CAPUT IV.

DE
VARIATIONE FORMULARUM INTEGRALIUM COMPLICATARUM DUAS VARIABILES INVOLVENTIUM.

Problema 8.

105.

Posito $v = \int \mathfrak{V} \partial x$, existente $\mathfrak{V}$ functione quacunque binarum variabilium x, y earumque differentialium

$$\partial y = p \partial x, \quad \partial p = q \partial x, \quad \partial q = r \partial x, \quad \text{etc.}$$

si V denotet functionem quamcunque ipsius v, investigare variationem formulae integralis complicatae $\int V \partial x$.

Solutio.

Quia quantitas v ipsa est formula integralis $\int \mathfrak{V} \partial x$, formula $\int V \partial x$ est utique complicata. Cum igitur functio V solam quantitatem v involvere ponatur, statuamus $\partial V = L \partial v$, tum vero pro functione $\mathfrak{V}$ sit ejus differentiale

$$\partial \mathfrak{V} = \mathfrak{M} \partial x + \mathfrak{N} \partial y + \mathfrak{P} \partial p + \mathfrak{Q} \partial q + \mathfrak{R} \partial r + \text{etc.}$$

His positis cum variatio quaesita sit

$$\delta \int V \partial x = \int \delta (V \partial x) = \int (\delta V \partial x + V \partial \delta x),$$

et per reductionem supra adhibitam

$$\delta \int V \partial x = V \delta x + \int (\partial x \delta V - \partial V \delta x).$$

Cum autem per hypothesin sit $\partial V = L \partial v$, erit etiam pro variatione $\delta V = L \delta v$, verum ob $v = \int \mathfrak{V} \partial x$ erit primo $\partial v = \mathfrak{V} \partial x$, ideoque $\partial V = L \mathfrak{V} \partial x$, tum vero

$$\delta v = \delta \int \mathfrak{V} \partial x = \mathfrak{V} \delta x + \int (\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x),$$

ac propterea

$$\delta V = L \mathfrak{V} \delta x = L \int (\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x),$$

hincque

$$\delta \int V \partial x = V \delta x + \int [L \mathfrak{V} \partial x \delta x + L \partial x \int (\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x) - L \mathfrak{V} \partial x \delta x],$$

seu $\delta \int V \partial x = V \delta x - \int L \partial x \int (\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x)$.

Ex praecedente autem capite patet esse

$$\begin{aligned} \int(\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x) = \delta \int \mathfrak{V} \partial x - \mathfrak{V} \partial x = \int \omega \partial x \Big(\mathfrak{N} - \frac{\partial \mathfrak{P}}{\partial x} + \frac{\partial\partial \mathfrak{Q}}{\partial x^2} - \frac{\partial^3 \mathfrak{R}}{\partial x^3} + \frac{\partial^4 \mathfrak{S}}{\partial x^4} - \text{etc.}\Big) \\ + \omega \Big(\mathfrak{P} - \frac{\partial \mathfrak{Q}}{\partial x} + \frac{\partial\partial \mathfrak{R}}{\partial x^2} - \frac{\partial^3 \mathfrak{S}}{\partial x^3} + \text{etc.}\Big) \\ + \frac{\partial \omega}{\partial x} \Big(\mathfrak{Q} - \frac{\partial \mathfrak{R}}{\partial x} + \frac{\partial\partial \mathfrak{S}}{\partial x^2} - \text{etc.}\Big) \\ + \frac{\partial\partial \omega}{\partial x^2} \Big(\mathfrak{R} - \frac{\partial \mathfrak{S}}{\partial x} + \text{etc.}\Big) \\ \text{etc.} \end{aligned}$$

sumto elemento ∂x constante et posito brevitatis ergo $\omega = \delta y - p \delta x$. Verum cum hinc substitutio molestias pariat, praestabit ex primo fonte rem repetere; cum igitur ex differentiali et variatione quantitatis $\mathfrak{V}$ fiat

$$\begin{aligned} \partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x = \partial x (\mathfrak{M} \delta x + \mathfrak{N} \delta y + \mathfrak{P} \delta p + \mathfrak{Q} \delta q + \mathfrak{R} \delta r + \text{etc.}) \\ - \delta x (\mathfrak{M} \partial x + \mathfrak{N} \partial y + \mathfrak{P} \partial p + \mathfrak{Q} \partial q + \mathfrak{R} \partial r + \text{etc.}) \end{aligned}$$

ob $\partial y = p \partial x$, $\partial p = q \partial x$, $\partial q = r \partial x$, $\partial r = s \partial x$, etc. erit

$$\begin{aligned} \partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x = \mathfrak{N} \partial x (\delta y - p \delta x) + \mathfrak{P} \partial x (\delta p - q \delta x) \\ + \mathfrak{Q} \partial x (\delta q - r \delta x) + \text{etc.} \end{aligned}$$

Verum ob ∂x constans, ex §. 79. fit

$$\delta y - p \delta x = \omega, \quad \delta p - q \delta x = \frac{\partial \omega}{\partial x}, \quad \delta q - r \delta x = \frac{\partial\partial \omega}{\partial x^2},$$

$$\delta r - s \delta x = \frac{\partial^3 \omega}{\partial x^3}, \quad \text{etc.}$$

sicque habebitur

$$\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x = \mathfrak{N} \omega \partial x + \mathfrak{P} \partial \omega + \mathfrak{Q} \frac{\partial \partial \omega}{\partial x} + \mathfrak{R} \frac{\partial^3 \omega}{\partial x^2} + \mathfrak{S} \frac{\partial^4 \omega}{\partial x^3} + \text{etc.}$$

cujus quidem integrale praebet superiorem expressionem. Ponatur nunc integrale $\int L \partial x = I$, eritque

$$\delta \int V \partial x = V \delta x + I \int (\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x) - \int I (\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x).$$

Nunc vero facile colligitur fore

$$\begin{aligned} \int I (\partial x \delta \mathfrak{V} - \partial \mathfrak{V} \delta x) = & \int \omega \partial x \left(I\mathfrak{N} - \frac{\partial . I\mathfrak{P}}{\partial x} + \frac{\partial \partial . I\mathfrak{Q}}{\partial x^2} - \frac{\partial^3 . I\mathfrak{R}}{\partial x^3} + \text{etc.}\right) \\ & + \omega \left(I\mathfrak{P} - \frac{\partial . I\mathfrak{Q}}{\partial x} + \frac{\partial \partial . I\mathfrak{R}}{\partial x^2} - \text{etc.}\right) \\ & + \frac{\partial \omega}{\partial x} \left(I\mathfrak{Q} - \frac{\partial . I\mathfrak{R}}{\partial x} + \text{etc.}\right) \text{ etc.} \end{aligned}$$

unde facta substitutione concluditur variatio quaesita

$$\begin{aligned} \delta \int V \partial x = V \delta x + I \int \omega \partial x & \left(\mathfrak{N} - \frac{\partial \mathfrak{P}}{\partial x} + \frac{\partial \partial \mathfrak{Q}}{\partial x^2} - \frac{\partial^3 \mathfrak{R}}{\partial x^3} + \text{etc.}\right) \\ - \int \omega \partial x & \left(I\mathfrak{N} - \frac{\partial . I\mathfrak{P}}{\partial x} + \frac{\partial \partial . I\mathfrak{Q}}{\partial x^2} - \frac{\partial^3 I\mathfrak{R}}{\partial x^3} + \text{etc.}\right) \\ + I \omega & \left(\mathfrak{P} - \frac{\partial \mathfrak{Q}}{\partial x} + \frac{\partial \partial \mathfrak{R}}{\partial x^2} - \frac{\partial^3 \mathfrak{S}}{\partial x^3} + \text{etc.}\right) \\ - \omega & \left(I\mathfrak{P} - \frac{\partial . I\mathfrak{Q}}{\partial x} + \frac{\partial \partial . I\mathfrak{R}}{\partial x^2} - \frac{\partial^3 . I\mathfrak{S}}{\partial x^3} + \text{etc.}\right) \\ + \frac{I \partial \omega}{\partial x} & \left(\mathfrak{Q} - \frac{\partial \mathfrak{R}}{\partial x} + \frac{\partial \partial \mathfrak{S}}{\partial x^2} - \text{etc.}\right) \\ - \frac{\partial \omega}{\partial x} & \left(I\mathfrak{Q} - \frac{\partial . I\mathfrak{R}}{\partial x} + \frac{\partial \partial . I\mathfrak{S}}{\partial x^2} - \text{etc.}\right) \\ + \frac{I \partial \partial \omega}{\partial x^2} & \left(\mathfrak{R} - \frac{\partial \mathfrak{S}}{\partial x} + \text{etc.}\right) \\ - \frac{\partial \partial \omega}{\partial x^2} & \left(I\mathfrak{R} - \frac{\partial . I\mathfrak{S}}{\partial x} + \text{etc.}\right) \\ + \frac{I \partial^3 \omega}{\partial x^3} & \left(\mathfrak{S} - \text{etc.}\right) \\ - \frac{\partial^3 \omega}{\partial x^3} & \left(I\mathfrak{S} - \text{etc.}\right) + \text{etc.} \end{aligned}$$

Si hic partes binae priores differentiatae iterum integrentur, reliquarum facta reductione, impetrabimus loco ∂I valorem $L \partial x$ restituendo

$$\begin{aligned}\delta\int V\partial x = V\delta x + \int L\partial x\int\omega\partial x\,(\mathfrak{N} & - \frac{\partial\mathfrak{P}}{\partial x} + \frac{\partial\partial\mathfrak{Q}}{\partial x^2} - \frac{\partial^3\mathfrak{R}}{\partial x^3} + \text{etc.})\\ + \int\omega\partial x\,(L\mathfrak{P} & - \frac{L\partial\mathfrak{Q} - \partial L\mathfrak{Q}}{\partial x} + \frac{L\partial\partial\mathfrak{R} + \partial . L\partial\mathfrak{R} + \partial\partial . L\mathfrak{R}}{\partial x^2} - \text{etc.})\\ + \omega\,(L\mathfrak{Q} & - \frac{L\partial\mathfrak{R} - \partial . L\mathfrak{R}}{\partial x} + \frac{L\partial\partial\mathfrak{S} + \partial . L\partial\mathfrak{S} + \partial\partial . L\mathfrak{S}}{\partial x^2} - \text{etc.})\\ + \frac{\partial\omega}{\partial x}\,(L\mathfrak{R} & - \frac{L\partial\mathfrak{S} - \partial . L\mathfrak{S}}{\partial x} + \text{etc.})\\ + \frac{\partial\partial\omega}{\partial x^2}\,(L\mathfrak{S} & - \text{etc.}) + \text{etc.}\end{aligned}$$

quae forma videtur simplicissima et ad usum maxime accomodata.

Corollarium 1.

106. Si ejusmodi relatio inter x et y quaeratur, ut integrale $\int V\partial x$ maximum minimumve evadat, variationis partes integrales nihilo aequari oportet, quod in genere fieri nequit; sed ad terminum, quousque integrale $\int V\partial x$ extenditur, spectari oportet, pro quo si ponamus fieri $I = \int L\partial x = A$, ex priori forma colligimus hanc aequationem

$$0 = (A - I)\mathfrak{N} - \frac{\partial . (A - I)\mathfrak{P}}{\partial x} + \frac{\partial\partial . (A - I)\mathfrak{Q}}{\partial x^2} - \frac{\partial^3 . (A - I)\mathfrak{R}}{\partial x^3} + \text{etc.}$$

Corollarium 2.

107. Quomodocunque autem haec aequatio pro quovis casu oblato tractetur, semper tandem eo est deveniendum ut formula integralis $I = \int L\partial x$ per differentiationem exturbari debeat, qua operatione simul quantitatem A inde extrudi evidens est; sicque aequatio resultans non amplius a termino integrationis pendebit.

Corollarium 3.

108. Quod si in genere pro variatione formulae integralis $\int V\partial x$ invenienda, valorem $\int L\partial x = I$ toti integrali respondentem ponamus $= A$, variatio quaesita ita exprimetur

$$\delta\int V\partial x = V\delta x + \int\omega\partial x\Big[(A-I)\mathfrak{N} - \frac{\partial.(A-I)\mathfrak{P}}{\partial x} + \frac{\partial\partial.(A-I)\mathfrak{Q}}{\partial x^2} - \frac{\partial^3.(A-I)\mathfrak{R}}{\partial x^3} + \text{etc.}\Big]$$
$$+ \omega\left(L\mathfrak{Q} - \frac{L\partial\mathfrak{R} - \partial.L\mathfrak{R}}{\partial x} + \frac{L\partial\partial\mathfrak{S} + \partial.L\partial\mathfrak{S} + \partial\partial.L\mathfrak{S}}{\partial x^2} - \text{etc.}\right)$$
$$+ \frac{\partial\omega}{\partial x}\left(L\mathfrak{R} - \frac{L\partial\mathfrak{S} - \partial.L\mathfrak{S}}{\partial x} + \text{etc.}\right)$$
$$+ \frac{\partial\partial\omega}{\partial x^2}(L\mathfrak{S} - \text{etc.}) + \text{etc.}$$

ubi $A - I$ est valor formulae $\int L\partial x$ a termino integrationis extremo ad quemvis locum indefinitum medium retro sumtus.

Scholion.

109. In solutione hujus problematis compendium se obtulit, quo etiam analysis in superiori capite adhibita non mediocriter contrahi potest. Cum enim ibi (§. 79.) pervenissemus ad

$$\delta\int V\partial x = V\delta x + \int(\partial x\delta V - \partial V\delta x),\quad \text{ob}$$
$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}\quad \text{et}$$
$$\delta V = M\delta x + N\delta y + P\delta q + Q\delta r + R\delta s + \text{etc.}$$

erit

$$\partial V = \partial x(M + Np + Pq + Qr + Rs + \text{etc.}),$$

hincque colligitur

$$\partial x\delta V - \partial V\delta x$$
$$= \partial x[N(\delta y - p\delta x) + P(\delta p - q\delta x) + Q(\delta q - r\delta x) + \text{etc.}].$$

Jam si brevitatis gratia ponatur $\delta y - p\delta x = \omega$, erit differentiando

$$\delta(p\partial x) - q\partial x\delta x - p\delta\partial x = \partial\omega;\quad \text{at}$$
$$\delta(p\partial x) = p\partial\delta x + \delta p\,\partial x,\quad \text{ergo}$$
$$\delta p - q\delta x = \frac{\partial\omega}{\partial x}.$$

Simili modo hanc formulam differentiando ob

$$\partial p = q\partial x \quad \text{et} \quad \partial q = r\partial x \quad \text{fit}$$
$$q\partial\delta x + \delta q\partial x - q\partial\delta x - \partial q\delta x = \partial x(\delta q - r\delta x) = \partial\,.\frac{\partial\omega}{\partial x},$$

unde perspicuum est

posito $\delta y - p\delta x = \omega$,

fore $\delta p - q\delta x = \frac{\partial\omega}{\partial x}$,

$$\delta q - r\delta x = \frac{1}{\partial x}\partial \cdot \frac{\partial\omega}{\partial x} = \frac{\partial\partial\omega}{\partial x^2}, \text{ sumto } \partial x \text{ constante,}$$

$$\delta r - s\delta x = \frac{1}{\partial x}\partial \cdot \frac{1}{\partial x}\partial \cdot \frac{\partial\omega}{\partial x} = \frac{\partial^3\omega}{\partial x^3},$$

etc.

Quocirca erit

$$\partial x\,\delta V - \partial V\,\delta x$$

$$= \partial x\left(N\omega + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial x^3} + \frac{S\partial^4\omega}{\partial x^4} + \text{etc.}\right),$$

siquidem differentiale ∂x constans accipiatur.

Problema 9.

110. Si fuerit $v = \int \mathfrak{V}\partial x$, existente

$$\partial\mathfrak{V} = \mathfrak{M}\partial x + \mathfrak{N}\partial y + \mathfrak{P}\partial p + \mathfrak{Q}\partial q + \mathfrak{R}\partial r + \text{etc.}$$

tum vero sit V functio quaecunque non solum quantitates

$$x,\ y,\ p = \frac{\partial y}{\partial x},\quad q = \frac{\partial p}{\partial x},\quad r = \frac{\partial q}{\partial x},\ \text{etc.}$$

sed etiam ipsam formulam integralem $v = \int \mathfrak{V}\partial x$ implicans, investigare variationem formulae integralis complicatae $\int V\partial x$.

Solutio.

Quoniam V est functio quantitatum v, x, y, p, q, r, etc. sumatur ejus differentiale quod sit

$$\partial V = L\partial v + M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

ac habebitur variatio ipsius V ita expressa

$$\delta V = L\delta v + M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}$$

tum vero notetur, ob

$$\partial v = \mathfrak{V}\partial x,\quad \partial y = p\partial x,\quad \partial p = q\partial x,\ \text{etc. esse}$$

$$\partial V = \partial x\,(L\mathfrak{V} + M + Np + Pq + Qr + Rs + \text{etc.}) \quad \text{et}$$
$$\delta\mathfrak{V} = \mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}$$

Praeterea habemus
$$\delta v = \int(\mathfrak{V}\partial\delta x + \partial x\delta\mathfrak{V}) = \mathfrak{V}\delta x + \int(\partial x\delta\mathfrak{V} - \partial\mathfrak{V}\delta x),$$
unde posito $\delta y - p\delta x = \omega$, erit per ante inventa
$$\delta v = \mathfrak{V}\delta x + \int\partial x\left(\mathfrak{N}\omega + \frac{\mathfrak{P}\partial\omega}{\partial x} + \frac{\mathfrak{Q}\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}\partial^3\omega}{\partial x^3} + \text{etc.}\right),$$
ubi commoditatis ergo sumsimus ∂x constans.

His praeparatis cum variatio quaesita sit
$$\delta\int V\partial x = V\delta x + \int(\partial x\delta V - \partial V\delta x),$$
ut reductione supra inventa uti possimus, ponamus
$$\partial V = L\partial v + \partial W,$$
ut sit
$$\delta V = L\delta v + \delta W \quad \text{et}$$
$$\delta W = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$
Quocirca nanciscemur hanc formam
$$\delta\int V\partial x = V\delta x + \int(L\partial x\delta v - L\partial v\delta x) + \int(\partial x\delta W - \partial W\delta x),$$
ubi est
$$\partial x\delta W - \partial W\delta x = \partial x\left(N\omega + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial x^3} + \text{etc.}\right).$$
Tum vero est
$$\partial x\delta v - \partial v\delta x = \partial x\int\partial x\left(\mathfrak{N}\omega + \frac{\mathfrak{P}\partial\omega}{\partial x} + \frac{\mathfrak{Q}\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}\partial^3\omega}{\partial x^3} + \text{etc.}\right)$$
ob $\partial v\delta x = \mathfrak{V}\partial x\delta x$. Quibus substitutis colligitur variatio quaesita
$$\delta\int V\partial x = V\delta x + \int L\partial x\int\partial x\left(\mathfrak{N}\omega + \frac{\mathfrak{P}\partial\omega}{\partial x} + \frac{\mathfrak{Q}\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}\partial^3\omega}{\partial x^3} + \text{etc.}\right)$$
$$+ \int\partial x\left(N + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial x^3} + \text{etc.}\right).$$

Quo jam hanc formam ulterius reducamus, ponamus integrale $\int L\partial x = I$ ita sumtum, ut pro initio, unde integrale $\int V\partial x$ capitur, evanescat, pro fine autem ubi integrale $\int V\partial x$ terminatur, fiat $I = A$, sicque fiet

$$\delta\int V\partial x = V\delta x + A\int\partial x\,(\mathfrak{N}\omega + \frac{\mathfrak{P}\partial\omega}{\partial x} + \frac{\mathfrak{Q}\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}\partial^3\omega}{\partial x^3} + \text{etc.})$$
$$- \int I\,\partial x\,(\mathfrak{N}\omega + \frac{\mathfrak{P}\partial\omega}{\partial x} + \frac{\mathfrak{Q}\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}\partial^3\omega}{\partial x^3} + \text{etc.})$$
$$+ \int\partial x\,(N\omega + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial v^3} + \text{etc.})$$

ad quam formam contrahendam statuamus

$$N + (A - I)\,\mathfrak{N} = N',$$
$$P + (A - I)\,\mathfrak{P} = P',$$
$$Q + (A - I)\,\mathfrak{Q} = Q',$$
$$R + (A - I)\,\mathfrak{R} = R',$$
$$\text{etc.}$$

ut prodeat forma illi, quam supra tractavimus, similis

$$\delta\int V\partial x = V\delta x + \int\partial x\,(N'\omega + \frac{P'\partial\omega}{\partial x} + \frac{Q'\partial\partial\omega}{\partial x^2} + \frac{R'\partial^3\omega}{\partial x^3} + \text{etc.}),$$

ubi ergo si post signum integrale differentialia ipsius ω elimenentur, perveniemus secundum §. 86. ad hanc expressionem

$$\delta\int V\partial x = \int\omega\partial x\,(N' - \frac{\partial P'}{\partial x} + \frac{\partial\partial Q'}{\partial x^2} - \frac{\partial^3 R'}{\partial x^3} + \frac{\partial^4 S'}{\partial x^4} - \text{etc.})$$
$$+ V\delta x + \omega\,(P' - \frac{\partial Q'}{\partial x} + \frac{\partial\partial R'}{\partial x^2} - \frac{\partial^3 S'}{\partial x^3} + \text{etc.})$$
$$+ \text{Const.} + \frac{\partial\omega}{\partial x}\,(Q' - \frac{\partial R'}{\partial x} + \frac{\partial\partial S'}{\partial x^2} - \text{etc.})$$
$$+ \frac{\partial\partial\omega}{\partial x^2}\,(R' - \frac{\partial S'}{\partial x} + \text{etc.})$$
$$+ \frac{\partial^3\omega}{\partial x^3}\,(S' - \text{etc.}) + \text{etc.}$$

Constanti autem per integrationem invectae ejusmodi valor tribui debet, ut pro initio integrationis formulae $\int V\partial x$ partes absolutae ad nihilum redigantur, siquidem prima pars integralis ita sumatur, ut pro eodem initio evanescat; tum vero universam expressionem ad finem integrationis, produci oportet pro quo jam posuimus fieri

$$\int L\,\partial x = I = A.$$

Corollarium 1.

111. In parte integrali variabilitas per totam integrationis extensionem debet comprehendi, in partibus autem absolutis, sufficit respexisse ad initium ac finem integrationis, pro utroque autem termino conditiones variationis praescriptae suppeditant valores ∂x, ω, $\frac{\partial\omega}{\partial x}$, $\frac{\partial\partial\omega}{\partial x^2}$, etc. Ac postquam ex conditionibus initii constans rite fuerit determinata, tum superest, ut singula membra ad finem integrationis accommodentur.

Corollarium 2.

112. Pro initio igitur integrationis ubi $I = 0$, erit primo

$$N' = N + A\mathfrak{N},\quad P' = P + A\mathfrak{P},\quad Q' = Q + A\mathfrak{Q},$$
$$R' = R + A\mathfrak{R},\ \text{etc.}$$

pro differentialibus vero ob $\partial I = L\partial x$ erit

$$\frac{\partial N'}{\partial x} = \frac{\partial N}{\partial x} + \frac{A\partial\mathfrak{N}}{\partial x} - L\mathfrak{N},$$

et ita de reliquis; similique modo pro differentialibus secundis

$$\frac{\partial\partial N'}{\partial x^2} = \frac{\partial\partial N}{\partial x^2} + \frac{A\partial\partial\mathfrak{N}}{\partial x^2} - \frac{2L\partial\mathfrak{N}}{\partial x} - \frac{\mathfrak{N}\partial L}{\partial x}.$$

Corollarium 3.

113. Pro fine autem integrationis, ubi $I = A$ fit

$$N' = N,\quad P' = P,\quad Q' = Q,\quad R' = R,\ \text{etc.}$$

valores vero differentiales ita se habebunt

$$\frac{\partial N'}{\partial x} = \frac{\partial N}{\partial x} + L\mathfrak{N},\quad \frac{\partial P'}{\partial x} = \frac{\partial P}{\partial x} - L\mathfrak{P},\quad \frac{\partial Q'}{\partial x} = \frac{\partial Q}{\partial x} - L\mathfrak{Q},\ \text{etc.}$$

secundi vero gradus hoc modo

$$\frac{\partial\partial N'}{\partial x^2} = \frac{\partial\partial N}{\partial x^2} - \frac{2L\partial N}{\partial x} - \frac{\mathfrak{N}\partial L}{\partial x},$$
$$\frac{\partial\partial P'}{\partial x^2} = \frac{\partial\partial P}{\partial x^2} - \frac{2L\partial\mathfrak{P}}{\partial x} - \frac{\mathfrak{P}\partial L}{\partial x},$$

et ita porro.

Scholion 1.

114. Quamquam natura variationum atque etiam quaestionum eo pertinentium jam satis est explicata, tamen hujus argumenti tam dignitas quam novitas ampliorem illustrationem requirere videntur, cum ne superfluum quidem foret eadem saepius inculcari. Cum igitur ante geometria et hujus calculi applicatione ad maxima et minima usi simus, ad hanc doctrinam magis explanandam, hic rem generalius pro sola Analysi contemplabimur. Primo igitur spectatur relatio quaecunque inter binas variabiles x et y, sive ea sit cognita, sive demum definienda, indeque formata consideratur formula integralis quaecunque $\int V\partial x$, quae intra certos terminos comprehensa, seu integratione a dato initio ad datum finem extensa, utique certum quendam valorem recipere debet. Tum illa relatio inter x et y, quaecunque fuerit, quomodocunque infinite parum immutetur, ut singulis x, variationibus quibuscunque δx auctis, jam respondeant eaedem y, variationibus quoque quibuscunque δy auctae, ubi quidem observandum est, tam in initio quam in fine rationum harum variationum per conditiones quaestionum dari, in medio autem istas variationes ita generaliter assumi, ut nulla plane lege inter se connectantur. Tum ex hac relatione variata ejusdem formulae integralis $\int V\partial x$ ab eodem initio ad eundem finem expansus, seu intra eosdem terminos contentus, definiri concipitur; ac tota jam quaestio in hoc versatur, ut hujus postremi valoris variati excessus supra priorem illum valorem formulae $\int V\partial x$ investigetur. Qui excessus cum per $\delta\int V\partial x$, quae forma ipsa est variatio formulae $\int V\partial x$, indicetur, hujus quaestionis solutionem hactenus dedimus ita late patentem, ut omnes casus quibus quantitas V est functio quaecunque non solum ipsarum x, y, p, q, r, s, etc. sed etiam insuper formulam quandam integralem $v = \int \mathfrak{V}\partial x$ utcunque involvens, in se complectatur.

Scholion 2.

115. Quod in praecedente capite tacite assumsimus de quantitate constante variationi inventae adjicienda, quippe quam pars integralis variationis sponte involvit, hoc in istius problematis solutione accuratius exponere est visum. Cum scilicet in hujusmodi quaestionibus, quae ad formulas integrales reducuntur, perpetuo ad terminos integrationis sit respiciendum, siquidem integrale nihil aliud est nisi summa elementorum a termino dato seu initio ad alium terminum seu finem continuatorum, haec consideratio prorsus essentialis est omni integrationi, sine qua idea valoris integralis ne consistere quidem potest. Quamobrem constitutis integrationis terminis initio scilicet et fine, statim ac variationis pars integralis ita est accepta ut pro initio evadat nulla, tum ejusmodi constantem adjici oportet, ut etiam partes absolutae pro eodem initio destruantur, sicque universa variationis expressio ad nihilum redigatur. Quod cum fuerit factum, ad finem integrationis demum progredi licet, ut hoc pacto vera variatio formulae integralis positae ab initio ad finem extensae obtineatur. Haec autem variationum doctrina ad duplicis generis quaestiones accommodari potest; dum in altero relatio inter variabiles x et y data assumitur, et formulae integralis itidem datae $\int V \partial x$ variatio investigatur, postquam per totam integrationis extensionem variabilibus x et y variationes quaecunque fuerint tributae, in altero autem genere ipsa illa variabilium x et y relatio quaeritur, ut formulae integralis $\int V \partial x$ variatio certa proprietate sit praedita; quemadmodum si ea formula maximum minimumve valorem recipere debeat, hanc variationem in nihilum abire necesse est. Ubi iterum duo casus se offerunt, prout maximum minimumve locum habere debet, vel quaecunque variationes ipsis x et y tribuantur, vel si tantum hae variationes certae cuidam legi adstringantur. Ex quo manifestum est, hanc Theoriam multo latius patere, quam quidem ea adhuc in usum est vocata.

Problema 10.

116. Si functio V praeter binas variabiles x, y, cum suis valoribus differentialibus

$$p = \frac{\partial y}{\partial x}, \quad q = \frac{\partial p}{\partial x}, \quad r = \frac{\partial q}{\partial x}, \quad \text{etc.}$$

etiam duas pluresve formulas integrales

$$v = \int \mathfrak{V} \partial x, \quad v' = \int \mathfrak{V}' \partial x, \quad \text{etc.}$$

involvat, ut sit

$$\partial \mathfrak{V} = \mathfrak{M} \partial x + \mathfrak{N} \partial y + \mathfrak{P} \partial p + \mathfrak{Q} \partial q + \mathfrak{R} \partial r + \text{etc.}$$
$$\partial \mathfrak{V}' = \mathfrak{M}' \partial x + \mathfrak{N}' \partial y + \mathfrak{P}' \partial p + \mathfrak{Q}' \partial q + \mathfrak{R}' \partial r + \text{etc.}$$

atque differentiali sumto

$$\partial V = L \partial v + L' \partial v' + M \partial x + N \partial y + P \partial p + Q \partial q + \text{etc.}$$

invenire variationem formulae integralis $\int V \partial x$.

Solutio.

Si hujus problematis solutio eodem modo instituatur ac praecedentis, mox patebit, calculum a geminata formula integrali

$$v = \int \mathfrak{V} \partial x \text{ et } v' = \int \mathfrak{V}' \partial x$$

non turbari, neque etiam si plures ejusmodi involverentur. Quare tota solutio tandem huc redibit, ut constitutis integrationis terminis, primo integralia

$$\int L \partial x = I \text{ et } \int L' \partial x = I'$$

ita sint capienda, ut pro initio integrationis evanescant, tum vero pro fine integrationis fiat $I = A$ et $I' = A'$; quibus quantitatibus inventis statuatur porro

$$N + (A - I) \mathfrak{N} + (A' - I') \mathfrak{N}' = N',$$
$$P + (A - I) \mathfrak{P} + (A' - I') \mathfrak{P}' = P',$$

$$Q + (A - I)\,\mathfrak{Q} + (A' - I')\,\mathfrak{Q}' = Q',$$
$$R + (A - I)\,\mathfrak{R} + (A' - I')\,\mathfrak{R}' = R',$$
etc.

eritque variatio quaesita, dum utrique variabili x et y variationes quaecunque tribuuntur, ex praecedentis solutione:

$$\begin{aligned}\delta\int V\partial x = \int\omega\partial x\,(N' - \tfrac{\partial P'}{\partial x} + \tfrac{\partial\partial Q'}{\partial x^2} - \tfrac{\partial^3 R'}{\partial x^3} + \tfrac{\partial^4 S'}{\partial x^4} - \text{etc.})\\ + V\delta x + \omega\,(P' - \tfrac{\partial Q'}{\partial x} + \tfrac{\partial\partial R'}{\partial x^2} - \tfrac{\partial^3 S'}{\partial x^3} + \text{etc.})\\ + \text{Const.} + \tfrac{\partial\omega}{\partial x}\,(Q' - \tfrac{\partial R'}{\partial x} + \tfrac{\partial\partial S'}{\partial x^2} - \text{etc.})\\ + \tfrac{\partial\partial\omega}{\partial x^2}\,(R' - \tfrac{\partial S'}{\partial x} + \text{etc.})\\ + \tfrac{\partial^3\omega}{\partial x^3}\,(S' - \text{etc.}) + \text{etc.}\end{aligned}$$

ubi commoditatis gratia elementum ∂x constans est assumtum.

Corollarium.

117. Si ergo etiam plures hujusmodi formulae integrales $\int\mathfrak{V}\partial x$ in functionem V quomodocunque ingrediantur, expressio variationis quaesitae inde non mutatur, sed tantum quantitates N′, P′, Q′, R′, etc. ex iis rite definiri convenit.

Scholion.

118. Etsi formulae integrales

$$I = \int L\partial x, \quad I' = \int L'\partial x,$$

binas variabiles involvunt, ideoque valores fixos recipere non posse videntur, tamen perpendendum est, in omnibus hujusmodi quaestionibus semper certam quandam relationem inter binas variabiles x et y supponi, sive ea absolute detur, sive demum per calculum definiri debeat. Hac igitur ipsa relatione jam in usum vocata, ut quantitas y instar functionis ipsius x spectari possit, formulae illae integrales utique determinatos valores sortientur.

Problema 11.

119. Si functio $\mathfrak{V}$ praeter variabiles x et y, earumque valores differentiales p, q, r, s, etc. ipsam quoque formulam integralem $u = \int \mathfrak{v} \partial x$ involvat, ut ejus differentiale sit

$$\partial \mathfrak{V} = \mathfrak{L} \partial u + \mathfrak{M} \partial x + \mathfrak{N} \partial y + \mathfrak{P} \partial p + \mathfrak{Q} \partial q + \mathfrak{R} \partial r + \text{etc.}$$

existente

$$\partial \mathfrak{v} = \mathfrak{m} \partial x + \mathfrak{n} \partial y + \mathfrak{p} \partial p + \mathfrak{q} \partial q + \mathfrak{r} \partial r + \text{etc.}$$

tum vero sit V functio quaecunque ipsarum x, y, p, q, r, etc. insuperque formulae integralis $v = \int \mathfrak{V} \partial x$, ut sit

$$\partial V = L \partial v + M \partial x + N \partial y + P \partial p + Q \partial q + R \partial r + \text{etc.}$$

invenire variationem formulae integralis $\int V \partial x$.

Solutio.

Ex problemate 9. statim invenimus variationem formulae integralis $\int \mathfrak{V} \partial x = v$; constitutis enim integrationis terminis sumtoque integrali $\int \mathfrak{L} \partial x = \mathfrak{J}$, ita ut evanescente pro integrationis initio, pro fine fiat $\mathfrak{J} = \mathfrak{A}$, tum fiat brevitatis gratia

$$\mathfrak{N} + (\mathfrak{A} - \mathfrak{J})\, \mathfrak{n} = \mathfrak{N}', \quad \mathfrak{P} + (\mathfrak{A} - \mathfrak{J})\, \mathfrak{p} = \mathfrak{P}',$$
$$\mathfrak{Q} + (\mathfrak{A} - \mathfrak{J})\, \mathfrak{q} = \mathfrak{Q}', \text{ etc.}$$

erit ex illius problematis solutione

$$\delta v = \mathfrak{V} \delta x + \int \partial x \left(\mathfrak{N}' \omega + \frac{\mathfrak{P}' \partial \omega}{\partial x} + \frac{\mathfrak{Q}' \partial \partial \omega}{\partial x^2} + \frac{\mathfrak{R}' \partial^3 \omega}{\partial x^3} + \text{etc.} \right)$$

posito $\omega = \delta y - p \delta x$ et sumto ∂x constante.

Jam vero cum quaeratur $\delta \int V \partial x$, ob

$$\delta \int V \partial x = V \delta x + \int (\partial x \delta V - \partial V \delta x),$$

posito brevitatis ergo

$$\partial V = L\partial v + \partial W \quad \text{et} \quad \delta V = L\delta v + \delta W,$$

ut sit

$$\partial W = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

erit ut ibidem vidimus

$$\delta \int V\partial x = V\delta x + \int (L\partial x\delta v - L\partial v\delta x)$$
$$+ \int \partial x \left(N\omega + \frac{P\partial\omega}{\partial x} + \frac{Q\partial\partial\omega}{\partial x^2} + \frac{R\partial^3\omega}{\partial x^3} + \text{etc.}\right),$$

ubi si loco ∂v et δv valores modo inventi substituantur, erit

$$\partial x\delta v - \partial v\delta x = \partial x \int \partial x \left(\mathfrak{N}'\omega + \frac{\mathfrak{P}'\partial\omega}{\partial x} + \frac{\mathfrak{Q}'\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}'\partial^3\omega}{\partial x^3} + \text{etc.}\right).$$

Nunc ponatur $\int L\partial x = I$, integrali ita sumto ut evanescat in integrationis initio, in fine autem fiat $I = A$, et habebimus

$$\int L(\partial x\delta v - \partial v\delta x) = \int (A - I)\, \partial x \left(\mathfrak{N}'\omega + \frac{\mathfrak{P}'\partial\omega}{\partial x} + \frac{\mathfrak{Q}'\partial\partial\omega}{\partial x^2} + \frac{\mathfrak{R}'\partial^3\omega}{\partial x^3} + \text{etc.}\right).$$

Restituantur pro $\mathfrak{N}'$, $\mathfrak{P}'$, $\mathfrak{Q}'$, $\mathfrak{R}'$, etc. valores supra assumti, et ad calculum contrahendum ponatur

$$N + (A - I)\,\mathfrak{N} + (A - I)(\mathfrak{A} - \mathfrak{I})\,\mathfrak{n} = N',$$
$$P + (A - I)\,\mathfrak{P} + (A - I)(\mathfrak{A} - \mathfrak{I})\,\mathfrak{p} = P',$$
$$Q + (A - I)\,\mathfrak{Q} + (A - I)(\mathfrak{A} - \mathfrak{I})\,\mathfrak{q} = Q',$$
$$R + (A - I)\,\mathfrak{R} + (A - I)(\mathfrak{A} - \mathfrak{I})\,\mathfrak{r} = R',$$
etc.

ac manifestum est, fore variationem quaesitam

$$\delta \int V\partial x = V\delta x + \int \partial x \left(N'\omega + \frac{P'\partial\omega}{\partial x} + \frac{Q'\partial\partial\omega}{\partial x^2} + \frac{R'\partial^3\omega}{\partial x^3} + \text{etc.}\right),$$

quae forma porro evolvitur in eandem expressionem, quam sub finem problematis 9. (§. 110.) exhibuimus, quam ergo hic denuo opponere foret superfluum.

Corollarium 1.

120. Hic ergo formula integralis $\int V\partial x$, cujus variationem assignavimus ita est comparata, ut non solum functio V formulam

integralem $\int \mathfrak{V}\partial x$ involvat, sed etiam haec functio $\mathfrak{V}$ aliam formulam integralem $\int \mathfrak{v}\partial x$ in se complectatur; ubi quidem functio $\mathfrak{v}$ nullam amplius formulam integralem implicat.

Corollarium 2.

121. Sin autem et haec functio $\mathfrak{v}$ insuper formulam integralem in se involvat, jam satis perspicuum est, quomodo tum solutionem institui oporteat; siquidem tum valores N', P', Q', R'; etc. partes insuper recipient, a postrema formula integrali pendentes.

Scholion.

122. Quomodocunque ergo formula integralis $\int V\partial x$ fuerit complicata, praecepta hactenus exposita omnino sufficiunt ad ejus variationem investigandam, etiamsi forte complicatio fuerit infinita. Cum igitur omnes expressiones binas variabiles implicantes, quarum variationes unquam sint investigandae, vel a formulis integralibus sint liberae, vel unam pluresve in se complectantur, easque vel simplices vel complicatas utcunque, huic Calculi variationum parti, quae circa duas variabiles versatur, abunde satisfactum videtur, ut vix quicquam amplius desiderari queat. Quamobrem ad formulas trium variabilium progrediamur ac primo quidem tales, quarum relatio per geminam aequationem definiri ponitur, ut binae variabiles tanquam functiones tertiae spectari queant, sive haec duplex relatio sit cognita, sive ex ipsa variationis indole investiganda.