CAPUT IV.

APPLICATIO METHODI INTEGRANDI IN CAPITE PRAECEDENTE TRADITAE AD EXEMPLA.

Problema 156.

1189.

Proposita hac aequatione differentiali

$$X = a^n y + \frac{\partial^n y}{\partial x^n}$$

ejus integrale completum invenire.

Solutio.

Hic ergo est $P = a^n + z^n$ ubi primo observetur, si n sit numerus impar, factorem simplicem esse $a + z$, ex quo nascitur pars integralis

$$\frac{1}{\mathfrak{A}} e^{-ax} \int e^{ax} X \partial x,$$

existente $\mathfrak{A}$ valore ex forma $\frac{P}{a+z}$ emergente, si ponatur $z = -a$, qui ergo valor cum sit etiam $\frac{\partial P}{\partial z} = n z^{n-1}$, ob $n - 1$ numerum parem, erit $\mathfrak{A} = n a^{n-1}$, ideoque haec integralis pars

$$\frac{1}{n a^{n-1}} e^{-ax} \int e^{ax} X \partial x.$$

Reliqui factores omnes in hac forma continentur

$$aa - 2az \cos. \phi + zz, \text{ existente } \phi = \frac{(2i+1)\pi}{n},$$

ubi i denotat numerum integrum quemcunque et π angulum duobus rectis aequalem. Comparata hac forma cum Probl. 153. et

Coroll. 1. fit $f = -a$, et ob $z = a(\cos.\theta + \sqrt{-1}.\sin.\theta)$, ex forma $\frac{\partial P}{\partial z}$ colligitur

$$\mathfrak{P} = n a^{n-1} \cos.(n-1)\theta \quad \text{et} \quad \mathfrak{Q} = n a^{n-1} \sin.(n-1)\theta;$$

cum igitur sit

$$\cos. n\theta = -1 \quad \text{et} \quad \sin. n\theta = 0, \quad \text{erit}$$

$$\mathfrak{P} = -n a^{n-1} \cos.\theta \quad \text{et} \quad \mathfrak{Q} = n a^{n-1} \sin\theta.$$

Quare posito $f x \sin.\theta = -a x \sin.\theta = \Phi$, integralis pars ex quolibet factore duplici oriunda est

$$\frac{2 e^{a x \cos.\theta}}{n a^{n-1}} \left\{ \begin{array}{l} (-\cos.\theta\cos.\Phi - \sin.\theta\sin.\Phi) \int e^{-a x \cos.\theta} X \partial x \cos.\Phi \\ (-\cos.\theta\sin.\Phi + \sin.\theta\cos.\Phi) \int e^{-a x \cos.\theta} X \partial x \sin.\Phi \end{array} \right\}$$

seu

$$\frac{-2e^{a x \cos.\theta}}{n a^{n-1}} [\cos.(\theta - \Phi) \int e^{-a x \cos.\theta} X \partial x \cos.\Phi - \sin.(\theta - \Phi) \int e^{-a x \cos.\theta} X \partial x \sin.\Phi]$$

et pro Φ valore restituto

$$\frac{-2 e^{a x \cos.\theta}}{n a^{n-1}} \left\{ \begin{array}{l} \cos.(\theta + a x \sin.\theta) \int e^{-a x \cos.\theta} X \partial x \cos.(a x \sin.\theta) \\ + \sin.(\theta + a x \sin.\theta) \int e^{-a x \cos.\theta} X \partial x \sin.(a x \sin.\theta). \end{array} \right\}$$

Jam pro θ successive substituantur anguli $\frac{\pi}{n}$, $\frac{3\pi}{n}$, $\frac{5\pi}{n}$, $\frac{7\pi}{n}$, quamdiu ipso π sunt minores, omnesque hae formae in unam summam conjectae, quibus casu quo n est numerus impar insuper addi oportet formam primo inventam

$$\frac{1}{n a^{n-1}} e^{-a x} \int e^{a x} X \partial x,$$

dabunt integrale quaesitum.

Corollarium 1.

1190. Casu quidem quo n est numerus impar, ultimus valor ipsius θ foret π, quem autem hic omitti jussimus, inde autem ob

$a x \sin. \theta = 0$ et $\cos. \theta = -1$,

prodiret ultima pars integralis

$$\frac{2\, e^{-a x}}{n\, a^{n-1}} \int e^{a x} \mathrm{X}\, \partial x,$$

dupla ejus quam capi convenit, cujus ratio est, quod sumto $\theta = \pi$ formula $a a + 2 a z + z z$ non amplius ipsa est factor, sed ejus radix quadrata $a + z$, ex quo hunc casum seorsim erui necesse erat.

Corollarium 2.

1191. Si est $\mathrm{X} = 0$, formulae integrales abeunt in constantes arbitrarias, et ex factore

$$a a - 2 a z \cos. \theta + z z$$

oritur haec pars integralis

$$\frac{-2\, e^{a x \cos. \theta}}{n\, a^{n-1}} [\mathrm{A} \cos. (\theta + a x \sin. \theta) + \mathfrak{A} \sin. (\theta + a x \sin. \theta)],$$

quae reducitur ad hanc formam

$$\mathrm{A}\, e^{a x \cos. \theta} \cos. (\zeta + a x \sin. \theta),$$

denotante ζ angulum constantem quemcunque, uti jam supra invenimus.

Problema 157.

1192. Proposita hac aequatione differentiali

$$\mathrm{X} = a^n y - \frac{\partial^n y}{\partial x^n}$$

ejus integrale completum invenire.

Solutio.

Forma algebraica hinc nata $\mathrm{P} = a^n - z^n$ factorem semper habet $a - z$, unde nascitur pars integralis $\frac{1}{\mathfrak{A}} e^{a x} \int e^{-a x} \mathrm{X}\, \partial x$, existente $\mathfrak{A} = \frac{P}{z - a}$ posito $z = a$. Cum ergo sit quoque

$$\mathfrak{A} = \frac{\partial P}{\partial z} = -n z^{n-1}, \text{ erit } \mathfrak{A} = -n a^{n-1},$$

ideoque haec pars integralis

$$\frac{-1}{n a^{n-1}} e^{ax} \int e^{-ax} X \partial x.$$

Deinde si n sit numerus par, hincque $n - 1$ impar, factor quoque erit $a + z$, qui praebet intsgralis partem

$$\frac{1}{n a^{n-1}} e^{-ax} \int e^{ax} X \partial x.$$

Reliqui factores omnes ipsius P sunt duplicis formae $aa - 2az \cos.\theta + zz$, existente angulo $\theta = \frac{2i\pi}{n}$, qua cum generali supra usurpata $ff + 2fz\cos.\theta + zz$ comparata, fit $f = -a$, et ex forma $\frac{\partial P}{\partial z} = -n z^{n-1}$ quaeri oportet formulam $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$, posito $z = a(\cos.\theta + \sqrt{-1}.\sin.\theta)$, unde colligitur

$$\mathfrak{P} = -n a^{n-1} \cos.(n-1)\theta \text{ et } \mathfrak{Q} = -n a^{n-1} \sin.(n-1)\theta,$$

seu ob $\cos. n\theta = 1$ et $\sin. n\theta = 0$, fit

$$\mathfrak{P} = -n a^{n-1} \cos.\theta \text{ et } \mathfrak{Q} = +n a^{n-1} \sin.\theta.$$

Posito jam angulo $-ax\sin.\theta = \Phi$, ex §. 1177. oritur pars integralis

$$\frac{2e^{ax\cos.\theta}}{n a^{n-1}} \left\{ \begin{array}{l} (-\cos.\theta\cos.\Phi + \sin.\theta\sin.\Phi) \int e^{-ax\cos.\theta} X \partial x \cos.\Phi \\ +(-\cos.\theta\sin.\Phi + \sin.\theta\cos.\Phi) \int e^{-ax\cos.\theta} X \partial x \sin.\Phi \end{array} \right\},$$

quae ut ante reducitur ad hanc formam

$$\frac{-2e^{ax\cos.\theta}}{n a^{n-1}} \left\{ \begin{array}{l} \cos.(\theta + ax\sin.\theta) \int e^{-ax\cos.\theta} X \partial x \cos.(ax\sin.\theta) \\ +\sin.(\theta + ax\sin.\theta) \int e^{-ax\cos.\theta} X \partial x \sin.(ax\sin.\theta) \end{array} \right\}.$$

Hic jam pro θ successive scribantur anguli $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$, etc. quamdiu sunt minores quam π, haeque partes omnes cum primum inventa atque etiam altera, si n fuerit numerus par, in unam summam collectae dabunt integrale quaesitum seu valorem ipsius y.

Corollarium.

1193. Cum factor duplex generalis $aa - 2a\cos.\theta + zz$ casibus $\theta = 0$ et $\theta = \pi$ non praebeat ipsos factores simplices reales $a - z$ et $a + z$ sed eorum quadrata, haec ratio est, cur pars integralis inde eruta prodeat dupla ejus, quam capi oportet.

Problema 158.

1194. Proposita hac aequatione differentiali

$$X = y + \frac{\partial y}{\partial x} + \frac{\partial\partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \ldots\ldots + \frac{\partial^n y}{\partial x^n},$$

ejus integrale completum investigare.

Solutio.

Forma algebraica hinc nata est

$$P = 1 + z^2 + z^3 + z^4 + \ldots\ldots + z^n,$$

cujus omnes factores scrutari oportet. Cum igitur sit $P = \frac{1 - z^{n+1}}{1 - z}$, formae $1 - z^{n+1}$ factores capi convenit, excluso $1 - z$; unde primo patet, si fuerit $n + 1$ numerus par, factorem simplicem fore $1 + z$, ex quo nascitur pars integralis $\frac{1}{\mathfrak{A}} e^{-x} \int e^x X \partial x$, existente $\mathfrak{A} = \frac{P}{1+z} = \frac{1 - z^{n+1}}{1 + zz}$, posito $z = -1$. Erit ergo quoque $\mathfrak{A} = \frac{(n+1) z^n}{2z}$, ideoque $\mathfrak{A} = \frac{1}{2}(n + 1)$, ut haec pars integralis sit $\frac{2}{n+1} e^{-x} \int e^x X \partial x$.

Factorum autem duplicium forma est $1 - 2z\cos.\theta + zz$, sumto angulo $\theta = \pm\frac{2i\pi}{n+1}$, ita ut pro §. 1176. sit $f = -1$. Consideretur forma

$$\frac{\partial P}{\partial z} = \frac{1 - (n+1)z^n + nz^{n+1}}{(1-z)^2},$$

quae posito $z = \cos.\theta + \sqrt{-1}.\sin.\theta$, abire sumitur in $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$, sicque erit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{1 - (n+1)\cos.n\theta + n\cos.(n+1)\theta - (n+1)\sqrt{-1}.\sin.n\theta + n\sqrt{-1}.\sin.(n+1)\theta}{1 - 2\cos.\theta + \cos.2\theta - 2\sqrt{-1}.\sin.\theta + \sqrt{-1}.\sin.2\theta}.$$

Cum vero sit

$$\sin.(n+1)\theta = 0 \text{ et } \cos.(n+1)\theta = 1, \text{ erit}$$

$$\sin.n\theta = -\sin.\theta \text{ et } \cos.n\theta = \cos.\theta, \text{ ideoque}$$

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{n+1 - (n+1)\cos.\theta + (n+1)\sqrt{-1}.\sin.\theta}{-2\cos.\theta + 2\cos.\theta^2 - 2\sqrt{-1}.\sin.\theta(1-\cos\theta)}, \text{ seu}$$

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{n+1}{2(1-\cos.\theta)}.\frac{1 - \cos.\theta + \sqrt{-1}.\sin.\theta}{-\cos.\theta - \sqrt{-1}.\sin.\theta}:$$

multiplicetur hujus fractionis numerator et denominator per $-\cos.\theta + \sqrt{-1}.\sin.\theta$ et prodibit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-(n+1)[1 + \cos.\theta - 2\cos.\theta^2 - \sqrt{-1}.\sin.\theta(1 - 2\cos.\theta)]}{2(1-\cos.\theta)},$$

ita ut sit

$$\mathfrak{P} = -\tfrac{1}{2}(n+1)(1+2\cos.\theta) \text{ et } \mathfrak{Q} = \tfrac{1}{2}(n+1)\frac{\sin.\theta(1-2\cos.\theta)}{1-\cos.\theta},$$

unde fit

$$\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q} = \frac{(n+1)^2}{2(1-\cos.\theta)}.$$

Tum vero posito angulo $-x\sin.\theta = \Phi$, colligitur

$$\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi = \frac{-(n+1)[\cos.(\theta-\Phi) - \cos.(2\theta-\Phi)]}{2(1-\cos.\theta)},$$

$$\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi = \frac{+(n+1)[\sin.(\theta-\Phi) - \sin.(2\theta-\Phi)]}{2(1-\cos.\theta)},$$

cum autem sit

$$\cos.a - \cos.b = 2\sin.\frac{a+b}{2}\sin.\frac{b-a}{2}, \text{ et}$$

$$\sin.a - \sin.b = -2\sin.\frac{b-a}{2}\cos.\frac{a+b}{2},$$

fit hinc

$$\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi = \frac{-(n+1)\sin.\frac{1}{2}(3\theta - 2\Phi)}{2\sin.\frac{1}{2}\theta} \text{ et}$$

$$\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi = \frac{-(n+1)\cos.\frac{1}{2}(3\theta - 2\Phi)}{2\sin.\frac{1}{2}\theta},$$

ex quo integralis pars quaesita erit

$$\frac{-4}{n+1}e^{x\cos.\theta}\sin.\tfrac{1}{2}\theta\left\{\begin{array}{l}\sin.\frac{1}{2}(3\theta+2x\sin.\theta)\int e^{-x\cos.\theta}X\partial x\cos.(x\sin.\theta)\\ -\cos.\frac{1}{2}(3\theta+2x\sin.\theta)\int e^{-x\cos.\theta}X\partial x\sin.(x\sin.\theta)\end{array}\right\}$$

Pro θ ergo successive substituantur anguli

$$\frac{2\pi}{n+1},\ \frac{4\pi}{n+1},\ \frac{6\pi}{n+1},\ \text{etc.}$$

quamdiu sunt minores quam π, haecque partes omnes in unam summam colligantur, cui si $n+1$ sit numerus par, addatur insuper $\frac{2}{n+1}e^{-x}\int e^{x}X\partial x$, sicque obtinebitur valor ipsius y.

Corollarium 1.

1195. Si aequatio proposita in infinitum progrediatur, ut sit n numerus infinitus, anguli θ priores omnes sunt infinite parvi ideoque numero infiniti, quoad numerus par $2i$ ad $n+1$ rationem finitam habere incipiat, tum autem pro θ sequentur omnes anguli finiti in progressione arithmetica increscentes, cujus differentia est $\frac{2\pi}{n+1}$, usque ad π, quorum numerus itidem est infinitus.

Corollarium 2.

1196. Quamdiu angulus θ est infinite parvus, integralis pars ex eo oriunda hanc induit formam

$$\frac{-\theta\theta e^{x}}{n+1}\left[(3+2x)\int e^{-x}X\partial x - \int e^{-x}Xx\partial x\right],$$

quae cum per cubum infiniti sit divisa, etiam multitudo infinita hujusmodi formularum pro evanescente est habenda.

Corollarium 3.

1197. Quodsi fuerit $X = 0$, ut hujus aequationis

$$0 = y + \frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \ldots\ldots + \frac{\partial^n y}{\partial x^n}$$

integrale sit investigandum, erit ejus pars quaecunque

$$e^{x \cos. \theta} [A \sin. \tfrac{1}{2}(3\theta + 2x \sin. \theta) + \mathfrak{A} \cos. \tfrac{1}{2}(3\theta + 2x \sin. \theta)],$$

seu simplicius

$$A e^{x \cos. \theta} (\cos. \zeta + x \sin. \theta).$$

Cum igitur si n sit numerus infinitus, pro θ angulus quicunque accipi queat, erit istius aequationis integrale particulare quodcunque

$$y = A e^{x \cos. \theta} (\cos. x \sin. \theta + \zeta),$$

sumendo pro ζ etiam angulum quemcunque.

Scholion.

1198. Num autem hujus aequationis differentialis in infinitum excurrentis

$$X = y + \frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \frac{\partial^4 y}{\partial x^4} + \text{etc.}$$

denotante X functionem quamcunque ipsius x, integrale commodius exprimi possit, quam per partium illarum innumerabilium evanescentium summam, quaestio est altioris indaginis, neque adhuc ad hunc scopum Analyseos fines satis videntur promoti. Casibus quidem, quibus X est functio rationalis integra, puta

$$X = a + bx + cx^2 + dx^3 + ex^4 + \text{etc.}$$

res nullam habet difficultatem, cum sumto

$$y = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.} + v,$$

hi coëfficientes α, β, γ, etc. semper ita definiri queant ut facta substitutione prodeat talis aequatio

$$0 = v + \frac{\partial v}{\partial x} + \frac{\partial \partial v}{\partial x^2} + \frac{\partial^3 v}{\partial x^3} + \text{etc.}$$

cui particulariter satisfacit valor

$$v = A\, e^{x \cos. \theta} \cos. (x \sin. \theta + \zeta),$$

sumtis pro ζ et θ angulis quibuscunque. Verum ex dato ejusmodi valore ipsius X invenitur

$\alpha = a - b$, $\beta = b - 2c$, $\gamma = c - 3d$, $\delta = d - 4e$, $\varepsilon = e - 5f$, etc.

Verum in genere cum fiat

$$\frac{\partial X}{\partial x} = \frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

evidens est semper, posito $y = X - \frac{\partial X}{\partial x} + v$, aequationem illam transformari in hanc

$$0 = v + \frac{\partial v}{\partial x} + \frac{\partial \partial v}{\partial x^2} + \frac{\partial^3 v}{\partial x^3} + \text{etc.}$$

Corollarium.

1199. En ergo praeter expectationem integrationem completam hujus aequationis differentialis in infinitum excurrentis

$$X = y + \frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} + \frac{\partial^3 y}{\partial x^3} + \frac{\partial^4 y}{\partial x^4} + \text{etc.}$$

pro qua jam novimus esse

$$y = X - \frac{\partial X}{\partial x} + A\, e^{x \cos. \theta} \cos. (x \sin. \theta + \zeta),$$

quod postremum membrum ob angulos ζ et θ arbitrarios in infinitum multiplicari potest. Haecque forma maxime complicatae illi ex solutione oriundae aequivalere est censenda.

Problema 159.

1200. Proposita hac aequatione differentiali

$$X = y + \frac{n \partial y}{a \partial x} + \frac{n(n-1) \partial \partial y}{1 \cdot 2 \cdot a^2 \partial x^2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 a^3} \frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

ubi quidem n sit numerus integer affirmativus, ut terminorum numerus sit finitus, ejus integrale completum investigare.

Solutio.

Formula algebraica hinc consideranda fit

$$P = 1 + \frac{n}{1} \cdot \frac{z}{a} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{z^2}{a^2} + \text{etc.} = \left(1 + \frac{z}{a}\right)^n,$$

quae ergo meros habet factores simplices inter se aequales $z + a$. Cum igitur sit $\frac{P}{(a+z)^n} = \frac{1}{a^n}$, ex §. 1163. statim colligitur integrale quaesitum

$$y = a^n e^{-ax} \int \partial x \int \partial x \int \partial x \; . \; . \; . \; . \; . \int e^{ax} X \partial x,$$

quoad signorum integralium numerus aequetur exponenti n. Hanc autem formam sequenti modo in integralia simplicia resolvere licet, ope reductionis generalis qua esse novimus

$$\int \partial x \int V \partial x = x \int V \partial x - \int V x \partial x,$$

unde fit

$$\int \partial x \int e^{ax} X \partial x = x \int e^{ax} X \partial x - \int e^{ax} X x \partial x,$$

$$\int \partial x \int \partial x \int e^{ax} X \partial x = \tfrac{1}{2} x^2 \int e^{ax} X \partial x - x \int e^{ax} X x \partial x + \tfrac{1}{2} \int e^{ax} X x^2 \partial x,$$

$$\int \partial x \int \partial x \int \partial x \int e^{ax} X \partial x = \frac{x^3 \int e^{ax} X \partial x - 3x^2 \int e^{ax} X x \partial x + 3x \int e^{ax} X x^2 \partial x - \int e^{ax} X x^3 \partial x}{1 \cdot 2 \cdot 3},$$

etc.

Cum igitur signorum integralium numerus sit $= n$, concludimus fore

$$y = \frac{a^n e^{-x}}{1 \cdot 2 \ldots \ldots (n-1)} \Big[x^{n-1} \int e^{ax} X \partial x - \frac{(n-1)}{1} x^{n-2} \int e^{ax} X x \partial x + \frac{(n-1)(n-2)}{1 \cdot 2} x^{n-3} \int e^{ax} X x^2 \partial x - \text{etc.} \Big],$$

ubi cum singula integralia constantem arbitrariam implicent, manifestum est, hoc integrale esse completum.

Corollarium 1.

1201. Si ergo esset $X = 0$, aequationis differentialis propositae integrale completum foret

$$y = e^{-ax}(A x^{n-1} + B x^{n-2} + C x^{n-3} + D x^{n-4} + \text{etc.} \ldots + M x + N),$$

ubi constantium arbitrariarum A, B, C, etc. numerus utique est $= n$.

Corollarium 2.

1202. Si numerus n fuerit infinitus, simulque quantitas a capiatur infinita, ut sit $a = n c$, aequatio integranda in infinitum excurret, eritque

$$X = y + \frac{\partial y}{c \partial x} + \frac{\partial \partial y}{1.2 c^2 \partial x^2} + \frac{\partial^3 y}{1.2.3 c^3 \partial x^3} + \text{etc.}$$

aequatio autem integralis ad hunc casum applicata nullam lucem foeneratur.

Corollarium 3.

1203. Quaecunque autem y functio fuerit ipsius x, constat si loco x scribatur $x + \frac{1}{c}$, eam abire in

$$y + \frac{\partial y}{c \partial x} + \frac{\partial \partial y}{1.2 c^2 \partial x^2} + \frac{\partial^3 y}{1.2.3 c^3 \partial x^3} + \text{etc.}$$

quae cum esse debeat $= X$, vicissim patet, y aequari ei functioni ipsius x quae nascitur ex X, si ibi loco x scribatur $x - \frac{1}{c}$.

Scholion 1.

1204. Quod quo facilius appareat observo, si proposita fuerit quaecunque ejusmodi aequatio

$$X = A y + B \frac{\partial y}{\partial x} + C \frac{\partial \partial y}{\partial x^2} + D \frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

semper sine ulla integratione integrale particulare per approximationem hoc modo inveniri posse: statuatur

$$y = \alpha X + \beta \frac{\partial X}{\partial x} + \gamma \frac{\partial \partial X}{\partial x^2} + \delta \frac{\partial^3 X}{\partial x^3} + \text{etc.}$$

factaque substitutione habebitur

$$\begin{aligned} X = A\alpha X &+ A\beta \cdot \frac{\partial X}{\partial x} + A\gamma \cdot \frac{\partial\partial X}{\partial x^2} + A\delta \cdot \frac{\partial^3 X}{\partial x^3} + \text{etc.} \\ &+ B\alpha \quad\;\; + B\beta \qquad\;\; + B\gamma \\ &\qquad\qquad\; + C\alpha \qquad\;\; + C\beta \\ &\qquad\qquad\qquad\qquad\quad\;\; + D\alpha \end{aligned}$$

sicque coëfficientes α, β, γ, δ, etc. definiuntur, ut sit $\alpha = \frac{1}{A}$, reliqui vero

$$\beta = \frac{-B\alpha}{A} = \frac{-B}{A^2},$$

$$\gamma = \frac{-C\alpha - B\beta}{A} = \frac{-C}{A^2} + \frac{BB}{A^3},$$

$$\delta = \frac{-D\alpha - C\beta - B\gamma}{A} = \frac{-D}{A^2} + \frac{2BC}{A^3} - \frac{B^3}{A^4},$$

$$\varepsilon = \frac{-E\alpha - D\beta - C\gamma - D\delta}{A} = \frac{-E}{A^2} + \frac{2BD + C^2}{A^3} - \frac{3BBC}{A^4} + \frac{B^4}{A^5},$$

etc.

quae si accommodentur ad casum problematis, fiet

$$y = X - \frac{n\partial X}{1a\partial x} + \frac{n(n+1)\partial\partial X}{1.2a^2\partial x^2} - \frac{n(n+1)(n+2)\partial^3 X}{1.2.3a^3\partial x^3} + \text{etc.}$$

Hinc casu quo $n = \infty$ et $a = nc$ colligitur

$$y = X - \frac{\partial X}{1c\partial x} + \frac{\partial\partial X}{1.2c^2\partial x^2} - \frac{\partial^3 X}{1.2.3c^3\partial x^3} + \text{etc.}$$

quae expressio etsi in infinitum excurrens manifesto definit eam ipsius x functionem, quae nascitur ex X, si loco x scribatur $x - \frac{1}{c}$. Quodsi jam hanc novam functionem signo X′ indicimus, ponamusque $y = X' + v$, aequatio Corollarii 2. abit in hanc

$$0 = v + \frac{\partial v}{\partial x} + \frac{\partial\partial v}{1.2\partial x} + \frac{\partial^3 v}{1.2.3\partial x^3} + \text{etc.}$$

cujus integrale particulare quodcunque est $v = Ae^{-ncx}x^m$ existente n numero infinito, et m numero integro positivo.

Scholion 2.

1205. Haec me deducunt ad sequentem speculationem circa serierum summationem. Sit nempe series quaecunque

$$\begin{matrix} 1 & 2 & 3 & 4 & & & & & & x \\ A, & B, & C, & D, & . & . & . & . & . & T, \end{matrix}$$

cujus terminus indici x respondens sit T functio quaecunque ipsius x. Statuatur summa omnium horum terminorum

$$A + B + C + D + \ldots\ldots + T = y,$$

at perspicuum est y fore ejusmodi functionem ipsius x, ut si in ea loco x scribatur $x - 1$, proditura sit eadem illa summa y termino ultimo T mulctata, scilicet $y - T$. At loco x scribendo $x - 1$, functio y abit in

$$y - \frac{\partial y}{\partial x} + \frac{\partial\partial y}{1.2\,\partial x^2} - \frac{\partial^3 y}{1.2.3\,\partial x^3} + \text{etc.}$$

unde oritur haec aequatio

$$T = \frac{\partial y}{\partial x} - \frac{\partial\partial y}{1.2\,\partial x^2} + \frac{\partial^3 y}{1.2.3\,\partial x^3} - \frac{\partial^4 y}{1.2.3.4\,\partial x^4} + \text{etc.};$$

quae semel integrata posito $\int T\,\partial x = X$, fit

$$X = y - \frac{\partial y}{1.2\,\partial x} + \frac{\partial\partial y}{1.2.3\,\partial x^2} - \frac{\partial^3 y}{1.2.3.4\,\partial x^3} + \text{etc.}$$

quam quomodo integrari conveniat videamus, dum eam aliquanto generaliorem reddemus.

Problema 160.

1206. Proposita hac aequatione differentiali

$$X = \frac{ny}{a} + \frac{n(n-1)\,\partial y}{1.2\,a^2\,\partial x} + \frac{n(n-1)(n-2)\,\partial\partial y}{1.2.3\,a^3\,\partial x^2} - \text{etc.}$$

ejus integrale completum investigare.

Solutio.

Formetur inde haec quantitas algebraica

$$P = \frac{n}{a} - \frac{n(n-1)z}{1.\ 2\,a^2} + \frac{n(n-1)(n-2)zz}{1.\ 2.\ 3\,a^3} - \text{etc.} = \frac{1 - \left(1 - \frac{z}{a}\right)^n}{z},$$

seu $P = \dfrac{a^n - (a-z)^n}{a^n z}$, cujus factor duplex quicunque hanc habebit formam

$$aa - 2a(a-z)\cos. 2\zeta + (a-z)^2,$$

existente angulo $2\zeta = \frac{2i\pi}{n}$. Abit autem haec forma in

$$2aa(1-\cos. 2\zeta) - 2az(1-\cos. 2\zeta) + zz,$$

vel $4aa\sin\zeta^2 - 4az\sin.\zeta^2 + zz$,

quas cum generali $ff + 2fz\cos.\theta + zz$ comparata dat

$$f = 2a\sin.\zeta, \text{ et } \cos.\theta = -\sin.\zeta,$$

unde

$$\theta = 90^0 + \zeta, \text{ et } \sin.\theta = \cos.\zeta,$$

existente $\zeta = \frac{i\pi}{n}$. Jam ad partem integralis hinc ortam inveniendam consideretur forma

$$\frac{\partial P}{\partial z} = \frac{-a^n + [a + (n-1)z](a-z)^{n-1}}{a^n zz},$$

in qua posito

$$z = -f(\cos.\theta + \sqrt{-1}.\sin.\theta), \text{ seu}$$

$$z = 2a\sin.\zeta(\sin.\zeta - \sqrt{-1}.\cos.\zeta) = a(1-\cos. 2\zeta - \sqrt{-1}.\sin. 2\zeta),$$

ut sit

$$a - z = a(\cos. 2\zeta + \sqrt{-1}.\sin. 2\zeta),$$

prodit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-1 + [n-(n-1)(\cos. 2\zeta + \sqrt{-1}.\sin. 2\zeta)][\cos. 2(n-1)\zeta + \sqrt{-1}.\sin. 2(n-1)\zeta]}{-4aa\sin.\zeta^2(\cos. 2\zeta + \sqrt{-1}.\sin. 2\zeta)}$$

Cum autem sit

$$\cos. 2n\zeta = 1 \text{ et } \sin. 2n\zeta = 0, \text{ erit}$$

$$\cos. 2(n-1)\zeta = \cos. 2\zeta \text{ et } \sin. 2(n-1)\zeta = -\sin. 2\zeta,$$

ideoque

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-n + n(\cos. 2\zeta - \sqrt{-1}.\sin. 2\zeta)}{-4aa\sin.\zeta^2(\cos. 2\zeta + \sqrt{-1}.\sin. 2\zeta)},$$

quae reducitur ad hanc formam

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n}{4aa\sin.\zeta^2}(\cos.2\zeta-\sqrt{-1}.\sin.2\zeta-\cos.4\zeta+\sqrt{-}\sin.4\zeta),$$

unde concluditur

$$\mathfrak{P}=\frac{n}{4aa\sin.\zeta^2}(\cos.2\zeta-\cos.4\zeta)=\frac{n}{2aa\sin\zeta}\sin.3\zeta,$$

$$\mathfrak{Q}=\frac{-n}{4aa\sin.\zeta^2}(\sin.2\zeta-\sin.4\zeta)=\frac{n}{2aa\sin.\zeta}\cos.3\zeta,$$

sicque est

$$\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}=\frac{nn}{4a^4\sin.\zeta^2},$$

et posito

$$\Phi=2ax\sin.\zeta\cos.\zeta=ax\sin.2\zeta,\ \text{fiet}$$

$$\mathfrak{P}\cos.\Phi-\mathfrak{Q}\sin.\Phi=\frac{n}{2aa\sin.\zeta}\sin.(3\zeta-\Phi)\ \text{et}$$

$$\mathfrak{P}\sin.\Phi+\mathfrak{Q}\cos.\Phi=\frac{n}{2aa\sin.\zeta}\cos.(3\zeta-\Phi).$$

Quocirca integralis pars hinc oriunda erit

$$\frac{4aa\sin.\zeta}{n}e^{2ax\sin.\zeta^2}\left\{\begin{array}{l}\sin.(3\zeta-\Phi)\int e^{-2ax\sin.\zeta^2}X\partial x\cos.\Phi\\+\cos.(3\zeta-\Phi)\int e^{-2ax\sin.\zeta^2}X\partial x\sin.\Phi\end{array}\right\}$$

ubi pro ζ successive scribi debent hi anguli

$$\frac{\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n},\ \frac{4\pi}{n},\ \text{etc.}$$

quamdiu sunt angulo recto minores, at si n sit numerus par ad has partes insuper addi oportet

$$-\frac{2aa}{n}e^{2ax}\int e^{-2ax}X\partial x,$$

sicque colligetur verus valor ipsius y.

Corollarium 1.

1207. Si est $X=0$, pars integralis ex quolibet angulo $\zeta=\frac{i\pi}{n}$ nata induit hanc formam

$$e^{2ax\sin.\zeta^2}[A\sin.(3\zeta-ax\sin.2\zeta)+B\cos.(3\zeta-ax\sin.2\zeta)],$$

seu hanc

$$Ae^{2ax\sin.\zeta^2}\sin.(\alpha+ax\sin.2\zeta),$$

denotante α angulum quemcunque constantem.

Corollarium 2.

1208. Invento integrali particulari quocunque $y = V$ quod aequationi propositae satisfaciat, si ponamus deinceps $y = V + v$, orietur haec aequatio

$$0 = \frac{nv}{a} - \frac{n(n-1)\partial v}{1.2 a^2 \partial x} + \frac{n(n-1)(n-2)\partial\partial v}{1.2.3 a^3 \partial x^2} - \text{etc.}$$

ex quo integrale completum erit

$$y = V + A e^{2 a x \sin. \zeta^2} \sin. (\alpha + a x \sin. 2\zeta),$$

ultima hac parte secundum omnes valores ipsius ζ multiplicata.

Corollarium 3.

1209. Si sumamus $n = \infty$ et $a = n$, ut haec prodeat aequatio differentialis in infinitum excurrens

$$X = y - \frac{\partial y}{1.2\partial x} + \frac{\partial\partial y}{1.2.3\partial x^2} - \frac{\partial^3 y}{1.2.3.4\partial x^3} + \frac{\partial^4 y}{1.2.3.4.5\partial x^4} - \text{etc.}$$

erit y terminus summarorius progressionis, cujus terminus generalis indici x respondens est $T = \frac{\partial X}{\partial x}$. Quamdiu ergo angulus $\zeta = \frac{i\pi}{n}$ est infinite parvus, ob $\Phi = 2 i \pi x$, integralis pars quaelibet est

$$4 i\pi . e^{\frac{2ii\pi\pi}{n} x} \left\{ \begin{matrix} \sin. (\frac{3i\pi}{n} - 2 i\pi x) \int e^{\frac{-2ii\pi\pi}{n} x} X \partial x \cos. (2 i \pi x) \\ + \cos. (\frac{3i\pi}{n} - 2 i\pi x) \int e^{\frac{-2ii\pi\pi}{n} x} X \partial x \sin. (2 i \pi x) \end{matrix} \right\},$$

et omissis evanescentibus

$$4 i\pi [\cos. (2 i\pi x) \int X \partial x \sin. (2 i\pi x) - \sin. (2 i \pi x) \int X \partial x \cos. (2 i \pi x)],$$

si jam hic pro i successive omnes numeri integri 1, 2, 3, etc. substituantur, omnium formularum hoc modo resultantium summa dabit verum et completum valorem ipsius y.

Scholion.

1210. Pro aequatione autem proposita methodo ante indicata integrale particulare per seriem differentialium invenire licet, ponendo

$$y = AX + \frac{B\partial X}{\partial x} + \frac{C\partial\partial X}{\partial x^2} + \frac{D\partial^3 X}{\partial x^3} + \frac{E\partial^4 X}{\partial x^4} + \text{etc.}$$

facta enim substitutione reperitur

$$A = \frac{a}{n},\ B = \frac{n-1}{2n},\ C = \frac{nn-1}{12an},\ D = \frac{n.n-1}{24aan},$$

$$E = \frac{-(nn-1)(nn-19)}{720a^3n},\ \text{etc.}$$

cujus quidem seriei difficile est legem progressionis in genere assignare. Verum pro casu $n = \infty$ et $a = n$, qui imprimis in doctrina progressionum est notatu dignus, hi coëfficientes ita se habent.

$$A = 1,\ B = \tfrac{1}{2},\ C = \tfrac{1}{12},\ D = 0,\ E = \tfrac{-1}{720},\ \text{etc.}$$

unde ea ipsa forma oritur, quam olim in genere pro termino summatorio dedi. Concessio autem hoc termino summatorio, qui fit $= V$, probe notari convenit, aequationem $y = V$ tantum esse integrale particulare aequationis propositae, completum vero facile exhiberi, si modo ad V addantur omnes hujusmodi formulae $A \sin. (\alpha + 2i\pi x)$, pro i scribendo sucressive omnes numeros $1, 2, 3, 4$, etc. ubi pro quolibet angulus α pro arbitrio assumi potest. Quod autem singuli hi valores aequationi

$$0 = v - \frac{\partial v}{2\partial x} + \frac{\partial\partial v}{6\partial x^2} - \frac{\partial^3 v}{24\partial x^3} + \frac{\partial^4 v}{120\partial x^4} - \frac{\partial^5 v}{720\partial x^5} + \text{etc.}$$

satisfaciant, ita facillime ostenditur. Posito brevitatis gratia $2i\pi = m$, ut sit $v = \sin. (\alpha + mx)$, et facta substitutione fieri debet

$$0 = \begin{cases} \sin. (\alpha + mx)(1 - \frac{mm}{6} + \frac{m^4}{120} - \text{etc.} \\ \cos. (\alpha + mx)(-\frac{m}{2} + \frac{m^3}{24} - \frac{m^5}{720} + \text{etc.} \end{cases} = \begin{cases} \sin. (\alpha + mx)\frac{1}{m}\sin. m \\ \cos. (\alpha + mx)\frac{1}{m}(\cos. m - 1). \end{cases}$$

Cum autem sit $m = 2i\pi$, manifesto est tam $\sin m = 0$ quam $\cos. m - 1 = 0$.

Problema 151.

1211. Proposita hac aequatione differentiali

$$X = y + \frac{n(n-1)}{1.2} \cdot \frac{\partial\partial y}{a^2\partial x^2} + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} \cdot \frac{\partial^4 y}{a^4\partial x^4} + \text{etc.}$$

ejus integrale completum investigare.

Solutio.

Quantitas algebraica hinc formanda est

$$P = 1 + \frac{n(n-1)}{1.2} \cdot \frac{zz}{aa} + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} \cdot \frac{z^4}{a^4} + \text{etc.}$$

quae ad hanc formam manifesto reducitur

$$P = \tfrac{1}{2}(1 + \tfrac{z}{a})^n + \tfrac{1}{2} \cdot (1 - \tfrac{z}{a})^n = \frac{(a+z)^n + (a-z)^n}{2\,a^n},$$

cujus factor quicunque trinomialis est

$$(a+z)^2 - 2(aa - zz)\cos. 2\zeta + (a-z)^2,$$

sumendo

$$2\zeta = \frac{(2i+1)\pi}{n}, \text{ seu } \zeta = \frac{(2i+1)\pi}{2n}.$$

Haec autem forma abit in

$$2aa(1 - \cos. 2\zeta) + 2zz(1 + \cos. 2\zeta) = 4aa \sin. \zeta^2 + 4zz \cos. \zeta^2,$$

qui factor generalis repraesentetur hoc modo

$$aa\,\text{tang.}\,\zeta^2 + zz,$$

sicque comparatio cum forma generali

$$ff + 2fz\cos. \theta + zz$$

praebet

$$f = -a\,\text{tang.}\,\zeta \text{ et } \theta = 90^\circ,$$

unde fit

$$\Phi = -ax\,\text{tang.}\,\zeta, \ (1177.)$$

et valor pro z substituendus

$$-f(\cos. \theta + \sqrt{-1} \cdot \sin. \theta) = a\,\text{tang.}\,\zeta \cdot \sqrt{-1},$$

quo pacto

$$\frac{\partial P}{\partial z} = \frac{n(a+z)^{n-1} - n(a-z)^{n-1}}{2\,a^n}$$

abire ponitur in $\mathfrak{P}+\mathfrak{Q}\sqrt{-1}$,

unde fit

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\tfrac{n}{2a}[(1+\text{tang.}\zeta.\sqrt{-1})^{n-1}-(1-\text{tang.}\zeta.\sqrt{-1})^{n-1}]$$

$$=\frac{n}{2a\cos.\zeta^{n-1}}[\cos.(n-1)\zeta+\sqrt{-1}.\sin.(n-1)\zeta-\cos.(n-1)\zeta+\sqrt{-1}.\sin.(n-1)\zeta],$$

ideoque $\mathfrak{P}=0$ et $\mathfrak{Q}=\dfrac{n\sin.(n-1)\zeta}{a\cos.\zeta^{n-1}}$. At ob $n\zeta=\frac{2i+1}{2}\pi$, hincque $\cos.n\zeta=0$ et $\sin.n\zeta=\pm 1$, prout i fuerit numerus par vel impar, erit $\sin.(n-1)\zeta=\pm\cos.\zeta$, ideoque $\mathfrak{Q}=\dfrac{\pm n}{a\cos.\zeta^{n-2}}$. Quocirca ob $\cos.\theta=0$, integralis pars ex hoc factore oriunda est

$$\pm\frac{2a\cos.\zeta^{n-2}}{n}(\cos.\Phi\int X\partial x\sin.\Phi-\sin.\Phi\int X\partial x\cos.\Phi),$$

seu ob $\Phi=-ax\,\text{tang.}\zeta$,

$$\pm\frac{2a\cos.\zeta^{n-2}}{n}\left\{\begin{array}{l}\sin.(ax\,\text{tang.}\zeta)\int X\partial x\cos.(ax\,\text{tang.}\zeta)\\-\cos.(ax\,\text{tang.}\zeta)\int X\partial x\sin.(ax\,\text{tang.}\zeta)\end{array}\right\},$$

ubi pro ζ successive substituantur anguli

$$\frac{\pi}{2n},\ \frac{3\pi}{2n},\ \frac{5\pi}{2n},\ \frac{7\pi}{2n},$$

quamdiu sunt recto minores, pro quibus ibi alternatim $+$ et $-$ scribi oportet; haeque partes omnes in unam summam collectae dabunt valorem completum ipsius y, dummodo pro ultima parte ex angulo $\zeta=\frac{\pi}{2}$ oriunda, quod evenit si n numerus impar, ejus tantum semissis capiatur.

Corollarium 1.

1212. Accommodemus haec statim ad casum $n=\infty$ et $a=nc$, ut proposita sit haec aequatio differentialis

$$X = y + \frac{\partial\partial y}{1.2 c^2 \partial x^2} + \frac{\partial^4 y}{1.2.3.4 c^4 \partial x^4} + \frac{\partial^6 y}{1 \ldots 6 c^6 \partial x^6} + \text{etc. in infinitum.}$$

Cum igitur hic valores ipsius ζ sint infinite parvi, erit

$$\cos.\zeta = 1 \text{ et } \operatorname{tang}.\zeta = \zeta = \frac{(4i \pm 1)\pi}{2n},$$

hinc $a\,x \operatorname{tang}. = (4\,i \pm 1)\,c\,x \cdot \frac{\pi}{2}$,

pro quo angulo scribamus ω. Ergo pars integralis quaecunque

$$\pm 2c(\sin.\omega \int X\partial x \cos.\omega - \cos.\omega \int X\partial x \sin.\omega),$$

ubi signa ambigua sibi mutuo respondent.

Corollarium 2.

1213. Si tantum angulus $\frac{\pi}{2}cx$ ponatur $= \Phi$, integrale universum ita erit expressum

$$\begin{aligned}\frac{y}{2c} = &+ \sin.\Phi \int X\partial x \cos.\Phi - \cos.\Phi \int X\partial x \sin.\Phi \\ &- \sin.3\Phi \int X\partial x \cos.3\Phi + \cos.3\Phi \int X\partial x \sin.3\Phi \\ &+ \sin.5\Phi \int X\partial x \cos.5\Phi - \cos.5\Phi \int X\partial x \sin.5\Phi \\ &- \sin.7\Phi \int X\partial x \cos.7\Phi + \cos.7\Phi \int X\partial x \sin.7\Phi \\ &\text{etc.}\end{aligned}$$

quae formulae in infinitum sunt continuandae.

Corollarium 3.

1214. Si ponamus $c = b\sqrt{-1}$, ut habetur haec aequatio infinita

$$X = y - \frac{\partial\partial y}{1.2 b^2 \partial x^2} + \frac{\partial^4 y}{1 \ldots 4 b^4 \partial x^4} - \frac{\partial^6 y}{1 \ldots 6 b^6 \partial x^6} + \text{etc.}$$

ac jam angulum $\frac{\pi}{2} b\,x$ vocemus ψ, erit integrale completum

$$\frac{y}{b} = + e^{-\psi} \int e^{\psi} X \partial x - e^{\psi} \int e^{-\psi} X \partial x$$
$$- e^{-3\psi} \int e^{3\psi} X \partial x + e^{3\psi} \int e^{-3\psi} X \partial x$$
$$+ e^{-5\psi} \int e^{5\psi} X \partial x - e^{5\psi} \int e^{-5\psi} X \partial x$$
etc.

Scholion.

1215. Si pro aequatione Corollarii 1. methodo supra exposita quaeramus integrale particulare per differentialia ipsius X expressum, huncque in finem ponamus

$$y = A X - \frac{B \partial\partial X}{c^2 \partial x^2} + \frac{C \partial^4 X}{c^4 \partial x^4} - \frac{D \partial^6 X}{c^6 \partial x^6} + \frac{E \partial^8 X}{c^8 \partial x^8} - \text{etc.}$$

reperiemus hos coëfficientium valores

$$A = 1, \; B = \frac{1}{1.2}, \; C = \frac{5}{1 \ldots 4}, \; D = \frac{61}{1 \ldots 6},$$
$$E = \frac{1385}{1 \ldots 8}, \; F = \frac{50521}{1 \ldots 10}, \text{ etc.}$$

Hicque valor si ponatur $= V$, vocato angulo $\frac{\pi}{2} c x = \Phi$, erit integrale completum

$$y = V + A \sin.(\alpha + \Phi) + B \sin.(\beta + 3\Phi) + C \sin.(\gamma + 5\Phi)$$
$$+ D \sin.(\delta + 7\Phi) + \text{etc.}$$

Problema 162.

1216. Proposita aequatione differentiali

$$X = y + \frac{n(n-1)}{1.2 a^2} \cdot \frac{\partial y}{\partial x} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4 a^4} \cdot \frac{\partial \partial y}{\partial x^2}$$
$$+ \frac{n \ldots\ldots (n-5)}{1 \ldots 6 a^6} \cdot \frac{\partial^3 y}{\partial x^3} + \text{etc.}$$

ejus integrale completum investigare.

Solutio.

Quantitas algebraica hinc formanda

$$P = 1 + \frac{n(n-1)}{1.2\,a^2} z + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4\ \ a^4} . zz + \text{etc.}$$
$$= \tfrac{1}{2}\left(1 + \tfrac{\sqrt{z}}{a}\right)^n + \tfrac{1}{2}\left(1 - \tfrac{\sqrt{z}}{a}\right)^n,$$

cum ex praecedente nascatur, si ibi loco zz scribatur z, sumto angulo $\zeta = \frac{2i+1}{2n}\pi$, factor quicunque erit $aa\,\text{tang.}\,\zeta^2 + z$, ita ut hujus formae omnes factores simplices sint reales. Hoc ergo factore cum formula $\alpha + z$ comparato, erit $\alpha = aa\,\text{tang.}\,\zeta^2$, et sumto $\mathfrak{A} = \frac{P}{\alpha + z}$ posito $z = -\alpha$, erit integralis pars ex hoc factore oriunda $\frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x$. Quia vero P evanescit posito $z = -\alpha$, erit quoque $\mathfrak{A} = \frac{\partial P}{\partial z}$; at est differentiando

$$\frac{\partial P}{\partial z} = \frac{n}{4a\sqrt{z}}\left[\left(1 + \tfrac{\sqrt{z}}{a}\right)^{n-1} - \left(1 - \tfrac{\sqrt{z}}{a}\right)^{n-1}\right].$$

Quia igitur poni oportet $\frac{\sqrt{z}}{a} = \text{tang.}\,\zeta . \sqrt{-1}$, erit

$$1 + \frac{\sqrt{z}}{a} = \frac{\cos.\zeta + \sqrt{-1}.\sin.\zeta}{\cos.\zeta} \quad \text{et} \quad 1 - \frac{\sqrt{z}}{a} = \frac{\cos.\zeta - \sqrt{-1}.\sin.\zeta}{\cos.\zeta}$$

hincque

$$\mathfrak{A} = \frac{n}{4\,aa\,\text{tang.}\,\zeta . \sqrt{-1}} . \frac{2\sqrt{-1}.\sin.(n-1)\zeta}{\cos\ \zeta^{n-1}} = \frac{n\sin.(n-1)\zeta}{2\,aa\sin.\zeta\cos.\zeta^{n-2}}.$$

Jam observetur esse $\sin.\,n\zeta = \sin.(2i+1)\frac{\pi}{2} = \pm 1$, (ubi signum superius valet si i numerus par, inferius si impar,) tum vero $\cos.\,n\zeta = 0$, unde fit $\sin.(n-1)\zeta = \pm\cos.\zeta$, ex quo conficitur

$$\mathfrak{A} = \frac{\pm n}{2\,a\,a\sin.\zeta\cos.\zeta^{n-3}},$$

et pars integralis quaesita habebitur

$$\pm\frac{2\,a\,a\sin.\zeta\cos.\zeta^{n-3}}{n} e^{-aa\,\text{tang.}\zeta^2.x}\int e^{aa\,\text{tang}\,\zeta^2.x} X \partial x.$$

Nunc igitur ipsi ζ successive tribuantur hi valores

$$\frac{\pi}{2n},\ \frac{3\pi}{2n},\ \frac{5\pi}{2n},\ \frac{7\pi}{2n},\ \text{etc.}$$

quoad angulum rectum non superent, atque omnes istae partes in unam summam collectae, dabunt integrale completum seu valorem ipsius y.

Corollarium 1.

1217. Si ponamus $n = \infty$ et $a = nc$, aequatio proposita in infinitum excurrit, eritque

$$X = y + \frac{\partial y}{1.2 c^2 \partial x} + \frac{\partial \partial y}{1.2.3.4 c^4 \partial x^2} + \frac{\partial^3 y}{1 \ldots 6 c^6 \partial x^3} + \text{etc.}$$

et forma algebraica inde nata

$$P = 1 + \frac{z}{1.2 c^2} + \frac{zz}{1.2.3.4 c^4} + \frac{z^3}{1 \ldots 6 c^6} + \text{etc.} = \tfrac{1}{2} e^{\frac{\sqrt{z}}{c}} + \tfrac{1}{2} e^{\frac{-\sqrt{z}}{c}},$$

quae omnes factores simplices habet reales, et ob ζ infinite parvum erit tang. $\zeta = \zeta = \frac{2i+1}{2n}\pi$, indeque factorum forma generalis

$$z + \frac{(2i+1)^2}{4}\pi\pi cc, \text{ seu } 1 + \frac{4z}{(2i+1)^2 \pi\pi cc}.$$

Corollarium 2.

1218. Ponatur brevitatis gratia angulus

$\frac{2i+1}{2}\pi = \theta$, erit

$aa \tang. \zeta^2 = \theta\theta cc$, tum vero

$\cos. \zeta = 1$ et $\frac{aa \sin. \zeta}{n} = \theta cc$,

ex quo integralis pars quaecunque erit

$$\pm 2\theta cc e^{-\theta\theta cc x} \int e^{\theta\theta ccx} X \partial x,$$

ubi pro θ successive omnes hos angulos scribi oportet

$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$, etc.

Corollarium 3.

1219. Perinde hic est sive cc negative sive positive capiatur, hinc istius aequationis differentialis infinitae

$$X = y + \frac{\partial y}{1.2 b \partial x} + \frac{\partial \partial y}{1.2.3.4 bb \partial x^2} + \frac{\partial^3 y}{1 \ldots 6 b^3 \partial x^3} + \text{etc.}$$

integrale erit

$$y = 2\,\theta\,b\,e^{-\theta\theta bx}\int e^{\theta\theta bx}\,X\,\partial x,$$

loco θ scribendo successive omnes hos angulos, ambiguitate signi jam sublata

$$+\frac{\pi}{2},\ -\frac{3\pi}{2},\ +\frac{5\pi}{2},\ -\frac{7\pi}{2},\ +\ \text{etc.}$$

unde si $X = 0$, integrale particulare quodvis est

$$y = A\,e^{-\theta\theta bx}.$$

Problema 163.

1220. Proposita aequatione differentiali

$$X = \frac{n\,\partial y}{a\,\partial x} + \frac{n(n-1)(n-2)\,\partial\partial y}{1.\,2.\,3.\,a^3\,\partial x^2} + \frac{n(n-1)(n-2)(n-3)(n-4)\,\partial^3 y}{1.\quad 2.\quad 3.\quad 4.\quad 5\,a^5\,\partial x^3} + \text{etc.}$$

ejus integrale completum investigare.

Solutio.

Etsi haec aequatio in ∂x ducta sponte semel integratur, praestat tamen hanc formam retinere, unde fit

$$P = \frac{nz}{a} + \frac{n(n-1)(n-2)\,zz}{1.\quad 2.\quad 3\quad a^3} + \frac{n(n-1)(n-2)(n-3)(n-4)\,z^3}{1.\quad 2.\quad 3.\quad 4.\quad 5\quad a^5} + \text{etc.}$$

quae manifesto ita exhiberi potest

$$P = \frac{\sqrt{z}}{2}\left[\left(1 + \frac{\sqrt{z}}{a}\right)^n - \left(1 - \frac{\sqrt{z}}{a}\right)^n\right],$$

cujus quidem statim unus factor se offert z; reliqui vero in hac forma continentur

$$\left(1 + \frac{\sqrt{z}}{a}\right)^2 - 2\left(1 - \frac{z}{aa}\right)\cos.\,2\,\zeta + \left(1 - \frac{\sqrt{z}}{a}\right)^2,$$

sumto angulo

$$2\,\zeta = \frac{2\,i\,\pi}{n} \quad \text{seu} \quad \zeta = \frac{i\,\pi}{n},$$

haec vero forma abit in

$$2\,(1 - \cos.\,2\,\zeta) + \frac{2\,z}{aa}\,(1 + \cos.\,2\,\zeta),$$

unde patet in genere factorem fore $a\,a\,\text{tang.}\,\zeta^2 + z$, quae etiam primum illum z complectitur sumto $i = 0$. Hinc posito $a\,a\,\text{tang.}\,\zeta^2 = \alpha$, integralis pars huic factori respondens erit

$$\tfrac{1}{\mathfrak{A}}\, e^{-\alpha x} \int e^{\alpha x}\, X\, \partial x,$$

si posito

$$z = -a\,a\,\text{tang.}\,\zeta^2 \text{ seu } \sqrt{z} = a\,\text{tang.}\,\zeta\,\sqrt{-1},$$

capiatur

$$\mathfrak{A} = \frac{\partial P}{\partial x} = \frac{1}{4\sqrt{z}}\left[(1 + \tfrac{\sqrt{z}}{a})^n - (1 - \tfrac{\sqrt{z}}{a})^n\right]$$
$$+ \frac{n}{4a}\left[(1 + \tfrac{\sqrt{z}}{a})^{n-1} + (1 - \tfrac{\sqrt{z}}{a})^{n-1}\right].$$ At

$$(1 + \tfrac{\sqrt{z}}{n})^n = \frac{\cos. n\zeta + \sqrt{-1}\,.\sin. n\zeta}{\cos. \zeta^n}, \text{ et } (1 - \tfrac{\sqrt{z}}{a})^n = \frac{\cos. n\zeta - \sqrt{-1}\,.\sin. n\zeta}{\cos \zeta^n},$$

quamobrem fiet

$$\mathfrak{A} = \frac{\sin. n\zeta}{2\,a\,\text{tang}\,\zeta\cos. \zeta^n} + \frac{n\cos. (n-1)\zeta}{2\,a\cos. \zeta^{n-1}} = \frac{+ n}{2\,a\cos. \zeta^{n-2}}$$

ob $\sin. n\zeta = 0$ et $\cos. n\zeta = \pm 1$, prout numerus i fuerit vel par vel impar. Quocirca integralis pars quaecunque ita erit expressa

$$\pm \frac{2\,a\cos. \zeta^{n-2}}{n}\, e^{-\alpha x} \int e^{\alpha x}\, X\, \partial x,$$

existente $\alpha = a\,a\,\text{tang.}\,\zeta^2$. Jam angulo ζ successive tribuantur hi valores

$$\frac{0\pi}{n}, \frac{1\pi}{n}, \frac{2\pi}{u}, \frac{3\pi}{n},$$

quoad angulum rectum $\frac{\pi}{2}$ non excedant, haeque formulae omnes cum suis signis in unam summam conjectae dabunt valorem completum pro y.

Corollarium 1.

1221. Prima igitur integralis pars nascitur ex angulo $\zeta = 0$, unde ea erit $+ \frac{2a}{n} \int X\, \partial x$, cujus autem loco ob rationes supra allegatas circa factores simplices, ejus tantum dimidium sumi debet, ut haec prima pars sit $= \frac{2}{n} \int X \partial x$, quod etiam inde patet, quod posito $z = 0$ fiat manifesto $\frac{P}{z} = \frac{n}{a}$.

Corollarium 2.

1222. Idem tenendum esset de parte ultima, siquidem ex valore $\zeta = \frac{\pi}{2}$ nascatur, quod evenit si n sit numerus par. Quia vero hoc casu fit $\cos. \zeta = 0$, haec tota integralis pars per se evanescit.

Corollarium 3.

1223. Si esset $X = 0$, quaelibet pars integralis foret $A e^{-aa \,tang.\, \zeta^2 x}$, denotante A quantitatem constantem arbitrariam; foretque adeo haec aequatio

$$y = A e^{-aa \,tang.\, \zeta^2 x}$$

integrale particulare aequationis, dummodo capiatur angulus

$$\zeta = \frac{i\pi}{n}.$$

Scholion.

1224. Hinc posito $n = \infty$ et $a = n\sqrt{b}$, integrari potest haec aequatio differentialis in infinitum excurrens

$$\frac{X}{\sqrt{b}} = \frac{\partial y}{1.b\partial x} + \frac{\partial\partial y}{1.2.3\,bb\,\partial x^2} + \frac{\partial^3 y}{1\ldots 5\,b^3\,\partial x^3} + \frac{\partial^4 y}{1\ldots 7\,b^4\,\partial x^4} + \text{etc.}$$

vel etiam haec per unam integrationem ex ista nata

$$\sqrt{b}\,.\int X\,\partial x = \frac{y}{1} + \frac{\partial y}{1.2.3\,b\,\partial x} + \frac{\partial\partial y}{1\ldots 5\,b^2\,\partial x^2} + \frac{\partial^3 y}{1\ldots 7\,b^3\,\partial x^3} + \text{etc.}$$

Cum enim sit angulus $\zeta = \frac{i\pi}{n}$ infite parvus, erit

$$\cos. \zeta = 1, \text{ et } a \,tang.\, \zeta = a\zeta = i\pi\sqrt{b},$$

ideoque

$$\alpha = aa \,tang.\, \zeta^2 = ii\pi\pi b,$$

habebitur pars integralis quaecunque

$$\pm 2\sqrt{b}\,.\,e^{-ii\pi\pi bx}\int e^{ii\pi\pi bx}X\partial x,$$

unde parte prima ex $i = 0$ nata ad dimidium reducta, ab rationes supra allegatas, erit integrale completum

$$\frac{y}{\sqrt{b}} = \int X\partial x - 2e^{-\pi\pi bx}\int e^{\pi\pi bx}X\partial x + 2e^{-4\pi\pi bx}\int e^{4\pi\pi bx}X\partial x$$
$$- 2e^{-9\pi\pi bx}\int e^{9\pi\pi bx}X\partial x + 2e^{-16\pi\pi bx}\int e^{16\pi\pi bx}X\partial x - \text{etc.}$$

Exemplum.

1225. Sit $n = 6$ et $a = 1$, ut integranda proponatur haec aequatio

$$X = \frac{6\partial y}{\partial x} + \frac{20\partial\partial y}{\partial x^2} + \frac{6\partial^3 y}{\partial x^3}, \text{ seu}$$
$$\int X\partial x = 6y + \frac{20\partial y}{\partial x} + \frac{6\partial\partial y}{\partial x^2}.$$

Valores ergo pro angulo ζ et inde pendentes sunt

$$\zeta = 0,\ 30^\circ,\ 60^\circ$$
$$\cos.\ \zeta = 1,\ \frac{\sqrt{3}}{2},\ \frac{1}{2},$$
$$\alpha = 0,\ \frac{1}{3},\ 3,$$

ex quibus colligitur integrale quaesitum

$$y = \frac{1}{6}\int X\partial x - \frac{3}{16}e^{-\frac{1}{3}x}\int e^{\frac{1}{3}x}X\partial x + \frac{1}{48}e^{-3x}\int e^{3x}X\partial x,$$

quod etiam aequationi satisfacere tentanti patebit.