CAPUT II.

DE VARIATIONE FORMULARUM DIFFERENTIALIUM DUAS VARIABILES INVOLVENTIUM.

Theorema 1.

37.

Variatio differentialis semper aequalis est differentiali variationis, seu est $\delta\partial V = \partial\delta V$, *quaecunque fuerit quantitas* V, *quae dum per differentialia crescit, etiam variationem recipit.*

Demonstratio.

Quantitas variabilis V spectari potest tanquam applicata curvae cujuspiam, quae suis differentialibus per eandem curvam progrediatur, suis variationibus vero in aliam curvam illi proximam transiliat. Dum autem in ejusdem curvae punctum proximum promovetur, fit ejus valor $= V + \partial V$, qui sit $= V'$, ideoque $\partial V = V' - V$; ex quo variatio ipsius ∂V hoc est $\delta\partial V$ erit $= \delta V' - \delta V$. Verum $\delta V'$ est valor proximus, in quem δV suo differentiali auctum abit, ita ut sit $\delta V' = \delta V + \partial\delta V$, seu $\delta V' - \delta V = \partial\delta V$; unde evidens est fore $\delta\partial V = \partial\delta V$, seu variationem differentialis esse aequalem differentiali variationis, prorsus uti Theorema affirmat.

Corollarium 1.

38. Hinc variatio differentialis secundi $\partial\partial V$ ita definitur,

ut sit $\delta\partial\partial V = \partial\delta . \partial V$, at cum sit $\delta\partial V = \partial\delta V$, aequalitas erit inter has formulas

$$\delta\partial\partial V = \partial\delta\partial V = \partial\partial\delta V,$$

Corollarium 2.

39. Eodem modo pro differentialibus tertii ordinis erit

$$\delta\partial^3 V = \partial\delta\partial\partial V = \partial\partial\delta\partial V = \partial^3 \delta V,$$

et pro differentialibus quarti ordinis variatio ita se habebit ut sit

$$\delta\partial^4 V = \partial\delta\partial^3 V = \partial\partial\delta\partial\partial V = \partial^3 \delta\partial V = \partial^4\delta V,$$

similique modo pro altioribus gradibus.

Corollarium 3.

40. Si igitur variatio desideretur differentialis cujuscunque gradus, signum variationis δ, ubicunque libuerit, inter signa differentiationis ∂ inseri potest; in ultimo autem loco positum declarat, variationem differentialis cujusvis gradus aequalem esse differentiali ejusdem gradus ipsius variationis.

Corollarium 4.

41. Cum igitur sit $\delta\partial^n V = \partial^n\delta V$, res semper eo reducitur, ut variationis quantitatis V seu ipsius δV differentialia cujusque gradus capi possint; atque in hac reductione praecipua vis hujus novi calculi est constituenda.

Scholion 1.

42. Vis demonstrationis in hoc potissimum est sita, quod δV abeat in $\delta V'$, si quantitas V suo differentiali increscit, quod quidem ex natura differentialium per se est manifestum; interim tamen juvabit id per Geometriam illustrasse. Pro curva quacunque EF sint coordinatae $AX = x$ et $XY = y$, in qua si per in-

Fig. 3.

tervallum infinite parvum YY' progrediamur, erit in differentialibus

$$AX' = x + \partial x \quad \text{et} \quad X'Y' = y + \partial y,$$

ideoque

$$\partial x = AX' - AX \quad \text{et} \quad \partial y = X'Y' - XY.$$

Nunc concipiamus aliam curvam ef illi proximam, cujus puncta y et y' cum illius punctis Y et Y' comparentur, ad quae propterea per variationes transitus fiat; ac sumtis simili modo coordinatis erit

$$Ax = x + \delta x \quad \text{et} \quad xy = y + \delta y,$$

ideoque

$$\delta x = Ax - AX \quad \text{et} \quad \delta y = xy - XY,$$

tum vero erit

$$Ax' = x + \partial x + \delta(x + \partial x) \quad \text{et}$$
$$x'y' = y + \partial y + \delta(y + \partial y),$$

quatenus a puncto Y' per variationem in punctum y' transilimus. Verum ad idem punctum y' quoque ex puncto y per differentiationem pervenimus, unde colligitur

$$Ax' = x + \delta x + \partial(x + \delta x) \quad \text{et}$$
$$x'y' = y + \delta y + \partial(y + \delta y).$$

His jam valoribus cum illis collatis, prodit

$$x + \partial x + \delta x + \delta\partial x = x + \delta x + \partial x + \partial\delta x \quad \text{et}$$
$$y + \partial y + \delta y + \delta\partial y = y + \delta y + \partial y + \partial\delta y,$$

unde manifesto sequitur fore

$$\delta\partial x = \partial\delta x \quad \text{et} \quad \delta\partial y = \partial\delta y.$$

Quae si attentius consideremus, principium, cui demonstratio innititur, huc redire comperimus, ut si quantitas variabilis primo per differentiationem deinde vero per variationem proferatur, idem proveniat, ac si ordine inverso primo per variationem tum vero per differentiationem promoveretur. Veluti in figura ex puncto Y primo per differentiationem pervenitur in Y', hinc vero per variationem in y': inverso autem ordine primum ex puncto Y per variationem pervenitur in y, hinc vero per differentiationem in punctum y', idem quod ante.

Scholion 2.

43. Theorema hoc latissime patet, neque enim ad casum duarum variabilium tantum restringitur, sed veritati est etiam consentaneum, quotcunque variabiles in calculum ingrediantur, quandoquidem in demonstratione solius illius variabilis cujus tam differentiale quam variatio consideratur, ratio habetur sine ullo respectu ad
Fig. 4. reliquas variabiles. Ne autem hic ulli dubio locus relinquatur, consideremus superficiem quamcunque, cujus punctum quodvis Z per coordinatas ternas

$$AX = x,\ XY = y,\ \text{et}\ YZ = z$$

definiatur, a quo si ad aliud punctum proximum Z' in eadem superficie progrediamur, hae coordinatae suis differentialibus increscent. Tum vero aliam quamcunque superficiem concipiamus proximam, cujus puncta z et z' cum illis Z et Z' conferantur, quod fit per variationem. His positis perspicuum est, duplici modo ad punctum z' perveniri posse, altero per variationem ex puncto Z' altero per differentiale ex puncto z, sicque fore

$$Ax' = AX' + \delta \,.\, AX' = Ax + \partial \,.\, Ax,$$
$$x'y' = X'Y' + \delta \,.\, X'Y' = xy + \partial \,.\, xy,$$
$$y'z' = Y'Z' + \delta \,.\, Y'Z' = yz + \partial \,.\, yz,$$

quod etiam de omnibus aliis quantitatibus variabilibus ad haec puncta referendis valet. Hinc autem luculenter sequitur fore

$$\delta\partial x = \partial\delta x,\ \delta\partial y = \partial\delta y,\ \delta\partial z = \partial\delta z.$$

Scholion 3.

44. Memorabile prorsus est, quod casu differentialium altioris ordinis signum variationis δ pro lubitu inter signa differentiationis ∂ inscribi possit, atque hinc intelligere licet, hanc permutabilitatem locum quoque esse habituram, etiamsi signum variationis δ perinde ac differentiationis ∂ aliquoties repetatur, quod fortasse in aliis speculationibus usu venire posset. Verum in praesenti instituto repetitio variationis δ nullo modo locum habere potest, quoniam lineam vel superficiem tantum cum unica alia sibi proxima comparamus; etsi enim haec generalissime consideratur, ut omnes possibiles itidem proximas in se complectatur, tamen tanquam unica spectatur, neque postquam e principali in proximam transiliverimus, novus transitus in aliam conceditur. Hinc ergo ejusmodi speculationes, quibus variationum variationes essent quaerendae, omnino excluduntur. Vicissim autem hic variationum differentialia cujusque ordinis admitti debent, et cum in formulis differentialibus, quae quidem significatum habent finitum, ratio differentialium tantum spectetur, quae si binae variabiles sint x et y, hujusmodi positionibus

$$\partial y = p\partial x, \quad \partial p = q\partial x, \quad \partial q = r\partial x, \quad \text{etc.}$$

ad formas finitas revocari solent, harum quantitatum p, q, r, etc. variationes potissimum assignari necesse est.

Problema 1.

45. Datis binarum variabilium x et y variationibus δx et δy, formulae differentialis $p = \frac{\partial y}{\partial x}$ variationem definire.

Solutio.

Cum sit

$$\delta\partial y = \partial\delta y \quad \text{et} \quad \delta\partial x = \partial\delta x,$$

variatio quaesita δp per notas differentiationis regulas reperitur, dummodo loco signi differentiationis ∂ scribatur signum variationis δ, unde cum oriatur

$$\delta p = \frac{\partial x \delta \partial y - \partial y \delta \partial x}{\partial x^2},$$

erit per conversionem ante demonstratam

$$\delta p = \frac{\partial x \partial \delta y - \partial y \partial \delta x}{\partial x^2},$$

ubi cum δx et δy sint variationes ipsarum x et y, hincque $\delta x + \partial \delta x$ et $\delta y + \partial \delta y$ variationes ipsarum $x + \partial x$ et $y + \partial y$, notandum est fore uti jam observavimus

$$\partial \delta x = \delta(x + \partial x) - \delta x \text{ et } \partial \delta y = \delta (y + \partial y) - \delta y.$$

Idem invenitur ex primis principiis, cum enim valor ipsius variatus sit $p + \delta p$, isque prodeat, si loco x et y earum valores variati, qui sunt $x + \delta x$ et $y + \delta y$, substituantur, erit

$$p + \delta p = \frac{\partial (y + \delta y)}{\partial (x + \delta x)} = \frac{\partial y + \partial \delta y}{\partial x + \partial \delta x},$$

unde ob $p = \frac{\partial y}{\partial x}$ fit

$$\delta p = \delta \cdot \frac{\partial y}{\partial x} = \frac{\partial y + \partial \delta y}{\partial x + \partial \delta x} - \frac{\partial y}{\partial x} = \frac{\partial x \partial \delta y - \partial y \partial \delta x}{\partial x^2},$$

quoniam in denominatore particula $\partial x \partial \delta x$ prae ∂x^2 evanescit.

Corollarium 1.

46. Si dum per differentialia progredimur, variabiles x et y continuo auctas designemus per x', x'', x''', etc. y', y'', y''', etc. ut sit

$$x' = x + \partial x \text{ et } y' = y + \partial y, \text{ erit}$$
$$\partial \delta x = \delta x' - \delta x \text{ et } \partial \delta y = \delta y' - \delta y,$$

hincque

$$\delta p = \delta \cdot \frac{\partial y}{\partial x} = \frac{\partial x (\delta y' - \delta y) - \partial y (\delta x' - \delta x)}{\partial x^2}.$$

Corollarium 2.

47. Quoniam variationes ambarum variabilium x et y neutiquam a se invicem pendent, sed prorsus arbitrio nostro relinquuntur, si ipsi x nullas tribuamus variationes ut sit

$$\delta x = 0 \text{ et } \delta x' = 0, \text{ erit}$$
$$\delta p = \frac{\partial \delta y}{\partial x} = \frac{\delta y' - \delta y}{\partial x}.$$

Corollarium 3.

48. Si praeterea unicae variabili y variationem δy tribuamus, ut sit $\delta y' = 0$, erit $\delta p = -\frac{\delta y}{\partial x}$, quae hypothesis minime naturae refragatur, quia curvam proximam ita cum principali congruentem assumi licet, ut in unico tantum puncto ab ea discrepet.

Scholion.

49. Vulgo in solutione problematum isoperimetricorum aliorumque ad id genus pertinentium, curva variata ita congruens statui solet, ut tantum in uno quasi elemento discrepet. Ita si quaerenda sit curva EF certa quadam maximi minimive proprietate Fig. 5.
gaudens, unicum punctum Y in locum proximum y transferri solet, ut curva variata $\mathrm{EM}y\mathrm{Y'F}$ tantum in intervallo minimo $\mathrm{MY'}$ a quaesita deflectat ita, ut positis

$$\mathrm{AX} = x \text{ et } \mathrm{XY} = y,$$

sit pro variata curva

$$\mathrm{A}x = x + \delta x \text{ et } xy = y + \delta y, \text{ seu}$$
$$\delta x = \mathrm{A}x - \mathrm{AX} \text{ et } \delta y = xy - \mathrm{XY},$$

pro sequentibus vero punctis, ad quae differentialia ducunt, sit ubique

$$\delta x' = 0,\ \delta y' = 0,\ \delta x'' = 0,\ \delta y'' = 0, \text{ etc.}$$

itemque pro antecedentibus. Quin etiam ad calculi commodum variatio $\mathrm{X}x = \delta x$ nulla sumi solet, ut omnis variatio ad solum elementum δy perducatur, quo casu utique habebitur $\delta p = -\frac{\delta y}{\partial x}$, haecque unica variatio utique sufficit ad problemata hujus generis, quae quidem fuerint tractata, resolvenda. Verum si, uti hic instituimus, haec problemata latius extendimus, ut curva quaesita circa initium et finem certas determinationes recipere queat, utique necessarium est calculum variationum quam generalissime absolvere, atque in omnibus curvae punctis variationes indefinitas coordinatis tribuere. Quod etiam maxime est necessarium, si hujusmodi investigationes ad lineas curvas non continuas accommodare velimus.

Problema 2.

50. Datis binarum variabilium x et y variationibus δx et δy, si ponatur $\partial y = p\partial x$ et $\partial p = q\partial x$, invenire variationem quantitatis q, seu valorem ipsius δq.

Solutio.

Cum sit $q = \frac{\partial p}{\partial x}$, erit pro valore variato

$$q + \delta q = \frac{\partial (p + \delta p)}{\partial (x + \delta x)} = \frac{\partial p + \partial \delta p}{\partial x + \partial \delta x},$$

unde auferendo quantitatem $q = \frac{\partial p}{\partial x}$ relinquitur

$$\delta q = \frac{\partial x \partial \delta p - \partial p \partial \delta x}{\partial x^2},$$

quae variatio ergo etiam ex differentiatione formulae $q = \frac{\partial p}{\partial x}$ resultat, si more consueto differentiatio instituatur, loco vero signi differentialis ∂ scribatur signum variationis δ; ubi quidem meminisse juvabit esse

$$\delta \partial x = \delta \partial x \quad \text{et} \quad \delta \partial p = \partial \delta p.$$

Supra autem invenimus, ob $p = \frac{\partial y}{\partial x}$ esse

$$\delta p = \frac{\partial x \partial \delta y - \partial y \partial \delta x}{\partial x^2},$$

unde porro per consuetam differentiationem valor ipsius $\partial \delta p$ scilicet differentiale ipsius δp colligitur.

Corollarium 1.

51. Cum sit $\frac{\partial y}{\partial x} = p$ et $\frac{\partial p}{\partial x} = q$, erit primo

$$\delta p = \frac{\partial \delta y}{\partial x} - \frac{p \partial \delta x}{\partial x},$$

tum vero

$$\delta q = \frac{\partial \delta p}{\partial x} - \frac{q \partial \delta x}{\partial x}.$$

Pro usu autem futuro praestat hic particulam $\partial \delta p$ relinqui, quam ejus valorem ex praecedente formula erui.

Corollarium 2.

52. Interim tamen cum prior per differentiationem det

$$\partial \delta p = \frac{\partial \partial \delta y}{\partial x} - \frac{\partial \partial x \partial \delta y}{\partial x^2} - \frac{p \partial \partial \delta x}{\partial x} - q \partial \delta x + \frac{p \partial \partial x \partial \delta x}{\partial x^2},$$

hoc valore substituto prodit

$$\delta q = \frac{\partial \partial \delta y}{\partial x^2} - \frac{\partial \partial x \partial \delta y}{\partial x^3} - \frac{p \partial \partial \delta x}{\partial x^2} - \frac{2q \partial \delta x}{\partial x} + \frac{p \partial \partial x \partial \delta x}{\partial x^3}.$$

Corollarium 3.

53. Quod si soli variabili y variationes tribuantur, ut particulae δx et quae inde derivantur evanescant, habebimus

$$\delta p = \frac{\partial \delta y}{\partial x} \quad \text{et} \quad \delta q = \frac{\partial \delta p}{\partial x^2} = \frac{\partial \partial \delta y}{\partial x^2} - \frac{\partial \partial x \partial \delta y}{\partial x^3},$$

ac si differentiale ∂x constans accipiatur, erit $\delta q = \frac{\partial \partial \delta y}{\partial x^2}$.

Scholion 1.

54. Quo haec facilius intelligantur, consideremus in curva EF, per relationem inter variabiles $AX = x$ et $XY = y$, plura Fig. 5.

puncta Y, Y', Y'', etc. secundum differentialia continuo promota, ut sit

$$AX = x,\quad AX' = x + \partial x,\quad AX'' = x + 2\partial x + \partial\partial x,$$
$$AX''' = x + 3\partial x + 3\partial\partial x + \partial^3 x,$$
$$XY = y,\quad X'Y' = y + \partial y,\quad X''Y'' = y + 2\partial y + \partial\partial y,$$
$$X'''Y''' = y + 3\partial y + 3\partial\partial y + \partial^3 y,$$

quae differentialia cujusque ordinis indicantes ita brevitatis gratia repraesententur

$$AX = x,\quad AX' = x',\quad AX'' = x'',\quad AX''' = x''',\ \text{etc.}$$
$$XY = y,\quad X'Y' = y',\quad X''Y'' = y'',\ X'''Y''' = y''',\ \text{etc.}$$

quibus singulis suae variationes nullo modo a se invicem pendentes tribui concipiantur, ita ut omnes istae variationes

$$\delta x,\quad \delta x',\quad \delta x'',\quad \delta x''',\quad \text{etc.}$$
$$\delta y,\quad \delta y',\quad \delta y'',\quad \delta y''',\quad \text{etc.}$$

a lubitu nostro pendentes tanquam cognitae spectari queant. His jam ita constitutis differentialia cujusque ordinis variationum in hunc modum repraesentabuntur, ut sit

$$\partial\delta x = \delta x' - \delta x,\quad \partial\partial\delta x = \delta x'' - 2\delta x' + \delta x,$$
$$\partial^3\delta x = \delta x''' - 3\delta x'' + 3\delta x' - \delta x,$$
$$\partial\delta y = \delta y' - \delta y,\quad \partial\partial\delta y = \delta y'' - 2\delta y' + \delta y,$$
$$\partial^3\delta y = \delta y''' - 3\delta y'' + 3\delta y' - \delta y.$$

Quodsi jam unicum punctum curvae Y variari sumamus, erit

$$\partial\delta x = -\delta x,\quad \partial\partial\delta x = +\delta x,\quad \partial^3\delta x = -\delta x,\quad \text{etc.}$$
$$\partial\delta y = -\delta y,\quad \partial\partial\delta y = +\delta y,\quad \partial^3\delta y = -\delta y,\quad \text{etc.}$$

hincque

$$\delta p = -\frac{\delta y}{\partial x} + \frac{p\delta x}{\partial x} \quad \text{et}$$
$$\delta q = \frac{\partial y}{\partial x^2} + \frac{\partial\partial x\delta y}{\partial x^3} - \frac{p\delta x}{\partial x^2} + \frac{2q\delta x}{\partial x} - \frac{p\partial\partial x\delta x}{\partial x^3},$$

ubi omissis partibus reliquarum respectu evanescentibus, erit

$$\delta q = \delta y \cdot \frac{1}{\partial x^2} - \delta x \cdot \frac{p}{\partial x^2}.$$

Denique si soli applicatae $XY = y$ variatio tribuatur, habebitur

$$\delta p = -\frac{1}{\partial x}\delta y \quad \text{et} \quad \delta q = \frac{1}{\partial x^2} \cdot \delta y.$$

Scholion 2.

55. Hinc patet si in unico curvae puncto variatio statuatur, insigniter contra recepta differentialium principia impingi, cum variationum differentialia superiora neutiquam prae inferioribus evanescant sed jugiter eundem valorem retineant, atque adeo variationes quantitatum p et q in infinitum excrescant, siquidem infinite parva δx et δy ex eodem ordine quo differentialia ∂x et ∂y assumantur. Quin etiam hinc in calculo maxime cavendum est ne in enormes errores praecipitemur, cum calculi praecepta legi continuitatis innitantur, qua lineae curvae continuo puncti fluxu describi concipiuntur, ita ut in earum curvatura nusquam saltus agnoscatur. Quodsi autem unicum curvae punctom Y in y diducatur, reliquo curvae tractu praeter bina quasi elementa My et yY' invariato relicto, evidens est curvaturae ingentem irregularitatem induci, cum vulgares ca culi regulae non amplius applicari queant. Cui incommodo ut occurramus tutissimum erit remedium, ut singulis curvae punctis mente saltem variationes tribuantur, quae continuitatis quapiam lege contineantur, neque ante irregularitas in calculo admittatur, quam omnes differentiationes et integrationes fuerint peractae; hocque modo saltem species continuitatis in calculo retineatur. Quamvis ergo variationum differentialia Fig. 5.

$$\partial\delta y, \quad \partial\partial\delta y, \quad \partial^3\delta y, \quad \text{etc. item}$$
$$\partial\delta x, \quad \partial\partial\delta x, \quad \partial^3\delta x, \quad \text{etc.}$$

forte in facta hypothesi ad simplices variationes revocare liceat, tamen expedit illas formas in calculo retineri ad easque sequentes integrationes accommodari, atque huc etiam redeunt operationes, quas olim, cum idem argumentum de inveniendis curvis maximi minimive proprietate praeditis tractassem, expedire docueram.

Problema 3.

56. Datis binarum variabilium x et y variationibus δx et δy, rationum inter differentialia cujuscunque gradus variationes investigare.

Solutio.

Quaestio huc redit ut positis continuo

$$\partial y = p\partial x,\ \partial p = q\partial x,\ \partial q = r\partial x,\ \partial r = s\partial x,\ \text{etc.}$$

quantitatum p, q, r, s, etc. variationes assignentur, cum ad has quantitates omnes differentialium cujuscunque ordinis rationes, quae quidem finitis valoribus continentur, reducantur. Ac de harum quidem duabus primis p et q jam vidimus esse

$$\delta p = \frac{\partial\delta y}{\partial x} - \frac{p\partial\delta x}{\partial x} \text{ et } \delta q = \frac{\partial\delta p}{\partial x} - \frac{q\partial\delta x}{\partial x}.$$

Quoniam igitur porro est

$$r = \frac{\partial q}{\partial x} \text{ et } s = \frac{\partial r}{\partial x},\ \text{etc.}$$

harum variationes simili modo per differentiationis regulas inveniuntur

$$\delta r = \frac{\partial\delta q}{\partial x} - \frac{r\partial\delta x}{\partial x},\ \delta s = \frac{\partial\delta r}{\partial x} - \frac{s\partial\delta x}{\partial x},\ \text{etc.}$$

ubi si lubuerit loco $\partial\delta p$, $\partial\delta q$, $\partial\delta r$, etc. differentialia variationum δp, δq, δr, etc. ante inventarum substitui possunt. Hoc autem non solum in formulas nimis prolixas induceret, sed etiam uti ex sequentibus patebit, ne quidem est necessarium, cum hinc multo facilius omnes deductiones, quibus opus erit, institui queant.

Corollarium 1.

57. Si soli variabili y variationes tribuantur, seu manentibus abscissis x tantum applicatae y suis variationibus augeantur, habebimus

$$\delta p = \frac{\partial \delta y}{\partial x}, \quad \delta q = \frac{\partial \delta p}{\partial x}, \quad \delta r = \frac{\partial \delta q}{\partial x}, \quad \delta s = \frac{\partial \delta r}{\partial x}.$$

Corollarium 2.

58. Quodsi praeterea omnia ipsius x incrementa ∂x aequalia capiantur, seu elementum ∂x constans statuatur, substitutis differentialibus praecedentium formularum in sequentibus, obtinebitur

$$\delta p = \frac{\partial \delta y}{\partial x}, \quad \delta q = \frac{\partial\partial \delta y}{\partial x^2}, \quad \delta r = \frac{\partial^3 \delta y}{\partial x^3}, \quad \delta s = \frac{\partial^4 \delta y}{\partial x^4}; \text{ etc.}$$

Corollarium 2.

59. Si solis abscissis x variationes tribuantur, ut variatio δy cum omnibus derivatis evanescat, simulque elementum ∂x constans capiatur, singulae hae variationes ita se habebunt

$$\delta p = \frac{-p\,\partial\,\delta x}{\partial x}, \quad \delta q = \frac{-p\,\partial\partial\,\delta x}{\partial x^2} - \frac{2q\,\partial\delta x}{\partial x},$$

$$\delta r = \frac{-p\,\partial^3\,\partial x}{\partial x^3} - \frac{3q\,\partial\partial\,\delta x}{\partial x^2} - \frac{3r\,\partial\,\delta x}{\partial x},$$

$$\delta s = \frac{-p\,\partial^4\,\delta x}{\partial x^4} - \frac{4q\,\partial^3\,\delta x}{\partial x^3} - \frac{6r\,\partial\partial\,\delta x}{\partial x^3} - \frac{4s\,\partial\,\delta x}{\partial x},$$

etc.

Corollarium 4.

60. Etiamsi ergo hoc casu elementum ∂x constans accipiatur, tamen hic occurrunt differentialia cujusque ordinis variationis δx, cujus rei ratio est, quod variationes valorum ipsius x continuo ulterius promotorum x', x'', etc. neutiquam a differentialibus pendere statuuntur.

Scholion.

61. Quando autem placuerit soli variabili x variationes tribuere, tum omnino praestat variabiles x et y inter se permutari, atque hujusmodi potius positionibus uti

$$\partial x = p\partial y, \quad \partial p = q\partial y, \quad \partial q = r\partial y, \text{ etc.}$$

quibus species differentialium tollatur, tum vero sumto elemento ∂y constante, similes formulae simpliciores pro variationibus quantitatum p, q, r, etc. reperiuntur, atque Corollario 2. Caeterum quo calculus ad omnes casus accommodari queat, semper expedit utrique variabili suas variationes tribui, etsi enim tum formae multo perplexiores prodeant, praecipue si evolvantur, tamen calculum prosequendo tam egregia se offerunt compendia, ut in fine calculus vix fiat operosior, neque hujus prolixitatis taedeat. Ad problemata ergo magis generalia ad hoc caput pertinentia progrediamur.

Problema 4.

62. Datis duarum variabilium x et y variationibus δx et δy, formulae cujuscunque finitae V tam ex illis variabilibus ipsis quam earum differentialibus cujuscunque ordinis conflatae variationem invenire.

Solutio.

Cum V sit quantitas valorem habens finitum, ponendo

$$\partial y = p\partial x, \quad \partial p = q\partial x, \quad \partial q = r\partial x, \quad \partial r = s\partial x, \quad \text{etc.}$$

differentialia inde tollentur, prodibitque pro V functio ex quantitatibus finitis formata x, y, p, q, r, s, etc. Quaecunque ergo sit ratio compositionis, ejus differentiale semper hujusmodi habebit formam

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + S\partial s + \text{etc.}$$

horum membrorum numero existente eo majore, quo altiora differentialia ingrediuntur in V. Quodsi vero hujus formulae V variatio δV fuerit indaganda, ea obtinetur si loco quantitatum variabilium x, y, p, q, r, etc. eaedem suis variationibus auctae substituantur, et a forma resultante ipsa quantitas V auferatur, ex quo intelligitur, variationem ope consuetae differentiationis inveniri signo tantum differentialis ∂ in signum variationis δ mutato. Quare cum differentiale supra jam sit exhibitum, impetrabimus variationem quaesitam

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.}$$

quamadmodum autem variationes δp, δq, δr, δs, etc. per variationes sumtas δx et δy determinentur, jam supra est ostensum.

Corollarium 1.

63. Si hic substituamus valores ante inventos, obtinebimus variationem quaesitam ita expressam

$$\delta V = M\delta x + N\partial y + \frac{1}{\partial x}(P\partial\delta y + Q\partial\delta p + R\partial\delta q + S\partial\delta r + \text{etc.}) - \frac{\partial\delta x}{\partial x}(Pp + Qq + Rr + Ss + \text{etc.}).$$

Corollarium 2.

64. Si variabili x nulla plane tribuatur variatio, atque insuper elementum ∂x constans accipiatur, tum quantitatis propositae V variatio ita prodibit expressa

$$\delta V = N\delta y + \frac{P\partial\delta y}{\partial x} + \frac{Q\partial\partial\delta y}{\partial x^2} + \frac{R\partial^3\delta y}{\partial x^3} + \frac{S\partial^4\delta y}{\partial x^4} + \text{etc.}$$

Scholion.

65. In his formis saltem species homogeneitatis in differentialibus spectatur, siquidem δx et δy ad ordinem differentialium

referantur, quod longe secus eveniret, si eo casu quo unicum curvae punctum variatur, statim vellemus loco differentialium variationum valores supra (§. 54.) exhibitos substituere, quo quippe pacto idea integrationis, qua hae formulae deinceps indigent, excluderetur. Caeterum patet quomodo inventio variationum ad consuetam differentiationem revocetur, dum totum discrimen in hoc tantum est situm, ut loco variationum δp, δq, δr, etc. valores jam ante assignati, quos quidem ipsos quoque per consuetam differentiationem elicuimus, substituantur. Conveniet autem hanc operationem aliquot exemplis illustrari, quo clarius indoles totius hujus tractationis percipiatur.

Exemplum 1.

66. *Formulae subtangentem exprimentis* $\frac{y\partial x}{\partial y}$ *variationem invenire.*

Ob $\partial y = p\,\partial x$ haec formula fit $\frac{y}{p}$, unde ejus variatio $\frac{\delta y}{p} - \frac{y\,\delta p}{pp}$, ubi loco δp valore substituto, fit ea

$$\frac{\delta y}{p} - \frac{y\partial\delta y}{pp\partial x} + \frac{y\partial\delta x}{p\partial x} = \frac{\partial x}{\partial y}\cdot\delta y - \frac{y\partial x}{\partial y^2}\cdot\partial\delta y + \frac{y}{\partial y}\,\partial\delta x,$$

quae postrema forma immediate ex differentiatione formulae propositae nascitur.

Exemplum 2.

67. *Formulae ipsam tangentem exprimentis* $\frac{y\sqrt{(\partial x^2 + \partial y^2)}}{\partial y}$ *variationem invenire.*

Posito $\partial y = p\partial x$ praebet hanc formam finitam

$$\frac{y}{p}\sqrt{(1 + pp)},$$

unde variatio quaesita est

$$\frac{\delta y}{p}\sqrt{(1 + pp)} - \frac{y\,\delta p}{pp\sqrt{(1 + pp)}}:$$

quae transformatur in hanc

$$\frac{\sqrt{(\partial x^2 + \partial y^2)}}{\partial y}\,\delta y - \frac{y\,\partial x}{\partial y^2 \sqrt{(\partial x^2 + \partial y^2)}}\,(\partial x \partial \delta y - \partial y \partial \delta x).$$

Exemplum 3.

68. *Formulae radium curvedinis exprimentis* $\frac{(\partial x^2 + \partial y)^{\frac{3}{2}}}{\partial x \partial \partial y}$ *variationem definire.*

Posito $\partial y = p\partial x$ et $\partial p = q\partial x$ haec formula transit in hanc $\frac{(1+pp)^{\frac{3}{2}}}{q}$, cujus propterea variatio est

$$\frac{2p\delta p}{q}\sqrt{(1+pp)} - \frac{\delta q}{qq}(1+pp)^{\frac{3}{2}},$$

ubi quidem substitutioni valorum ante inventorum non immoror.

Problema 5.

69. Datis duarum quantitatum variabilium x et y variationibus δx et δy, formulae tam ex illis variabilibus quam earum differentialibus cujuscunque ordinis conflatae, sive fuerit infinita sive infinite parva, variationem investigare.

Solutio.

Positis ut hactenus $\partial y = p\partial x$, $\partial p = q\partial x$, $\partial q = r\partial x$, etc. formula semper reducetur ad hujusmodi formam $V\partial x^n$, ubi V sit functio finita quantitatum x, y, p, q, r, etc. exponens vero n sive positivus sive negativus, ita ut priori casu formula sit infinite parva, posteriori vero infinite magna. Ponamus igitur differentiationem ordinariam dare

$$\partial V = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

unde simul ejus variatio habetur. Cum igitur formae propositae variatio sit

$$nV\partial x^{n-1}\,\partial\delta x + \partial x^n \delta V,$$

erit utique haec variatio quam quaerimus

$$nV\partial x^{n-1}\,\partial\delta x + \partial x^n\,(M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}),$$

ubi ex superioribus hos valores substitui oportet

$$\delta p = \frac{\partial\delta y - p\partial\delta x}{\partial x};\quad \delta q = \frac{\partial\delta p - q\partial\delta x}{\partial x},$$
$$\delta r = \frac{\partial\delta q - r\partial\delta x}{\partial x},\quad \delta s = \frac{\partial\delta r - s\partial\delta x}{\partial x},$$

quae cum per se sint perspicua, nulla ampliori explicatione indigent; simulque hoc caput penitus absolutum videtur.