CAPUT V.

DE

INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM HUJUS FORMAE

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \frac{Ex^4\partial^4 y}{\partial x^4} + \text{etc.}$$

Problema 164.

1226.

Proposita aequatione differentiali hujus formae

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cxx\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \ldots + \frac{Nx^n\partial^n y}{\partial x^n},$$

definire functionem ipsius x, per quam ea multiplicata fiat integrabilis.

Solutio.

Attendenti mox patebit, simplicem potestatem ipsius x hoc praestare. Sit igitur integrabilis haec aequatio

$$Xx^\lambda\partial x = Ax^\lambda y\partial x + Bx^{\lambda+1}\partial y + \frac{Cx^{\lambda+2}\partial\partial y}{\partial x^2} + \ldots + \frac{Nx^{\lambda+n}\partial^n y}{\partial x^{n-1}},$$

cujus integrale sit

$$\int Xx^\lambda\partial x = A'x^{\lambda+1}y + \frac{B'x^{\lambda+2}\partial y}{\partial x} + \frac{C'x^{\lambda+3}\partial\partial y}{\partial x^2} + \ldots + \frac{M'x^{\lambda+n}\partial^{n-1}y}{\partial x^{n-1}}.$$

Cum igitur hujus differentiale illi debeat esse aequale, sequentes nanciscemur determinationes

$A = (\lambda+1)A'$, hinc $(\lambda+1)A' = A$

$B = (\lambda+2)B' + A'$, $(\lambda+1)(\lambda+2)B' = (\lambda+1)B - A$

$C = (\lambda+3)C' + B'$, $(\lambda+1)(\lambda+2)(\lambda+3)C' = (\lambda+1)(\lambda+2)C - (\lambda+1)B + A$

$D = (\lambda+4)D' + C'$, $(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)D' = (\lambda+1)(\lambda+2)(\lambda+3)D$
$- (\lambda+1)(\lambda+2)C + (\lambda+1)B - A$

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.
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$N = M'$

integralis enim termini sequentes, qui involuerent differentialis gradum $\partial^n y$ altioresque, evanescere debent, quia alioquin integratio non successisset. Cum igitur in integrali littera N' evanescat, pervenimus ad hanc aequationem

$$0 = A - (\lambda+1)B + (\lambda+1)(\lambda+2)C - (\lambda+1)(\lambda+2)(\lambda+3)D \ldots\ldots \pm (\lambda+1)(\lambda+2)\ldots\ldots(\lambda+n)N$$

ex qua aequatione exponens λ potestatis quaesitae x^λ definiri debet. Formetur ergo talis expressio algebraica

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \ldots\ldots + N(z-1)(z-2)(z-3)\ldots\ldots(z-n),$$

hujusque quaerantur omnes factores simplices, ut sit

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z) \text{ etc.}$$

factorum horum numero existente $= n$. Jam ex quolibet factore $z+\alpha$ ad nihilum reducto, valor $z = -\alpha$ dabit potestatem x^α, per quam proposita aequatio multiplicata integrabilis evadit, ita ut ejus integrale sit futurum

$$x^{-\alpha-1}\int x^\alpha X \partial x = A'y + \frac{B'x\partial y}{\partial x} + \frac{C'x^2\partial\partial y}{\partial x^2} + \frac{D'x^3\partial^3 y}{\partial x^3} + \ldots + \frac{Nx^{n-1}\partial^{n-1}y}{\partial x^{n-1}}$$

ubi differentialium gradus unitate est inferior. Ita autem haec aequatio integrata per propositam determinatur, ut sit

$$A = (\alpha + 1) A'$$
$$B = (\alpha + 2) B' + A'$$
$$C = (\alpha + 3) C' + B'$$
$$D = (\alpha + 4) D' + C'$$
etc.

donec perveniatur ad ultimum coëfficientem N, qui utrobique est idem.

Corollarium 1.

1227. Quia aequatio integrata similis est ipsi propositae, ea per certam potestatem ipsius x multiplicata denuo fiet integrabilis. Ad hanc enim potestatem inveniendam considerare oportet hanc formam algebraicam

$$Q = A' + B'(z-1) + C'(z-1)(z-2) + D'(z-1)(z-2)(z-3) + \ldots$$
$$\ldots + N(z-1)(z-2)\ldots(z-n+1),$$

cujus si fuerit factor simplex quicunque $z + \mu$, erit $x^{+\mu}$ illa potestas ipsius x, aequationem integrabilem reddens.

Corollarium 2.

1228. Quodsi ipsa aequatio proposita per potestatem $x^{+\alpha}$ multiplicata reddita fuerit integrabilis, hic probe notari convenit, quantitatem Q ex integrata formatam ita pendere a priori P ex ipsa proposita formata, ut sit $Q = \frac{P}{\alpha + z}$, quandoquidem per hypothesin $\alpha + z$ est factor ipsius P.

Scholion 1.

1229. Ad hanc insignem proprietatem demonstrandam, quod scilicet sit $P = (\alpha + z) Q$, tantum opus est, ut quantitas Q per

$\alpha + z$ multiplicetur; verum quo conclusio clarius in oculos occurrat, pro singulis terminis ipsius Q multiplicator bipartito est repraesentandus; ac pro primo quidem termino loco $\alpha + z$ scribatur $(\alpha + 1) + (z - 1)$, pro secundo $(\alpha + 2) + (z - 2)$, pro tertio $(\alpha + 3) + (z - 3)$, pro quarto $(\alpha + 4) + (z - 4)$ etc. ita ut cujusque termini productum binis partibus exhibeatur, quam operationem hic totam apponam.

$$Q = A' + B'(z-1) + C'(z-1)(z-2) + D'(z-1)(z-2)(z-3) + \text{etc.}$$

Multipl.	$\alpha + 1$	$\alpha + 2$	$\alpha + 3$	$\alpha + 4$
	$z - 1$	$z - 2$	$z - 3$	$z - 4$

$$\text{Prod.}\ (\alpha+1)A' + A'(z-1) \quad + B'(z-1)(z-2) \quad + C'(z-1)(z-2)(z-3)\ \text{etc.}$$
$$+ (\alpha+2)B'(z-1) + (\alpha+3)C'(z-1)(z-2) + (\alpha+4)D'(z-1)(z-2)(z-3)$$

In solutione autem vidimus esse

$$(\alpha+1)A' = A, \quad (\alpha+2)B' + A' = B, \quad (\alpha+3)C' + B' = C, \quad \text{etc.}$$

quocirca hoc productum hac forma exprimetur

$$A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

cui valor ipsius P est aequalis, sicque demonstrata est insignis illa proprietas memorata, quod sit $Q = \frac{P}{\alpha + z}$.

Corollarium 3.

1230. Quodsi ergo valor ipsius P in factores simplices resolutus ita repraesentetur

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)\ \text{etc.}$$

et ex factore $\alpha + z$ aequatio proposita per x^2 multiplicata integretur, tum vero ex integratata simili modo valor Q formetur, erit

$$Q = N(\beta + z)(\gamma + z)(\delta + z)\ \text{etc.}$$

Corollarium 4.

1231. Aequatio ergo integrata, postquam ad formam propositae fuerit perducta, ut posito

$$x^{-\alpha-1} \int x^{\alpha} X \partial x = X'$$

habeatur

$$X' = A' y + \frac{B' x \partial x}{\partial x} + \frac{C' x^2 \partial \partial y}{\partial x^2} + \frac{D' x^3 \partial^3 y}{\partial x^3} + \text{etc.}$$

haec denuo tntegrabilis reddetur, si multiplicetur per quampiam harum potestatum x^{β}, x^{γ}, x^{δ}, etc. quae etiam ipsam propositam integrabilem reddidissent.

Scholion 2.

1232. Antequam continuationem harum integrationum ulterius prosequar, conveniet eum casum formae propositae generalis seorsim evolvi, quo prius aequationis membrum X in nihilum abit. Hoc enim casu hoc commodi usu venit, ut statim sine integrationibus repetitis integrale completum exhiberi queat, idque simili modo quo supra in Capite II. sum usus. Huic quidem casui quia jam multo facilius tractari potest, proprium caput assignare nolui, ne praecepta nimis multiplicari videantur.

Problema 166.

1233. Proposita hac aequatione differentiali cujuscunque ordinis

$$0 = A y + \frac{B x \partial y}{\partial x} + \frac{C x^2 \partial \partial y}{\partial x^2} + \frac{D x^3 \partial^3 y}{\partial x^3} + \text{etc.}$$

ubi variabilis y cum suis differentialibus nusquam plus una dimensione, altera vero x adeo nullam obtineat, ejus integrale completum inveuire.

Solutio.

Particulariter huic aequationi satisfieri perspicuum est, si y certae ipsius x potestati aequetur, ponamus ergo esse $y = x^{\mu}$, et facta substitutione, cum ubique per x^μ diviserimus, perveniemus ad hanc aequationem

$$0 = A + \mu B + \mu(\mu - 1)C + \mu(\mu - 1)(\mu + 2)D + \text{etc.}$$

unde exponentem μ determinari oportet. Vel si secundum praeceptum praecedentis Problematis ex aequatione proposita hanc formemus formulam algebraicam

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

eamque in factores simplices resolvamus, ut sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

evidens est posito $\mu = z - 1$, primae aequationi satisfieri sumendo

$$\mu = -\alpha - 1, \text{ vel } \mu = -\beta - 1, \text{ vel } \mu = -\gamma - 1, \text{ etc.}$$

ita ut quisque factor suppeditet integrale particulare. Cum igitur factorum numerus aequetur gradui differentialium summo, hinc colligetur integrale completum aequationis propositae

$$y = \mathfrak{A}x^{-\alpha-1} + \mathfrak{B}x^{-\beta-1} + \mathfrak{C}x^{-\gamma-1} + \mathfrak{D}x^{-\delta-1} + \text{etc.}$$

ubi tantum observari convenit, si factorum illorum simplicium duo pluresve fuerint inter se aequales, integralis formam simili modo immutari debere, quo supra Capite II. sum usus. Scilicet cum aequationes ibi tractatae ad praesentem formam revocentur, si ibi loco x scribatur lx, inde has regulas haurimus.

1) Si forma P factorem habeat $(\alpha + z)^2$, inde nascitur pars integralis

$$x^{-\alpha-1}(\mathfrak{A} + \mathfrak{B}\,lx).$$

2) Si forma P factorem habeat $(\alpha + z)^3$, pars integralis inde orta est

$$x^{-\alpha-1}[\mathfrak{A} + \mathfrak{B}\,lx + \mathfrak{C}(lx)^2].$$

3) Si forma P factorem habeat $(\alpha + z)^4$, pars integralis inde orta erit

$$x^{-\alpha-1}[\mathfrak{A} + \mathfrak{B}\,lx + \mathfrak{C}(lx)^2 + \mathfrak{D}(lx)^3].$$

Si factores occurrant imaginarii, partes inde oriundae per solitam imaginariorum reductionem facile ad formam realem revocabuntur, uti in corollariis docebo.

Corollarium 1.

1234. Si forma P duos habeat factores simplices imaginarios in formula $ff + 2fz\cos.\theta + zz$ contentos, hac cum producto $(\alpha + z)(\beta + z)$ comparata erit

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et } \beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

unde fit

$$x^{-\alpha} = x^{-f\cos.\theta}\, x^{-f\sqrt{-1}.\sin.\theta} = x^{-f\cos.\theta}\, e^{-\sqrt{-1}.f\sin.\theta\, l x}.$$

Est vero

$$e^{-u\sqrt{-1}} = \cos.u - \sqrt{1}.\sin.u,$$

ideoque habetur

$$x^{-\alpha-1} = x^{-f\cos.\theta}\left(\frac{\cos.(f\sin.\theta\, lx) - \sqrt{-1}.\sin.(f\sin.\theta\, l x)}{x}\right).$$

Quare cum $x^{-\beta-1}$ simili modo exprimatur, mutato signo ipsius $\sqrt{-1}$, ex factore duplici $ff + 2fz\cos.\theta + zz$ haec nascitur pars integralis

$$x^{-f\cos.\theta-1}[\mathfrak{A}\cos.(f\sin.\theta\, lx) + \mathfrak{B}\sin.(f\sin.\theta\, l x)],$$

quae etiam ita potest repraesentari

$$\mathfrak{A}\, x^{-f\cos.\theta-1}\cos.(\mathfrak{a} + f\sin.\theta\, l x),$$

denotante $\mathfrak{a}$ angulum constantem arbitrarium.

Corollarium 2.

1235. Simili modo si factores aequales involuat, ut sit

$$(\alpha + z)^2(\beta + z)^2 = (ff + 2fz\cos.\theta + zz)^2,$$

litterae α et β eosdem quos ante sortientur valores imaginarios, ex quorum reductione colligitur haec pars integralis inde oriunda

$$x^{-f\cos.\phi-1}\,[\mathfrak{A}\cos.(\mathfrak{a}+f\sin.\phi\, lx)+\mathfrak{B}\, lx\,.\cos.(\mathfrak{b}+f\sin.\phi\, lx)],$$

quatuor constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{a}$ et $\mathfrak{b}$ continens.

Corollarium 3.

1136. Hinc ergo evidens est, quomodo ex factoribus formae P, sive sint simplices sive duplices, sive inaequales sive aequales, singulas integralis partes assignari indeque totum integrale completum formari conveniat.

Scholion.

1237. Totum ergo negotium huc reeit, ut quantitas algebraica ex aequatione differentiali formata

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

in suos factores reales vel simplices vel duplices resolvetur, in quo plerumque maxima difficultas versatur, quoniam hujusmodi formae minus tractari sunt solitae. Cum vero haec resolutio isti aequationi cum generali, quam hoc capite evolvere suscepi, est communis, quicquid hic praestare licuerit, potius in aequatione generali ostendi conveniet, ad quam resolvendam propterea revertor. Id tantum hic observasse necesse duco, quod si pro aequatione generali

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cxx\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \text{etc.}$$

undecunque innotuerit integrale particulare, puta $y = V$ existente V certa functione ipsius x, tum posito $y = V + v$ perveniri ad hanc aequationem

$$0 = Av + \frac{Bx\partial v}{\partial x} + \frac{Cxx\partial\partial v}{\partial x^2} + \frac{Dx^3\partial^3 v}{\partial x^3} + \text{etc.}$$

cujus integrale completum per praecepta hujus problematis inventum si loco v scribatur, habebitur integrale completum illius aequationis, quo pacto certe insigne calculi compendium obtinetur.

Problema 166.

1238. Proposita aequatione differentiali gradus cujuscunque n hujus formae

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \ldots + \frac{Nx^n\partial^n y}{\partial x^n},$$

ejus integrale per integrationem n vicibus repetitam invenire.

Solutio.

Ex hac aequatione formetur haec quantitas algebraica

$$P = A + B(z-1) + C(z-1)(z-2) + \ldots + N(z-1)(z-2)\ldots(z-n),$$

cujus quaerantur omnes factores simplices, nullo habito respectu sive sint reales sive imaginarii, ut ea hoc modo exprimatur

$$P = N(\alpha + z)(\beta + z)(\gamma + z)\ldots(\mu + z)(\nu + z),$$

factorum numero existente $= n$. Quo facto, initio hujus capitis vidimus, quemlibet factorem puta $\alpha + z$ praebere potestatem x^α, per quam nostra aequatio fiat integrabilis, atque adeo ostendimus, integrale inde ortum, si compendii gratia ponamus

$$x^{-\alpha-1}\int x^\alpha X\partial x = X',$$

fore

$$X' = A'y + \frac{B'x\partial y}{\partial x} + \frac{C'x^2\partial\partial y}{\partial x^2} + \ldots + \frac{Nx^{n-1}\partial^{n-1}y}{\partial x^{n-1}},$$

ita ut sit $A' = \frac{A}{\alpha+1}$, caeterique coëfficientes ita se habeant, uti ibidem docuimus; hic autem sufficiet ad primum potissimum respexisse. Absoluta jam Prima integratione, si eadem lege ex aequatione semel integrata formemus quantitatem

$$P' = A' + B'(z-1) + C'(z-1)(z-2) + \ldots + N(z-1)(z-2)\ldots(z-n+1),$$

cujus resolutio in factores jam ex prima forma P constat, postquam §. 1229 demonstravi esse

$$P' = N(\beta + z)(\gamma + z)(\delta + z) \ldots (\mu + z)(\nu + z),$$

ita ut sit $P' = \frac{P}{\alpha + z}$. Hinc ergo simili modo factor $\beta + z$ suppeditabit multiplicatorem x^β, quo haec aequatio integrabilis redditur, ac posito

$$x^{-\beta-1} \int x^\beta X' \partial x = X'',$$

ut sit

$$X'' = x^{-\beta-1} \int e^{\beta-\alpha-1} \partial x \int x^\alpha X \partial x,$$

integrale erit

$$X'' = A'' y + \frac{B'' x \partial y}{\partial x} + \frac{C'' x^2 \partial \partial y}{\partial x^2} + \ldots + \frac{N x^{n-2} \partial^{n-2} y}{\partial x^{n-2}},$$

existente hic $A'' = \frac{A'}{\beta + 1} = \frac{A}{(\alpha+1)(\beta+1)}$. Quodsi hoc modo tot integrationes successive absolvantur, quot unitates continentur in indice n, sicque omnes factores simplices formae P in usum vocentur, tandem ad hanc pervenietur aequationem $X^{(n)} = A^{(n)} y$, quae est ipsa integralis desiderata. Cum autem hic futurum sit

$$A^{(n)} = \frac{A}{(\alpha+1)(\beta+1)(\gamma+1) \ldots (\nu+1)},$$

evidens est denominatorem nasci ex forma $\frac{P}{N}$, si loco z scribatur unitas; tum autem sumto $z = 1$, prima forma manifesto dat $P = A$, ita ut denominator iste fiat $= \frac{A}{N}$, ideoque $A^{(n)} = N$, quod etiam inde patet, quod omnium aequationum ultimi termini habeant eundem coëfficientem N, quo ergo in postrema integrali ipse primus terminus y debet esse affectus. Deinde vero est

$$X^{(n)} = x^{-\nu-1} \int x^{\nu-\mu-1} \partial x \int x^{\mu-\lambda-1} \partial x \ldots \int x^{\beta-\alpha-1} \partial x \int x^\alpha X \partial x;$$

ubi cum numeri α, β, γ, etc. utcunque inter se permutari possunt, integrale quaesitum etiam hoc modo repraesentari potest:

$$N y = x^{-\alpha-1} \int x^{\alpha-\beta-1} \partial x \int x^{\beta-\gamma-1} \partial x \ldots \int x^{\mu-\nu-1} \partial x \int x^\nu X \partial x$$

Corollarium 1.

1239. Totum ergo negotium huc redit, ut forma algebraica $P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$ in suos factores simplices resolvatur, quibus inventis ut sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z) \ldots \ldots (\nu + z),$$

hinc integrale quaesitum facile exhibetur, et quidem pro factorum varia permutatione pluribus modis, qui autem omnes eundem valorem exprimunt, uti ex sequentibus clarius patebit.

Corollarium 2.

1240. Cum haec forma integralis inventa tot involuat integrationes, quoti gradus fuerit aequatio differentialis proposita, totidem quoque constantes arbitrariae ingerentur, quemadmodum indoles integrationis completae postulat.

Scholion.

1241. Quoniam integrale inventum pluribus integrationibus est involutum, ad usum faciliorem conveniet hanc formam in partes resolvi, quae singulae unicum tantum signum integrale contineant. Hanc autem resolutionem simili modo instituere licet, quo supra sumus usi, atque hic quidem totum negotium ad hujusmodi formulam revocatur

$$\int x^{m-n-1} \partial x \int x^n X \partial x,$$

quae manifesto ita reducitur, ut sit

$$\frac{1}{m-n} x^{m-n} \int x^n X \partial x - \frac{1}{m-n} \int x^m X \partial x:$$

ubi tamen observandum est, si fuerit $m = n$, peculiari reductione opus esse, hocque casu fore

$$\int \frac{\partial x}{x} \int x^n X \partial x = l x . \int x^n X \partial x - \int x^n X \partial x \, l x.$$

Hac ergo regula utemur in resolutione sequentium problematum, quibus successive omnes gradus differentialium percurramus, praetermisso quidem gradu primo, cum aequationis

$$X = A y + \frac{N x \partial y}{\partial x}, \text{ ob}$$

$$P = A + N(z - 1) = N(\alpha + z),$$

integrale sit

$$N y = x^{-\alpha - 1} \int x^{\alpha} X \partial x,$$

quod nulla reductione indiget.

Problema 167.

1242. Proposita hac aequatione differentiali secundi gradus

$$X = A y + \frac{B x \partial y}{\partial x} + \frac{N x^2 \partial \partial y}{\partial x^2},$$

ejus integrale per formulas integrales simplices evoluere.

Solutio.

Cum sit

$$P = A + B(z - 1) + N(z - 1)(z - 2),$$

statuatur

$$P = N(\alpha + z)(\beta + z),$$

eritque integrale per methodum praecedentem inventum $N y = X''$ existente

$$x^{\beta + 1} X'' = \int x^{\beta - \alpha - 1} \partial x \int x^{\alpha} X \partial x,$$

quae forma evoluitur in hanc

$$\frac{1}{\beta - \alpha} x^{\beta - \alpha} \int x^{\alpha} X \partial x - \frac{1}{\beta - \alpha} \int x^{\beta} X \partial x,$$

sicque erit

$$N y = \frac{x^{-\alpha - 1}}{\beta - \alpha} \int x^{\alpha} X \partial x + \frac{x^{-\beta - 1}}{\alpha - \beta} \int x^{\beta} X \partial x.$$

Hinc autem casum excipi oportet, quo $\beta = \alpha$, tum enim fit

$$x^{\alpha+1} X'' = \int \frac{\partial x}{x} \int x^{\alpha} X \partial x = l x \int x^{\alpha} X \partial x - \int x^{\alpha} X \partial x \, l x$$

pro hoc igitur casu habebimus

$$N y = x^{-\alpha-1} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x, \text{ seu}$$

$$N y = x^{-\alpha-1} (l x \int x^{\alpha} X \partial x - \int x^{\alpha} X \partial x \, l x),$$

ubi quidem prior forma praeferenda videtur.

Corollarium 1.

1243. Si ambo factores simplices sint imaginarii, ponatur

$$(\alpha + z)(\beta + z) = f f + 2 f z \cos.\theta + z z,$$

eritque

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et}$$
$$\beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

indeque

$$\beta - \alpha = -2 f \sqrt{-1}.\sin.\theta.$$

Tum vero

$$x^{\alpha} = x^{f\cos.\theta}[\cos.(f\sin.\theta.lx) + \sqrt{-1}.\sin.(f\sin.\theta.lx)]$$
$$x^{-\alpha} = x^{-f\cos.\theta}[\cos.(f\sin.\theta lx) - \sqrt{-1}.\sin.(f\sin.\theta lx)],$$

quae formulae mutato signo ipsius $\sqrt{-1}$ ad x^{β} et $x^{-\beta}$ transferuntur.

Corollarium 2.

1244. Ponatur brevitatis gratia angulus

$$f \sin.\theta \, l x = \Phi,$$

et facta substitutione habebimus

$$Nxy = \frac{x^{-f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta} X \partial x (\cos.\Phi + \sqrt{-1}.\sin.\Phi)}{-2 f \sqrt{-1}.\sin.\theta}$$
$$+ \frac{x^{-f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta} X \partial x (\cos.\Phi - \sqrt{-1}.\sin.\Phi)}{2 f \sqrt{-1}.\sin.\theta}$$

ubi partes imaginariae se sponte destruunt, fietque

$$Nxy = \frac{x^{-f\cos.\theta}}{f\sin.\theta}(\sin.\Phi\int x^{f\cos.\theta}X\partial x\cos.\Phi - \cos.\Phi\int x^{f\cos.\theta}X\partial x\sin.\Phi).$$

Corollarium 3.

1245. Forma ergo haec realis modo inventa aequivalet illi imaginaria implicanti

$$Nxy = \frac{x^{-\alpha}}{\beta-\alpha}\int x^{\alpha}X\partial x + \frac{x^{-\beta}}{\alpha-\beta}\int x^{\beta}X\partial x,$$

si fuerit

$$(\alpha+z)(\beta+z) = ff + 2fz\cos.\theta + zz,$$

ponaturque $\Phi = f\sin.\theta\, lx$, quae reductio semel facta etiam in sequentibus usum praestabit.

Problema 168.

1246. Proposita hac aequatione differentiali tertii gradus

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cxx\partial\partial y}{\partial x^2} + \frac{Nx^3\partial^3 y}{\partial x^3},$$

ejus integrale per formulas integrales simplices evoluere.

Solutio.

Cum hic sit

$$P = A + B(z-1) + C(z-1)(z-2) + N(z-1)(z-2)(z-3),$$

ponatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z),$$

et cum per integrationem generalem prodeat $Ny = X'''$, notetur esse $X''' = x^{-\gamma-1}\int x^{\gamma}X''\partial x$, siquidem valorem ipsius X'' ex binis factoribus $\alpha+z$ et $\beta+z$ jam invenimus; hinc enim per problema praecedens habetur

$$X'' = \frac{x^{-\alpha-1}}{\beta-\alpha}\int x^{\alpha}X\partial x + \frac{x^{-\beta-1}}{\alpha-\beta}\int x^{\beta}X\partial x:$$

unde colligitur

$$\int x^\gamma X'' \partial x = \frac{x^{\gamma-\alpha}\int x^\alpha X \partial x}{(\beta-\alpha)(\gamma-\alpha)} - \frac{\int x^\gamma X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{x^{\gamma-\beta}\int x^\beta X \partial x}{(\alpha-\beta)(\gamma-\beta)} - \frac{\int x^\gamma X \partial x}{(\alpha-\beta)(\gamma-\beta)}.$$

Est vero

$$\frac{1}{(\beta-\alpha)(\gamma-\alpha)} + \frac{1}{(\alpha-\beta)(\gamma-\beta)} = \frac{-1}{(\alpha-\gamma)(\beta-\gamma)}$$

quod quemadmodum cum per se liquet, tum vero ex Theoremate §. 1169. demonstrato perspicitur. Quocirca integrale quaesitum ita obtinetur expressum

$$N x y = \frac{x^{-\alpha}\int x^\alpha X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{x^{-\beta}\int x^\beta X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{x^{-\gamma}\int x^\gamma X \partial x}{(\alpha-\gamma)(\beta-\gamma)}.$$

Corollarium 1.

1247. Si forma P duos habeat factores aequales, ut sit $\beta = \alpha$, quia tum est

$$X'' = x^{-\alpha-1}\int\frac{\partial x}{x}\int x^\alpha X \partial x, \text{ erit}$$

$$x^\gamma X''\partial x = \frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int\frac{\partial x}{x}\int x^\alpha X\partial x - \frac{1}{\gamma-\alpha}\int x^{\gamma-\alpha-1}\partial x\int x^\alpha X\partial x, \text{ at}$$

$$\int x^{\gamma-\alpha-1}\partial x\int x^\alpha X\partial x = \frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^\alpha X\partial x - \frac{1}{\gamma-\alpha}\int x^\gamma X\partial x;$$

unde colligitur

$$Nxy = \frac{x^{-\alpha}}{\gamma-\alpha}\int\frac{\partial x}{x}\int x^\alpha X\partial x - \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^\alpha X\partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^\gamma X\partial x.$$

Corollarium 2.

1248. Similis forma oritur, si sumatur $\gamma = \beta$, tum enim fit

$$\int x^\gamma X'' \partial x = \frac{x^{\beta-\alpha}}{(\beta-\alpha)^2}\int x^\alpha X \partial x - \frac{\int x^\beta X \partial x}{(\beta-\alpha)^2} + \frac{1}{\alpha-\beta}\int\frac{\partial x}{x}\int x^\beta X \partial x$$

ideoque

$$N x y = \frac{x^{-\beta}}{\alpha-\beta}\int\frac{\partial x}{x}\int x^\beta X \partial x - \frac{x^{-\beta}\int x^\beta X \partial x}{(\beta-\alpha)^2} + \frac{x^{-\alpha}\int x^\alpha X \partial x}{(\beta-\alpha)^2}.$$

Corollarium 3.

1249. Quodsi autem omnes tres factores inter se fuerint aequales $\alpha = \beta = \gamma$, erit

$$\int x^\gamma X'' \partial x = \int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^\alpha X \partial x;$$

ideoque hoc casu integrale ita succincte exprimitur

$$N x y = x^{-\alpha}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^\alpha X \partial x.$$

Corollarium 4.

1250. Si duo factores sint imaginarii, scilicet

$$(\alpha + z)(\beta + z) = ff + 2fz \cos.\theta + zz, \text{ ob}$$

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et}$$

$$\beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

postremum quidem nostri integralis membrum manet reale ob

$$(\alpha - \gamma)(\beta - \gamma) = \gamma\gamma - 2\gamma f \cos.\theta + ff,$$

at bina priora fient, posito $\Phi = f \sin.\theta\, l x$,

$$\frac{x^{-f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta} X \partial x (\cos.\Phi + \sqrt{-1}.\sin.\Phi)}{-2f\sqrt{-1}.\sin.\theta[\gamma - f(\cos.\theta + \sqrt{-1}.\sin.\theta)]}$$

$$+\frac{x^{-f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta} X \partial x (\cos.\Phi - \sqrt{-1}.\sin.\Phi)}{2f\sqrt{-1}.\sin.\theta[\gamma - f(\cos.\theta - \sqrt{-1}.\sin.\theta)]},$$

quae reducuntur ad hanc formam realem

$$\frac{x^{-f\cos\theta}[\gamma\,\mathrm{si}.\Phi - f\,\mathrm{si}(\theta+\Phi)]\int x^{-f\cos.\theta}\,X\,\partial x\,\mathrm{cs}.\Phi - x^{-f\cos.\theta}[\gamma\,\mathrm{cs}.\Phi - f\,\mathrm{cs}.(\theta+\Phi)]\int x^{f\cos.\theta}\,X\,\partial x\,\mathrm{si}.\Phi}{f\sin.\theta\,(\gamma\gamma - 2\gamma f\cos.\theta + ff)}$$

Scholion.

1251. Quod ad factores imaginarios attinet, integralium indenatorum reductio facilius in genere instituetur, unde in his differentialium gradibus determinatis, ei non amplius immorabor. Factores autem aequales hic data opera pro singulis gradibus accuratius persequi est visum; quia supra nimis cito ad evolutionem generalem properanti in insignem errorem illabi contigit, quem statim feliciter evitassem, si eadem methodo ibi essem usus. Hujusmodi autem vitium circa factores imaginarios hic non est pertimescendum, cum in hoc negotio nihil sub specie infinite parvi negligendum occurrat. Ex hoc autem fonte errores illi, quos supra commisi, sunt nati: quod vitium subtile quo clarius ob oculos ponatur, una cum necesaria emendatione hic evoluam. Quaestio scilicet pro casu praesenti huc redit, ut valor harum duarum formularum

$$\frac{x^{-\alpha}\int x^{\alpha}\,X\,\partial x}{(\beta-\alpha)\,(\gamma-\alpha)} + \frac{x^{-\beta}\int x^{\beta}\,X\,\partial x}{(\alpha-\beta)\,(\gamma-\beta)}$$

definiatur, casu quo $\beta = \alpha$ et ambo membra in infinitum excrescent; hunc in finem pono $\beta = \alpha + \omega$, existente ω particula evanescente, et cum sit

$$x^{\beta} = x^{\alpha}\,x^{\omega} = x^{\alpha}\,e^{\omega\,l\,x} = x^{\alpha}\,(1 + \omega\,l\,x),$$

hincque

$$x^{-\beta} = x^{-\alpha}\,(1 - \omega\,l\,x),$$

habebimus

$$\frac{x^{-\alpha}\int x^{\alpha}\,X\,\partial x}{\omega\,(\gamma-\alpha)} - \frac{x^{-\alpha}\,(1-\omega lx)\int x^{\alpha}\,X\,\partial x\,(1+\omega lx)}{\omega\,(\gamma-\beta)}.$$

Quia nunc est

$$\frac{1}{\gamma-\alpha} = \frac{1}{\gamma-\beta+\omega} = \frac{1}{\gamma-\beta} - \frac{\omega}{(\gamma-\beta)^2},$$

prius membrum induit hanc formam

$$\frac{x^{-\alpha}\int x^{\alpha}\,\mathrm{X}\,\partial x}{\omega(\gamma-\beta)}-\frac{x^{-\alpha}\int x^{\alpha}\,\mathrm{X}\,\partial x}{(\gamma-\beta)^2};$$

posterius vero evolutum istam

$$\frac{-x^{-\alpha}\int x^{\alpha}\,\mathrm{X}\,\partial x}{\omega(\gamma-\beta)}+\frac{x^{-\alpha}\,l\,x\int x^{\alpha}\,\mathrm{X}\,\partial x-x^{-\alpha}\int x^{\alpha}\,\mathrm{X}\,\partial x\,l\,x}{\gamma-\beta},$$

sicque valor quaesitus casu $\beta=\alpha$ concluditur

$$\frac{x^{-\alpha}(lx\int x^{\alpha}\mathrm{X}\,\partial x-\int x^{\alpha}\,\mathrm{X}\,\partial x\,lx)}{\gamma-\alpha}-\frac{x^{-\alpha}\int x^{\alpha}\mathrm{X}\,\partial x}{(\gamma-\alpha)^2},\ \text{seu}$$

$$\frac{x^{-\alpha}}{\gamma-\alpha}\int\frac{\partial x}{x}\int x^{\alpha}\,\mathrm{X}\,\partial x-\frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^{\alpha}\,\mathrm{X}\,\partial x,$$

cujus formulae posterius membrum, quod in vitiosa illa methodo erat omissnm, inde resultat, quod hic ad discrimen inter expressiones $\gamma-\alpha$ et $\gamma-\beta$ respeximus, quam necessariam cautionem supra neglexinus.

Problema 169.

1252. Proposita hac aequatione differentiali quarti gradus

$$\mathrm{X}=\mathrm{A}\,y+\frac{\mathrm{B}\,x\,\partial x}{\partial x}+\frac{\mathrm{C}\,x\,x\,\partial\partial y}{\partial x^2}+\frac{\mathrm{D}\,x^3\,\partial^3 y}{\partial x^3}+\frac{\mathrm{N}\,x^4\,\partial^4 y}{\partial x^4},$$

ejus integrale per formulas integrales simplices evoluere.

Solutio.

Formata hinc expressione algebraica

$$\mathrm{P}=\mathrm{A}+\mathrm{B}(z-1)+\mathrm{C}(z-1)(z-2)+\mathrm{D}(z-1)(z-2)(z-3)$$
$$+\mathrm{N}(z-1)(z-2)(z-3)(z-4),$$

statuatur

$$\mathrm{P}=\mathrm{N}(\alpha+z)(\beta+z)(\gamma+z)(\delta+z),$$

et per praecepta generalia est

$N y = X^{IV}$, existente $X^{IV} = x^{-\delta-1} \int X''' x^{\delta} \partial x$,

siquidem X''' ex tribus prioribus factoribus determinetur, quemadmodum in problemate praecedente est factum. Valorem scilicet ibi pro $N x y$ inventum hic per $x^{\delta-1} \partial x$ multiplicari oportet, unde oritur

$$\int x^{\delta} X''' \partial x = + \frac{x^{\delta-\alpha} \int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} - \frac{\int x^{\delta} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}$$
$$+ \frac{x^{\delta-\beta} \int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)} - \frac{\int x^{\delta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{x^{\delta-\gamma} \int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} - \frac{\int x^{\delta} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}$$

ubi ob rationes supra demonstratas tres postremi termini coalescunt in $+ \frac{\int x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$, ita ut sit integrale quaesitum

$$N x y = \frac{x^{-\alpha} \int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} + \frac{x^{-\beta} \int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{x^{-\gamma} \int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} + \frac{x^{-\delta} \int x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$$

siquidem omnes factores sint inter se inaequales. Casus autem quibus duo pluresve sunt aequales, in corollariis explorabimus.

Corollarium 1.

1253. Si fuerint duo factores aequales nempe $\delta = \gamma$, seu si sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)^2$$

ex eadem forma pro X''' ante inventa oritur integrale

$$\mathrm{N}\,x\,y = \frac{x^{-\alpha}\int x^{\alpha}\,\mathrm{X}\,\partial x}{(\beta-\alpha)(\gamma-\alpha)^2} - \frac{x^{-\gamma}\int x^{\gamma}\,\mathrm{X}\,\partial x}{(\beta-\alpha)(\gamma-\alpha)^2}$$

$$+ \frac{x^{-\beta}\int x^{\beta}\,\mathrm{X}\,\partial x}{(\alpha-\beta)(\gamma-\beta)^2} - \frac{x^{-\gamma}\int x^{\gamma}\,\mathrm{X}\,\partial x}{(\alpha-\beta)(\gamma-\beta)^2}$$

$$+ \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma}\,\mathrm{X}\,\partial x}{(\alpha-\gamma)(\beta-\gamma)};$$

ubi membra negativa ita repraesentari possunt

$$\frac{x^{-\gamma}\int x^{\gamma}\,\mathrm{X}\,\partial x}{(\alpha-\beta)}\left(\frac{1}{(\gamma-\alpha)^2} - \frac{1}{(\gamma-\beta)^2}\right).$$

Corollarium 2.

1254. Si fuerint tres factores aequales, ut sit

$$\mathrm{P} = \mathrm{N}(\alpha+z)(\beta+z)^3,$$

ideoque $\delta = \gamma = \beta$, ex formula §. 1248. inventa colligitur integrale

$$\mathrm{N}\,x\,y = \frac{x^{-\alpha}\int x^{\alpha}\,\mathrm{X}\,\partial x}{(\beta-\alpha)^3} - \frac{x^{-\beta}\int x^{\beta}\,\mathrm{X}\,\partial x}{(\beta-\alpha)^3}$$

$$- \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^{\beta}\,\mathrm{X}\,\partial x}{(\beta-\alpha)^2} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\beta}\,\mathrm{X}\,\partial x}{\alpha-\beta}.$$

Corollarium 3.

1255. Si omnes quatuor factores fuerint aequales, ut sit

$$\mathrm{P} = \mathrm{N}(\alpha+z)^4, \text{ existente } \delta = \gamma = \beta = \alpha,$$

ex forma pro tribus aequalibus §. 1249. inventa fit integrale

$$\mathrm{M}\,x\,y = x^{-\alpha}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\alpha}\,\mathrm{X}\,\partial x.$$

Corollarium 4.

1256. Si habeatur $\beta = \alpha$ et $\delta = \gamma$, ut bini factores sint aequales scilicet $P = N(\alpha + z)^2 (\gamma + z)^2$, ex § 1247, ubi factores erant $(\alpha + z)^2 (\gamma + z)$, colligitur integrale

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{\partial x}{x}\int x^\alpha X\,\partial x - \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^{\gamma-\alpha-1}\partial x\int x^\alpha X\partial x$$

$$-\frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int x^\alpha X\,\partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^3}\int x^\gamma X\partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{\partial x}{x}\int x^\gamma X\partial x;$$

quae ob

$$\int x^{\gamma-\alpha-1}\partial x\int x^\alpha X\,\partial x = \frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^\alpha X\,\partial x - \frac{1}{\gamma-\alpha}\int x^\gamma X\,\partial x$$

contrahitur in hanc formam

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{\partial x}{x}\int x^\alpha X\,\partial x + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{\partial x}{x}\int x^\gamma X\,\partial x$$

$$-\frac{2\,x^{-\alpha}}{(\gamma-\alpha)^3}\int x^\alpha X\,\partial x - \frac{2\,x^{-\gamma}}{(\alpha-\gamma)^3}\int x^\gamma X\,\partial x.$$

Problema 170.

1257. Proposita hac aequatione differentiali quinti gradus

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \frac{Ex^4\partial^4 y}{\partial x^4} + \frac{Nx^5\partial^5 y}{\partial x^5},$$

ejus integrale per formulas integrales simplices evoluere.

Solutio.

Cum hic sit quantitas algebraica formanda

$$P = A + B(z-1) + C(z-1)(z-2) + \ldots\ldots$$

$$\ldots + N(z-1)(z-2)(z-3)(z-4)(z-5)$$

statuatur

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)(\varepsilon + z),$$

ac si hi factores omnes sint inter se inaequales, ex integrali praecedente novam instituendo integrationem prodibit integrale quaesitum

$$N x y = \frac{x^{-\alpha} \int x^{\alpha} X \partial x}{(\beta - \alpha)(\gamma - \alpha)(\delta - \alpha)(\varepsilon - \alpha)} + \frac{x^{-\beta} \int x^{\beta} X \partial x}{(\alpha - \beta)(\gamma - \beta)(\delta - \beta)(\varepsilon - \beta)}$$
$$+ \frac{x^{-\gamma} \int x^{\gamma} X \partial x}{(\alpha - \gamma)(\beta - \gamma)(\delta - \gamma)(\varepsilon - \gamma)} + \frac{x^{-\delta} \int x^{\delta} X \partial x}{(\alpha - \delta)(\beta - \delta)(\gamma - \delta)(\varepsilon - \delta)}$$
$$+ \frac{x^{-\varepsilon} \int x^{\varepsilon} X \partial x}{(\alpha - \varepsilon)(\beta - \varepsilon)(\gamma - \varepsilon)(\delta - \varepsilon)}.$$

Casus quo duo pluresve factores sunt aequales, in corollariis evolvemus.

Corollarium 1.

1258. Si fuerint duo factores aequales, ut sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)^2, \text{ ideoque } \varepsilon = \delta$$

ex praecedente problemate colligitur integrale

$$N x y = + \frac{x^{-\alpha} \int x^{\alpha} X \partial x - x^{-\delta} \int x^{\delta} X \partial x}{(\beta - \alpha)(\gamma - \alpha)(\delta - \alpha)^2}$$
$$+ \frac{x^{-\beta} \int x^{\beta} X \partial x - x^{-\delta} \int x^{\delta} X \partial x}{(\alpha - \beta)(\gamma - \beta)(\delta - \beta)^2}$$
$$+ \frac{x^{-\gamma} \int x^{\gamma} X \partial x - x^{-\delta} \int x^{\delta} X \partial x}{(\alpha - \gamma)(\beta - \gamma)(\delta - \gamma)^2}$$
$$+ \frac{x^{-\delta} \int \frac{\partial x}{x} \int x^{\delta} X \partial x}{(\alpha - \delta)(\beta - \delta)(\gamma - \delta)}$$

Corollarium 2.

1259. Si fuerint tres factores aequales, ut sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)^3, \text{ ideoque}$$
$$\varepsilon = \delta = \gamma$$

ex corollario 1. problematis praecedentis colligitur

$$N x y = \frac{x^{-\alpha}\int x^{\alpha} X \partial x - x^{-\gamma}\int x^{\gamma} X \partial x}{(\beta - \alpha)(\gamma - \alpha)^3} - \frac{x^{-\gamma}\int \frac{\partial x}{x} \int x^{\gamma} X \partial x}{(\beta - \alpha)(\gamma - \alpha)^2}$$
$$+ \frac{x^{-\beta}\int x^{\beta} X \partial x - x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha - \beta)(\gamma - \beta)^3} - \frac{x^{-\gamma}\int \frac{\partial x}{x} \int x^{\gamma} X \partial x}{(\alpha - \beta)(\gamma - \beta)^2}$$
$$+ \frac{x^{-\gamma}\int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^{\gamma} X \partial x}{(\alpha - \gamma)(\beta - \gamma)}.$$

Corollarium 3.

1260. Si quatuor factores sint aequales, ut sit

$$P = N(\alpha + z)(\beta + z)^4, \text{ ideoque}$$

$\varepsilon = \delta = \gamma = \beta$, erit per §. 1254.

$$N x y = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta - \alpha)^4} - \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\beta - \alpha)^4} - \frac{x^{-\beta}\int \frac{\partial x}{x} \int x^{\beta} X \partial x}{(\beta - \alpha)^3}$$
$$- \frac{x^{-\beta}\int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^{\beta} X \partial x}{(\beta - \alpha)^2} + \frac{x^{-\beta}\int \frac{\partial x}{x} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^{\beta} X \partial x}{\alpha - \beta}.$$

Ac si omnes quinque sint inter se aequales seu

$$P = N(\alpha + z)^5, \text{ erit integrale}$$
$$N x y = x^{-\alpha}\int \frac{\partial x}{x} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x.$$

Corollarium 4.

1261. Si P habeat duos factores quadratos, ut sit

$P = N(\alpha + z)(\beta + z)^2(\gamma + z)^2$, ideoque $\delta = \gamma$ et $\varepsilon = \beta$, erit ex §. 1253. integrale, reductione necessaria facta,

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\partial x - x^{-\beta}x^{\beta}X\partial x}{(\beta-\alpha)^2(\gamma-\alpha)^2} - \frac{x^{-\gamma}x^{\gamma}X\partial x + x^{-\beta}x^{\beta}X\partial x}{(\beta-\alpha)(\alpha-\gamma)^2(\beta-\gamma)}$$

$$+ \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^{\beta}X\partial x}{(\alpha-\beta)(\gamma-\beta)^3} - \frac{x^{-\gamma}\int x^{\gamma}X\partial x + x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)(\beta-\gamma)^3}$$

$$+ \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\partial x + x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\gamma)(\beta-\gamma)^3},$$

quae porro redigitur ad hanc formam

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}X\partial x}{(\beta-\alpha)^2(\gamma-\alpha)^2} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^{\beta}X\partial x}{(\alpha-\beta)^2(\gamma-\beta)^2} - \frac{x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)^2(\gamma-\beta)^2} - \frac{2x^{-\beta}\int x^{\beta}X\partial x}{(\alpha-\beta)(\gamma-\beta)^3}$$

$$+ \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)^2} - \frac{x^{-\gamma}\int x^{\gamma}X\partial x}{(\alpha-\gamma)^2(\beta-\gamma)^2} - \frac{2x^{-\gamma}\int x^{\gamma}X\partial x}{(\alpha-\gamma)(\beta-\gamma)^3}.$$

Corollarium 5.

1262. Si P habeat et factorem quadratum et cubicum, ut sit $P = N(\alpha+z)^2(\gamma+z)^3$, ideoque $\beta = \alpha$ et $\varepsilon = \delta = \gamma$, ex §. 1254. colligitur integrale

$$Nxy = \frac{x^{-\alpha}\int\frac{\partial x}{x}\int x^{\alpha}X\partial x}{(\gamma-\alpha)^3} - \frac{3x^{-\alpha}\int x^{\alpha}X\partial x}{(\gamma-\alpha)^4}$$

$$+ \frac{x^{-\gamma}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\gamma}X\partial x}{(\alpha-\gamma)^2} - \frac{2x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma}X\partial x}{(\alpha-\gamma)^3} - \frac{3x^{-\gamma}\int x^{\gamma}X\partial x}{(\alpha-\gamma)^4}.$$

Scholion.

1263. Ex his formulis parum constat, quemadmodum eas ulterius pro majori factorum numero continuari oporteat, si quidem factorum aliquot inter se fuerint aequales; integralium enim partes, quae factoribus inaequalibus respondent, legem servant manifestam. Quae autem partibus aequalibus respondent, adhibita certa reductione commodius exprimi possunt. Veluti pro casu corollarii 1. si brevitatis gratia ponatur $\alpha - \delta = p$, $\beta - \delta = q$ et $\gamma - \delta = r$, forma $x^{-\delta} \int x^{\delta} X \partial x$ ducta est in

$$\frac{1}{(p-q)(r-p)pp} + \frac{1}{(p-q)(q-r)qq} + \frac{1}{(r-p)(q-r)rr}, \text{ seu}$$
$$\frac{(q-r)qqrr + (r-p)pprr + (p-q)ppqq}{(p-q)(q-r)(r-p)ppqqrr},$$

cujus fractionis numerator est

$$-(p-q)(q-r)(r-p)(pq + pr + qr),$$

ita ut haec fractio reducatur ad istam

$$\frac{-pq - pr - qr}{ppqqrr} = -\frac{1}{pqr}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)$$

Quando ergo est

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)^2,$$

integrale ita se habet

$$Nxy = \frac{x^{-\alpha} \int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2} + \frac{x^{-\beta} \int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2} + \frac{x^{-\gamma} \int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2}$$
$$+ \frac{x^{-\delta} \int \frac{\partial x}{x} x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)} - \frac{x^{-\delta} \int x^{\delta} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\left(\frac{1}{\alpha-\delta} + \frac{1}{\beta-\delta} + \frac{1}{\gamma-\delta}\right).$$

Pro casu autem $P = M(\alpha + z)(\beta + z)(\gamma + z)^3$ habebitur

$$Nxy = \frac{x^{-\alpha} \int x^{\alpha} X \partial x}{(\beta-\alpha)(\gamma-\alpha)^3} + \frac{x^{-\beta} \int x^{\beta} X \partial x}{(\alpha-\beta)(\gamma-\beta)^3}$$
$$+ \frac{x^{-\gamma} \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)} - \frac{x^{-\gamma} \int \frac{\partial x}{x} \int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{\alpha-\gamma} + \frac{1}{\beta-\gamma}\right)$$
$$+ \frac{x^{-\gamma} \int x^{\gamma} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{(\alpha-\gamma)^2} + \frac{1}{(\alpha-\gamma)(\beta-\gamma)} + \frac{1}{(\beta-\gamma)^2}\right).$$

Tum vero pro casu $P = N(\alpha + z)(\beta + z)^4$ fit

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta - \alpha)^4} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\beta} X \partial x}{\alpha - \beta}$$

$$- \frac{x^{-\beta}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\beta} X \partial x}{\alpha - \beta}\cdot\frac{1}{\alpha - \beta} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^{\beta} X \partial x}{\alpha - \beta}\cdot\frac{1}{(\alpha - \beta)^2}$$

$$- \frac{x^{-\beta}\int x^{\beta} X \partial x}{\alpha - \beta}\cdot\frac{1}{(\alpha - \beta)^3}.$$

At pro casu $P = (\alpha + z)(\beta + z)^2(\gamma + z)^2$ integrale ita se habet

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\beta - \alpha)^2(\gamma - \alpha)^2} + \frac{x^{-\beta}\int\frac{\partial x}{x}\int x^{\beta} X \partial x}{(\alpha - \beta)(\gamma - \beta)^2}$$

$$- \frac{x^{-\beta}\int x^{\beta} X \partial x}{(\alpha - \beta)(\gamma - \beta)^2}\left(\frac{1}{\alpha - \beta} + \frac{2}{\gamma - \beta}\right) + \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha - \gamma)(\beta - \gamma)^2}$$

$$- \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha - \gamma)(\beta - \gamma)^2}\left(\frac{1}{\alpha - \gamma} + \frac{2}{\beta - \gamma}\right).$$

Denique pro casu $P = N(\alpha + z)^2(\gamma + z)^3$ est

$$Nxy = \frac{x^{-\alpha}\int\frac{\partial x}{x}\int x^{\alpha} X \partial x}{(\gamma - \alpha)^3} - \frac{x^{-\alpha}\int x^{\alpha} X \partial x}{(\gamma - \alpha)^3}\cdot\frac{3}{\gamma - \alpha}$$

$$+ \frac{x^{-\gamma}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha - \gamma)^2} - \frac{x^{-\gamma}\int\frac{\partial x}{x}\int x^{\gamma} X \partial x}{(\alpha - \gamma)^2}\cdot\frac{2}{\alpha - \gamma}$$

$$+ \frac{x^{-\gamma}\int x^{\gamma} X \partial x}{(\alpha - \gamma)^2}\cdot\frac{3}{(\alpha - \gamma)^2},$$

unde indoles harum formularum jam magis fit perspicua, simulque patet partem integralis ex aliquot factoribus oriundam non pendere ab aequalitate reliquorum. Quocirca jam problema generale aggredi licebit.

Problema 171.

1264. Proposita aequatione differentiali cujuscunque gradus hujus formae

$$X = Ay + \frac{Bx\partial y}{\partial x} + \frac{Cx^2\partial\partial y}{\partial x^2} + \frac{Dx^3\partial^3 y}{\partial x^3} + \dots + \frac{Nx^n\partial^n y}{\partial x^n},$$

ex qua forma algebraica hac lege formata

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \dots$$
$$\dots + N(z-1)(z-2)\dots\dots(z-n),$$

omnes factores habeat inter se inaequales; valorem ipsius y completum per formulas integrales simplices exhibere.

Solutio.

Sint primo formae P omnes factores simplices reales

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\dots\dots(\nu+z),$$

factorum numero existente $= n$, et ex antecedentibus patet, ex quolibet factore nasci integralis partem. Ad has partes inveniendas, eliciantur sequentes valores

1.) posito $z = -\alpha$ sit $\mathfrak{A} = \frac{P}{\alpha+z}$, seu $\mathfrak{A} = \frac{\partial P}{\partial z}$,

2.) posito $z = -\beta$ sit $\mathfrak{B} = \frac{P}{\beta+z}$, seu $\mathfrak{B} = \frac{\partial P}{\partial z}$,

3.) posito $z = -\gamma$ sit $\mathfrak{C} = \frac{P}{\gamma+z}$, seu $\mathfrak{C} = \frac{\partial P}{\partial z}$,

etc.

Cum igitur sit

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\dots\dots(\nu-\alpha) = \frac{\mathfrak{A}}{N},$$

littera N ex superioribus formis per divisionem tolletur, fietque integrale quaesitum

$$xy = \frac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}X\partial x + \frac{1}{\mathfrak{B}}x^{-\beta}\int x^{\beta}X\partial x + \frac{1}{\mathfrak{C}}x^{-\gamma}\int x^{\gamma}X\partial x + \text{etc.}$$

quoad singuli factores fuerint exhausti.

Quodsi jam forma P factores habeat imaginarios, partium inde ortarum imaginariarum ad realitatem reductio sequenti modo instituetur. Quoniam bini factores simplices imaginarii praebent factorem duplicem realem, ponamus

$$(\alpha + z)(\beta + z) = ff + 2fz \cos.\theta + zz,$$

ita ut sit

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et } \beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

unde primum valores literarum $\mathfrak{A}$ et $\mathfrak{B}$ definiantur, quarum cum utraque derivetur ex forma $\frac{\partial P}{\partial z}$, illa posito $z = -\alpha$ haec vero posito $z = -\beta$, in ipsa forma $\frac{\partial P}{\partial z}$ loco z ubique scribatur

$$-f(\cos\theta \pm \sqrt{-1}.\sin.\theta),$$

prodeatque $\mathfrak{P} \pm \mathfrak{Q}\sqrt{-1}$. Ac perspicuum est fore

$$\mathfrak{A} = \mathfrak{P} + \mathfrak{Q}\sqrt{-1} \text{ et } \mathfrak{B} = \mathfrak{P} - \mathfrak{Q}\sqrt{-1},$$

ubi notandum est, quantitates $\mathfrak{P}$ et $\mathfrak{Q}$ esse reales. Deinde cum sit

$$x^{m+n\sqrt{-1}} = x^m e^{n\sqrt{-1}.lx} = x^m[\cos.(nlx) + \sqrt{-1}.\sin.(nlx)],$$

si brevitatis ergo ponamus angulum $f\sin.\theta\, lx = \Phi$, erit

$$x^{\alpha} = x^{f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi),$$
$$x^{-\alpha} = x^{-f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi),$$
$$x^{\beta} = x^{f\cos.\theta}(\cos.\Phi - \sqrt{-1}.\sin.\Phi),$$
$$x^{-\beta} = x^{-f\cos.\theta}(\cos.\Phi + \sqrt{-1}.\sin.\Phi).$$

Quare pro binis partibus

$$\frac{1}{\mathfrak{A}} x^{-\alpha}\int x^{\alpha} X \partial x + \frac{1}{\mathfrak{B}} x^{-\beta}\int x^{\beta} X \partial x,$$

ob $\mathfrak{A}\mathfrak{B} = \mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}$ habebimus

$$+\frac{x^{-f\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\left\{\begin{matrix}(\mathfrak{P}-\mathfrak{Q}\sqrt{-1})(\cos.\Phi-\sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta}(\cos.\Phi+\sqrt{-1}.\sin.\Phi)X\partial x \\ (\mathfrak{P}+\mathfrak{Q}\sqrt{-1})(\cos.\Phi+\sqrt{-1}.\sin.\Phi)\int x^{f\cos.\theta}(\cos.\Phi-\sqrt{-1}.\sin.\Phi)X\partial x\end{matrix}\right\}$$

quae forma ob partes imaginarias se tollentes reducitur ad hanc

$$\frac{\left\{\begin{array}{l} 2\, x^{-f\cos.\theta}\, (\mathfrak{P} \cos.\Phi - \mathfrak{Q} \sin.\Phi) \int x^{f\cos.\theta}\, X\, \partial x \cos.\Phi \\ +\, 2\, x^{-f\cos.\theta}\, (\mathfrak{Q} \cos.\Phi + \mathfrak{P} \sin.\Phi) \int x^{f\cos.\theta}\, X\, \partial x \sin.\Phi \end{array}\right\}}{\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}}.$$

Talisque forma ad integrale accedit, quoties forma P hujusmodi habet factorem duplicem $ff + 2fz \cos.\theta + zz$.

Corollarium 1.

1265. Etsi autem factorum simplicium ipsius P quidam sunt imaginarii, eorum qui sunt reales evolutio inde non perturbatur, sed ex singulis partes in integrale inferendae a natura reliquorum factorum minime pendent.

Corollarium 2.

1266. Pars integralis ex binis factoribus imaginariis seu uno factore duplici oriunda aliquanto succinctius repraesentari potest, si ponatur

$$\mathfrak{P} = \mathfrak{O} \cos.\zeta \text{ et } \mathfrak{Q} = \mathfrak{O} \sin.\zeta,$$

sic enim ea fiet

$$\frac{2}{\mathfrak{O}} x^{-f\cos.\theta} [\cos.(\zeta + \Phi) \int x^{f\cos.\theta}\, X\, \partial x \cos.\Phi$$
$$+ \sin.(\zeta + \Phi) \int x^{f\cos.\theta}\, X\, \partial x \sin.\Phi],$$

ubi ζ et θ sunt anguli constantes, Φ vero variabilis ob $\Phi = f \sin.\theta \,.\, l x$.

Problema 172.

1267. Si pro aequatione differentiali in praecedente problemate proposita quantitas algebraica P inde formata duos habeat factores simplices aequales, integralis partem inde oriundam investigare.

Solutio.

In forma ergo ante exhibita

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

ponamus esse $\beta = \alpha$, quoniam vero tum utraque integralis pars oritur infinita, altera signo $+$, altera signo $-$ affecta, ita ut junctim sumtae partem constituant finitam, ad hanc eliciendam statuamus $\beta = \alpha - \omega$, denotante ω quantitatem evanescentem, eritque

$$\mathfrak{A} = - N\omega(\gamma - \alpha)(\delta - \alpha)(\varepsilon - \alpha) \text{ etc. et}$$

$$\mathfrak{B} = + N\omega(\gamma - \beta)(\delta - \beta)(\varepsilon + \beta) \text{ etc.}$$

Ponatur jam

$$\frac{P}{(\alpha + z)(\beta + z)} = \frac{P}{(\alpha + z)^2} = Q,$$

ut sit

$$Q = N(\gamma + z)(\delta + z)(\varepsilon + z) \text{ etc.}$$

ac manifestum est fieri

$$\mathfrak{A} = -\omega Q, \text{ posito } z = -\alpha \text{ et}$$

$$\mathfrak{B} = +\omega Q, \text{ posito } z = -\beta = -\alpha + \omega,$$

unde intelligitur valorem ipsius Q posteriorem excedere priorem suo differentiali ∂Q, si fiat

$$z = -\alpha \text{ et } \partial z = \omega,$$

ita ut sit

$$\mathfrak{B} = \omega\left(Q + \omega\frac{\partial Q}{\partial z}\right), \text{ posito } z = -\alpha,$$

hincque

$$\frac{1}{\mathfrak{B}} = \frac{1}{\omega Q} - \frac{\partial Q}{QQ\partial z} = \frac{1}{\omega Q} + \frac{1}{\partial z}\partial \cdot \frac{1}{Q}, \text{ existente } \frac{1}{\mathfrak{A}} = -\frac{1}{\omega Q}:$$

tum vero cum sit

$$x^{\beta} = x^{\alpha} x^{-\omega} = x^{\alpha}(1 - \omega l x) \text{ et } x^{-\beta} = x^{-\alpha}(1 + \omega l x),$$

binae partes integralis quaesitae erunt

$$-\frac{1}{\omega Q} x^{-\alpha} \int x^{\alpha} X \partial x$$
$$+ \left(\frac{1}{\omega Q} + \frac{1}{\partial z} \partial \cdot \frac{1}{Q}\right) x^{-\alpha} (1 + \omega lx) \int x^{\alpha} X \partial x (1 - \omega lx),$$

ubi cum membra per ω divisa se destruant, resultat

$$\frac{1}{Q} x^{-\alpha} (lx \int x^{\alpha} X \partial x - \int x^{\alpha} X \partial x \, lx) + \frac{1}{\partial z} \partial \cdot \frac{1}{Q} \cdot x^{-\alpha} \int x^{\alpha} X \partial x$$

seu

$$\frac{1}{Q} x^{-\alpha} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x + \frac{1}{\partial z} \partial \cdot \frac{1}{Q} \cdot x^{-\alpha} \int x^{\alpha} X \partial x,$$

siquidem tam in valore $\frac{1}{Q}$ quam in $\frac{1}{\partial z} \partial \cdot \frac{1}{Q}$ ubique loco z scribatur $-\alpha$. Cum vero sit $Q = \frac{P}{(\alpha + z)^2}$, hi valores inde facile inveniuntur.

Corollarium 1.

1268. Quodsi ergo quantitas algebraica P ex aequatione differentiali formata factorem habeat quadratum $(\alpha + z)^2$, inde in integrale transferenda est haec portio

$$\frac{(\alpha+z)^2}{P} x^{-\alpha} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x + \frac{1}{\partial z} \partial \cdot \frac{(\alpha+x)^2}{P} \cdot x^{-\alpha} \int x^{\alpha} X \partial x,$$

posito $z = -\alpha$; dum si hic factor $\alpha + z$ esset solitarius integralis pars inde oriunda foret

$$\frac{\alpha+z}{P} x^{-\alpha} \int x^{\alpha} X \partial x, \text{ posito } z = -\alpha.$$

Corollarium 2.

1269. Cum sit $Q = \frac{P}{(\alpha+z)^2}$, casu $z = -\alpha$ fiet $Q = \frac{\partial \partial P}{2 \partial z^2}$; verum quia hic ipsi z jam valor determinatus est tributus, hinc $\frac{\partial Q}{\partial z}$ colligere non licet, sed prima est utendum qua fit $\frac{\partial Q}{\partial z} = \frac{(\alpha + z) \partial P - 2 P \partial z}{(\alpha + z)^3 \partial z}$, cujus fractionis cum numerator et denominator casu $z = -\alpha$ evanescat, erit pro eodem casu

$$\frac{\partial Q}{\partial z} = \frac{(\alpha + z) \partial \partial P - \partial z \partial P}{3 (\alpha + z)^2 \partial z^2} = \frac{(\alpha + z) \partial^3 P}{6 (\alpha + z) \partial z^3} = \frac{\partial^3 P}{6 \partial z^3}.$$

Corollarium 3.

1270. Hoc valore invento, quia est eodem casu $z = -\alpha$, quantitas $Q = \frac{\partial \partial P}{2 \partial z^2}$ erit

$$\frac{1}{\partial z} \partial \cdot \frac{1}{Q} = -\frac{\partial Q}{QQ \partial z} = -\frac{2 \partial z \partial^3 P}{3 \partial \partial P^2}, \text{ seu}$$

$$\frac{1}{\partial z} \partial \cdot \frac{1}{Q} = \frac{2 \partial z}{3} \partial \cdot \frac{1}{\partial \partial P},$$

ex quibus formulis, si factores ipsius P non sint evoluti, partes integralis facilius reperiuntur.

Problema 173.

1271. Si pro aequatione differentiali praecedente quantitas algebraica P inde formata factorem habeat cubicum $(\alpha + z)^3$, integralis partem inde oriundam investigare.

Solutio.

Ponamus ergo esse

$$P = (\alpha + z)^2 (\gamma + z) R, \text{ existente } \gamma = \alpha - \omega,$$

ubi ω pro quantitate evanescente assumitur. Quod ergo ante erat Q, id hinc fit $Q = (\gamma + z) R$, et facto $z = -\alpha$, erit $Q = -\omega R$, si etiam in R ponatur $z = -\alpha$. Deinde cum sit

$$\frac{\partial Q}{\partial z} = R + \frac{(\gamma + z) \partial R}{\partial z} = R - \frac{\omega \partial R}{\partial z},$$

eodem casu erit

$$\frac{1}{\partial z} \partial \cdot \frac{1}{Q} = -\frac{1}{\omega \omega R} + \frac{\partial R}{\omega RR \partial z} = -\frac{1}{\omega \omega} \cdot \frac{1}{R} - \frac{1}{\omega \partial z} \partial \cdot \frac{1}{R}.$$

Quocirca ex factore quadrato $(\alpha + z)^2$ per praecedens problema haec obtinetur integralis pars

$$-\frac{1}{\omega R} x^{-\alpha} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x - \left(\frac{1}{\omega \omega R} + \frac{1}{\omega \partial z} \partial \cdot \frac{1}{R}\right) x^{-\alpha} \int x^{\alpha} X \partial x,$$

cujus ambo membra in infinitum excrescunt ob $\omega = 0$.

Adjicìamus autem partem ex tertio factore

$$\gamma + z = \alpha - \omega + z,$$

oriundam, quae ob $\frac{P}{\gamma+z} = (\alpha+z)^2 R$ est

$$\frac{1}{(\alpha+z)^2 R} x^{-\gamma} \int x^{\gamma} X \partial x, \text{ posito } z = -\gamma = -\alpha+\omega.$$

Quod si jam R ut ante is fuerit valor, qui oritur posito $z = -\alpha$, augendo hunc valorem particula ω, loco $\frac{1}{R}$ scribi debet

$$\frac{1}{R} + \frac{\omega}{\partial z}\partial \cdot \frac{1}{R} + \frac{\omega^3}{1.2\partial z^2}\partial\partial \cdot \frac{1}{R} \text{ etc.}$$

si quidem valorem $z = -\alpha$ et hic retineamus: unde haec integralis pars ob $\alpha+z = \omega$ erit

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega\partial z}\partial \cdot \frac{1}{R} + \frac{1}{2\partial z^2}\partial\partial \cdot \frac{1}{R}\right) x^{-\alpha+\omega} \int x^{\alpha-\omega} X \partial x;$$

sicque manifestum est, illum ipsius $\frac{1}{R}$ valorem usque ad secundam potestatem ipsius ω continuari debuisse, atque eadem lege hic alteram partem x involuentem exprimi conveniet. Ad quod observo, si habeatur hujusmodi formula $x^{\omega}\int x^{-\omega} V \partial x$ secundum potestates ipsius ω evoluenda, id hac ratione commodissime fieri. Posito

$$v = x^{\omega} \int x^{-\omega} V \partial x, \text{ ut sit } x^{-\omega} v = \int x^{-\omega} V \partial x,$$

erit differentiando $\partial v - \frac{\omega v \partial x}{x} = V \partial x$, quare posito

$$v = T + \omega T' + \omega^2 T'' + \omega^3 T''' + \text{ etc.}$$

habebitur, terminos secundum potestates ipsius ω disponendo,

$$\left.\begin{array}{l} \partial T + \omega \partial T' + \omega\omega \partial T'' + \omega^3 \partial T''' + \text{etc.} \\ - V\partial x - \omega T \frac{\partial x}{x} - \omega\omega T' \frac{\partial x}{x} - \omega^3 T'' \frac{\partial x}{x} - \text{etc.} \end{array}\right\} = 0$$

ideoque

$$T = \int V \partial x, \; T' = \int \frac{\partial x}{x} \int V \partial x, \; T'' = \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int V \partial x, \text{ etc.}$$

Consequenter cum in applicatione sit $V = x^{\alpha} X$, erit pars integralis ex factore $\gamma + z = \alpha - \omega + z$ nata

$$\left(\frac{1}{\omega\omega R} + \frac{1}{\omega\partial z}\partial \cdot \frac{1}{R} + \frac{1}{2\partial z^2}\partial\partial \cdot \frac{1}{R}\right) x^{-\alpha}\Big(\int x^{\alpha} X \partial x + \omega \int \frac{\partial x}{x} \int x^{\alpha} X \partial x$$
$$+ \omega^2 \int \frac{\partial x}{x} \int \frac{\partial x}{x} \int x^{\alpha} X \partial x\Big),$$

qua cum parte ex $(\alpha+z)^2$ nata junctim sumta, omnia membra infinita se mutuo destruunt, et pro quantitatis $P=(\alpha+z)^3 R$ factore cubico $(\alpha+z)^3$ in integrale ingreditur haec pars

$$\frac{1}{R}x^{-\alpha}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^\alpha X\,\partial x+\frac{1}{\partial z}\partial\cdot\frac{1}{R}\cdot x^{-\alpha}\int\frac{\partial x}{x}\int x^\alpha X\,\partial x$$
$$+\frac{1}{2\,\partial z^2}\partial\partial\cdot\frac{1}{R}\cdot x^{-\alpha}\int x^\alpha X\,\partial x,$$

si modo in quantitate $R=\frac{P}{(\alpha+z)^3}$ ubique scribatur $z=-\alpha$.

Corollarium 1.

1272. Methodus in solutione hujus problematis adhibita facile ad quotcunque factores aequales extendi potest. Si enim fuerit $(\alpha+z)^m$ factor quantitatis P, atque in hac fractione $\frac{(\alpha+z)^m}{P}$ suisque differentialibus, postquam fuerint evoluta, ponatur $z=-\alpha$, partes integralis inde natae ita se habebunt

Factor quant. P	$\alpha+z$	$(\alpha+z)^2$	$(\alpha+z)^3$
Pars integralis.	$\frac{\alpha+z}{P}x^{-\alpha}\int x^\alpha X\partial x$	$\frac{(\alpha+z)^2}{P}x^{-\alpha}\int\frac{\partial x}{x}\int x^\alpha X\partial x$ $\frac{1}{\partial z}\partial.\frac{(\alpha+z)^2}{P}.x^{-\alpha}\int x^\alpha X\partial x$	$\frac{(\alpha+z)^3}{P}x^{-\alpha}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int x^\alpha X\partial x$ $\frac{1}{\partial z}\partial.\frac{(\alpha+z)^3}{P}.x^{-\alpha}\int\frac{\partial x}{x}\int x^\alpha X\partial x$ $\frac{1}{2\partial z^2}\partial\partial.\frac{(\alpha+z)^3}{P}.x^{-\alpha}\int x^\alpha X\partial x.$

Corollarium 2.

1273. Si fuerint duo pluresve factores duplices inter se aequales, sumtis

$$\alpha=f(\cos.\theta+\sqrt{-1}.\sin.\theta)\text{ et}$$
$$\beta=f(\cos.\theta-\sqrt{-1}.\sin.\theta),$$

partes pro $(\alpha \mp z)^2$ et $(\beta + z)^2$ seorsim evolutae methodo supra adhibita non difficulter conjungentur, et ad realitatem reducentur.

Scholion.

1274. Simili methodo, qua hoc caput est pertractatum, in evolutione capitis III. hujus sectionis uti oportebat, neque tum ullum periculum in errores prolabendi fuisset pertimescendum. Superfluum autem nunc foret, errores ibi commissos hic emendare, cum non solum methodus plane esset eadem, sed etiam aequatio hic tractata facile in formam ibi consideratam transmutari queat et vicissim. Quodsi enim in aequatione capitis III.

$$X = Ay + \frac{B\,\partial y}{\partial x} + \frac{C\,\partial\partial y}{\partial x^2} + \frac{D\,\partial^3 y}{\partial x^3} + \frac{E\,\partial^4 y}{\partial x^4} + \text{etc.}$$

statuatur $x = lv$, ut sit $\partial x = \frac{\partial v}{v}$, functio autem X abeat in functionem ipsius v quae sit V, proveniet aequatio ejus formae quam hic tractavimus. Dum autem ibi elementum ∂x pro constanti est habitum, ad hanc conditionem exuendam ponamus

$$\partial y = p\,\partial x,\ \partial p = q\,\partial x,\ \partial q = r\,\partial x,\ \partial q = s\,\partial x,\ \text{etc.}$$

ut haec aequatio resultet

$$X = V = Ay + Bp + Cq + Dr + Es + Ft + \text{etc.}$$

Nunc autem ob $\partial x = \frac{\partial v}{v}$ adipiscimur, elemento ∂v constante sumto

$$p = \frac{\partial y}{\partial x} = \frac{v\,\partial y}{\partial v},$$

$$q = \frac{\partial p}{\partial x} = \frac{vv\,\partial\partial y}{\partial v^2} + \frac{v\,\partial y}{\partial v},$$

$$r = \frac{\partial q}{\partial x} = \frac{v^3\,\partial^3 y}{\partial v^3} + \frac{3v^2\,\partial\partial y}{\partial v^2} + \frac{v\,\partial y}{\partial v},$$

$$s = \frac{\partial r}{\partial x} = \frac{v^4\,\partial^4 y}{\partial v^4} + \frac{6v^3\,\partial^3 y}{\partial v^3} + \frac{7vv\,\partial\partial y}{\partial v^2} + \frac{v\,\partial y}{\partial v},$$

$$t = \frac{\partial s}{\partial x} = \frac{v^5\,\partial^5 y}{\partial v^5} + \frac{10v^4\,\partial^4 y}{\partial v^4} + \frac{25v^3\,\partial^3 y}{\partial v^3} + \frac{15vv\,\partial\partial y}{\partial v^2} + \frac{v\,\partial y}{\partial v},\ \text{etc.}$$

Quare aequatio inter v et y erit haec:

$$V = Ay + \frac{Bv\partial z}{\partial v} + \frac{Cvv\partial\partial y}{\partial v^2} + \frac{Dv^3\partial^3 y}{\partial v^3} + \frac{Ev^4\partial^4 y}{\partial v^4} + \frac{Fv^5\partial^5 y}{\partial v^5} + \text{etc.}$$
$$\begin{array}{llll} +C & +3D & +6E & +10F \\ +D & +7E & +25F & \\ +E & +15F & & \\ +F & & & \end{array}$$

cujus integrationem hic docuimus. Imprimis autem notandum est quantitatem algebraicam P hinc formandam

$$P = A + (B + C + D + E + F)(z - 1)$$
$$+ (C + 3D + 7E + 15F)(z - 1)(z - 2)$$
$$+ (D + 6E + 25F)(z - 1)(z - 2)(z - 3) + \text{etc.}$$

ad hanc formam reduci

$$P = A + B(z - 1) + C(z - 1)^2 + D(z - 1)^3 + E(z - 1)^4$$
$$+ F(z - 1)^5 + \text{etc.}$$

quae quantitas algebraica ab illa, qua in capite III. ad integrationem sumus usi, hoc tantum differt, quod ibi littera z id quod hic formula $z - 1$ expressimus; ex quo etiam ambarum integratio facillime altera ad alteram reducitur.

Conclusio libri primi.

1275. Atque haec fere sunt, quae ad librum primum de calculo integrali pertinere sunt visa, ubi methodum tradere institui, functiones unius variabilis ex data quacunque differentialium cujusque ordinis relatione investigandi; quod opus mihi equidem ita pertractasse videor, ut vix quicquam eorum, quae adhuc de hoc argumento ab aliis sunt inventa ab aliis sunt inventa et in medium allata, sit praetermissum.