partial W}{\partial x}\right),$$

hincque porro per y differentiando $V = (\frac{\partial\partial W}{\partial x \partial y})$; unde patet integrale W ita comparatum esse, ut fiat $V = (\frac{\partial\partial W}{\partial x \partial y})$.

Corollarium 2.

161. Cum duplex integratio sit instituenda, utraque quantitas arbitraria introducitur; altera autem integratio loco constantis functionem quamcunque ipsius x quae sit X, altera autem functionem quamcunque ipsius y, quae sit Y invehit, ita ut completum integrale sit

$$\int\int V \partial x \partial y = W + X + Y.$$

Corollarium 3.

162. Hoc etiam per ipsam resolutionem confirmatur, fit enim primo

$$\int V \partial y = (\frac{\partial W}{\partial x}) + (\frac{\partial X}{\partial x}), \text{ ob } (\frac{\partial Y}{\partial x}) = 0.$$

Tum vero fit $V = (\frac{\partial\partial W}{\partial x \partial y})$, quia neque X neque $\frac{\partial X}{\partial x}$ ab y pendet. Quare si fuerit $(\frac{\partial\partial W}{\partial x \partial y}) = V$, erit integrale completum

$$\int\int V \partial x \partial y = W + X + Y.$$

Scholion 1.

163. Omnino autem necessarium est, ut indoles hujusmodi formularum integralium duplicatarum $\int\int V \partial x \partial y$ accuratius examini subjicietur, quod commodissime per Theoriam superficierum praestari poterit. Sint ergo ut hactenus x et y binae coordinatae or-
Fig. 7. thogonales in basi assumtae, $AX = x$, $XY = y$, cui in Y normaliter insistat tertia ordinata $YZ = z$ ad superficiem usque porrecta. Si jam binae illae coordinatae x et y suis differentialibus crescant $XX' = \partial x$ et $YY' = \partial y$, inde basi oritur parallelogrammum elementare $YxyY' = \partial x \partial y$, cui elementum formulae integralis conve-

nit. Ita si de soliditate a superficie inclusa sit quaestio, ejus elementum erit $= z\partial x\partial y$, ideoque tota soliditas $= \int\int z\partial x\partial y$; si superficies ipsa quaeratur, posito $\partial z = p\partial x + p'\partial y$, erit ejus elementum huic rectangulo $\partial x\partial y$ imminens

$$= \partial x\partial y \surd (1 + pp + p'p'),$$

ideoque ipsa superficies

$$= \int\int \partial x\partial y \surd (1 + pp + p'p'),$$

ex quo generatim intelligitur ratio formulae integralis duplicatae $\int\int V\partial x\partial y$. Quod si jam talis formulae valor quaeratur, qui dato spatio in basi veluti ADYX respondeat, primo sumta x constante investigetur integrale simplex $\int V\partial y$, ac tum ipsi y assignetur magnitudo XY ad curvam DY porrecta, quae ex hujus curvae natura aequabitur certae functioni ipsius x. Sic igitur $\partial x\int V\partial y$ exprimet formulae propositae elementum rectangulo $XYxX' = y\partial x$ conveniens, cujus integrale denuo sumtum $\int\partial x\int V\partial y$ et ex sola variabili x constans, tandem dabit valorem toti spatio ADYX respondentem, siquidem utraque integratio adjectione constantis rite determinetur.

Scholion 2.

164. Ita se habere debet evolutio hujusmodi formularum integralium duplicatarum, si ad figuram in basi datam veluti ADYX fuerit accommodanda; sin autem utramque integrationem indefinite expedire velimus, ut primo sumta x constante quaeramus integrale $\int V\partial y$, quod rectangulo elementari $XYyX' = y\partial x$ convenire est intelligendum, siquidem in ∂x ducatur, deinde vero in integratione formulae $\int\partial x\int V\partial y$ quantitatem $y = XY$ eandem manere concipiamus, sola x pro variabili sumta, tum valor prodibit rectangulo indefinito $APYX = xy$ respondens, si quidem constantes per

utramque integrationem ingressae debite definiantur. At si spatii istius reliqui termini praeter lineas XY et PY ut indefiniti spectentur, integrale $\int\int V\partial x\partial y$ recipiet binas functiones X + Y indefinitas, illam ipsius x, hanc vero ipsius y. Quodsi ergo ad calculum maximorum et minimorum haec deinceps accommodare velimus, quoniam maximi minimive proprietas, quae in spatium quodpiam datum ADYX competere debet, simulquoque cuivis spatio indefinito veluti APYX conveniat necesse est, duplicem illam integrationem modo hic exposito indefinite administrari conveniet.

Problema 20.

165. Si V sit formula quaecunque ex ternis variabilibus x, y, z earumque differentialibus composita, invenire variationem formulae integralis duplicatae $\int\int V\partial x\partial y$, dum quantitati z, quae ut functio binarum x et y spectetur, variationes quaecunque tribuuntur.

Solutio.

Ad speciem differentialium tollendam statuamus

$$p = \left(\frac{\partial z}{\partial x}\right),\quad p' = \left(\frac{\partial z}{\partial y}\right),$$

$$q = \left(\frac{\partial p}{\partial x}\right),\quad q' = \left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right),\quad q'' = \left(\frac{\partial p'}{\partial y}\right),$$

$$r = \left(\frac{\partial q}{\partial x}\right),\quad r' = \left(\frac{\partial q}{\partial y}\right) = \left(\frac{\partial q'}{\partial x}\right),\quad r'' = \left(\frac{\partial q'}{\partial x}\right) = \left(\frac{\partial q''}{\partial x}\right),\quad r''' = \left(\frac{\partial q''}{\partial y}\right),$$

ut V fiat functio quantitatum finitarum x, y, z, p, p', q, q', q'', r, r', r'', r''', etc. Tum ponatur ejus differentiale

$$\begin{aligned}\partial V = L\partial x + M\partial y + N\partial z + P\,\partial p &+ Q\,\partial q + R\,\partial r\\ P'\partial p' &+ Q'\,\partial q' + R'\,\partial r'\\ &+ Q''\partial q'' + R''\partial r''\\ &\qquad\qquad + R'''\partial r'''\end{aligned}$$

etc.

ex quo cum simul ejus variatio δV innotescat, ex problemate praecedente colligitur variatio quaesita

$$\delta \iint V \partial x \partial y = \iint \partial x \partial y \left\{ \begin{array}{l} N\delta z + P\,\delta p + Q\,\delta q + R\,\delta r + \text{etc.} \\ \qquad + P'\delta p' + Q'\,\delta q' + R'\,\delta r' \\ \qquad\qquad + Q''\delta q'' + R''\,\delta r'' \\ \qquad\qquad\qquad + R'''\delta r''' \\ \qquad\qquad \text{etc.} \end{array} \right.$$

Quodsi jam uti §. 154. fecimus, ponamus variationem $\delta z = \omega$, quam ut functionem quamcunque binarum variabilium x et y spectare licet, indidem istam variationem concludimus fore

$$\delta \iint V \partial x \partial y = \iint \partial x \partial y \left\{ \begin{array}{l} N\omega + P\left(\frac{\partial \omega}{\partial x}\right) + Q\left(\frac{\partial\partial \omega}{\partial x^2}\right) + R\left(\frac{\partial^3 \omega}{\partial x^3}\right) + \text{etc.} \\ \qquad + P'\left(\frac{\partial \omega}{\partial y}\right) + Q'\left(\frac{\partial\partial \omega}{\partial x \partial y}\right) + R'\left(\frac{\partial^3 \omega}{\partial x^2 \partial y}\right) \\ \qquad\qquad + Q''\left(\frac{\partial\partial \omega}{\partial y^2}\right) + R''\left(\frac{\partial^3 \omega}{\partial x \partial y^2}\right) \\ \qquad\qquad\qquad + R'''\left(\frac{\partial^3 \omega}{\partial y^3}\right) \\ \qquad\qquad \text{etc.} \end{array} \right.$$

Corollarium 1.

166. Si ergo utriusque functionis z et $\delta z = \omega$ indoles, seu ratio compositionis ex binis variabilibus x et y esset data, tum per praecepta ante exposita variatio formulae integralis duplicatae $\iint V \partial x \partial y$ assignari posset; quomodocunque quantitas V ex variabilibus x, y, z earumque differentialibus fuerit conflata.

Corollarium 2.

167. Totum scilicet negotium redibit ad evolutionem formulae integralis duplicatae inventae, quae cum pluribus constet partibus, singulas partes per duplicem integrationem, uti ante explicatum, tractari conveniet.

Scholion.

168. Quando autem ratio functionis z non constat, eaque demum ex conditione variationis elici debet, ita ut ipsa variatio $\delta z = \omega$ nullam plane determinationem patiatur, quemadmodum fit si formula $\int\int V\partial x\partial y$ valorem maximum minimumve obtinere debeat; tum omnino necessarium est, ut singula variationis inventae $\delta\int\int V\partial x\partial y$ membra ita reducantur, ut ubique post signum integrationis duplicatum non valores differentiales variationis $\delta z = \omega$ sed haec ipsa variatio occurrat; cujusmodi reductione jam supra in formulis binas tantum variabiles involventibus sumus usi. Talis autem reductio, cum pro formulis integralibus duplicatis minus sit consueta, accuratiorem explicationem postulat. Quem in finem observo, hujusmodi reductione perveniri ad formulas simpliciter integrales, in quibus alterutra tantum quantitatum x et y pro variabili habeatur, altera ut constante spectata, ad quod indicandum, ne signa praeter necessitatem multiplicentur, talis forma $\int T\partial x$ denotabit integralae formulae differentialis $T\partial x$, dum quantitas y pro constanti habetur; similique modo intelligendum est in hac forma $\int T\partial y$ solam quantitatem y ut variabilem considerari, quod eo magis perspicuum est, cum hac conditione omissa, hae formulae nullum plane significatum essent habiturae. Neque ergo in posterum opus est declarari, si T ambas variabiles x et y complectatur, utra earum in formulis integralibus simplicibus $\int T\partial x$ vel $\int T\partial y$, sive variabilis accipiatur, cum ea sola, cujus differentiale exprimitur, pro variabili sit habenda. In formulis autem duplicatis $\int\int V\partial x\partial y$ perpetuo tenendum est, alteram integrationem ad solius x, alteram vero ad solius y variabilitatem adstringi, perindeque esse, utra integratio prior instituatur.

Problema 21.

169. Variationem formulae integralis duplicatae $\int\int V\partial x\partial y$,

in praecedente problematae inventam, ita transformare, ut post signum integrale duplicatum ubique ipsa variatio $\delta z = \omega$ occurrat, exturbatis ejus differentialibus.

Solutio.

Quo haec transformatio latius pateat, sint T et v functiones quaecunque binarum variabilium x et y, et consideretur haec formula duplicata $\int\int T\partial x\partial y\left(\frac{\partial v}{\partial x}\right)$, quae separata integrationum varietate ita repraesentetur $\int\partial y\int T\partial x\left(\frac{\partial v}{\partial x}\right)$, ut in integratione $\int T\partial x\left(\frac{\partial v}{\partial x}\right)$ sola quantitas x ut variabilis spectetur. Tum autem erit $\partial x\left(\frac{\partial v}{\partial x}\right) = \partial v$, quia y pro constante habetur, ideoque fiet

$$\int T\partial v = Tv - \int v\partial T,$$

ubi cum in differentiali ∂T solius variabilis x ratio habetur, ad hoc declarandum loco ∂T scribi convenit $\partial x\left(\frac{\partial T}{\partial x}\right)$, ita ut sit

$$\int T\partial x\left(\frac{\partial v}{\partial x}\right) = Tv - \int v\partial x\left(\frac{\partial T}{\partial x}\right),$$

hincque nostra formula ita prodeat reducta

$$\int\int T\partial x\partial y\left(\frac{\partial v}{\partial x}\right) = \int Tv\partial y - \int\int v\partial x\partial y\left(\frac{\partial T}{\partial x}\right).$$

Simili modo permutatis variabilibus consequemur

$$\int\int T\partial x\partial y\left(\frac{\partial v}{\partial y}\right) = \int Tv\partial x - \int\int v\partial x\partial y\left(\frac{\partial T}{\partial y}\right).$$

Hoc jam quasi lemmate praemisso, variationis in praecedente problemate inventae reductio ita se habebit

$$\int\int P\,\partial x\partial y\left(\frac{\partial \omega}{\partial x}\right) = \int P\,\omega\partial y - \int\int \omega\partial x\partial y\left(\frac{\partial P}{\partial x}\right),$$

$$\int\int P'\partial x\partial y\left(\frac{\partial \omega}{\partial y}\right) = \int P'\omega\partial x - \int\int \omega\partial x\partial y\left(\frac{\partial P'}{\partial y}\right).$$

Porro pro sequentibus membris sit primo $\left(\frac{\partial \omega}{\partial x}\right) = v$, ideoque $\left(\frac{\partial\partial \omega}{\partial x^2}\right) = \left(\frac{\partial v}{\partial x}\right)$, unde colligitur

$$\int\!\!\int Q\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x^2}\right)=\int Q\partial y\left(\frac{\partial\omega}{\partial x}\right)-\int\!\!\int\partial x\partial y\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial\omega}{\partial x}\right),$$

ac postremo membro similiter reducto, fit

$$\int\!\!\int Q\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x^2}\right)=\int Q\partial y\left(\frac{\partial\omega}{\partial x}\right)-\int\omega\partial y\left(\frac{\partial Q}{\partial x}\right)+\int\!\!\int\omega\partial x\partial y\left(\frac{\partial\partial Q}{\partial x^2}\right).$$

Per eandem substitutionem habebimus $\left(\frac{\partial\partial\omega}{\partial x\partial y}\right)=\left(\frac{\partial v}{\partial y}\right)$, hincque

$$\int\!\!\int Q'\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x\partial y}\right)=\int Q'\partial x\left(\frac{\partial\omega}{\partial x}\right)-\int\!\!\int\partial x\partial y\left(\frac{\partial\omega}{\partial x}\right)\left(\frac{\partial Q'}{\partial y}\right),\ \text{seu}$$

$$\int\!\!\int Q'\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x\partial y}\right)=\int Q'\partial x\left(\frac{\partial\omega}{\partial x}\right)-\int\omega\partial y\left(\frac{\partial Q'}{\partial y}\right)+\int\!\!\int\omega\partial x\partial y\left(\frac{\partial\partial Q'}{\partial x\partial y}\right),$$

quae forma ob

$$\int Q'\partial x\left(\frac{\partial\omega}{\partial x}\right)=Q'\omega-\int\omega\partial x\left(\frac{\partial Q'}{\partial x}\right),$$

abit in hanc

$$\int\!\!\int Q'\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x\partial y}\right)=Q'\omega-\int\omega\partial x\left(\frac{\partial Q'}{\partial x}\right)+\int\!\!\int\omega\partial x\partial y\left(\frac{\partial\partial Q'}{\partial x\partial y}\right),$$
$$-\int\omega\partial y\left(\frac{\partial Q'}{\partial x}\right)$$

tum vero pro tertia forma hujus ordinis naciscimur

$$\int\!\!\int Q''\partial x\partial y\left(\frac{\partial\partial\omega}{\partial x^2}\right)=\int Q''\partial x\left(\frac{\partial\omega}{\partial y}\right)-\int\omega\partial x\left(\frac{\partial Q''}{\partial y}\right)+\int\!\!\int\omega\partial x\partial y\left(\frac{\partial\partial Q''}{\partial y^2}\right).$$

Porro ob $\left(\frac{\partial^3\omega}{\partial x^3}\right)=\left(\frac{\partial\partial v}{\partial x^2}\right)$, manente $v=\left(\frac{\partial\omega}{\partial x}\right)$, fiet

$$\int\!\!\int R\partial x\partial y\left(\frac{\partial\partial v}{\partial x^2}\right)=\int R\partial y\left(\frac{\partial v}{\partial x}\right)-\int v\partial y\left(\frac{\partial R}{\partial x}\right)+\int\!\!\int v\partial x\partial y\left(\frac{\partial\partial R}{\partial x^2}\right)\ \text{et}$$

$$\int\!\!\int v\partial x\partial y\left(\frac{\partial\partial R}{\partial x^2}\right)=\int\omega\partial y\left(\frac{\partial\partial R}{\partial x^2}\right)-\int\!\!\int\omega\partial x\partial y\left(\frac{\partial^3 R}{\partial x^3}\right),$$

ita ut sit

$$\int\!\!\int R\partial x\partial y\left(\frac{\partial^3\omega}{\partial x^3}\right)=\int R\partial y\left(\frac{\partial\partial\omega}{\partial x^2}\right)-\int\partial y\left(\frac{\partial\omega}{\partial x}\right)\left(\frac{\partial R}{\partial x}\right)+\int\omega\partial y\left(\frac{\partial\partial R}{\partial x^2}\right)$$
$$-\int\!\!\int\omega\partial x\partial y\left(\frac{\partial^3 R}{\partial x^3}\right).$$

Deinde ob $\left(\frac{\partial^3\omega}{\partial x^2\partial y}\right)=\left(\frac{\partial\partial v}{\partial x\partial y}\right)$, erit

$$\int\!\!\int R'\partial x\partial y\left(\frac{\partial\partial v}{\partial x\partial y}\right)=R'v-\int v\partial x\left(\frac{\partial R'}{\partial x}\right)+\int\!\!\int v\partial x\partial y\left(\frac{\partial\partial R'}{\partial x\partial y}\right)$$
$$-\int v\partial y\left(\frac{\partial R'}{\partial y}\right),$$

et quia hic

$$\int\int v\partial x\partial y\left(\frac{\partial\partial R}{\partial x\partial y}\right)=\int\omega\partial y\left(\frac{\partial\partial R'}{\partial x\partial y}\right)-\int\int\omega\partial x\partial y\left(\frac{\partial^3 R'}{\partial x^2\partial y}\right),$$

concludimus fore

$$\int\int R'\partial x\partial y\left(\frac{\partial^3\omega}{\partial x^2\partial y}\right)=R'\left(\frac{\partial\omega}{\partial x}\right)-\int\left(\frac{\partial\omega}{\partial x}\right)\partial x\left(\frac{\partial R'}{\partial x}\right)+\int\omega\partial y\left(\frac{\partial\partial R'}{\partial x\partial y}\right)$$
$$-\int\left(\frac{\partial\omega}{\partial x}\right)\partial y\left(\frac{\partial R'}{\partial y}\right)-\int\int\omega\partial x\partial y\left(\frac{\partial^3 R'}{\partial x^2\partial y}\right).$$

Tandem permutandis x et y hinc colligimus

$$\int\int R''\partial x\partial y\left(\frac{\partial^3\omega}{\partial x\partial y^2}\right)=R''\left(\frac{\partial\omega}{\partial y}\right)-\int\left(\frac{\partial\omega}{\partial y}\right)\partial y\left(\frac{\partial R''}{\partial y}\right)+\int\omega\partial x\left(\frac{\partial\partial R''}{\partial x\partial y}\right)$$
$$-\int\left(\frac{\partial\omega}{\partial y}\right)\partial x\left(\frac{\partial R''}{\partial x}\right)-\int\int\omega\partial x\partial y\left(\frac{\partial^3 R'''}{\partial x\partial y^2}\right) \text{ et}$$

$$\int\int R'''\partial x\partial y\left(\frac{\partial^3\omega}{\partial y^3}\right)=\int R'''\partial x\left(\frac{\partial\partial\omega}{\partial y^2}\right)-\int\left(\frac{\partial\omega}{\partial y}\right)\partial x\left(\frac{\partial R'''}{\partial y}\right)+\int\omega\partial x\left(\frac{\partial\partial R'''}{\partial y^2}\right)$$
$$-\int\int\omega\partial x\partial y\left(\frac{\partial^3 R'''}{\partial y^3}\right).$$

Quos valores si substituamus, reperimus

$$\delta\int\int V\partial x\partial y=\int\int\omega\partial x\partial y\left\{\begin{array}{l}N-\left(\frac{\partial P}{\partial x}\right)+\left(\frac{\partial\partial Q}{\partial x^2}\right)-\left(\frac{\partial^3 R}{\partial x^3}\right)+\text{etc.}\\ \quad-\left(\frac{\partial P'}{\partial y}\right)+\left(\frac{\partial\partial Q'}{\partial x\partial y}\right)-\left(\frac{\partial^3 R'}{\partial x^2\partial y}\right)\\ \qquad+\left(\frac{\partial\partial Q''}{\partial y^2}\right)-\left(\frac{\partial^3 R''}{\partial x\partial y^2}\right)\\ \qquad\quad-\left(\frac{\partial^3 R'''}{\partial y^3}\right)\end{array}\right.$$

$$+\int P\,\omega\partial y\quad+\int Q\,\partial y\left(\frac{\partial\omega}{\partial x}\right)-\int\omega\partial y\left(\frac{\partial Q}{\partial x}\right)+Q'\omega$$
$$+\int P'\omega\partial x\quad-\int\omega\,\partial x\left(\frac{\partial Q'}{\partial x}\right)-\int\omega\partial y\left(\frac{\partial Q'}{\partial y}\right)$$
$$+\int Q''\partial x\left(\frac{\partial\omega}{\partial y}\right)-\int\omega\partial x\left(\frac{\partial Q''}{\partial y}\right)$$
$$+\int R\,\partial y\left(\frac{\partial\partial\omega}{\partial x^2}\right)+R'\left(\frac{\partial\omega}{\partial x}\right)-\int\left(\frac{\partial\omega}{\partial x}\right)\partial x\left(\frac{\partial R'}{\partial x}\right)-\int\left(\frac{\partial\omega}{\partial y}\right)\partial y\left(\frac{\partial R'}{\partial y}\right)+\int R'''\partial x\left(\frac{\partial\partial\omega}{\partial y^2}\right)$$
$$-\int\left(\frac{\partial\omega}{\partial x}\right)\partial y\left(\frac{\partial R}{\partial x}\right)+R''\left(\frac{\partial\omega}{\partial y}\right)-\int\left(\frac{\partial\omega}{\partial x}\right)\partial y\left(\frac{\partial R'}{\partial y}\right)-\int\left(\frac{\partial\omega}{\partial y}\right)\partial x\left(\frac{\partial R''}{\partial x}\right)-\int\left(\frac{\partial\omega}{\partial y}\right)\partial x\left(\frac{\partial R'''}{\partial y}\right)$$
$$+\int\omega\partial y\left(\frac{\partial\partial R}{\partial x^2}\right)\quad+\int\omega\,\partial y\left(\frac{\partial\partial R'}{\partial x\partial y}\right)+\int\omega\,\partial x\left(\frac{\partial\partial R''}{\partial x\partial y}\right)+\int\omega\partial x\left(\frac{\partial\partial R'''}{\partial y^2}\right).$$

Corollarium 1.

170. Hujus expressionis pars prima satis est perspicua,

reliquae vero partes commode ita disponi possunt, ut earum ratio comprehendatur

$$\int \omega \partial y \left\{ \begin{matrix} P - (\frac{\partial Q}{\partial x}) + (\frac{\partial \partial R}{\partial x^2}) \\ - (\frac{\partial Q'}{\partial y}) + (\frac{\partial \partial R'}{\partial x \partial y}) \text{ etc.} \\ + (\frac{\partial \partial R''}{\partial y^2}) \end{matrix} \right\} + \int \omega \partial x \left\{ \begin{matrix} P' - (\frac{\partial Q''}{\partial y}) + (\frac{\partial \partial R'''}{\partial y^2}) \\ - (\frac{\partial Q'}{\partial x}) + (\frac{\partial \partial R''}{\partial x \partial y}) \text{ etc.} \\ + (\frac{\partial \partial R'}{\partial x^2}) \end{matrix} \right\}$$

$$+ \int (\frac{\partial \omega}{\partial x}) \partial y \left\{ \begin{matrix} Q - (\frac{\partial R}{\partial x}) \text{ etc.} \\ - (\frac{\partial R'}{\partial y}) \end{matrix} \right\} + \int (\frac{\partial \omega}{\partial y}) \partial x \left\{ \begin{matrix} Q'' - (\frac{\partial R''}{\partial y}) \text{ etc.} \\ - (\frac{\partial R'}{\partial x}) \end{matrix} \right\}$$

$$+ \int (\frac{\partial \partial \omega}{\partial x^2}) \partial y \, (R - \text{etc.}) + \int (\frac{\partial \partial \omega}{\partial y^2}) \partial x \, (R''' - \text{etc.})$$

$$+ \omega \left\{ \begin{matrix} Q' - (\frac{\partial R'}{\partial x}) \text{ etc.} \\ - (\frac{\partial R''}{\partial y}) \end{matrix} \right\} \begin{matrix} + (\frac{\partial \omega}{\partial x}) (R' - \text{etc.}) \\ + (\frac{\partial \omega}{\partial y}) (R'' - \text{etc.}). \end{matrix}$$

Corollarium 2.

171. Hic levi attentione adhibita mox patebit, quomodo istae partes ulterius continuari debeant, si forte quantitas V differentialia altiorum graduum complectatur.

Corollarium 3.

172. In harum formularum integralium aliis, quae differentiali ∂y sunt affectae, quantitas x constans sumitur, cui tribuitur valor termino integrationis conveniens; aliis vero quae differentiali ∂x sunt affectae, y est constans et termino integrationis aequalis, unde patet in terminis integrationum tam x quam y recipere valorem constantem.

Scholion.

173. Haec ergo variationis formula ad eum casum est accommodata, quo utriusque integrationis termini tribuunt tam ipsi x quam ipsi y valores constantes. Veluti si de superficie fuerit
Fig. 7. quaestio, formula integralis $\int\int V \partial x \partial y$ ad rectangulum APYX in

basi assumtum est referenda; ejusque valor ita definiri debet, ut sumtis $x = 0$ et $y = 0$, qui sunt valores initiales, evanescat, quo facto statui oportet $x = AX$ et $y = AP$, qui sunt valores finales; atque ad eandem legem ipsa variatio inventa est expedienda. Quodsi jam ea quaeratur superficies, in qua formulae $\int\int V\partial x\partial y$ hoc modo definitae valor fiat maximus vel minimus, ante omnia necesse est, ut pars variationis prima duplicem integrationem involvens ad nihilum redigatur, quomodocunque variatio $\delta z = \omega$ accipiatur, unde haec nascetur aequatio

$$\begin{aligned} 0 = N &- \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial\partial Q}{\partial x^2}\right) - \left(\frac{\partial^3 R}{\partial x^3}\right) + \text{etc.} \\ &- \left(\frac{\partial P'}{\partial y}\right) + \left(\frac{\partial\partial Q'}{\partial x\partial y}\right) - \left(\frac{\partial^3 R'}{\partial x^2\partial y}\right) \\ &\qquad + \left(\frac{\partial\partial Q''}{\partial y^2}\right) - \left(\frac{\partial^3 R''}{\partial x\partial y^2}\right) \\ &\qquad\qquad - \left(\frac{\partial^3 R'''}{\partial y^3}\right) \end{aligned}$$

qua natura superficiei hac indole praeditae exprimetur. Constantes autem per duplicem integrationem ingressae ita determinari debent, ut reliquis variationis partibus satisfiat.

Scholion 2.

174. Quo haec investigatio in se maxime abstrusa exemplo illustretur, ponamus ejusmodi superficiem investigari debere, quae inter omnes alias eandem soliditatem includentes sit minima. Hunc in finem efficiendum est ut haec formula integralis duplicata

$$\int\int\partial x\partial y\,[z + a\surd(1 + pp + p'p')],$$

maximum minimumve evadat. Cum ergo sit

$$V = z + a\surd(1 + pp + p'p'), \text{ erit}$$
$$L = 0,\ M = 0,\ N = 1,$$

atque

$$P = \frac{ap}{\surd(1 + pp + p'p')} \text{ et } P' = \frac{ap'}{\surd(1 + pp + p'p')},$$

ideoque

$$\partial V = N\partial z + P\partial p + P'\partial p',$$

existente

$$\partial z = p\partial x + p'\partial y.$$

Quare superficiei quaesitae natura hac aequatione exprimetur

$$N - \left(\frac{\partial P}{\partial x}\right) - \left(\frac{\partial P'}{\partial y}\right) = 0, \quad \text{seu} \quad 1 = \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial P'}{\partial y}\right).$$

Est vero

$$\left(\frac{\partial P}{\partial x}\right) = \frac{a}{(1 + pp + p'p')^{\frac{3}{2}}} \left[(1 + p'p')\left(\frac{\partial p}{\partial x}\right) - pp'\left(\frac{\partial p'}{\partial x}\right)\right],$$

$$\left(\frac{\partial P'}{\partial y}\right) = \frac{a}{(1 + pp + p'p')^{\frac{3}{2}}} \left[(1 + pp)\left(\frac{\partial p'}{\partial y}\right) - pp'\left(\frac{\partial p}{\partial y}\right)\right],$$

ubi notetur esse $\left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right)$. Ex quo ista obtinetur aequatio

$$\frac{(1 + pp + p'p')^{\frac{3}{2}}}{a} = (1 + p'p')\left(\frac{\partial p}{\partial x}\right) - 2pp'\left(\frac{\partial p}{\partial y}\right)$$
$$+ (1 + pp)\left(\frac{\partial p'}{\partial y}\right),$$

quam autem quomodo tractari oporteat, haud patet, etiamsi facile perspiciatur, in ea aequationem pro superficie sphaerica

$$zz = cc - xx - yy,$$

quin etiam cylindrica $zz = cc - yy$ contineri.