# CAPUT V.

## DE VARIATIONE FORMULARUM INTEGRALIUM TRES VARIABILES INVOLVENTIUM, ET DUPLICEM RELATIONEM IMPLICANTIUM.

### Problema 12.

123.

Proposita formula quacunque ternas variabiles $x$, $y$, $z$ cum suis differentialibus cujuscunque gradus involvente, ejus variationem definire ex variationibus omnium trium variabilium oriundam.

### Solutio.

Sit W formula ista proposita, cujus primo quaeratur valor variatus $W + \delta W$, qui oritur si loco $x$, $y$, $z$ scribantur ipsarum valores variati

$$x + \delta x, \quad y + \delta y, \quad z + \delta z,$$

similiterque pro earum differentialibus

$$\partial x + \partial \delta x, \quad \partial y + \partial \delta y, \quad \partial z + \partial \delta z,$$

et ita porro: a quo si ipsa formula W auferatur, remanebit ejus variatio $\delta W$. Ex quo intelligitur hanc variationem per consuetam differentiationem obtineri si modo loco signi differentiationis $\partial$, signum variationes capi oporteat, perinde esse, in quonam loco inter differentiationis signa signum variationis $\delta$ collocetur, quemadmodum supra demonstravimus; unde signum variationis perpetuo in postremo loco poni poterit, quod cum ad formulas integrales progrediemur, commodissimum videtur, sicut ex iis quae hactenus de formulis integralibus binas variabiles involventibus sunt tradita, satis est manifestum.

### Corollarium 1.

124. Quoniam $z$ perinde ac $y$ tanquam functio ipsius $x$ spectari potest, si ponatur

$$\frac{\partial y}{\partial x} = p \quad \text{et} \quad \frac{\partial z}{\partial x} = \mathfrak{p}, \quad \text{erit}$$

$$\delta p = \frac{\partial \delta y - p \partial \delta x}{\partial x} \quad \text{et} \quad \delta \mathfrak{p} = \frac{\partial \delta z - \mathfrak{p} \partial \delta x}{\partial x},$$

similique modo formulae hinc derivatae a superioribus non discrepant.

### Corollarium 2.

125. Ponamus

$$\delta y - p \delta x = \omega \quad \text{et} \quad \delta z - \mathfrak{p} \delta x = \mathfrak{w},$$

eritque

$$\partial \delta y - p \partial \delta x - q \partial x \delta x = \partial \omega \quad \text{et} \quad \partial \delta z - \mathfrak{p} \partial \delta x - \mathfrak{q} \partial x \delta x = \partial \mathfrak{w},$$

si scilicet statuamus

$$\frac{\partial p}{\partial x} = q \quad \text{et} \quad \frac{\partial \mathfrak{p}}{\partial x} = \mathfrak{q},$$

unde patet fore

$$\delta p - q \delta x = \frac{\partial \omega}{\partial x} \quad \text{et} \quad \delta \mathfrak{p} - \mathfrak{q} \delta x = \frac{\partial \mathfrak{w}}{\partial x}.$$

### Corollarium 3.

126. Si ulterius statuamus

$$\frac{\partial q}{\partial x} = r; \quad \frac{\partial \mathfrak{q}}{\partial x} = \mathfrak{r}; \quad \frac{\partial r}{\partial x} = s; \quad \frac{\partial \mathfrak{r}}{\partial x} = \mathfrak{s} \quad \text{etc.}$$

erit simili modo sumto $\partial x$ constante

$$\delta q - r\delta x = \frac{\partial\partial\omega}{\partial x^2}; \quad \delta\mathfrak{q} - \mathfrak{r}\delta x = \frac{\partial\partial\mathfrak{w}}{\partial x^2},$$
$$\delta r - s\delta x = \frac{\partial^3\omega}{\partial x^3}; \quad \delta\mathfrak{r} - \mathfrak{s}\delta x = \frac{\partial^3\mathfrak{w}}{\partial x^3},$$

sicque deinceps.

## Scholion 1.

127. Sive ergo formula varianda habuerit valorem finitum sive infinitum, sive evanescentem, ope horum praeceptorum ejus variatio perinde ac supra inveniri potest, neque enim haec praecepta a superioribus discrepant, nisi quod hic duplicis generis valores differentiales, alteri litteris latinis, $p$, $p$, $r$, $s$ etc. alteri germanicis $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ etc. indicati, introduci debeant, cujus rei ratio in eo est sita, quod hic utraque variabilis $y$ et $z$ tanquam functio ipsius $x$ spectari potest. Sin autem unica aequatio inter ternas coordinatas daretur, vel quaereretur, litterae hic introductae $p$ et $\mathfrak{p}$ nullos habiturae essent valores certos, cum salva illa aequatione fractiones $\frac{\partial y}{\partial x}$ et $\frac{\partial z}{\partial x}$ omnes omnino valores recipere possent. Omissis autem his litteris, ipsisque differentialibus in calculo relictis, etiam pro hoc casu regula in solutione exposita variationem declarabit.

## Scholion 2.

128. Supra jam notavi, hunc casum trium variabilium, quarum relatio gemina aequatione definitur, sollicite esse distinguendum ab eo, ubi relatio unica aequatione definiri assumitur. Discrimen hoc ex Geometria clarissime illustratur, ubi ternae variabiles vicem ternarum coordinatarum gerunt; totidem autem in calculo adhiberi oportet non solum quando quaestio circa superficies versatur, sed etiam quando lineae curvae non in eodem plano sitae sunt explorandae. Atque hoc quidem casu posteriori determinatio lineae curvae duas aequationes inter ternas coordinatas postulat, ita ut binae quaevis tanquam functiones tertiae spectari possint.

Superficiei autem natura jam unica aequatione inter ternas coordinatas definitur, ita ut unaquaeque tanquam functio binarum reliquarum spectari queat, unde ingens discrimen in ipsa tractatione oritur. Praesens igitur caput inservire poterit ejusmodi lineis curvis indagandis quae non in eodem plano sitae maximi minimive quapiam gaudeant proprietate.

### Problema 13.

129. Si V fuerit functio quaecunque trium variabilium $x$, $y$, $z$, earum insuper differentialia cujusque ordinis implicans, eaeque variabiles variationes quascunque recipiant, invenire variationem formulae integralis $\int V\partial x$.

### Solutio.

Quaecunque differentialia in functionem V ingrediantur, ea his factis substitutionibus

$$\partial y = p\partial x;\ \partial p = q\partial x;\ \partial q = r\partial x;\ \partial r = s\partial x \text{ etc.}$$
$$\partial z = \mathfrak{p}\,\partial x;\ \partial\mathfrak{p} = \mathfrak{q}\,\partial x;\ \partial\mathfrak{q} = \mathfrak{r}\,\partial x;\ \partial\mathfrak{r} = \mathfrak{s}\,\partial x \text{ etc.}$$

tollentur, et quantitas V erit functio quantitatum finitarum $x$, $y$, $z$, $p$, $q$, $r$, $s$ etc. $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ etc. Ejus ergo differentiale hujusmodi habebit formam

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + S\partial s + \text{etc.}$$
$$+ \mathfrak{N}\partial z + \mathfrak{P}\partial\mathfrak{p} + \mathfrak{Q}\partial\mathfrak{q} + \mathfrak{R}\partial\mathfrak{r} + \mathfrak{S}\partial\mathfrak{s} + \text{etc.}$$

unde mutatis signis differentiationis $\partial$ in $\delta$, simul habebitur variatio $\delta V$. Ex supra autem demonstratis etiam pro hoc casu trium variabilium habebitur

$$\delta\int V\partial x = \int(V\,\partial\delta x + \partial x\delta V) = V\delta x + \int(\partial x\delta V - \partial V\delta x).$$

At facta substitutione fiet

$$\frac{\partial x\delta V - \partial V\delta x}{\partial x} = M\delta x + N\,\delta y + P\delta p + Q\,\delta q + R\,\delta r + \text{etc.}$$
$$+ \mathfrak{N}\delta z + \mathfrak{P}\delta\mathfrak{p} + \mathfrak{Q}\delta\mathfrak{q} + \mathfrak{R}\delta\mathfrak{r} + \text{etc.}$$

$$— M\delta x — Np\delta x — Pq\delta x — Qr\delta x — Rs\delta x — \text{etc.}$$
$$— \mathfrak{N}\mathfrak{p}\delta x — \mathfrak{P}\mathfrak{q}\delta x — \mathfrak{Q}\mathfrak{r}\delta x — \mathfrak{R}\mathfrak{s}\delta x — \text{etc.}$$

Quodsi jam brevitas gratia statuamus

$$\delta y — p\delta x = \omega \quad \text{et} \quad \delta z — \mathfrak{p}\delta x = \mathfrak{w}$$

sumto elemento $\partial x$ constante, ex §§. 125. et 126. erit

$$\delta p — q\delta x = \frac{\partial\omega}{\partial x}; \quad \delta\mathfrak{p} — \mathfrak{q}\delta x = \frac{\partial\mathfrak{w}}{\partial x};$$
$$\delta q — r\delta x = \frac{\partial\partial\omega}{\partial x^2}; \quad \delta\mathfrak{q} — \mathfrak{r}\delta x = \frac{\partial\partial\mathfrak{w}}{\partial x^2};$$
$$\delta r — s\delta x = \frac{\partial^3\omega}{\partial x^3}; \quad \delta\mathfrak{r} — \mathfrak{s}\delta x = \frac{\partial^3\mathfrak{w}}{\partial x^3};$$

etc.

unde variatio quaesita hoc modo commode exprimetur

$$\delta\int V\partial x = V\delta x + \int\partial x \left\{ \begin{array}{l} N\omega + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial x^3} + \text{etc.} \\ \mathfrak{N}\mathfrak{w} + \frac{\mathfrak{P}\partial\mathfrak{w}}{\partial x} + \frac{\mathfrak{Q}\partial\partial\mathfrak{w}}{\partial x^2} + \frac{\mathfrak{R}\partial^3\mathfrak{w}}{\partial x^3} + \text{etc.} \end{array} \right\}$$

quae ut supra ad hanc formam reducitur

$$\begin{aligned} \delta\int V\partial x = & + \int\omega\partial x \left(N — \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} — \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} — \text{etc.}\right) \\ & + \int\mathfrak{w}\partial x \left(\mathfrak{N} — \frac{\partial\mathfrak{P}}{\partial x} + \frac{\partial\partial\mathfrak{Q}}{\partial x^2} — \frac{\partial^3\mathfrak{R}}{\partial x^3} + \frac{\partial^4\mathfrak{S}}{\partial x^4} — \text{etc.}\right) \\ & + V\delta x + \omega\left(P — \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} — \frac{\partial^3 S}{\partial x^3} + \text{etc.}\right) \\ & + \text{Const.} + \mathfrak{w}\left(\mathfrak{P} — \frac{\partial\mathfrak{Q}}{\partial x} + \frac{\partial\partial\mathfrak{R}}{\partial x^2} — \frac{\partial^3\mathfrak{S}}{\partial x^3} + \text{etc.}\right) \\ & + \frac{\partial\omega}{\partial x}\left(Q — \frac{\partial R}{\partial x} + \frac{\partial\partial S}{\partial x^2} — \text{etc.}\right) \\ & + \frac{\partial\mathfrak{w}}{\partial x}\left(\mathfrak{Q} — \frac{\partial\mathfrak{R}}{\partial x} + \frac{\partial\partial\mathfrak{S}}{\partial x^2} — \text{etc.}\right) \\ & + \frac{\partial\partial\omega}{\partial x^2}\left(R — \frac{\partial S}{\partial x} + \text{etc.}\right) \\ & + \frac{\partial\partial\mathfrak{w}}{\partial x^2}\left(\mathfrak{R} — \frac{\partial\mathfrak{S}}{\partial x} + \text{etc.}\right) \\ & + \frac{\partial^3\omega}{\partial x^3}\left(S — \text{etc.}\right) \\ & + \frac{\partial^3\mathfrak{w}}{\partial x^3}\left(\mathfrak{S} — \text{etc.}\right) + \text{etc.} \end{aligned}$$

cujus indoles ex superioribus satis est manifesta; eademque circa constantis additionem sunt observanda.

### Corollarium 1.

130. In hac solutione ambae variabiles $y$ et $z$ tanquam functiones ipsius $x$ spectantur, sive jam sint cognitae, sive demum ex variationis indole definiendae. Neque etiam formula integralis $\int V\partial x$ certum esset habitura valorem, nisi tam $y$ quam $z$ per $x$ determinari conciperetur.

### Corollarium 2.

131. Si formula $V\partial x$ per se sit integrabilis, nulla assumta relatione inter ternas variabiles, variatio integralis $\int V\partial x$ nullas quoque formulas integrales involvere potest; ideoque necesse est, ut tum sit

$$\text{et } N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0,$$

$$\text{et } \mathfrak{N} - \frac{\partial \mathfrak{P}}{\partial x} + \frac{\partial\partial \mathfrak{Q}}{\partial x^2} - \frac{\partial^3 \mathfrak{R}}{\partial x^3} + \frac{\partial^4 \mathfrak{S}}{\partial x^4} - \text{etc.} = 0.$$

### Corollarium 3.

132. Vicissim etiam si hae duae aequationes locum habeant, hoc certum erit criterium, formulam differentialem $V\partial x$ per se integrationem admittere, nulla inter variabiles stabilita relatione.

### Exemplum.

133. *Quo hoc criterium magis illustremus, sumamus ejusmodi formulam per se integrabilem, sitque*

$$\int V\partial x = \frac{z\partial y}{x\partial z} = \frac{pz}{x\mathfrak{p}},$$

*unde fit*

$$V = \frac{-pz}{xx\mathfrak{p}} + \frac{p}{x} + \frac{zq}{x\mathfrak{p}} - \frac{zp\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}},$$

Ex cujus differentiatione colligimus $N = 0$, et

$$P = \frac{-z}{xx\mathfrak{p}} + \frac{1}{x} - \frac{z\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}}; \quad Q = \frac{z}{x\mathfrak{p}}; \text{ porro}$$

$$\mathfrak{R} = \frac{-p}{xx\mathfrak{p}} + \frac{q}{x\mathfrak{p}} - \frac{p\mathfrak{q}}{x\mathfrak{pp}},$$

$$\mathfrak{P} = \frac{pz}{xx\mathfrak{pp}} - \frac{zq}{x\mathfrak{pp}} + \frac{2zp\mathfrak{q}}{x\mathfrak{p}^3}, \quad \text{et} \quad \mathfrak{Q} = \frac{-zp}{x\mathfrak{pp}}.$$

Jam pro prima aequatione ob $N = 0$ fieri oportet

$$-\frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} = 0, \quad \text{seu} \quad P - \frac{\partial Q}{\partial x} = \text{Const.}$$

cujus veritas ex differentiatione ipsius Q statim fit perspicua.

Pro altera aequatione

$$\mathfrak{R} - \frac{\partial\mathfrak{P}}{\partial x} + \frac{\partial\partial\mathfrak{Q}}{\partial x^2} = 0,$$

quia hinc est

$$\int \mathfrak{R}\partial x = \mathfrak{P} - \frac{\partial\mathfrak{Q}}{\partial x},$$

primo necesse est ut integrabilis existat haec formula

$$\mathfrak{R}\partial x = \frac{-p\partial x}{xx\mathfrak{p}} + \frac{q\partial x}{x\mathfrak{p}} - \frac{p\partial x}{x\mathfrak{pp}},$$

unde ob $q\partial x = \partial p$ manifesto fit

$$\int \mathfrak{R}\partial x = \frac{p}{x\mathfrak{p}}.$$

Superest ergo ut sit

$$\frac{\partial\mathfrak{Q}}{\partial x} = \mathfrak{P} - \int\mathfrak{R}\partial x = \frac{pz}{xx\mathfrak{pp}} - \frac{zq}{x\mathfrak{pp}} + \frac{2zp\mathfrak{q}}{x\mathfrak{p}^3} - \frac{p}{x\mathfrak{p}}.$$

Verum differentiando $\mathfrak{Q} = \frac{-zp}{x\mathfrak{pp}}$, utrinque perfecta aequalitas resultat.

### Scholion 1.

134. Quodsi ergo quaestio huc redeat, ut formulae integrali $\int V\partial x$ valor maximus minimusve sit conciliandus, tum ante omnia in ejus variatione ambas partes integrales idque seorsim nihilo aequari oportet, propterea quod utcunque variationes constituantur; variatio $\delta\int V\partial x$ semper debeat evanescere, unde duae emergunt aequationes istae

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0 \quad \text{et}$$

$$\mathfrak{N} - \frac{\partial \mathfrak{P}}{\partial x} + \frac{\partial\partial \mathfrak{Q}}{\partial x^2} - \frac{\partial^3 \mathfrak{R}}{\partial x^3} + \frac{\partial^4 \mathfrak{S}}{\partial x^4} - \text{etc.} = 0,$$

quibus duplex relatio inter ternas variabiles $x$, $y$, $z$ ita exprimitur, ut deinceps tam $y$ quam $z$ recte tanquam functio ipsius $x$ spectari possit. Quando autem hae aequationes sunt differentiales idque altioris gradus, totidem utrinque constantes arbitrariae per integrationes in calculum invehuntur, quoti gradus utraque fuerit differentialis. Has vero constantes deinceps ita definiri oportet, ut conditionibus tam pro initio quam pro fine integrationis formulae $\int V\partial x$ praescriptis satisfiat, quod negotium eo redit, ut praeterea variationis partes absolutae ad nihilum redigantur. Primo scilicet constans ita definiri debet, ut conditionibus pro initio praescriptis satisfiat, ubi quidem ex quaestionis indole particulae

$$\omega, \; \mathfrak{w}, \; \frac{\partial \omega}{\partial x}, \; \frac{\partial \mathfrak{w}}{\partial x}, \; \frac{\partial\partial \omega}{\partial x^2}, \; \frac{\partial\partial \mathfrak{w}}{\partial x^2} \; \text{etc.}$$

definitos valores sortiri solent. Tum vero cum idem circa finem integrationis usu veniat, ex singulis constantes per integrationem ingressae determinabuntur.

### Scholion 2.

135. Plurimum conducet hic observasse, membra, quibus variatio $\delta\int V\partial x$ exprimitur, sponte in duas classes dispesci, in quarum altera litterae tantum eae conspiciuntur, quae ad variabilitatem ipsius $y$, seu ad ejus habitum respectu $x$ referuntur, idque ita ac si quantitas $z$ constans esset assumta, altera vero classis similes literas a variabilitate ipsius $z$ tantum pendentes continet, quasi quantitas $y$ esset constans. Ex quo colligere licet, si etiam quarta variabilis $v$ accedat, quae ut functio ipsius $x$ quoque spectari queat, tum ad illas duas classes tertiam insuper esse adjiciendam, quae similia membra a variabilitate solius $v$ pendentia complectatur. Quocirca solutio hic data spectari potest, quasi ad quotcunque va-

riabiles extendatur, dummodo tot inter eas aequationes dari concipiantur, ut omnes pro functionibus unius haberi queant. Etsi ergo hoc caput tantum tres variabiles prae se fert, tamen ad quodcunque pertinere est intelligendum, si modo ejusmodi conditiones proponantur, ut tandem per unam reliquae omnes determinentur. Talem autem conditionem formulae integrales hujus formae $\int V\partial x$ necessario involvunt; quotcunque enim variabiles in quantitatem V ingrediantur, expressio $\int V\partial x$ certum valorem definitum omnino obtinere nequit, nisi omnes variabiles tanquam functiones unius $x$ spectari queant. Longe aliter autem est comparata ratio earum formularum integralium, quae ad duas pluresve variabiles a se invicem minime pendentes referuntur.

### Problema 14.

136. Si functio V praeter tres variabiles $x$, $y$, $z$, earumque differentialia cujuscunque gradus, insuper involvat formulam integralem $v = \int \mathfrak{V}\partial x$, ubi $\mathfrak{V}$ sit functio quaecunque earundem variabilium $x$, $y$, $z$, cum suis differentialibus, investigare variationem formulae integralis $\int V\partial x$.

### Solutio.

Ut species saltem differentialium e calculo tollatur, ponamus ut ante

$$\partial y = p\partial x,\quad \partial p = q\partial x,\quad \partial q = r\partial x,\quad \partial r = s\partial x, \text{ etc.}$$
$$\partial z = \mathfrak{p}\partial x,\quad \partial \mathfrak{p} = \mathfrak{q}\partial x,\quad \partial \mathfrak{q} = \mathfrak{r}\partial x,\quad \partial \mathfrak{r} = \mathfrak{s}\partial x, \text{ etc.}$$

ac functione V differentiata prodeat

$$\partial V = L\partial v + M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$
$$+ \mathfrak{N}\partial z + \mathfrak{P}\partial \mathfrak{p} + \mathfrak{Q}\partial \mathfrak{q} + \mathfrak{R}\partial \mathfrak{r} + \text{etc.}$$

tum vero ob $\partial v = \mathfrak{V}\partial x$ sit

$$\partial \mathfrak{V} = M'\partial x + N'\partial y + P'\partial p + Q'\partial q + R'\partial r + \text{etc.}$$
$$+ \mathfrak{N}'\partial z + \mathfrak{P}'\partial \mathfrak{p} + \mathfrak{Q}'\partial \mathfrak{q} + \mathfrak{R}'\partial \mathfrak{r} + \text{etc.}$$

ubi ob defectum litterarum iisdem accentu distinctis utor. Hinc autem simul earundem quantitatum V et $\mathfrak{V}$ variationes habentur. Jam cum quaeratur variatio $\delta\int V\partial x$, habebimus primo quidem ut ante

$$\delta\int V\partial x = V\delta x + \int(\partial x\delta V - \partial V\delta x)\,,$$

ubi cum valor ipsius V non discrepet a praecedente, nisi quod hic ad ejus differentiale $\partial V$ accedat pars $L\partial v = L\mathfrak{V}\partial x$, et ad variationem $\delta V$ haec pars $L\partial v = L\,\delta\int\mathfrak{V}\partial x$; etiam variatio quaesita $\delta\int V\partial x$ forma ante inventa exprimetur, si modo ad eam adjiciatur hoc membrum

$$\int L(\partial x\delta\int\mathfrak{V}\partial x - \mathfrak{V}\partial x\delta x) = \int L\partial x\,(\delta\int\mathfrak{V}\partial x - \mathfrak{V}\delta x).$$

Quia vero formula integralis $\int\mathfrak{V}\partial x$ eadem est quae in problemate praecedente est tractata, si ut ibi fecimus, statuamus

$$\delta y - p\delta x = \omega \quad \text{et} \quad \delta z - \mathfrak{p}\delta x = \mathfrak{w}\,,$$

elemento $\partial x$ constante assumto habebimus

$$\delta\int\mathfrak{V}\partial x - \mathfrak{V}\delta x = \int\partial x \left\{ \begin{array}{l} N'\omega + \frac{P'\partial\omega}{\partial x} + \frac{Q'\partial\partial\omega}{\partial x^2} + \frac{R'\partial^3\omega}{\partial x^3} + \text{etc.} \\ \mathfrak{N}'\mathfrak{w} + \frac{\mathfrak{P}'\partial\mathfrak{w}}{\partial x} + \frac{\mathfrak{Q}'\partial\partial\mathfrak{w}}{\partial x^2} + \frac{\mathfrak{R}'\partial^3\mathfrak{w}}{\partial x^3} + \text{etc.} \end{array} \right.$$

Ponamus jam integrale $\int L\partial x = I$, si scilicet ita capiatur, ut pro initio integrationis evanescat, tum vero pro termino finali integrationis fiat $I = A$, quo facto pro tota integrationis extensione erit

$$\int L\partial x\,(\delta\int\mathfrak{V}\partial x - \mathfrak{V}\delta x) = \int(A - I)\,\partial x \left\{ \begin{array}{l} N'\omega + \frac{P'\partial\omega}{\partial x} + \frac{Q'\partial\partial\omega}{\partial x^2} + \text{etc.} \\ \mathfrak{N}'\mathfrak{w} + \frac{\mathfrak{P}'\partial\mathfrak{w}}{\partial x} + \frac{\mathfrak{Q}'\partial\partial\mathfrak{w}}{\partial x^2} + \text{etc.} \end{array} \right.$$

Nunc igitur introducamus sequentes abbreviationes

$$\begin{array}{ll} N + (A - I)\,N' = N^\circ, & \mathfrak{N} + (A - I)\,\mathfrak{N}' = \mathfrak{N}^\circ, \\ P + (A - I)\,P' = P^\circ, & \mathfrak{P} + (A - I)\,\mathfrak{P}' = \mathfrak{P}^\circ, \\ Q + (A - I)\,Q' = Q^\circ, & \mathfrak{Q} + (A - I)\,\mathfrak{Q}' = \mathfrak{Q}^\circ, \\ R + (A - I)\,R' = R^\circ, & \mathfrak{R} + (A - I)\,\mathfrak{R}' = \mathfrak{R}^\circ, \\ \text{etc.} & \text{etc.} \end{array}$$

atque manifestum est variationem quaesitam ita expressam iri

$$\delta \int V \partial x = V \delta x + \int \partial x \left\{ \begin{array}{l} N^\circ \omega + \frac{P^\circ \partial \omega}{\partial x} + \frac{Q^\circ \partial\partial \omega}{\partial x^2} + \frac{R^\circ \partial^3 \omega}{\partial x^3} + \text{etc.} \\ \mathfrak{N}^\circ \mathfrak{w} + \frac{\mathfrak{P}^\circ \partial \mathfrak{w}}{\partial x} + \frac{\mathfrak{Q}^\circ \partial\partial \mathfrak{w}}{\partial x^2} + \frac{\mathfrak{R}^\circ \partial^3 \mathfrak{w}}{\partial x^3} + \text{etc.} \end{array} \right.$$

quae etiam ut ante evolvitur in hanc formam

$$\begin{aligned} \delta \int V \partial x = & + \int \omega \partial x \left(N^\circ - \frac{\partial P^\circ}{\partial x} + \frac{\partial\partial Q^\circ}{\partial x^2} - \frac{\partial^3 R^\circ}{\partial x^3} + \frac{\partial^4 S^\circ}{\partial x^4} - \text{etc.}\right) \\ & + \int \mathfrak{w} \partial x \left(\mathfrak{N}^\circ - \frac{\partial \mathfrak{P}^\circ}{\partial x} + \frac{\partial\partial \mathfrak{Q}^\circ}{\partial x^2} - \frac{\partial^3 \mathfrak{R}^\circ}{\partial x^3} + \frac{\partial^4 \mathfrak{S}^\circ}{\partial x^4} - \text{etc.}\right) \\ & + V \delta x \quad + \omega \left(P^\circ - \frac{\partial Q^\circ}{\partial x} + \frac{\partial\partial R^\circ}{\partial x^2} - \frac{\partial^3 S^\circ}{\partial x^3} + \text{etc.}\right) \\ & + \text{Const.} \quad + \mathfrak{w} \left(\mathfrak{P}^\circ - \frac{\partial \mathfrak{Q}^\circ}{\partial x} + \frac{\partial\partial \mathfrak{R}^\circ}{\partial x^2} - \frac{\partial^3 \mathfrak{S}^\circ}{\partial x^3} + \text{etc.}\right) \\ & + \frac{\partial \omega}{\partial x} \left(Q^\circ - \frac{\partial R^\circ}{\partial x} + \frac{\partial\partial S^\circ}{\partial x^2} - \text{etc.}\right) \\ & + \frac{\partial \mathfrak{w}}{\partial x} \left(\mathfrak{Q}^\circ - \frac{\partial \mathfrak{R}^\circ}{\partial x} + \frac{\partial\partial \mathfrak{S}^\circ}{\partial x^2} - \text{etc.}\right) \\ & + \frac{\partial\partial \omega}{\partial x^2} \left(R^\circ - \frac{\partial S^\circ}{\partial x} + \text{etc.}\right) \\ & + \frac{\partial\partial \mathfrak{w}}{\partial x^2} \left(\mathfrak{R}^\circ - \frac{\partial \mathfrak{S}^\circ}{\partial x} + \text{etc.}\right) \\ & + \frac{\partial^3 \omega}{\partial x^3} \left(S^\circ - \text{etc.}\right) \\ & + \frac{\partial^3 \mathfrak{w}}{\partial x^3} \left(\mathfrak{S}^\circ - \text{etc.}\right) + \text{etc.} \end{aligned}$$

ubi neminem offendat signum nihili litteris suffixum, siquidem non exponentem denotat, sed tantum ad has litteras ab iisdem nude positis distinguendas adhibetur.

### Corollarium 1.

137. Si igitur formula integralis $\int V \partial x$ habere debeat valorem maximum vel minimum, variationis inventae bina membra priora statim nihilo aequalia statui oportet, unde duae resultant aequationes differentiales, quibus indefinita relatio utriusque variabilis $y$ et $z$ ad $x$ definitur.

### Corollarium 2.

138. Etiamsi hic conditionum, quae forte pro initio et fine integrationis proponantur, nondum ratio habetur, tamen ea jam occulte in calculum ingreditur, quia litterae I et A terminos integrationis respiciunt. Interim tamen eae in ipsa aequationum differentialium tractatione iterum ex calculo expelluntur; dum enim formula integralis $\int L\partial x = I$ eliditur, simul quantitas constans A egreditur.

### Corollarium 3.

139. Expeditis autem aequationibus his duabus differentialibus, idque generalissime, ut totidem constantes arbitrariae in calculum invehantur, quot integrationes institui oportuit, tum demum ad conditiones utriusque termini integrationis formulae $\int V\partial x$ est attendendum, quandoquidem hinc ex reliquis variationis membris absolutis illae constantes determinari debent.

### Scholion.

140. Solutio hujus problematis ita est comparata ut jam satis sit perspicuum, quemadmodum etiam formulas magis complicatas, veluti si functio V plures formulas integrales involvat, vel si quoque $\mathfrak{V}$ formulas novas integrales complectatur, expediri conveniat. Quin etiam nunc est manifestum, si hujusmodi formulae integrales plures tribus variabiles contineant, quomodo tum variationes inveniri oporteat, atque adeo non solum taediosum sed etiam superfluum foret si copiosius hoc argumentum persequi vellem. Ad partem igitur hujus doctrinae alteram multo abstrusiorem progredior, ubi etiam relationibus inter variabiles constitutis duae pluresve a se invicem minime pendentes in calculo relinquuntur.