CAPUT III.

DE VARIATIONE FORMULARUM INTEGRALIUM SIMPLICIUM DUAS VARIABILES INVOLVENTIUM.

Definitio 6.

70.

Formulam integralem simplicem hic appello, quae nulla alia integralia in se involvit, sed simpliciter integrale refert formulae differentialis, praeter binas variabiles quaecunquae earum differentialia complectentis.

Corollarium 1.

71. Si ergo x et y sint binae variabiles, formula integralis $\int W$ erit simplex, si expressio W praeter has variabiles tantum earum differentialia, cujuscunque fuerint ordinis, contineat, neque praeterea alias formulas integrales in se implicet.

Corollarium 2.

72. Quod si ergo statuamus

$$\partial x = p \partial x, \quad \partial p = q \partial x, \quad \partial q = r \partial x, \text{ etc.}$$

ut species differentialium tollatur, quoniam integratio requirit formulam differentialem, expressio illa W semper reducetur ad hujusmodi formam $V \partial x$, existente V functione quantitatum x, y, p, q, etc.

Corollarium 3.

73. Dum igitur formula integralis simplex sit hujusmodi $\int V\partial x$, ubi V est functio quantitatum x, y, p, q, r, etc. ejus indolem commodissime differentiale ejus repraesentabit, si dicamus esse

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

Scholion.

74. Distinguo hic formulas integrales simplices a complicatis, in quibus integratio proponitur ejusmodi formularum differentialium, quae jam ipsae unam pluresve formulas integrales involvunt. Veluti si littera s denotet integrale

$$\int \sqrt{(\partial y^2 + \partial y^2)} = \int \partial x \sqrt{(1 + pp)},$$

atque quantitas V praeter illas quantitates etiam hanc s contineat, formula integralis $\int V\partial x$ merito censetur complicata, cujus variatio singularia praecepta postulat deinceps exponenda. Hoc autem capite primo methodum formularum integralium simplicium variationes inveniendi tradere constitui.

Theorema 2.

75. *Variatio formulae integralis $\int W$ semper aequalis est integrali variationis ejusdem formulae differentialis, cujus integrale proponitur; seu est $\delta \int W = \int \delta W$.*

Demonstratio.

Cum variatio sit excessus, quo valor variatus cujusque quantitatis superat ejus valorem naturalem, perpendamus formulae propositae $\int W$ valorem variatum, quem obtinet, si loco variabilium x et y earundem valores suis variationibus δx et δy aucti substituantur. Cum autem tum quantitas W abeat in $W + \delta W$, formae propositae valor variatus erit

$$\int (W + \delta W) = \int W + \int \delta W,$$

unde cum sit

$$\delta \int W = \int (W + \delta W) - \int W,$$

habebimus

$$\delta \int W = \int \delta W,$$

unde patet variationem integralis aequari integrali variationis.

Idem etiam hoc modo ostendi potest. Ponatur $\int W = w$, ita ut quaerenda sit variatio δw. Quia ergo sumtis differentialibus est $\partial w = W$, capiantur nunc variationes, eritque

$$\delta \partial w = \delta W = \partial \delta w,$$

ob $\delta \partial w = \partial \delta w$. Nunc vero aequatio $\partial \delta w = \delta W$ denno integrata praebet

$$\delta w = \int \delta W = \delta \int W.$$

Corollarium 1.

76. Proposita ergo hac formula integrali $\int V \partial x$, ejus variatio $\delta \int V \partial x$ erit

$$\int \delta (V \partial x) = \int (V \delta \partial x + \partial x \delta V),$$

hincque ob $\delta \partial x = \partial \delta x$ habebitur

$$\delta \int V \partial x = \int V \partial \delta x + \int \partial x \delta V.$$

Corollarium 2.

77. Posito $\delta x = \omega$ ut sit $\partial \delta x = \partial \omega$, quia est

$$\int V \partial \omega = V \omega - \int \omega \partial V,$$

in priori membro differentiale variationis ∂x exuitur, fietque

$$\delta \int V \partial x = V \delta x - \int \partial V \delta x + \int \partial x \delta V,$$

ubi prima pars ab integratione est immunis.

Scholion.

78. Quemadmodum supra ostendimus, signa differentiationis ∂ cum signo variationis δ expressioni cuicunque praefixa inter se pro lubitu permutari posse, ita nunc videmus signum integrationis $\int$ cum signo variationis δ permutari posse, cum sit

$$\delta \int W = \int \delta W.$$

Atque hoc etiam ad integrationes repetitas patet, ut si proposita fuerit talis formula $\int\int W$, ejus variatio his modis exhiberi possit

$$\delta \int\int W = \int \delta \int W = \int\int \delta W,$$

ideoque variatio formularum integralium ad variationes expressionum nullam amplius integrationem involventium reducatur, pro quibus inveniendis jam supra praecepta sunt tradita.

Problema 6.

79. Propositis binarum variabilium x et y variationibus δx et δy, si positis

$$\partial y = p\partial x, \quad \partial p = q\partial x, \quad \partial q = r\partial x, \quad \text{etc.}$$

fuerit V functio quaecunque quantitatum x, y, p, q, r, etc. formulae integralis $\int V\partial x$ variationem investigare.

Solutio.

Modo vidimus (§. 77.) hujus formulae integralis variationem ita exprimi, ut sit

$$\delta \int V\partial x = V\delta x - \int \partial V\delta x + \int \partial x\delta V.$$

Jam ad variationem δV elidendam, cum sit V functio quantitatum x, y, p, q, r, etc. statuamus ejus differentiale esse

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

ac simili modo ejus variatio ita erit expressa

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}$$

quibus valoribus substitutis consequimur variationem quaesitam

$$\delta\int V\partial x = V\delta x + \int\partial x\,(M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.})$$
$$-\int\delta x\,(M\partial x + N\partial y + P\partial P + Q\partial q + R\partial r + \text{etc.})$$

ubi cum partes ab M pendentes se destruant, erit partibus secundum litteras N, P, Q, R, etc. separatis variatio

$$\delta\int V\partial x = V\delta x + \int N\,(\partial x\delta y - \partial y\delta x) + \int P\,(\partial x\delta p - \partial p\delta x)$$
$$+ \int Q\,(\partial x\delta q - \partial q\delta x) + \int R\,(\partial x\delta r - \partial r\delta x) + \text{etc.}$$

ubi est uti supra invenimus

$$\partial x\delta p = \partial\delta y - p\partial\delta x, \quad \partial x\delta q = \partial\delta p - q\partial\delta x,$$
$$\partial x\delta r = \partial\delta q - r\partial\delta x, \quad \text{etc.}$$

quibus valoribus substitutis ob $\partial y = p\partial x$ obtinetur

$$\delta\int V\partial x = V\delta x + \int N\partial x\,(\delta y - p\delta x) + \int P\partial\,.\,(\delta y - p\delta x)$$
$$+ \int Q\partial\,.\,(\delta p - q\delta x) + \int R\partial\,.\,(\delta q - r\delta x) + \text{etc.}$$

Ad hanc expressionem ulterius reducendam, notetur esse

$$\delta p - q\delta x = \frac{\partial\delta y - p\partial\delta x - \partial p\delta x}{\partial x} = \frac{\partial\,.\,(\delta y - p\delta x)}{\partial x},$$
$$\delta q - r\delta x = \frac{\partial\delta p - q\partial\delta x - \partial q\delta x}{\partial x} = \frac{\partial\,.\,(\delta p - q\delta x)}{\partial x},$$
$$\delta r - s\delta x = \frac{\partial\delta q - r\partial\delta x - \partial r\delta x}{\partial x} = \frac{\partial\,.\,(\delta q - r\delta x)}{\partial x},$$

etc.

quo pacto quaevis formula ad praecedentem reducitur; unde si brevitas gratia ponamus $\delta y - p\delta x = \omega$, erit ut sequitur

$$\delta y - p\delta x = \omega,$$
$$\delta p - q\delta x = \frac{1}{\partial x}\partial\,\omega,$$
$$\delta q - r\delta x = \frac{1}{\partial x}\partial\,.\,\frac{\partial\omega}{\partial x},$$
$$\delta r - s\delta x = \frac{1}{\partial x}\partial\,.\,\frac{1}{\partial x}\partial\,.\,\frac{\partial\omega}{\partial x},$$

etc.

sicque variationibus litterarum derivatarum p, q, r, etc. ex calculo exclusis adipiscimur variationem quaesitam

$$\delta\int V\delta x = V\delta x + \int N\partial x \,.\, \omega + \int P\partial\omega + \int Q\partial \,.\, \frac{\partial\omega}{\partial x} + \int R\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{\partial\omega}{\partial x}$$
$$+ \int S\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{\partial\omega}{\partial x} + \int T\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{1}{\partial x}\partial \,.\, \frac{\partial\omega}{\partial x} + \text{etc.}$$

cujus formae lex progressionis est manifesta, cujuscunque gradus differentialia in formulam V ingrediantur.

Corollarium 1.

80. Hujus igitur variationis pars prima $V\delta x$ a signo integrationis est immunis, atque adeo solam variationem δx involvit, reliquae vero partes utramque perpetuo eodem modo junctam et in littera

$$\omega = \delta y - p\delta x,$$

comprehensam continet.

Corollarium 2.

81. Secunda pars

$$\int N\partial x \,.\, \omega = \int N\omega\partial x$$

commodius exprimi nequit, tertia vero $\int P\partial\omega$ commodius ita exprimi videtur, ut sit

$$\int P\partial\omega = P\omega - \int\omega\partial P,$$

ac post signum integrale jam ipsa quantitas ω reperiatur.

Corollarium 3.

82. Quarta pars $\int Q\partial \,.\, \frac{\partial\omega}{\partial x}$ simili modo reducitur ad

$$Q\,\frac{\partial\omega}{\partial x} - \int\partial Q \,.\, \frac{\partial\omega}{\partial x},$$

hocque membrum posterius, cum sit $\int\frac{\partial Q}{\partial x} \,.\, \partial\omega$, porro praebet

$$\frac{\partial Q}{\partial x}\,\omega - \int\omega\,\partial \,.\, \frac{\partial Q}{\partial x},$$

ita ut tertia pars resolvatur in haec membra

$$Q \cdot \frac{\partial\omega}{\partial x} - \frac{\partial Q}{\partial x} \cdot \omega + \int \omega \partial \frac{\partial Q}{\partial x}.$$

Corollarium 4.

83. Quinta pars

$$\int R \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x}$$

reducitur primo ad

$$R \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x} - \int \frac{\partial R}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x},$$

tum vero posterius membrum ad

$$\frac{\partial R}{\partial x} \cdot \frac{\partial\omega}{\partial x} - \int \frac{1}{\partial x} \partial \cdot \frac{\partial R}{\partial x} \cdot \partial\omega,$$

hocque tandem ulterius ad

$$\frac{1}{\partial x} \partial \cdot \frac{\partial R}{\partial x} \cdot \omega - \int \omega \partial \frac{1}{\partial x} \cdot \partial \frac{\partial R}{\partial x};$$

ita ut haec quinta pars jam ita exprimatur

$$R \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x} - \frac{\partial R}{\partial x} \cdot \frac{\partial\omega}{\partial x} + \frac{1}{\partial x} \partial \cdot \frac{\partial R}{\partial x} \cdot \omega - \int \omega \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial R}{\partial x},$$

Corollarium 5.

84. Simili modo sexta pars

$$\int S \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x}$$

ita reperitur expressa

$$S \cdot \frac{1}{\partial x} \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x} - \frac{\partial S}{\partial x} \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial\omega}{\partial x} + \frac{1}{\partial x} \partial \cdot \frac{\partial S}{\partial x} \cdot \frac{\partial\omega}{\partial x}$$
$$- \frac{1}{\partial x} \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial S}{\partial x} \cdot \omega + \int \omega \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{1}{\partial x} \partial \cdot \frac{\partial S}{\partial x}.$$

Problema 7.

85. Positis

$$\partial y = p\partial x, \quad \partial p = q\partial x, \quad \partial q = r\partial x, \quad \partial r = s\partial x, \text{ etc.}$$

si V fuerit functio quaecunque quantitatum $x, y, p, q, r, s,$ etc. ita ut sit

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + S\partial s + \text{etc.}$$

formulae integralis $\int V\partial x$ variationem ex utriusque variabilis x et y variatione natam ita exprimere, ut post signum integrale nulla occurrunt variationum differentialia.

Solutio.

In corollariis praecedentis problematis jam omnia ita sunt ad hunc scopum praeparata, ut nihil aliud opus sit, nisi transformationes singularum partium in ordinem redigantur, quo pacto duplicis generis partes obtinentur; uno continente formulas integrales, quas quidem omnes in eandem summam colligere licet, altero partes absolutas quas ita in membra distribuemus, ut secundum ipsas variationes δx et δy earumque differentialia cujusque gradus procedant. Posita autem brevitatis gratia formula $\delta y - p\delta x = \omega$ variatio quaesita ita se habebit

$$\begin{aligned}\delta\int V\partial x = & \int \omega\,\partial x\Big(N - \frac{\partial P}{\partial x} + \frac{1}{\partial x}\partial . \frac{\partial Q}{\partial x} - \frac{1}{\partial x}\partial . \frac{1}{\partial x}\partial . \frac{\partial R}{\partial x} + \frac{1}{\partial x}\partial . \frac{1}{\partial x}\partial . \frac{1}{\partial x}\partial . \frac{\partial S}{\partial x} - \text{etc.}\Big) \\ & + V\delta x + \omega\Big(P - \frac{\partial Q}{\partial x} + \frac{1}{\partial x}\partial . \frac{\partial R}{\partial x} - \frac{1}{\partial x}\partial . \frac{1}{\partial x}\partial . \frac{\partial S}{\partial x} + \text{etc.}\Big) \\ & + \frac{\partial \omega}{\partial x}\Big(Q - \frac{\partial R}{\partial x} + \frac{1}{\partial x}\partial . \frac{\partial S}{\partial x} - \text{etc.}\Big) \\ & + \frac{1}{\partial x}\partial . \frac{\partial \omega}{\partial x}\Big(R - \frac{\partial S}{\partial x} + \text{etc.}\Big) \\ & + \frac{1}{\partial x}\partial . \frac{1}{\partial x}\partial . \frac{\partial \omega}{\partial x}(S - \text{etc.}) + \text{etc.}\end{aligned}$$

cujus formae indoles ex sola inspectione statim est manifesta, ut uberiori illustratione non sit opus.

Corollarium 3.

86. Haec expressio multo simplicior redditur, si elementum ∂x capiatur constans, quo quidem amplitudo expressionis nequaquam restringitur, tum enim fiet

$$\delta\int V\partial x = \int\omega\partial x\,(N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.})$$
$$+ V\delta x + \omega\,(P - \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} - \frac{\partial^3 S}{\partial x^3} + \text{etc.})$$
$$+ \frac{\partial\omega}{\partial x}(Q - \frac{\partial R}{\partial x} + \frac{\partial\partial S}{\partial x^2} - \text{etc.})$$
$$+ \frac{\partial\partial\omega}{\partial x^2}(R - \frac{\partial S}{\partial x} + \text{etc.})$$
$$+ \frac{\partial^3\omega}{\partial x^3}(S - \text{etc.}) + \text{etc.}$$

Corollarium 2.

87. Si quaestio sit de linea curva, prima pars integralis valorem per totam curvam ab initio usque ad terminum, ubi coordinatae x et y subsistunt, congregat, simul omnes variationes in singulis curvae punctis factas complectens, dum reliquae partes absolutae tantum per variationes in extremitate curvae factas definiuntur.

Corollarium 3.

88. Si curvam coordinatis x et y definitam ut datam spectemus, aliaque curva ab ea infinit eparum discrepans consideretur, dum in singulis punctis utrique coordinatae variationes quaecunque tribuantur, expressio inventa indicat, quantum formulae integralis $\int V\partial x$ valor ex curva variata collectus superat ejusdem valorem ex ipsa curva data desumtum.

Corollarium 4.

89. Cum sit $\omega = \delta y - p\delta x$, patet hanc quantitatem ω evanescere, si in singulis punctis variationes δx et δy ita accipiantur, ut sit

$$\delta y : \delta x = p : 1 = \partial y : \partial x.$$

Hoc igitur casu curva variata plane non discrepat a data, ac tota variatio formulae $\int V \partial x$ reducitur ad $V \delta x$.

Scholion 1.

90. Variatio haec pro formula integrali $\int V \partial x$ inventa statim sappeditat regulam, quam olim tradidi pro curva invenienda in qua valor ejusdem formulae integralis sit maximus vel minimus. Illa enim regula postulat, ut haec expressio

$$N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.}$$

nihilo aequalis statuatur. Hic autem statim evidens est, ad id, ut variatio formulae $\int V \partial x$ evanescat, quemadmodum natura maximorum et minimorum exigit, ante omnia requiri, ut prima pars signo integrali contenta evanescat, ex quo fit

$$N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0.$$

Praeterea vero etiam partes absolutas nihilo aequari oportet, in quo applicatio ad utrumque curvae terminum continetur. Ipsa enim curvae natura per illam aequationem exprimitur, quae cum ob differentialia altioris gradus totidem integrationes totidemque constantes arbitrarias assumat, harum constantium determinationi illae partes absolutae inserviunt, ut tam in initio quam in fine quaesita curva certis conditionibus respondeat, veluti ad datas lineas curvas terminetur. Ac si aequatio illa fuerit differentialis ordinis quarti vel adeo altioris, partium quoque absolutarum numerus augetur, quibus effici potest, ut curva quaesita non solum utrinque ad datas lineas terminetur, sed ibidem quoque certa directio, quin etiam si ad altiora differentialia assurgat, certa curvaminis lex praescribi queat. Semper autem applicationem faciendo pulcherrime evenire solet, ut ipsa quaestionum indoles ejusmodi conditiones involvat, quibus per partes absolutas commodissime satisfieri possit.

Scholion 2.

91. Quanta autem mysteria in hac forma, quam pro variatione formulae integralis $\int V \partial x$ invenimus, lateant, in ejus applicatione ad maxima et minima multo luculentius declarare licet, hic tantum observo, partem integralem necessario in istam variationem ingredi. Cum enim rem in latissimo sensu simus complexi, ut in singulis curvae punctis utrique variabili x et y variationes quascunque nulla plane lege inter se connexas tribuerimus, fieri omnino nequit, ut variatio toti curvae conveniens non simul ab omnibus variationibus intermediis pendeat, quippe quibus aliter constitutis necesse est, ut inde totius curvae variatio mutationem perpetuamur. Atque in hoc variatio formularum integralium potissimum dissidet a variatione ejusmodi exgressionum; quales in superiori capite consideravimus, quae unice a variatione ultimis elementis tributa pendet. Ex quo luculenter sequitur, si forte quantitas V ita fuerit comparata, ut formula differentialis $V\partial x$ integrationem admittat, nulla stabilita relatione inter variabiles x et y sique integralis $\int V \partial x$ sit functio absoluta quantitatum x, y, p, q, r, etc. tum etiam ejus variationem tantum a variatione extremorum elementorum pendere posse, sicque partem variationis integralem plane in nihilum abire debere, ex quo sequens insigne Theorema colligitur.

Theorema 3.

92. *Posito* $\partial y = p \partial x$, $\partial p = q \partial x$, $\partial q = r \partial x$, $\partial r = s \partial x$, *etc. si V fuerit ejusmodi functio ipsarum* $x, y, p, q, r, s,$ *etc. ut posito ejus differentiali*

$$\partial V = M \partial x + N \partial y + P \partial p + Q \partial q + R \partial r + S \partial s + \text{etc.}$$

fuerit

$$N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0,$$

sumto elemento ∂x constante, tum formula differentialis $V\partial x$ per se erit integrabilis, nulla stabilita relatione inter variabiles x et y; ac vicissim.

Demonstratio.

Si fuerit

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0,$$

tum formulae integralis $\int V\partial x$ variatio nullam implicat formulam integralem, ideoque pro quovis situ coordinatarum x et y a solis variationibus, quae ipsis in extremo termino tribuuntur, pendet, quod fieri neutiquam posset, si formula $V\partial x$ integrationem respueret, propterea quod tum variatio insuper ab omnibus variationibus intermediis simul necessario penderet; unde sequitur quoties aequatio illa locum habet, toties formulam $V\partial x$ integrationem admittere; ita ut $\int V\partial x$ futura sit certa ac definita functio quantitatum x, y, p, q, r, s, etc. Vicissim autem quoties formula differentialis $V\partial x$ integrationem admittit, ejusque propterea integrale $\int V\partial x$ est vera functio quantitatum x, y, p, q, r, s, etc. toties quoque ejus variatio tantum ab extremis variationibus ipsarum x et y pendet, neque variationes intermediae jam ullo modo afficere possunt. Ex quo necesse est ut variationis pars integralis supra inventa evanescat, id quod fieri nequit, nisi fuerit

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0,$$

sicque Theorema propositum etiam inversum veritati est consentaneum.

Corollarium 1.

93. En ergo insigne criterium, cujus ope formula differentialis duarum variabilium, cujuscunque gradus differentialia in eam ingrediantur, dijudicari potest, utrum sit integrabilis nec ne? Multo

latius ergo patet illo criterio satis noto, quo formularum differentialium primi gradus integrabilitas dignosci solet.

Corollarium 2.

94. Primo ergo si quantitas V sit tantum functio ipsarum x et y nullam differentialium rationem involvens, ut sit

$$\partial V = M\partial x + N\partial y,$$

tum formula differentialis $V\partial x$ integrationem non admittit, nisi sit $N = 0$, hoc est nisi V sit functio ipsius x tantum, quod quidem per se est perspicuum.

Corollarium 3.

95. Proposita autem hujusmodi formula differentiali $v\partial x + u\partial y$, ea cum forma $V\partial x$ ob $\partial y = p\partial x$ comparata, dat $V = u + pu$, ideoque

$$M = \left(\frac{\partial v}{\partial x}\right) + p\left(\frac{\partial u}{\partial x}\right), \quad N = \left(\frac{\partial v}{\partial y}\right) + p\left(\frac{\partial u}{\partial y}\right),$$

et $P = u$, quandoquidem quantitates v et u nulla differentialia implicare sumuntur, Erit ergo

$$\partial P = \partial u = \partial x \left(\frac{\partial u}{\partial x}\right) + \partial y \left(\frac{\partial u}{\partial y}\right).$$

Quara cum criterium integrabilitatis postulet ut sit

$$N - \frac{\partial P}{\partial x} = 0,$$

erit pro hoc casu

$$\left(\frac{\partial v}{\partial y}\right) + p\left(\frac{\partial u}{\partial y}\right) - \left(\frac{\partial u}{\partial x}\right) - p\left(\frac{\partial u}{\partial y}\right) = 0,$$

seu $\left(\frac{\partial v}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\right)$,

quod est criterium jam vulgo cognitum.

Scholion 1.

96. Demonstratio hujus Theorematis omnino est singularis,

cum ex doctrina variationum sit petita, quae tamen ab hoc argumento prorsus est aliena; vix vero alia via patet ad ejus demonstrationem pertingendi. Tum vero hic accuratior cognitio functionum diligenter est animadvertenda, qua ostendimus, formulam integralem $\int V \partial x$ neutiquam ut functionem quantitatum x, y, p, q, r, etc. spectari posse, nisi revera integrationem admittat. Natura enim functionum semper hanc proprietatem habet adjunctam, ut statim atque quantitatibus, quae eam ingrediuntur, valores determinati tribuuntur, ipsa functio ex iis formata determinatum adipiscatur valorem; veluti haec functio xy, si ponamus $x = 2$ et $y = 3$, valorem accipit $= 6$. Longe secus autem evenit in formula integrali $\int y \partial x$, cujus valor pro casu $x = 2$ et $y = 3$ neutiquam assignari potest, nisi inter y et x certa quaedam relatio statuatur; tum autem ea formula abit in functionem unicae variabilis. Formularum ergo integralium, quae integrari nequeunt, natura sollicite a natura functionum distingui debet, cum functiones, statim atque quantitatibus variabilibus, ex quibus conflantur, determinati valores tribuuntur, ipsae quoque determinatos valores recipiant, etiamsi variabiles nullo modo a se invicem pendeant; quod minime evenit in formulis integralibus, quippe quarum determinatio omnes plane valores intermedios simul includit. Imprimis autem huic discrimini universa doctrina de maximis et minimis, ad quam hic attendimus, innititur, ubi formulas, quibus maximi minimive proprietates conciliari debet, necessario ejusmodi integrales esse oportet, quae per se integrationem non admittant.

Scholion 2.

97. Ad majorem Theorematis illustrationem ejusmodi formulam integralem $\int V \partial x$ consideremus, quae per se sit integrabilis, ponamusque exempli gratia

$$\int V \partial x = \frac{x \partial y}{y \partial x} = \frac{xp}{y},$$

ita ut sit

$$V = \frac{p}{y} - \frac{xpp}{yy} + \frac{xq}{y},$$

atque ideo haec formula differentialis

$$\left(\frac{p}{y} - \frac{xpp}{yy} + \frac{xq}{y}\right) \partial x,$$

sit absolute integrabilis; ac videamus, an Theorema nostrum hanc integrabilitatem declaret? Quantitatem ergo V differentiemus, et differentiali cum forma

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q$$

comparato, obtinebimus

$$M = \frac{-pp}{yy} + \frac{q}{y}, \quad N = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy},$$

$$P = \frac{1}{y} - \frac{2xp}{yy} \text{ et } Q = \frac{x}{y}.$$

Cum nunc secundum Theorema fieri debeat

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} = 0,$$

primo colligimus differentiando

$$\frac{\partial P}{\partial x} = \frac{-3p}{yy} + \frac{4xpp}{y^3} - \frac{2xq}{yy} \text{ et } \frac{\partial Q}{\partial x} = \frac{1}{y} - \frac{xp}{yy},$$

tum vero

$$\frac{\partial\partial Q}{\partial x^2} = \frac{-2p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy}.$$

Ergo

$$\frac{\partial P}{\partial x} - \frac{\partial\partial Q}{\partial x^2} = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy},$$

cui valori quantitas N utique est aequalis.

Scholion 3.

98. Caeterum quando formula differentialis $V\partial x$ integrationem per se admittit, ideoque posito

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

secundum Theorema est

$$N - \frac{\partial P}{\partial x} + \frac{\partial\partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.} = 0,$$

hinc alia insignia consectaria deducuntur. Primo enim cum per ∂x multiplicando et integrando fiat

$$\int N\partial x - P + \frac{\partial Q}{\partial x} - \frac{\partial\partial R}{\partial x^2} + \frac{\partial^3 S}{\partial x^3} - \text{etc.} = A,$$

patet etiam formulam $N\partial x$ absolute esse integrabilem. Deinde cum hinc porro fiat

$$\int \partial x (\int N\partial x - P) + Q - \frac{\partial R}{\partial x} + \frac{\partial\partial S}{\partial x^2} - \text{etc.} = Ax + B.$$

etiam formula

$$\partial x (\int N\partial x - P),$$

integrationem admittit. Postea etiam simili modo integrabilis erit haec forma

$$\partial x [\int \partial x (\int N\partial x - P) + Q],$$

tum vero etiam haec

$$\partial x [\int \partial x (\int \partial x (\int N\partial x - P) + Q) - R],$$

et ita porro. Unde sequens Theorema non minus notatu dignum et in praxi utilissimum colligimus.

Theorema 4.

99. *Positis $\partial y = p\partial x$, $\partial p = q\partial x$, $\partial q = r\partial x$, $\partial r = s\partial x$, etc. si V ejusmodi fuerit functio ipsarum x, y, p, q, r, s, etc, ut formula differentialis $V\partial x$ per se sit integrabilis, tum posito*

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + S\partial s + \text{etc.}$$

etiam sequentes formulae differentiales per se integrationem admittent:

I. Formula $N\partial x$ erit per se integrabilis;

tum posito $P - \int N\partial x = \mathfrak{P}$,

II. Formula $\mathfrak{P}\partial x$ erit per se integrabilis;
porro posito $Q - \int \mathfrak{P}\partial x = \mathfrak{Q}$,

III. Formula $\mathfrak{Q}\partial x$ erit per se integrabilis;
deinde posito $R - \int \mathfrak{Q}\partial x = \mathfrak{R}$,

IV. Formula $\mathfrak{R}\partial x$ erit per se integrabilis;
ulterius posito $S - \int \mathfrak{R}\partial x = \mathfrak{S}$,

V. Formula $\mathfrak{S}\partial x$ erit per se integrabilis;
et ita porro.

Demonstratio.

Hujus Theorematis veritas jam in praecedente §. est evicta, unde simul patet, si omnes hae formulae integrationem admittant, etiam principalem $V\partial x$ absolute fore integrabilem.

Corollarium 1.

100. Cum V sit functio quantitatum

$$x,\ y,\ p = \frac{\partial y}{\partial x},\quad q = \frac{\partial p}{\partial x},\quad r = \frac{\partial q}{\partial x},\ \text{etc.}$$

quantitates per differentiationem inde derivatae M, N, P, Q, R, etc. etiam ita exhiberi possunt, ut sit

$$M = \left(\frac{\partial V}{\partial x}\right);\quad N = \left(\frac{\partial V}{\partial y}\right);\quad P = \left(\frac{\partial V}{\partial p}\right),\quad Q = \left(\frac{\partial V}{\partial q}\right),\ \text{etc.}$$

unde ob primam formulam patet, si fuerit formula $V\partial x$ integrabilis, tum etiam formulam $\left(\frac{\partial V}{\partial y}\right)\partial x$ fore integrabilem.

Corollarium 2.

101. Deinde ergo quoque ob eandem rationem formula haec $\left(\frac{\partial\partial V}{\partial y^2}\right)\partial x$, hincque porro istae

$$\left(\frac{\partial^3 V}{\partial y^3}\right)\partial x,\quad \left(\frac{\partial^4 V}{\partial y^4}\right)\partial x,\ \text{etc.}$$

omnes per se integrationem admittent.

Corollarium 3.

102. Quia tot tantum litterae P, Q, R, etc. adsunt, quoti gradus differentialia in formula $V\partial x$ reperiuntur, et sequentes omnes evanescunt, litterae germanicae inde derivatae $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{S}$, etc. tandem evanescere vel in functiones solius quantitatis x abire debent, quia alioquin sequentes integrabilitates locum habere non possent.

Exemplum.

103. *Sit V ejusmodi functio, ut fiat*

$$\int V\partial x = \frac{y\,(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{x\partial x\partial\partial y}.$$

Factis substitutionibus

$$\partial y = p\partial x,\quad \partial p = q\partial x,\quad \partial q = r\partial x,\ \text{etc.}$$

pro hoc exemplo functio V ita exprimetur

$$V = \frac{p\,(1+pp)^{\frac{3}{2}}}{xq} - \frac{y\,(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3yp\sqrt{(1+pp)}}{x} - \frac{yr\,(1+pp)^{\frac{3}{2}}}{xqq},$$

unde per differentiationem elicuimus sequentes valores

$$N = \frac{-(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3p\sqrt{(1+pp)}}{x} - \frac{r\,(1+pp)^{\frac{3}{2}}}{xqq},$$

$$P = \frac{(1+4pp)\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y\,(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3\,ypr\sqrt{(1+pp)}}{xqq},$$

$$Q = \frac{-p\,(1+pp)^{\frac{3}{2}}}{xqq} + \frac{y\,(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr\,(1+pp)^{\frac{3}{2}}}{xq^3},$$

$$R = \frac{-y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Jam igitur primo integrabilem esse oportet formulam $N\partial x$ seu

$$-\frac{\partial x(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3p\partial x\sqrt{(1+pp)}}{x} - \frac{\partial q(1+pp)^{\frac{3}{2}}}{xqq},$$

unde statim patet integrale hoc fore

$$\int N\partial x = \frac{(1+pp)^{\frac{3}{2}}}{xq}.$$

Jam pro secunda formula hinc nanciscimur

$$\mathfrak{P} = P - \int N\partial x = \frac{3pp\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypr\sqrt{(1+pp)}}{xqq},$$

ita ut integranda sit haec formula

$$\mathfrak{P}\partial x = \frac{3p\partial y\sqrt{(1+pp)}}{xq} - \frac{3yp\partial x\sqrt{(1+pp)}}{xxq} + \frac{3y\partial x(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3yp\partial q\sqrt{(1+pp)}}{xqq},$$

cujus integrale, vel saltem ejus pars ex postremo membro manifesto colligitur $\frac{3yp\sqrt{(1+pp)}}{qx}$, cujus differentiale cum totam formulam exhauriat erit

$$\int \mathfrak{P}\,\partial x = \frac{3yp\sqrt{(1+pp)}}{xq}.$$

Nunc pro tertia formula habebimus

$$\mathfrak{Q} = Q - \int \mathfrak{P}\partial x = \frac{-p(1+pp)^{\frac{3}{2}}}{xqq} + \frac{y(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr(1+pp)^{\frac{5}{2}}}{xq^3} - \frac{3yp\sqrt{(1+pp)}}{xq},$$

unde per ∂x multiplicando, ob $\partial x = \frac{\partial p}{q}$ in ultimo membro fit

$$\mathfrak{Q}\partial x = \frac{-\partial y(1+pp)^{\frac{3}{2}}}{x\,qq} + \frac{y\partial x(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2y\partial q(1+pp)^{\frac{3}{2}}}{xq^3} - \frac{3yp\partial p\sqrt{(1+pp)}}{xqq},$$

cujus penultimum membrum declarat integrale

$$\int \mathfrak{Q}\partial x = \frac{-y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Quarta porro formula ita erit comparata

$$\mathfrak{R} = R - \int \mathfrak{Q}\partial x = 0,$$

unde perspicuum est, non solum hanc $\mathfrak{R}\partial x$ sed etiam sequentes omnes fore integrabiles.

Scholion.

104. Theoremata haec eo pulchriora videntur, quod eorum demonstratio ejusmodi principio innititur, cujus ratio hinc prorsus est aliena; propterea quod in his veritatibus nullum amplius vestigium variationum apparet; ex quo nullum est dubium quin demonstratio etiam ex alio fonte magis naturali hauriri queat.