CAPUT III.

DE

INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM HUJUS FORMAE

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \text{etc.}$$

Problema 147.

1138.

Proposita aequatione differentiali

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \ldots + \frac{N\partial^n y}{\partial x^n},$$

sumto elemento ∂x constante, et significante X functionem quamcunque ipsius x, invenire functionem ipsius x, per quam haec aequatio multiplicata integrabilis evadat.

Solutio.

Sit $P\partial x$ iste multiplicator quem quaerimus, et cum prius membrum X eo integrabile reddatur, ejus rationem ex altero membro definiri oportet. Facile autem intelligitur, formam hujus multiplicatoris P ejusmodi fore $e^{\lambda x}$, ita ut quantitas λ definiri debeat. Sit ergo $e^{\lambda x}\partial x$ multiplicator, atque hanc formam

$$e^{\lambda}\partial x\left(Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \ldots + \frac{N\partial^n y}{\partial x^n}\right),$$

integrabilem esse oportet, cujus integrale propterea statuatur

$$e^{\lambda x}\left(A' y + \frac{B' \partial y}{\partial x} + \frac{C' \partial \partial y}{\partial x^2} + \ldots\ldots + \frac{M' \partial^{n-1} y}{\partial x^{n-1}}\right),$$

ita ut hujus differentiale cum illa forma congruere debet, quod cum sit

$$e^{\lambda x}\partial x \left\{ \begin{array}{l} \lambda A' y + \frac{\lambda B' \partial y}{\partial x} + \frac{\lambda C' \partial\partial y}{\partial x^2} + \ldots\ldots + \frac{\lambda M' \partial^{n-1} y}{\partial x^n} \\ \quad + \frac{A' \partial y}{\partial x} + \frac{B' \partial \partial y}{\partial x^2} + \ldots\ldots\ldots\ldots\ldots + \frac{M' \partial^n y}{\partial x^n} \end{array} \right\},$$

necesse est sit

$$A' = \frac{A}{\lambda},\ B' = \frac{B - A'}{\lambda},\ C' = \frac{C - B'}{\lambda},\ D' = \frac{D - C'}{\lambda},\ \ldots.$$

$$\ldots.\ M' = \frac{M - L'}{\lambda},\ \text{atque}\ M' = N. \quad \text{Hinc erit}$$

$$A' = \frac{A}{\lambda},$$

$$B' = \frac{B}{\lambda} - \frac{A}{\lambda^2},$$

$$C' = \frac{C}{\lambda} - \frac{B}{\lambda^2} + \frac{A}{\lambda^3},$$

$$D' = \frac{D}{\lambda} - \frac{C}{\lambda^2} + \frac{B}{\lambda^3} - \frac{A}{\lambda^4},$$

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$$M' = \frac{M}{\lambda} - \frac{L}{\lambda^2} + \frac{K}{\lambda^3} \ldots\ldots \pm \frac{A}{\lambda^n},\ \text{et}$$

$$0 = \frac{N}{\lambda} - \frac{M}{\lambda^2} + \frac{L}{\lambda^3} \ldots\ldots \mp \frac{A}{\lambda^{n+1}},$$

ubi ex ultima aequatione quantitas λ erui debet, quae aequatio induit hanc formam

$$A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 \ldots\ldots \pm N\lambda^n = 0$$

unde cum λ sortiatur n valores, totidem quoque multiplicatores, inveniuntur.

Videamus, quomodo hae determinationes pro singulis valoribus exponentis n se habeant.

I. Si $n = 1$; erit $A - B\lambda = 0$, tum vero

$$A' = \frac{A}{\lambda} = B.$$

II. Si $n = 2$; erit $A - B\lambda + C\lambda^2 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda \text{ et } B' = \frac{B\lambda - A}{\lambda^2} = C.$$

III. Si $n = 3$; erit $A - B\lambda + C\lambda^2 - D\lambda^3 = 0$ tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2,$$

$$B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda \text{ et}$$

$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D.$$

IV. Si $n = 4$; erit $A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2 - E\lambda^3,$$

$$B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda + E\lambda^2,$$

$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D - E\lambda,$$

$$D' = \frac{D\lambda^3 - C\lambda^2 + B\lambda - A}{\lambda^4} = E.$$

V. Si $n = 5$; erit $A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 - F\lambda^5 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2 - E\lambda^3 + F\lambda^4$$

$$B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda + E\lambda^2 - F\lambda^3$$

$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D - E\lambda + F\lambda^2$$

$$D' = \frac{D\lambda^3 - C\lambda^2 + B\lambda - A}{\lambda^4} = E - F\lambda$$

$$E' = \frac{E\lambda^4 - D\lambda^3 + C\lambda^2 - B\lambda + A}{\lambda^5} = F$$

atque ita porro.

Invento autem hoc multiplicatore $e^{\lambda x}\partial x$, prius aequationis membrum fit $\int e^{\lambda x} X \partial x$, et aequatio proposita, quae est differentialis gradus n, per integrationem reducitur ad hanc uno gradu simpliciorem

$$\int e^{\lambda x} X \partial x = e^{\lambda x}\left(A'y + B'\frac{\partial y}{\partial x} + C'\frac{\partial\partial y}{\partial x^2} + \ldots\ldots + M'\frac{\partial^{n-1} y}{\partial x^{n-1}}\right).$$

Corollarium 1.

1139. Intagratione ergo hac prima instituta, aequatio proposita uno gradu deprimitur, et definitis coëfficientibus A', B', C', etc. ex superioribus formulis, aequatio integralis hac forma exhiberi potest

$$e^{-\lambda x}\int e^{\lambda x} X \partial x = A'y + B'\frac{\partial y}{\partial x} + C'\frac{\partial\partial y}{\partial x^2} + \ldots\ldots + M'\frac{\partial^{n-1} y}{\partial x^{n-1}}.$$

Corollarium 2.

1140. Cum prius membrum $e^{-\lambda x}\int e^{\lambda x} X \partial x$ sit functio ipsius x constantem arbitrariam involvens, si ejus loco ponatur X', haec aequatio similem formam habet atque ipsa proposita, ideoque eadem methodo iterum integrari et ad gradum differentialitatis $n-2$ reduci potest, quae hujusmodi formam habebit

$$X'' = A''y + B''\frac{\partial y}{\partial x} + C''\frac{\partial\partial y}{\partial x^2} + \ldots\ldots\ldots + L''\frac{\partial^{n-2} y}{\partial x^{n-2}}.$$

Corollarium 3.

1141. Hoc modo ulterius progrediendo tandem ad aequationem differentialem primi gradus pervenietur

$$X^{(n-1)} = A^{(n-1)} y + B^{(n-1)}\frac{\partial y}{\partial x},$$

quae simili modo ad aequationem finitam $X^{(n)} = A^{(n)} y$ reducitur, qua relatio inter ipsas variabiles x et y exprimitur.

Scholion.

1142. Haec igitur est methodus hujusmodi aequationes differentiales altiorum graduum successive per gradus integrandi, ubi tot opus est integrationibus, quoti gradus differentialis fuerit ipsa aequatio proposita. Totum ergo negotium situm est in inventione successiva coëfficientium, quos ex praecedentibus ope multiplicatoris definiri oportet. In genere quidem lex, qua ii continuo ex antecedentibus determinantur, non ita est perspicua, ut inde forma integrabilis extremi perspici possit; verum quia ex capite superiori novimus, casu quo primum membrum X evanescit, etiam ultimum integrale lege satis simplici contineri, idem hic usu venire merito suspicamur, eamque legem facillime agnoscemus, si pedetentim a gradibus inferioribus ad altiores progrediamur. Ac primo quidem casu, quo aequatio est differentialis primi gradus $X = Ay + B\frac{\partial y}{\partial x}$, multiplicator erit $e^{\lambda x}\partial x$, posito $A - \lambda B = 0$, ut sit $\lambda = \frac{A}{B}$, et cum sit $A' = \frac{A}{\lambda} = B$, integrale erit

$$\int e^{\lambda x} X \partial x = B e^{\lambda x} y \text{ seu } e^{-\lambda x}\int e^{\lambda x} X \partial x = By.$$

Ad hanc similitudinem aequationes graduum altiorum evolvamus, ac formam integralis ultimi investigemus.

Problema 148.

1143. Proposita aequatione differentiali secundi gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2},$$

per duplicem integrationem relationem inter x et y investigare.

Solutio.

Sit $e^{\lambda x}\partial x$ multiplicator hanc aequationem per se integrabilem reddens, eritque $A - B\lambda + C\lambda^2 = 0$, tum sumatur

$$A' = \frac{A}{\lambda} = B - C\lambda \text{ et } B' = \frac{B\lambda - A}{\lambda^2} = C,$$

positoque

$$e^{-\lambda x}\int e^{\lambda x}X\,\partial x = X',$$

aequatio semel integrata est

$$X' = A'y + B'\frac{\partial y}{\partial x}.$$

Hujus jam multiplicator sit $e^{\mu x}\partial x$, eritque $A' - B'\mu = 0$, ac statuatur $A'' = \frac{A'}{\mu} = B'$; positoque $e^{-\mu x}\int e^{\mu x}X'\partial x = X''$ habebimus $X'' = A''y$, quae est aequatio bis integrata relationem quaesitam inter x et y exprimens.

Cum igitur hic sit $A'' = B'$ et $B' = C$, erit $A'' = C$. Deinde loco A' et B' substitutis valoribus, aequatio $A' - B'\mu = 0$ induit hanc formam

$$B - C\lambda - C\mu = 0, \text{ seu } B - C(\lambda + \mu) = 0,$$

ex qua cum sit $\lambda + \mu = \frac{B}{C}$, patet $\lambda + \mu$ aequari summae binarum radicum aequationis $A - B\lambda + C\lambda^2 = 0$. Quoniam igitur λ ejus una est radix, μ necessario ejus alteram radicem denotat. Quare si ex aequatione proposita, uti in capite praecedente fecimus, hanc formemus aequationem $A + Bz + Czz = 0$, ejus radices erunt $z = -\lambda$ et $z = -\mu$. Seu si factores ejus statuamus $C(\alpha + z)(\beta + z)$, litterae α et β praebebunt valores λ et μ. Hinc cum sit

$$X' = e^{-\alpha x}\int e^{\alpha x}X\,\partial x \text{ erit}$$

$$X'' = e^{-\beta x}\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x}X\,\partial x. \quad \text{At}$$

$$\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x}X\,\partial x = \frac{1}{\beta-\alpha}e^{(\beta-\alpha)x}\int e^{\alpha x}X\,\partial x - \frac{1}{\beta-\alpha}\int e^{\beta x}X\,\partial x;$$

unde concluditur

$$X'' = \frac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x}X\,\partial x + \frac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}X\,\partial x.$$

Quocirca aequationis propositae integrale completum est

$$Cy = \frac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x}X\,\partial x + \frac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}X\,\partial x$$

ubi litterae α et β ita sunt capiendae, ut sit

$$A + Bz + Czz = C(\alpha + z)(\beta + z).$$

Corollarium 1.

1144. Si bini hi factores sint aequales, seu $\beta = \alpha$, erit

$$X'' = e^{-\alpha x} \int \partial x \int e^{\alpha x} X \partial x = e^{-\alpha x} x \int e^{\alpha x} X \partial x - e^{-\alpha x} \int e^{\alpha x} X x \partial x,$$

ideoque casu

$$A + Bz + Czz = C(\alpha + z)^2,$$

aequationis nostrae integrale est

$$Cy = e^{-\alpha x}(x \int e^{\alpha x} X \partial x - \int e^{\alpha x} X x \partial x).$$

Corollarium 2.

1145. Si bini factores sint imaginarii, quod evenit si

$A + Bz + Czz = C(ff + 2fz \cos.\theta + zz)$, erit

$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta)$ et

$\beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta)$ hinc

$e^{\alpha x} = e^{fx\cos.\theta}(\cos.fx\sin.\theta + \sqrt{-1}.\sin.fx\sin.\theta)$ et

$e^{\beta x} = e^{fx\cos.\theta}(\cos.fx\sin.\theta - \sqrt{-1}.\sin.fx\sin.\theta)$; atque

$\beta - \alpha = -2\sqrt{-1}.f\sin.\theta$.

Corollarium 3.

1146. Quo haec facilius substituere queamus, sit brevitatis gratia

$$e^{fx\cos.\theta} = m, \quad \cos.fx\sin.\theta = p, \quad \text{et } \sin.fx\sin.\theta = q,$$

ut sit

$$e^{\alpha x} = mp + mq\sqrt{-1}, \quad \text{et } e^{\beta x} = mp - mq\sqrt{-1}.$$

Hinc fit

$$\int e^{\alpha x} X \partial x = \int mp X \partial x + \int mq X \partial x \sqrt{-1} \text{ et}$$
$$\int e^{\beta x} X \partial x = \int mp X \partial x - \int mq X \partial x \sqrt{-1}.$$

Tum vero est

$$e^{-\alpha x} = \frac{p - q\sqrt{-1}}{m}, \text{ et } e^{-\beta x} = \frac{p + q\sqrt{-1}}{m}.$$

Corollarium 4.

1147. Ex his colligimus

$$e^{-\alpha x} \int e^{\alpha x} X \partial x = \frac{p}{m} \int m p X \partial x - \frac{q\sqrt{-1}}{m} \int m p X \partial x + \frac{p\sqrt{-1}}{m} \int m q X \partial x + \frac{q}{m} \int m q X \partial x,$$

et sumto $\sqrt{-1}$ negativo, prodit $e^{-\beta x} \int e^{\beta x} X \partial x$, quae forma inde subtracta relinquit

$$-\frac{2q\sqrt{-1}}{m} \int m p X \partial x + \frac{2p\sqrt{-1}}{m} \int m q X \partial x;$$

hocque residuum dividi debet per

$$\beta - \alpha = -2\sqrt{-1} . f \sin. \theta.$$

Unde integrale colligitur

$$C y = \frac{q}{m \sin. \theta} \int m p X \partial x - \frac{p}{m \sin. \theta} \int m q X \partial x.$$

Corollarium 5.

1148. Restituantur pro m, p, q valores assumti, atque aequationis nostrae, si fuerit

$$A + B z + C z z = C(f f + 2 f z \cos. \theta + z z)$$

integrale erit

$$Cy = e^{-fx\cos.\theta}\left(\frac{\sin. fx\sin.\theta}{\sin. \theta} \int e^{fx\cos.\theta} X \partial x \cos. fx \sin.\theta - \frac{\cos. fx \sin.\theta}{\sin. \theta} \int e^{fx\cos.\theta} X \partial x \sin. fx \sin.\theta\right);$$

quae ergo expressio aequivalet illi, si α et β valores imaginarios obtineant.

Problema 149.

1149. Proposita aequatione differentiali tertii gradus

$$X = A y + B \frac{\partial y}{\partial x} + C \frac{\partial \partial y}{\partial x^2} + D \frac{\partial^3 y}{\partial x^3}$$

per triplicem integrationem ejus integrale completum invenire.

Solutio

Posito multiplicatore $e^{\lambda x}\partial x$, debet esse

$$A - B\lambda + C\lambda^2 - D\lambda^3 = 0,$$

tum sumatur

$$A' = B - C\lambda + D\lambda^2, \quad B' = C - D\lambda \quad \text{et} \quad C' = D,$$

positoque

$$e^{-\lambda x}\int e^{\lambda x} X\partial x = X',$$

aequatio semel integrata praebet

$$X' = A'y + B'\frac{\partial y}{\partial x} + C'\frac{\partial\partial y}{\partial x^2}.$$

Hujus porro multiplicator statuatur $e^{\mu x}\partial x$, ut sit

$$A' - B'\mu + C'\mu^2 = 0,$$

sumaturque

$$A'' = B' - C'\mu \quad \text{et} \quad B'' = C',$$

et posito

$$e^{-\mu x}\int e^{\mu x} X'\partial x = X'',$$

aequatio secunda integralis est

$$X'' = A''y + B''\frac{\partial y}{\partial x},$$

cujus multiplicator erit $e^{\nu x}\partial x$, sumendo $A'' - B''\nu = 0$, at posito $A''' = B''$, erit aequatio integralis tertia

$$e^{-\nu x}\int e^{\nu x} X''\partial x = A'''y = Dy,$$

quaeri ergo oportet quantitates λ, μ, ν. Est vero primo

$$A - \beta\lambda + C\lambda^2 - D\lambda^3 = 0, \text{ tum}$$

$$B - C(\lambda + \mu) + D(\lambda\lambda + \lambda\mu + \mu\mu) = 0,$$

et ob

$$A'' = C - D(\lambda + \mu) \quad \text{et} \quad B'' = D,$$

erit tertio

$$C - D(\lambda + \mu + \nu) = 0;$$

ex qua postrema aequalitate patet, $\lambda + \mu + \nu$ aequari summae radicum aequationis primae, cujus λ est una radix. Quod autem μ et ν sint reliquae radices, hoc modo ostenditur. Consideretur aequatio

$$A + Bz + Cz^2 + Dz^3 = 0,$$

cujus si una radix sit $z = -\lambda$, seu $\lambda + z$ unus factor, dividatur per eum aequatio, ac prodibit

$$Dz^2 + (C - D\lambda)z + B + C\lambda + D\lambda\lambda = 0,$$

quae est ipsa aequatio secunda $C'zz + B'z + A' = 0$, cujus radices sunt $z = -\mu$ et $z = -\nu$, seu factores $(\mu + z)(\nu + z)$, uti in problemate praecedente ostendimus. Quare si formulae

$$A + Bz + Cz^2 + Dz^3,$$

factores sint

$$D(\alpha + z)(\beta + z)(\gamma + z),$$

pro integrali ultimo inveniendo ponatur

$$e^{-\alpha x}\int e^{\alpha x} X \partial x = X', \quad e^{-\beta x}\int e^{\beta x} X' \partial x = X'', \text{ et}$$
$$e^{-\gamma x}\int e^{\gamma x} X'' \partial x = X''',$$

eritque $Dy = X'''$. Verum per reductionem integralium est, uti supra vidimus

$$X'' = \frac{1}{\beta - \alpha} e^{-\alpha x}\int e^{\alpha x} X \partial x + \frac{1}{\alpha - \beta} e^{-\beta x}\int e^{\beta x} X \partial x,$$

hincque porro

$$\int e^{\gamma x} X'' \partial x = \frac{1}{(\beta - \alpha)(\gamma - \alpha)} e^{(\gamma - \alpha)x}\int e^{\alpha x} X \partial x - \frac{1}{(\beta - \alpha)(\gamma - \alpha)}\int e^{\gamma x} X \partial x$$
$$+ \frac{1}{(\alpha - \beta)(\gamma - \beta)} e^{(\gamma - \beta)x}\int e^{\beta x} X \partial x - \frac{1}{(\alpha - \beta)(\gamma - \beta)}\int e^{\gamma x} X \partial x,$$

ubi bini postremi termini contrahuntur in

$$\frac{1}{(\alpha - \gamma)(\gamma - \beta)}\int e^{\gamma x} X \partial x.$$

Quamobrem integrale quaesitum est

$$Dy = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}.$$

Corollarium 1.

1150. Si formulae $A + Bz + Cz^2 + Dz^3$ duo factores fuerint aequales, puta $\gamma = \beta$, erit

$$\int e^{\beta x} X'' \partial x = \frac{1}{(\beta-\alpha)^2} e^{(\beta-\alpha)x} \int e^{\alpha x} X \partial x - \frac{1}{(\beta-\alpha)^2} \int e^{\beta x} X \partial x$$
$$+ \frac{1}{\alpha-\beta} x \int e^{\beta x} X \partial x - \frac{1}{\alpha-\beta} \int e^{\beta x} X x \partial x,$$

ideoque integrale hoc casu erit

$$Dy = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x - e^{-\beta x}\int e^{\beta x} X \partial x}{(\beta-\alpha)^2} + \frac{e^{-\beta x} x \int e^{\beta x} X \partial x - e^{-\beta x}\int e^{\beta x} X x \partial x}{\alpha-\beta}.$$

Corollarium 2.

1151. Si omnes tres factores sint aequales, seu $\alpha = \beta = \gamma$, erit

$$e^{\alpha x} X'' = \int \partial x \int e^{\alpha x} X \partial x = x \int e^{\alpha x} X \partial x - \int e^{\alpha x} X x \partial x, \text{ et}$$
$$e^{\alpha x} X''' = \int e^{\alpha x} X'' \partial x = \int \partial x \int \partial x \int e^{\alpha x} X \partial x, \text{ seu}$$
$$e^{\alpha x} X''' = \tfrac{1}{2} x x \int e^{\alpha x} X \partial x - x \int e^{\alpha x} X x \partial x + \tfrac{1}{2} \int e^{\alpha x} X x x \partial x,$$

unde integrale hoc casu erit

$$Dy = \tfrac{1}{2} e^{-\alpha x} (x x \int e^{\alpha x} X \partial x - 2 x \int e^{\alpha x} X x \partial x + \int e^{\alpha x} X x x \partial x),$$

seu

$$Dy = e^{-\alpha x} \int \partial x \int \partial x \int e^{\alpha x} X \partial x.$$

Scholion.

1152. In genere etiam nulla integralium reductione adhibita integrale nostrae aequationis ita exprimi potest, ut sit

$$Dy = e^{-\gamma x} \int e^{(\gamma-\beta)x} \partial x \int e^{(\beta-\alpha)x} \partial x \int e^{\alpha x} X \partial x,$$

posito

$$A + Bz + Cz^2 + Dz^3 = D(\alpha + z)(\beta + z)\gamma + z),$$

ubi imprimis notatu dignum occurrit, quod ternas litteras α, β, γ quomodocunque inter se permutare licet, ita ut haec integralis expressio sex modis variari possit. In problemate etiam praecedente, ubi duo tantum factores occurrunt

$$C(\alpha+z)(\beta+z) = A + Bz + Cz^2,$$

aequationis

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2}$$

integrale completum ita exhiberi potest

$$Cy = e^{-\beta x}\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x}X\,\partial x,$$

ac permutatis litteris α et β etiam hoc modo

$$Cy = e^{-\alpha x}\int e^{(\alpha-\beta)x}\partial x\int e^{\beta x}X\,\partial x.$$

Quarum formularum aequalitas si fuerit perspecta, id quod tentanti facile patebit, praecedentium quoque variationem declarat. Sit enim $e^{-\alpha x}\int e^{\alpha x}X\,\partial x = X'$, erit pro superiori formula

$$Dy = e^{-\gamma x}\int e^{(\gamma-\beta)x}\partial x\int e^{\beta x}X'\,\partial x,$$

cui cum aequalis sit ista

$$Dy = e^{-\beta x}\int e^{(\beta-\gamma)x}\partial x\int e^{\gamma x}X'\,\partial x,$$

erit etiam pro X' valore restituto

$$Dy = e^{-\beta x}\int e^{(\beta-\gamma)x}\partial x\int e^{(\gamma-\alpha)x}\partial x\int e^{\alpha x}X\,\partial x,$$

quae a prima hoc tantum differt, quod litterae β et γ sunt permutatae. Quod autem etiam litterae β et γ cum α permutari queant, hoc modo difficilius ostenditur, ex reductione autem in solutione adhibita, atque adeo ex ipsa solutionis indole per se est manifestum.

Problema 150.

1153. Proposita aequatione differentiali quarti gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + E\frac{\partial^4 y}{\partial x^4},$$

sumto elemento ∂x constante, et denotante X functionem quamcunque ipsius x, ejus integrale investigare.

Solutio.

In subsidium vocetur formula algebraica ex aequatione proposita facile formanda

$$A + Bz + Czz + Dz^3 + Ez^4 = P,$$

quae in factores suos simplices resolvatur, ut sit

$$P = E(\alpha + z)(\beta + z)(\gamma + z)(\delta + z),$$

et multiplicator aequationem nostram integrabilem reddens erit $e^{\lambda x}\partial x$, sumendo λ aequali uni litterarum α, β, γ, δ; sumatur ergo $\lambda = \alpha$, ut sit multiplicator $e^{\alpha x}\partial x$, atque posito $e^{-\alpha x}\int e^{\alpha x} X \partial x = X'$, aequatio semel integrata erit

$$X' = A'y + B'\frac{\partial y}{\partial x} + C'\frac{\partial\partial y}{\partial x^2} + D'\frac{\partial^3 y}{\partial x^3},$$

ubi A', B', C', D' ita determinantur, ut sit

$$A' = \frac{A}{\alpha},\ B' = \frac{-A + B\alpha}{\alpha^2},\ C' = \frac{C\alpha^2 - B\alpha + A}{\alpha^3},\ D' = \frac{D\alpha^3 - C\alpha^2 + B\alpha - A}{\alpha^4};$$

seu

$$A' = \frac{A}{\alpha},\ B' = \frac{B - A'}{\alpha},\ C' = \frac{C - B'}{\alpha},\ D' = \frac{D - C'}{\alpha},$$

vel etiam

$$A = A'\alpha,\ B = B'\alpha + A',\ C = C'\alpha + B',\ D = D'\alpha + C'.$$

Ex quibus determinationibus liquet, si ponatur

$$A' + B'z + C'z^2 + D'z^3 = Q,$$

hanc formulam Q nasci ex formula P, si haec per $\alpha + z$ dividatur, ita ut sit

$$Q = \frac{P}{\alpha + z} = E(\beta + z)(\gamma + z)(\delta + z).$$

Eodem ergo modo secundam integrationem instituamus ope multiplicatoris $e^{\beta x}\partial x$, et posito

$$e^{-\beta x}\int e^{\beta x} X' \partial x = X'',$$

erit, aequatio integralis

$$X'' = A'' y + B'' \frac{\partial y}{\partial x} + C'' \frac{\partial \partial y}{\partial x^2},$$

coëfficientibus A'', B'', C'' ita sumtis, ut sit

$$A'' + B'' z + C'' z^2 = \frac{P}{(\alpha + z)(\beta + z)} = E(\gamma + z)(\delta + z).$$

Hinc porro ope multiplicatoris $e^{\gamma x}\partial x$ integrando, si ponamus $e^{-\gamma x}\int e^{\gamma x} X'' \partial x = X'''$, inveniemus

$$X''' = A''' y + B''' \frac{\partial y}{\partial x},$$

existente

$$A''' + B''' z = \frac{P}{(\alpha + z)(\beta + z)(\gamma + z)} = E(\delta + z).$$

Ac tandem ope multiplicatoris $e^{\delta x}\partial x$, posita forma

$$e^{-\delta x}\int e^{\delta x} X''' \partial x = X'''',$$

integrale ultimum reperitur

$$X'''' = A'''' y \text{ existente } A'''' = E.$$

Haec igitur omnia colligendo, integrale quaesitum erit

$$E y = e^{-\delta x}\int e^{(\delta - \gamma)x}\partial x \int e^{(\gamma - \beta)x}\partial x \int e^{(\beta - \alpha)x}\partial x \int e^{\alpha x} X \partial x,$$

quae expressio jam sine ullis ambagibus ex resolutione formae principalis

$$P = A + B z + C z^2 + D z^3 + E z^4,$$

in factores scilicet

$$P = E(\alpha + z)(\beta + z)(\gamma + z)(\delta + z),$$

confici potest, ubi notandum quomodocunque ordo litterarum α, β, γ, δ permutetur, pro $E y$ semper eundem valorem prodire debere.

Corollarium 1.

1154. Cum sit $X' = e^{-\alpha x}\int e^{\alpha x} X \partial x$, erit uti jam vidimus

$$X'' = e^{-\beta x}\int e^{\beta x} X' \partial x = e^{-\beta x}\left(\frac{e^{(\beta-\alpha)x}}{\beta-\alpha}\int e^{\alpha x} X \partial x - \frac{1}{\beta-\alpha}\int e^{\beta x} X \partial x\right)$$

seu

$$X'' = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{\beta-\alpha} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{\alpha-\beta}.$$

Corollarium 2.

1155. Porro ob $X''' = e^{-\gamma x}\int e^{\gamma x} X'' \partial x$, erit simili mod reductionem instituendo

$$X''' = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\beta-\alpha)(\alpha-\gamma)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\beta)(\beta-\gamma)},$$

quae reducitur ad hanc formam

$$X''' = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)(\beta-\gamma)}$$

Corollarium 3.

1156. Hinc simili modo evolvitur valor X'''', ubi quider sufficeret primum membrum eruisse, quippe ex quo ob permutabili tatem reliqua sponte formantur. Hoc modo integrale nostrae aequa tionis reperietur hac forma expressum

$$Ey = \frac{e^{-\alpha x}\int e^{\alpha x} X \partial x}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X \partial x}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} + \frac{e^{-\delta x}\int e^{\delta x} X \partial x}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}$$

Scholion.

1157. Si duae pluresve radices sint aequales vel imagina riae, integralia inventa transformationem postulant, quam deincep investigabimus. Atque haec postrema quidem forma magis apta videtur, unde transformationes repetantur. Ita pro factorum aequalitate si sit $\delta = \gamma$, bina postrema membra tantum reductionem postulant, ad quam inveniendam ponatur $\delta = \gamma - \omega$, et penultimum membrum erit $-\frac{e^{-\gamma x}\int e^{\gamma x} X \partial x}{\omega(\alpha - \gamma)(\beta - \gamma)}$; pro ultimo autem notandum est, esse

$$\frac{1}{\alpha - \delta} = \frac{1}{\alpha - \gamma + \omega} = \frac{1}{\alpha - \gamma} - \frac{\omega}{(\alpha - \gamma)^2}, \text{ et } \frac{1}{\beta - \delta} = \frac{1}{\beta - \gamma} - \frac{\omega}{(\beta - \gamma)^2},$$

hincque

$$\frac{1}{(\alpha - \delta)(\beta - \delta)} = \frac{1}{(\alpha - \gamma)(\beta - \gamma)} + \frac{\omega(2\gamma - \alpha - \beta)}{(\alpha - \gamma)^2(\beta - \gamma)^2}, \text{ unde}$$

$$\frac{1}{(\alpha - \delta)(\beta - \delta)(\gamma - \delta)} = \frac{1}{\omega(\alpha - \gamma)(\beta - \gamma)} + \frac{2\gamma - \alpha - \beta}{(\alpha - \gamma)^2(\beta - \gamma)^2}.$$

Tum vero pro numeratore erit

$$e^{-\delta x} = e^{-\gamma x}(1 + \omega x), \text{ et } e^{\delta x} = e^{\gamma x}(1 - \omega x),$$

ideoque

$$e^{-\delta x}\int e^{\delta x} X \partial x = e^{-\gamma x}\int e^{\gamma x} X \partial x + \omega e^{-\gamma x} x \int e^{\gamma x} X \partial x$$
$$- \omega e^{-\gamma x}\int e^{\gamma x} X x \partial x,$$

atque hinc bina ultima membra ob terminos per ω divisos se destruentes, abeunt in hanc formam

$$\frac{(2\gamma - \alpha - \beta) e^{-\gamma x}\int e^{\gamma x} X \partial x}{(\alpha - \gamma)^2(\beta - \gamma)^2} + \frac{e^{-\gamma x} x \int e^{\gamma x} X \partial x - e^{-\gamma x}\int e^{\gamma x} X x \partial x}{(\alpha - \gamma)(\beta - \gamma)},$$

quae expressio etiam ex priori forma elicitur. Eodem modo problema in genere resolvi potest.

Problema 151.

1158. Proposita aequatione differentiali cujuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots + N\frac{\partial^n y}{\partial x^n},$$

sumto elemento ∂x constante, et denotante X functionem quamcunque ipsius x, ejus integrale investigare.

Solutio.

Formetur ex hac aequatione formula algebraica

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = P,$$

quae in factores simplices resolvatur, ut sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)\ldots\ldots(\nu + z),$$

quorum numerus est n. Quodsi jam simili modo per singulas in gationes continuo progrediamur, tandem ad hanc aequationem in gralem extremam perveniemus

$$Ny = e^{-\nu x}\int e^{(\nu-\mu)x}\partial x\int e^{(\mu-\lambda)x}\partial x\int\ldots\ldots\int e^{(\beta-\alpha)x}\partial x\int e^{\alpha x}X\partial$$

seu cum factores inter se permutare liceat, erit etiam

$$Ny = e^{-\alpha x}\int e^{(\alpha-\beta)x}\partial x\int e^{(\beta-\gamma)x}\partial x\int\ldots\ldots\int e^{(\mu-\nu)x}\partial x\int e^{\nu x}X\partial$$

Haec vero expressio per similes reductiones, quibus supra sum usi, in sequentes partes resolvi potest, ad quas commodius repr sentandas sit brevitatis gratia

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\ldots\ldots(\nu-\alpha) = \alpha',$$
$$(\alpha-\beta)(\gamma-\beta)(\delta-\beta)\ldots\ldots(\nu-\beta) = \beta',$$
$$(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)\ldots\ldots(\nu-\gamma) = \gamma'$$

$$\vdots$$

$(\alpha-\nu)(\beta-\nu)(\gamma-\nu)\ldots\ldots(\mu-\nu) = \nu'$, hincque erit

$$Ny = \tfrac{1}{\alpha'}e^{-\alpha x}\int e^{\alpha x}X\partial x + \tfrac{1}{\beta'}e^{-\beta x}\int e^{\beta x}X\partial x$$
$$+ \tfrac{1}{\gamma'}e^{-\gamma x}\int e^{\gamma x}X\partial x + \ldots\ldots + \tfrac{1}{\nu'}e^{-\nu x}\int e^{\nu x}X\partial x$$

Ne autem opus sit ad valores α', β', γ', etc. inveniendos, factores in se invicem multiplicare, cum sit

$$\frac{P}{N(\alpha+z)} = (\beta+z)(\gamma+z)(\delta+z)\ldots\ldots(\nu+z$$

evidens est, hanc formulam praebere valorem α', si in ea statuatur $z = -\alpha$; hoc autem casu fractionis $\frac{P}{N(\alpha + z)}$ tam numerator, quam denominator evanescit, ex quo ejus valor erit $\frac{\partial P}{N \partial z}$. Quare cum sit

$$P = A + B z + C z^2 + D z^3 + \ldots\ldots + N z^n, \text{ erit}$$
$$\frac{\partial P}{\partial z} = B + 2 C z + 3 D z^2 + \ldots\ldots + n N z^{n-1},$$

quae expressio vocetur Q, unde patet fore

$$\alpha' = \frac{Q}{N} \text{ posito } z = -\alpha,$$
$$\beta' = \frac{Q}{N} \text{ posito } z = -\beta,$$
$$\gamma' = \frac{Q}{N} \text{ posito } z = -\gamma,$$
etc.

Vel cum his valoribus substitutis aequatio integralis per N dividi queat, sequentes valores colligantur

$$B - 2 C \alpha + 3 D \alpha^2 - 4 E \alpha^3 \ldots\ldots \pm n N \alpha^{n-1} = \mathfrak{A},$$
$$B - 2 C \beta + 3 D \beta^2 - 4 E \beta^3 \ldots\ldots \pm n N \beta^{n-1} = \mathfrak{B},$$
$$B - 2 C \gamma + 3 D \gamma^2 - 4 E \gamma^3 \ldots\ldots \pm n N \gamma^{n-1} = \mathfrak{C},$$
$$\vdots$$
$$B - 2 C \nu + 3 D \nu^2 - 4 E \nu^3 \ldots\ldots \pm n N \nu^{n-1} = \mathfrak{N},$$

quibus constitutis erit integrale quaesitum

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x + \frac{1}{\mathfrak{C}} e^{-\gamma x} \int e^{\gamma x} X \partial x + \text{etc.}$$

quoad omnes factores fuerint in computum ducti.

Corollarium 1.

1159. Cum sit

$$\alpha' = \frac{\mathfrak{A}}{N}, \; \beta' = \frac{\mathfrak{B}}{N}, \; \gamma' = \frac{\mathfrak{C}}{N}, \text{ etc. erit}$$
$$\mathfrak{A} = N \alpha', \; \mathfrak{B} = N \beta', \; \mathfrak{C} = N \gamma', \text{ etc.}$$

Hinc ob

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = N(\alpha + z)(\beta + z)(\gamma + z)\ldots\ldots(\nu + z)$$

erit

$$\mathfrak{A} = \frac{P}{\alpha + z} \text{ posito } z = -\alpha,$$
$$\mathfrak{B} = \frac{P}{\beta + z} \text{ posito } z = -\beta,$$
$$\mathfrak{C} = \frac{P}{\gamma + z} \text{ posito } z = -\gamma,$$

et ita porro.

Corollarium 2.

1160. Regula ergo hujus aequationis propositae

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n}$$

integrale completum inveniendi ita se habet. Formetur inde formula algebraica haec

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = P,$$

cujus quaerantur omnes factores simplices, qui sint

$$\alpha + z,\ \beta + z,\ \gamma + z,\ \delta + z, \text{ etc.}$$

quorum multitudo numero n est aequalis, tum pro singulis factoribus sequentes quantitates constantes definiantur $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. ut sit

$$\mathfrak{A} = \frac{P}{\alpha + z} \text{ posito } z = -\alpha, \text{ seu}$$
$$\mathfrak{A} = B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 \ldots \pm nN\alpha^{n-1}$$
$$\mathfrak{B} = \frac{P}{\beta + z} \text{ posito } z = -\beta, \text{ seu}$$
$$\mathfrak{B} = B - 2C\beta + 3D\beta^2 - 4E\beta^3 \ldots \pm nN\beta^{n-1}$$
$$\mathfrak{C} = \frac{P}{z + \gamma} \text{ posito } z = -\gamma, \text{ seu}$$
$$\mathfrak{C} = B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 \ldots \pm nN\gamma^{n-1}$$

his omnibus inventis, erit integrale quaesitum

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x + \frac{1}{\mathfrak{C}} e^{-\gamma x} \int e^{\gamma x} X \partial x + \text{etc.}$$

quae forma tot constat partibus, quot fuerint factores simplices in formula P.

Corollarium 3.

1161. Cum hoc modo integrale tot constet partibus, quoti ordinis est aequatio differentialis proposita, et una quaeque pars per integrationem unam invehat constantem arbitrariam; manifestum est integrale ope hujus regulae inventum fore completum.

Scholion.

1162. Integratio ergo hujusmodi aequationum differentialium nulla amplius laborat difficultate, si modo formulae illius algebraicae P omnes factores simplices, seu quod eodem redit, hujus aequationis algebraicae

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n = 0,$$

omnes radices numero n assignari queant. Hic vero duplicis generis casus occurrunt, quibus haec integratio vehementer impeditur, quando scilicet vel duo pluresve eorum factorum simplicium inter se fiunt aequales, vel imaginarii, quo quidem posteriori casu hoc tantum incommodi accedit, quod partes quaepiam integralis inventi imaginaria involvant, quae autem facta reductione se mutuo destruunt. Priori vero casu partes ex factoribus aequalibus oriundae adeo fiunt infinitae, sed ita diversis signis affectae, ut conjunctim nihilominus quantitatem finitam referant, cujus valorem nonnisi per plures ambages elicere licet, ubi probe notandum est, utroque casu inventionem reliquarum integralis partium, quae factoribus inaequalibus conveniunt, neutiquam hinc turbari. Methodum autem huic fini accommodatam in sequenti problemate explicabo.

Problema 152.

1163. Proposita aequatione differentiali cujuscunque gradus

$$X = A\,y + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + E\frac{\partial^4 y}{\partial x^4} + \ldots + N\frac{\partial^n y}{\partial x^n}$$

si forma algebraica inde facta

$$P = A + B\,z + C\,z^2 + D\,z^3 + E\,z^4 + \ldots\ldots + N\,z^n$$

duos pluresve factores simplices inter se habeat aequales, partem integralis inde oriundam investigare.

Solutio.

Sint primo duo factores $\alpha + z$ et $\beta + z$ inter se aequales, seu $\beta = \alpha$, reliquus vero factor formae P sit $= Q$, ut habeatur

$$P = (\alpha + z)(\beta + z)\,Q = (\alpha + z)^2\,Q,$$

posito autem $z = -\alpha$, abeat Q in $\mathfrak{C}$. Jam initio saltem litterae α et β ut diversae spectentur, excepta quantitate $\mathfrak{C}$ quae utrinque sit eadem, atque pro binis integralis partibus ex his binis factoribus oriundis habebimus

$$\mathfrak{A} = (\beta - \alpha)\,\mathfrak{C} \text{ et } \mathfrak{B} = (\alpha - \beta)\,\mathfrak{C}.$$

Partes autem integralis inde oriundae littera v designentur, ut sit

$$(\beta - \alpha)\,\mathfrak{C}\,v = e^{-\alpha x}\int e^{\alpha x}\,X\,\partial x - e^{-\beta x}\int e^{\beta x}\,X\,\partial x;$$

unde differentiando colligimus

$$(\beta - \alpha)\,\mathfrak{C}\,\partial v = -\alpha\,e^{-\alpha x}\,\partial x\int e^{\alpha x}\,X\,\partial x + \beta\,e^{-\beta x}\,\partial x\int e^{\beta x}\,X\,\partial x;$$

ad hanc addatur prior per $\beta\,\partial x$ multiplicata, fietque

$$(\beta - \alpha)\,\mathfrak{C}\,\partial v + (\beta - \alpha)\,\mathfrak{C}\,\beta\,v\,\partial x = (\beta - \alpha)\,e^{-\alpha x}\,\partial x\int e^{\alpha x}\,X\,\partial x;$$

quae per $\beta - \alpha$ divisa, et per $e^{-\alpha x}$ multiplicata, ob $\beta = \alpha$, integrale praebet

$$\mathfrak{C}\,e^{\alpha x}\,v = \int\partial x\int e^{\alpha x}\,X\,\partial x.$$

Quocirca loco binarum partium ex factoribus aequalibus $\alpha + z$ et $\beta + z$ oriundarum scribi oportet hanc formulam

$$v = \tfrac{1}{\mathfrak{C}} e^{-\alpha x} \int \partial x \int e^{\alpha x} \mathrm{X} \partial x,$$

ubi $\mathfrak{C}$ oritur ex forma $\frac{P}{(\alpha + z)^2}$ posito $z = -\alpha$.

Ponamus jam formulam P tres habere factores simplices aequales, ut sit $\alpha + z = \beta + z = \gamma + z$, quosquidem initio ut diversos spectemus.

Ponamus ergo $P = (\alpha + z)(\beta + z)(\gamma + z) Q$, abeatque Q $\mathfrak{M}$, posito $z = -\alpha$, ac pro integralis partibus habebimus

$$\mathfrak{A} = (\beta - \alpha)(\gamma - \alpha)\mathfrak{M}, \quad \mathfrak{B} = (\alpha - \beta)(\gamma - \beta)\mathfrak{M},$$
$$\mathfrak{C} = (\alpha - \gamma)(\beta - \gamma)\mathfrak{M}.$$

Hinc si summam trium integralis partium, quam quaerimus, littera v denotemus, erit

$$\mathfrak{M} v = \frac{e^{-\alpha x} \int e^{\alpha x} \mathrm{X} \partial x}{(\beta - \alpha)(\gamma - \alpha)} + \frac{e^{-\beta x} \int e^{\beta x} \mathrm{X} \partial x}{(\alpha - \beta)(\gamma - \beta)} + \frac{e^{-\gamma x} \int e^{\gamma x} \mathrm{X} \partial x}{(\alpha - \gamma)(\beta - \gamma)}.$$

Cum nunc sit

$$\frac{1}{(\beta - \alpha)(\gamma - \alpha)} + \frac{1}{(\alpha - \beta)(\gamma - \beta)} + \frac{1}{(\alpha - \gamma)(\beta - \gamma)} = 0,$$

erit differentiando

$$\frac{\mathfrak{M} \partial v}{\partial x} = \frac{-\alpha e^{-\alpha x} \int e^{\alpha x} \mathrm{X} \partial x}{(\beta - \alpha)(\gamma - \alpha)} - \frac{\beta e^{-\beta x} \int e^{\beta x} \mathrm{X} \partial x}{(\alpha - \beta)(\gamma - \beta)} - \frac{\gamma e^{-\gamma x} \int e^{\gamma x} \mathrm{X} \partial x}{(\alpha - \gamma)(\beta - \gamma)},$$

ad quam si prima per α multiplicata addatur, fit

$$\mathfrak{M}\left(\frac{\partial v}{\partial x} + \alpha v\right) = \frac{e^{-\beta x} \int e^{\beta x} \mathrm{X} \partial x}{\gamma - \beta} + \frac{e^{-\gamma x} \int e^{\gamma x} \mathrm{X} \partial x}{\beta - \gamma}.$$

Haec aequatio denuo differentietur, ut prodeat

$$\mathfrak{M}\left(\frac{\partial \partial v}{\partial x^2} + \frac{\alpha \partial v}{\partial x}\right) = \frac{-\beta e^{-\beta x} \int e^{\beta x} \mathrm{X} \partial x}{\gamma - \beta} - \frac{\gamma e^{-\gamma x} \int e^{\gamma x} \mathrm{X} \partial x}{\beta - \gamma},$$

ad quam illa per $\beta = \alpha$ multiplicata, si addatur, oritur

$$\mathfrak{M}\left(\frac{\partial \partial v}{\partial x^2} + \frac{2\alpha\partial v}{\partial x} + \alpha\alpha v\right) = e^{-\gamma x}\int e^{\gamma x}\mathrm{X}\,\partial x = e^{-\alpha x}\int e^{\alpha x}\mathrm{X}\,\partial x;$$

unde jam omnia incommoda sunt sublata. Multiplicetur nunc per $e^{\alpha x}\partial x$, et integratio dabit

$$\mathfrak{M}\, e^{\alpha x}\left(\frac{\partial v}{\partial x} + \alpha v\right) = \int \partial x \int e^{\alpha x}\mathrm{X}\,\partial x,$$

quae per ∂x multiplicata denuo fit integrabilis, proditque

$$\mathfrak{M}\, e^{\alpha x} v = \int \partial x \int \partial x \int e^{\alpha x}\mathrm{X}\,\partial x.$$

Quocirca si forma P factorem habeat cubicum $(\alpha + z)^3$, quaeratur quantitas $\mathfrak{M}$, ut sit

$$\mathfrak{M} = \frac{\mathrm{P}}{(\alpha + z)^3}, \text{ posito } z = -\alpha;$$

et integralis pars hinc oriunda erit

$$\frac{1}{\mathfrak{M}} e^{-\alpha x}\int \partial x \int \partial x \int e^{\alpha x}\mathrm{X}\,\partial x.$$

Simili modo si formula P quatuor habeat factores aequales, ut sit $\mathrm{P} = (\alpha + z)^4\,\mathrm{Q}$, capiatur $\mathfrak{M} = \frac{\mathrm{P}}{(\alpha + z)^4}$ seu $\mathfrak{M} = \mathrm{Q}$, posito $z = -\alpha$, et integralis pars inde nata erit

$$\frac{1}{\mathfrak{M}} e^{-\alpha x}\int \partial x \int \partial x \int \partial x \int e^{\alpha x}\mathrm{X}\,\partial x:$$

sicque etiam casus, quibus formula P adhuc plures habet factores aequales, facile resolventur.

Nota. Tota haec solutio est vitiosa, propterea quod licet quantitates α, β, γ, etc. quae ponuntur aequales, ut diversae spectentur, tamen pro singulis membris quantitas $\mathfrak{M}$ eundem valorem retinere assumitur. Quodsi enim litterae α, β, γ, etc. infinite parum a se invicem discrepare concipiantur, etiam in valoribus littera $\mathfrak{M}$ indicatis differentiam infinite parvam agnoscere oportet, unde cum singulae partes integralis fiant infinitae, iisque evolutis membra infinita se mutuo tollant, ex differentiis infinite parvis litterae $\mathfrak{M}$ partes quoque finitae emergunt. Correctionem horum errorum petere licet ex seq. Probl. 154, dum factores aequales in aequationem peculiarem conjiciuntur. Malui autem hunc correctionis laborem industriae lectorum relinquere, quam hoc opus a tali errore liberare, saepe enim plus prodest errores, in quos etiam exercitatis incidere contingit, conservari, quo melius harum rerum studiosi addiscant, quanta circumspectione cavendum sit, ne in ratiocinando hallucinemur.

Corollarium 1.

1164. Notatu hic omnino est dignum, quod hae formulae

$$\partial v + \alpha v \partial x,\ \partial\partial v + 2\alpha \partial x \partial v + \alpha^2 v \partial x^2,$$
$$\partial^3 v + 3\alpha \partial x \partial\partial v + 3\alpha^2 \partial x^2 \partial v + \alpha^3 v^3 \partial x^3$$

et in genere haec

$$\partial^n v + \tfrac{n}{1}\alpha\partial x \partial^{n-1} v + \tfrac{n(n-1)}{1.2}\alpha^2 \partial x^2 \partial^{n-2} v + \tfrac{n(n-1)(n-2)}{1.2.3}\alpha^3 \partial x^3 \partial^{n-3} v + \text{etc.}$$

si semel per $e^{\alpha x}$ multiplicentur, successive toties integrationem admittant, quot unitates continet index n, ita ut postremum integrale sit $e^{\alpha x}$.

Corollarium 2.

1165. Ratio autem hujus phaenomeni inde est manifesta, quod si formula $e^{\alpha x} v$ continuo differentietur, sumto elemento ∂x constante, formulae illae differentiales per $e^{\alpha x}$ multiplicatae prodeant, ita ut sit

$$\partial^n . e^{\alpha x} v = e^{\alpha x}\left(\partial^n v + \tfrac{n}{1}\alpha\partial x \partial^{n-1} v + \tfrac{n(n-1)}{1.2}\alpha^2 \partial x^2 \partial^{n-2} v + \text{etc.}\right)$$

Corollarium 3.

1166. Aeque memoratu dignum est alterum phaenomenum, quod solutio ista nobis offert; sumtis scilicet numeris quibuscunque α, β, γ, δ, etc. sequentes aequalitates semper locum habere, ut sit

$$\frac{1}{\alpha-\beta} + \frac{1}{\beta-\alpha} = 0,$$
$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)} + \frac{1}{(\beta-\alpha)(\beta-\gamma)} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)} = 0,$$
$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{1}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} = 0,$$
etc.

quotcunque numeri hoc modo capiantur.

**

Corollarium 4.

1167. Si formula P in factores simplices resoluta ponatur

$$P = N(\alpha + z)(\beta + z)(\gamma + z) \ldots\ldots (\mu + z)(\nu + z)$$

expressio integralis prius inventa (1158.) quae erat

$$Ny = e^{-\alpha x}\int e^{(\alpha-\beta)x}\partial x\int e^{(\beta-\gamma)x}\partial x\int e^{(\gamma-\delta)x}\partial x \ldots\ldots \int e^{(\mu-\nu)x}\partial x\int e^{\nu x}X\partial x$$

ob factores aequales nulla implicatur difficultate, forma autem posterior, qua integrale in partes ex singulis factoribus ortas distributum exhibetur, et quae ad usum multo magis accommodata videtur, eo difficiliori egebat evolutione.

Scholion.

1168. Phaenomenum Corollario 3. observatum eo majorem attentionem meretur, quod etiam ad Arithmeticam vulgarem transferri potest, ubi usu adeo insigni non cariturum videtur, praecipue quod ejus demonstratio minime sit obvia, sed ex profundioribus Analyseos penetralibus repeti debeat, ex quo haud alienum fore arbitror, si huic insigni Theoremati arithmetico hic locum concedam idque eo magis, quod solutio problematis hic exposita sine demonstratione istius Theorematis minime foret perfecta.

Theorema Arithmeticum.

1169. Si habeantur numeri quotcunque a, b, c, d, etc. ex iisque dum a quolibet singuli reliqui subtrahantur, formentur sequentia producta

$$(a-b)(a-c)(a-d)(a-e) \text{ etc.} = \mathfrak{A}$$
$$(b-a)(b-c)(b-d)(b-e) \text{ etc.} = \mathfrak{B}$$
$$(c-a)(c-b)(c-d)(c-e) \text{ etc.} = \mathfrak{C}$$
$$(d-a)(d-b)(d-c)(d-e) \text{ etc.} = \mathfrak{D}$$

etc.

semper habebitur

$$\frac{1}{\mathfrak{A}} + \frac{1}{\mathfrak{B}} + \frac{1}{\mathfrak{C}} + \frac{1}{\mathfrak{D}} + \text{etc.} = 0.$$

Demonstratio.

Consideretur secundum principia in Introductione ad Analysin infinitorum tradita haec fractio

$$\frac{Z}{(z-a)(z-b)(z-c)(z-d)\text{ etc.}},$$

ubi Z denotet ejusmodi functionem rationalem integram ipsius z, in qua summa potestas ipsius z minor sit numero factorum denominatoris; haecque fractio resolvi poterit in has fractiones simplices, quibus ea junctim sumtis sit aequalis, scilicet

$$\frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c} + \frac{D}{z-d} + \text{etc.}$$

Ad quam resolutionem sumamus illum numeratorem $Z = z^n$, existente n numero integro minore quam denominator continet factores, atque hi numeratores ita definiuntur, ut sit

$$A = \frac{a^n}{(a-b)(a-c)(a-d)\text{ etc.}}$$

$$B = \frac{b^n}{(b-a)(b-c)(b-d)\text{ etc.}}$$

$$C = \frac{c^n}{(c-a)(c-b)(c-d)\text{ etc.}}$$

etc.

Cum igitur istae fractiones negative sumtae nempe

$$\frac{A}{a-z} + \frac{B}{b-z} + \frac{C}{c-z} + \frac{D}{d-z} + \text{etc.}$$

ad fractionem propositam adjectae in nihilum abeant, si z sit numerorum propositorum a, b, c, d, etc. ultimus, quorum adeo multitudo major est quam $n+1$, ponatur

$$(a-b)(a-c)(a-d) \ldots\ldots (a-z) = \mathfrak{A}$$
$$(b-a)(b-c)(b-d) \ldots\ldots (b-z) = \mathfrak{B}$$
$$(c-a)(c-b)(c-d) \ldots\ldots (c-z) = \mathfrak{C}$$
$$(d-a)(d-b)(d-c) \ldots\ldots (d-z) = \mathfrak{D}$$
etc.
$$(z-a)(z-b)(z-c) \ldots\ldots (z-y) = \mathfrak{Z}$$

ut fractio proposita sit $\frac{z^n}{\mathfrak{Z}}$. Atque hinc perspicuum est, summa omnium harum fractionum esse

$$\frac{a^n}{\mathfrak{A}} + \frac{b^n}{\mathfrak{B}} + \frac{c^n}{\mathfrak{C}} + \frac{d^n}{\mathfrak{D}} + \ldots\ldots + \frac{z^n}{\mathfrak{Z}} = 0$$

dum sit $n+1$ minor numero terminorum Sumto ergo $n =$ oritur casus Theorematis.

Corollarium 1.

1170. Haec si transferantur ad numeros $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, et supra (1160.) definitos, ubi aliquod leve discrimen in factorum constitutione probe est notandum, intelligetur esse

$$\frac{1}{\mathfrak{A}} + \frac{1}{\mathfrak{B}} + \frac{1}{\mathfrak{C}} + \frac{1}{\mathfrak{D}} + \text{etc.} = 0$$
$$-\frac{\alpha}{\mathfrak{A}} - \frac{\beta}{\mathfrak{B}} - \frac{\gamma}{\mathfrak{C}} - \frac{\delta}{\mathfrak{D}} - \text{etc.} = 0$$
$$+\frac{\alpha^2}{\mathfrak{A}} + \frac{\beta^2}{\mathfrak{B}} + \frac{\gamma^2}{\mathfrak{C}} + \frac{\delta^2}{\mathfrak{D}} + \text{etc.} = 0$$
$$-\frac{\alpha^3}{\mathfrak{A}} - \frac{\beta^3}{\mathfrak{B}} - \frac{\gamma^3}{\mathfrak{C}} - \frac{\delta^3}{\mathfrak{D}} - \text{etc.} = 0$$
etc.

donec perveniatur ad hanc formam

$$\pm\frac{\alpha^{n-1}}{\mathfrak{A}} \pm \frac{\beta^{n-1}}{\mathfrak{B}} \pm \frac{\gamma^{n-1}}{\mathfrak{C}} \pm \frac{\delta^{n-1}}{\mathfrak{D}} \pm \text{etc.}$$

cujus summa non amplius est evanescens, sed aequalis fractioni

Corollarium 2.

1171. Hoc etiam ex evolutione formae in Theoremate adhibitae colligere licet; etenim si ea statuatur

$$\frac{z^{n-1}}{(z-a)(z-b)(z-c)\ldots\ldots(z-y)},$$

existente omnium litterarum a, b, c, etc. numero $= n$, quia hic numerator z^{n-1} tot habet dimensiones, quot sunt factores in denominatore, pars integra in hac fractione contenta est unitas; quae etiam facta resolutione conservatur, et in applicatione ad casum memoratum abit in $\frac{1}{N}$.

Scholion.

1172. Post hujus Theorematis demonstrationem demum clare a posteriori ostendi potest, quemadmodum integrale supra (1160.) exhibitum aequationi differentiali ibidem propositae satisfaciat. Notatis enim expressionibus §. 1170, cum supra invenerimus integrale

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x + \text{etc.}$$

erit continuo differentiando

$$\frac{\partial y}{\partial x} = -\frac{\alpha}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x - \frac{\beta}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x - \text{etc.}$$

$$\frac{\partial \partial y}{\partial x^2} = \frac{\alpha^2}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x + \frac{\beta^2}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x + \text{etc.}$$

$$\frac{\partial^3 y}{\partial x^3} = \frac{-\alpha^3}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x - \frac{\beta^3}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x - \text{etc.}$$

etc.

usque ad

$$\frac{\partial^{n-1} y}{\partial x^{n-1}} = \pm \frac{\alpha^{n-1}}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x \pm \frac{\beta^{n-1}}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x \pm \text{etc.}$$

unde sequens forma differentialis resultat

$$\frac{\partial^n y}{\partial x^n} = \mp \frac{\alpha^n}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X \partial x \mp \frac{\beta^n}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X \partial x \mp \text{etc.}$$

$$\mp \left(\frac{\alpha^{n-1}}{\mathfrak{A}} + \frac{\beta^{n-1}}{\mathfrak{B}} + \frac{\gamma^{n-1}}{\mathfrak{C}} + \text{etc.} \right) X,$$

quod postremum membrum abit in $\frac{1}{N} X$.

Si jam omnes hae formae singulatim multiplicentur per quantitates A, B, C, D . . . . . , N, quoniam est

$$A - B\alpha + C\alpha^2 - D\alpha^3 + \ldots\ldots \mp N\alpha^n = 0,$$
$$A - B\beta + C\beta^2 - D\beta^3 + \ldots\ldots \mp N\beta^n = 0,$$

propterea quod $\alpha + z$, $\beta + z$, $\gamma + z$, etc. sunt factores formae

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n,$$

manifesto obtinebimus

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n} = X$$

quae est ipsa aequatio differentialis initio proposita.

Problema 153.

1173. Proposita aequatione differentiali cujuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n}$$

si expressio algebraica hinc formata

$$P = A + Bz + Cz^2 + Dz^3 \ldots\ldots\ldots + Nz^n$$

duos habeat factores simplices imaginarios factore duplici $ff + 2fz\cos.\theta + zz$ contentos, investigare partes integralis hinc oriundas.

Solutio.

Sint $\alpha + z$ et $\beta + z$ hi duo factores imaginarii, ut sit $\alpha = f(\cos.\,\theta + \sqrt{-1}.\sin.\,\theta)$ et $\beta = f(\cos.\,\theta - \sqrt{-1}.\sin.\,\theta)$, ob

$$(\alpha + z)(\beta + z) = ff + 2fz\cos.\,\theta + zz,$$

ac statuatur $P = (ff + 2fz\cos.\,\theta + zz)Q$, existente

$$Q = A' + B'z + C'z^2 + \ldots\ldots + N'z^{n-2}.$$

Cum igitur integralis partes ex binis illis factoribus simplicibus imaginariis ortae sint

$$\frac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}X\partial x + \frac{1}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}X\partial x = v,$$

hos valores imaginarios ad realitatem perduci oportet. Erunt autem $\mathfrak{A}$ et $\mathfrak{B}$ quantitates imaginariae resultantes ex forma

$$(f\cos.\,\theta \mp \sqrt{-1}.f\sin.\,\theta + z)Q,$$

si loco z scribatur

$$-f\cos.\,\theta \mp \sqrt{-1}.f\sin\,\theta.$$

At facta hac substitutione fit

$$Q = A' - B'f\cos.\,\theta + C'ff\cos.\,2\theta - D'f^3\cos.\,3\theta + \text{etc.}$$
$$\mp\sqrt{-1}.B'f\sin.\,\theta \pm \sqrt{-1}.C'ff\sin.\,2\theta \mp \sqrt{-1}.D'f^3\cos.\,3\theta \pm \text{etc.}$$

Ponamus brevitatis gratia

$$A' - B'f\cos.\,\theta + C'ff\cos.\,2\theta - D'f^3\cos.\,3\theta + \text{etc.} = \mathfrak{M} \text{ et}$$
$$-B'f\sin.\,\theta + C'ff\sin.\,2\theta - D'f^3\sin.\,3\theta + \text{etc.} = \mathfrak{N},$$

ut sit $Q = \mathfrak{M} \pm \mathfrak{N}\sqrt{-1}$, ubi signorum ambiguorum superius valet pro litteris α et $\mathfrak{A}$, inferius pro litteris β et $\mathfrak{B}$. Hinc ergo erit

$$\mathfrak{A} = -2\sqrt{-1}.f\sin.\,\theta(\mathfrak{M} + \mathfrak{N}\sqrt{-1}) \text{ et}$$
$$\mathfrak{B} = +2\sqrt{-1}.f\sin.\,\theta(\mathfrak{M} - \mathfrak{N}\sqrt{-1}),$$

ideoque

$$2v\sqrt{-1}\,.\,f\sin\theta = \frac{-e^{-\alpha x}\int e^{\alpha x}X\partial x}{\mathfrak{M}+\mathfrak{N}\sqrt{-1}} + \frac{e^{-\beta x}\int e^{\beta x}X\partial x}{\mathfrak{M}-\mathfrak{N}\sqrt{-1}}.$$

Est vero

$$e^{\alpha x} = e^{fx\cos.\theta}[\cos.(fx\sin.\theta)+\sqrt{-1}\,.\,\sin.(fx\sin.\theta)] \text{ et}$$
$$e^{\beta x} = e^{fx\cos.\theta}[\cos.(fx\sin.\theta)-\sqrt{-1}\,.\,\sin.(fx\sin.\theta)].$$

Sit brevitatis gratia angulus fx sin. $\theta = \Phi$ erit

$$2\sqrt{-1}\,.\,v(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})f\sin.\theta =$$
$$-(\mathfrak{M}-\mathfrak{N}\sqrt{-1})e^{-fx\cos.\theta}(\cos.\Phi-\sqrt{-1}\sin.\Phi)\int e^{fx\cos.\theta}X\partial x(\cos.\Phi+\sqrt{-1}.\sin.\Phi)$$
$$+(\mathfrak{M}+\mathfrak{N}\sqrt{-1})e^{-fx\cos.\theta}(\cos.\Phi+\sqrt{-1}.\sin.\Phi)\int e^{fx\cos.\theta}X\partial x(\cos.\Phi-\sqrt{-1}.\sin.\Phi)$$
$$= e^{-fx\cos.\theta}2\sqrt{-1}\,.\,(\mathfrak{M}\sin.\Phi+\mathfrak{N}\cos.\Phi)\int e^{fx\cos.\theta}X\partial x\cos.\Phi$$
$$-e^{-fx\cos.\theta}2\sqrt{-1}\,.\,(\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi)\int e^{fx\cos.\theta}X\partial x\sin.\Phi.$$

Quocirca habebimus integralis partem quaesitam

$$v = \frac{\left\{\begin{array}{l} +e^{-fx.\cos.\theta}(\mathfrak{M}\sin\Phi+\mathfrak{N}\cos.\Phi)\int e^{fx\cos.\theta}X\partial x\cos.\Phi \\ -e^{-fx\cos.\theta}(\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi)\int e^{fx\cos.\theta}X\partial x\sin.\Phi \end{array}\right\}}{(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})f\sin.\theta}$$

existente $\Phi = fx\sin.\theta$.

Corollarium 1.

1174. Praecipuum igitur opus hic consistit in inventione formulae imaginariae $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$, quae colligi debet ex quantitate Q, dum loco z scribitur valor imaginarius

$$-f(\cos.\theta+\sqrt{-1}\,.\,\sin.\theta),$$

unde hoc commodi nascitur, ut loco z^n scribi oporteat

$$(-f)^n(\cos.n\theta+\sqrt{-1}\,.\,\text{siu}.\,n\theta).$$

Corollarium 2.

1175. Cum sit $Q = \frac{P}{ff+2fz\cos.\theta+zz}$, etiam ex hac forma per candem substitutionem formula imaginaria $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$

inveniri potest, ubi autem notandum est, hac substitutione tam numeratorem P quam denominatorem evanescere. Ex quo manifestum est, valorem istius formulae rite obtineri ex hac fractione

$$\frac{\partial P}{2(f\cos.\theta+z)\partial z}=\frac{\partial P}{-2\sqrt{-1}.f\sin.\theta\partial z}.$$

Corollarium 3.

1176. Quoniam igitur est

$$\frac{\partial P}{\partial z}=B+2Cz+3Dz^2+4Ez^3+\ldots nNz^{n-1},$$

si statuamus

$$\mathfrak{P}=B-2Cf\cos.\theta+3Df^2\cos.2\theta-4Ef^3\cos.3\theta\ldots\pm nNf^{n-1}\cos.(n-1)\theta,$$

$$\mathfrak{Q}=-2Cf\sin\theta+3Df^2\sin.2\theta-4Ef^3\sin.3\theta\ldots\pm nNf^{n-1}\sin.(n-1)\theta,$$

ut facta substitutione fiat

$$\frac{\partial P}{\partial z}=\mathfrak{P}+\sqrt{-1}.\mathfrak{Q},$$

habebimus

$$\mathfrak{M}+\sqrt{-1}.\mathfrak{N}=\frac{\mathfrak{P}+\sqrt{-1}.\mathfrak{Q}}{-2\sqrt{-1}.f\sin.\theta}=\frac{-\mathfrak{Q}+\sqrt{-1}.\mathfrak{P}}{2f\sin.\theta},$$

ideoque

$$\mathfrak{M}=\frac{-\mathfrak{Q}}{2.f\sin.\theta}\text{ et }\mathfrak{N}=\frac{\mathfrak{P}}{2f\sin.\theta}.$$

Corollarium 4.

1177. Immediate ergo ex quantitate P indeque derivatis $\mathfrak{P}$ et $\mathfrak{Q}$, posito $fx\sin.\theta=\Phi$, integralis pars ex factore duplici $ff+2fz\cos.\theta+zz$ nata erit expressa

$$v=\frac{2e^{-fx\cos.\theta}}{\mathfrak{PP}+\mathfrak{QQ}}\left\{\begin{array}{l}(\mathfrak{P}\cos.\Phi-\mathfrak{Q}\sin.\Phi)\int e^{fx\cos.\theta}X\partial x\cos.\Phi\\+(\mathfrak{P}\sin.\Phi+\mathfrak{Q}\cos.\Phi)\int e^{fx\cos.\theta}X\partial x\sin.\Phi).\end{array}\right\}$$

Scholion.

1178. Quotcunque ergo forma

$$P = A + Bz + Cz^2 + Dz^3 + \text{etc.}$$

habuerit factores duplices, pro singulis ope horum praeceptorum partes integralis facile definiuntur, et quia hinc inventio partium, quae factoribus simplicibus conveniunt, sive ii sint inaequales, non turbatur, omnibus partibus in unam summam conjectis habebitur integrale completum aequationis differentialis propositae. Verum tamen haec praecepta non sufficiunt, si factorum duplicium bini pluresve inter se fuerint aequales; hujusmodi enim casus peculiarem exigunt evolutionem similem ejus, qua pro casu duorum pluriumve factorum simplicium inter se aequalium sum usus. Ne autem hanc tractionem nimis protraham, sufficiet casum pro duobus factoribus duplicibus inter se aequalibus evolvisse, cum inde methodus ad plures facile extendatur.

Problema 154.

1179. Proposita aequatione differentiali cujuscunque gradus

$$X = Ay + B\frac{\partial y}{\partial x} + C\frac{\partial\partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n},$$

si expressio algebraica inde formata

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n$$

habeat factorem duplicem quadratum

$$(ff + 2fz\cos.\theta + zz)^2,$$

partem integralis ei convenientem investigare.

Solutio.

Ponamus ergo $P = (ff + 2fz\cos.\theta + zz)^2 Q$, sitque

$$Q = A' + B'z + C'zz + \ldots + N'z^{n-4},$$

ac primo imaginaria non curantes statuamus

$$\alpha = f(\cos.\theta + \sqrt{-1}.\sin.\theta) \text{ et}$$
$$\beta = f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

ut sit

$$P = (\alpha + z)^2 (\beta + z)^2 Q.$$

Jam ex iis quae supra (1163.) de binis factoribus simplicibus aequalibus docuimus, ponamus formam

$$\frac{P}{(\alpha+z)^2} = (\beta + z)^2 Q, \text{ posito } z = -\alpha, \text{ abire in } \mathfrak{A},$$

at hanc formam

$$\frac{P}{(\beta+z)^2} = (\alpha + z)^2 Q, \text{ posito } z = -\beta, \text{ in } \mathfrak{B},$$

quibus quantitatibus $\mathfrak{A}$ et $\mathfrak{B}$ inventis, ibi ostendi fore integralis partes hinc oriundas

$$\frac{1}{\mathfrak{A}} e^{-\alpha x} \int \partial x \int e^{\alpha x} X \partial x + \frac{1}{\mathfrak{B}} e^{-\beta x} \int \partial x \int e^{\beta x} X \partial x = v,$$

quas, cum jam imaginaria involvant, ad realitatem reduci oportet. Faciamus ut in problemate praecedente

$$\mathfrak{M} = A' - B'f\cos.\theta + C'f^2\cos.2\theta - D'f^3\cos.3\theta + \text{etc.}$$
$$\mathfrak{N} = \quad - B'f\sin.\theta + C'f^2\sin.2\theta - D'f^3\sin.3\theta + \text{etc.}$$

ut quantitas Q, posito $z = -\alpha = -f(\cos.\theta + \sqrt{-1}.\sin.\theta)$, abeat in $\mathfrak{M} + \mathfrak{N}\sqrt{-1}$, at posito

$$z = -\beta = -f(\cos.\theta - \sqrt{-1}.\sin.\theta),$$

in $\mathfrak{M} - \mathfrak{N}\sqrt{-1}$.

Cum jam sit $(\beta - \alpha)^2 = (-2\sqrt{-1}.f\sin.\theta)^2 = -4ff\sin.\theta^2$, cui quoque $(\alpha - \beta)^2$ aequatur, erit

$$\mathfrak{A} = -4ff\sin.\theta^2(\mathfrak{M} + \mathfrak{N}\sqrt{-1}) \text{ et}$$
$$\mathfrak{B} = -4ff\sin.\theta^2(\mathfrak{M} - \mathfrak{N}\sqrt{-1}),$$

ideoque

$$-4ff\sin.\theta^2(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})v=(\mathfrak{M}-\mathfrak{N}\sqrt{-1})e^{-\alpha x}\int\partial x\int e^{\alpha x}X\partial x$$
$$+(\mathfrak{M}+\mathfrak{N}\sqrt{-1})e^{-\beta x}\int\partial x\int e^{\beta x}X\partial x.$$

At posito $fx\sin.\theta=\Phi$, est ut vidimus

$$e^{\alpha x}=e^{fx\cos.\theta}(\cos.\Phi+\sqrt{-1}\,.\,\sin.\Phi),$$
$$e^{-\alpha x}=e^{-fx\cos.\theta}(\cos.\Phi-\sqrt{-1}\,.\,\sin.\Phi),$$
$$e^{\beta x}=e^{fx\cos.\theta}(\cos.\Phi-\sqrt{-1}\,.\,\sin.\Phi),$$
$$e^{-\beta x}=e^{-fx\cos.\theta}(\cos.\Phi+\sqrt{-1}\,.\,\sin.\Phi),$$

unde illius aequationis alterum membrum abit in

$$+e^{-fx\cos.\theta}[\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi-\mathfrak{N}\sqrt{-1}\,.\,\cos.\Phi-\mathfrak{M}\sqrt{-1}\,.\,\sin.\Phi]$$
$$\times\int\partial x\int e^{fx\cos.\theta}X\partial x(\cos.\Phi+\sqrt{-1}\,.\,\sin.\Phi)$$
$$+e^{-fx\cos.\theta}[\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi+\mathfrak{N}\sqrt{-1}\,.\,\cos.\Phi+\mathfrak{M}\sqrt{-1}\,.\,\sin.\Phi]$$
$$\times\int\partial x\int e^{fx\cos.\theta}X\partial x(\cos.\Phi-\sqrt{-1}\,.\,\sin.\Phi),$$

ubi partes imaginariae sponte se destruunt, ita ut obtineatur

$$v=\frac{\left\{\begin{array}{l}-e^{-fx\cos.\theta}(\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi)\int\partial x\int e^{fx\cos.\theta}X\partial x\cos.\Phi\\-e^{-fx\cos.\theta}(\mathfrak{N}\cos.\Phi+\mathfrak{M}\sin.\Phi)\int\partial x\int e^{fx\cos.\theta}X\partial x\sin.\Phi\end{array}\right\}}{2(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})ff\sin.\theta^2}$$

seu hoc modo

$$v=\frac{-e^{-fx\cos.\theta}}{2(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N})ff\sin.\theta^2}\left\{\begin{array}{l}+(\mathfrak{M}\cos.\Phi-\mathfrak{N}\sin.\Phi)\int\partial x\int e^{fx\cos.\theta}X\partial x\cos.\Phi\\+(\mathfrak{M}\sin.\Phi+\mathfrak{N}\cos.\Phi)\int\partial x\int e^{fx\cos.\theta}X\partial x\sin\Phi\end{array}\right\},$$

quae expressio ab imaginariis penitus est liberata.

Nota. Etiam haec solutio insigni correctione indiget diligentiae lectorum relicta.

Corollarium 1.

1180. Quoniam formula imaginaria $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$ nascitur ex quantitate Q, si loco z scribatur

$$-f(\cos.\theta + \sqrt{-1}.\sin.\theta),$$

eadem positione quoque reperietur ex forma

$$\frac{P}{(ff + 2fz\cos.\theta + zz)^2},$$

verum hic tam numerator quam denominator prodit evanescens.

Corollarium 2.

1181. Orietur ergo quoque idem valor ex formula

$$\frac{\partial P}{4\partial z[f^3\cos.\theta + ffz(1 + 2\cos.\theta^2) + 3fzz\cos.\theta + z^3]},$$

ubi cum idem incommodum denuo recurrat, orietur quoque ex hac formula

$$\frac{\partial\partial P}{4\partial z^2[ff(1 + 2\cos.\theta^2) + 6fz\cos.\theta + 3zz]}.$$

Corollarium 3.

1182. Statuatur hic primo in denominatore

$$z = -f(\cos.\theta + \sqrt{-1}.\sin.\theta)$$

fietque haec formula

$$\frac{-\partial\partial P}{8ff\partial z^2\sin.\theta^2}.$$

Deinde cum sit

$$\frac{\partial\partial P}{2\partial z^2} = C + 3Dz + 6Ezz + \ldots\ldots + \frac{n(n-1)}{1.2}Nz^{n-2},$$

ponamus brevitatis gratia

$$\mathfrak{P} = C - 3Df\cos.\theta + 6Eff\cos.2\theta - \ldots\ldots \pm \frac{n(n-1)}{1.2}Nf^{n-2}\cos.(n-2)\theta,$$

$$\mathfrak{Q} = -3Df\sin.\theta + 6Eff\sin.2\theta - \ldots\ldots \pm \frac{n(n-1)}{1.2}Nf^{n-2}\sin.(n-2)\theta,$$

ut sit facta substitutione

$$\frac{\partial\partial P}{2\partial z^2} = \mathfrak{P} + \mathfrak{Q}\sqrt{-1},$$

ideoque

$$\mathfrak{M} + \mathfrak{N}\sqrt{-1} = \frac{-\mathfrak{P} - \mathfrak{Q}\sqrt{-1}}{4ff\sin.\theta^2},$$

et consequenter

$$\mathfrak{M} = \frac{-\mathfrak{P}}{4ff \sin.\theta^2} \text{ et } \mathfrak{N} = \frac{-\mathfrak{Q}}{4ff \sin.\theta^2}.$$

Quos ergo valores in parte integralis inventa substituere licet.

Corollarium 4.

1183. Facta autem substitutione, factor duplex quadratus $(ff + 2fz\cos.\theta + zz)^2$ hanc praebet integralis partem

$$v = \frac{2e^{-fx\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\left\{\begin{array}{l} +(\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi)\int\partial x\int e^{fx\cos.\theta}X\partial x\cos.\Phi \\ +(\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi)\int\partial x\int e^{fx\cos.\theta}X\partial x\sin.\Phi \end{array}\right\},$$

ubi Φ denotat angulum $fx\sin.\theta$.

Scholion.

1184. Si hanc expressionem cum ea, quam problemate praecedente invenimus, comparemus, vix actuali simili evolutione erit opus pro casibus magis complicatis. Ita si quantitas

$$P = A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Nz^n,$$

factorem habeat duplicem cubicum

$$(ff + 2fz\cos.\theta + zz)^3,$$

quantitates $\mathfrak{P}$ et $\mathfrak{Q}$ ita definiuntur, ut sit

$$\mathfrak{P} = D - 4Ef\cos.\theta + 10Fff\cos.2\theta - 20Gf^3\cos.3\theta + \ldots \pm \frac{n(n-1)(n-2)}{1.2.3}Nf^{n-3}\cos.(n-3)\theta,$$

$$\mathfrak{Q} = -4Ef\sin.\theta + 10Fff\sin.2\theta - 20Gf^3\sin.3\theta + \ldots \pm \frac{n(n-1)(n-2)}{1.2.3}Nf^{n-3}\sin.(n-3)\theta,$$

quibus inventis, erit integralis pars hinc nata

$$\frac{2e^{-fx\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\left\{\begin{array}{l} +(\mathfrak{P}\cos.\Phi - \mathfrak{Q}\sin.\Phi)\int\partial x\int\partial x\int e^{fx\cos.\theta}X\partial x\cos.\Phi \\ +(\mathfrak{P}\sin.\Phi + \mathfrak{Q}\cos.\Phi)\int\partial x\int\partial x\int e^{fx\cos.\theta}X\partial x\sin.\Phi \end{array}\right\},$$

neque jam ulterior progressio ulli amplius difficultati est obnoxia. Quocirca aequationis hoc capite propositae resolutionem ita concinne mihi equidem absolvisse videor, ut nihil amplius desiderari possit. Interim hoc argumentum maxime illustrabitur, si haec praecepta ad exempla particularia accommodabimus; cui instituto sequens caput est destinatum. Ante autem insignem proprietatem circa hujusmodi aequationes generales proponam, quae in Analysi ingentem usum habitura videtur.

Problema 155.

1135. Proposita aequatione differentiali cujuscunque gradus

$$X = A y + B \frac{\partial y}{\partial x} + C \frac{\partial \partial y}{\partial x^2} + D \frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N \frac{\partial^{m+n} y}{\partial x^{m+n}},$$

si formula algebraica inde nata

$$P = A + B z + C z^2 + D z^3 + \ldots\ldots\ldots + N z^{m+n}$$

duobus factoribus constet $P = QR$, ut sit

$$Q = \mathfrak{A} + \mathfrak{B} z + \mathfrak{C} z^2 + \ldots\ldots + \mathfrak{N} z^m \text{ et}$$
$$R = \mathfrak{a} + \mathfrak{b} z + \mathfrak{c} z^2 + \ldots\ldots + \mathfrak{n} z^n,$$

integrationem illius aequationis ad integrationem binarum aequationum simpliciorum revocare.

Solutio.

Si formam integralem primo (1158.) perpendamus, haud difficulter inde colligimus, postquam hanc aequationem integraverimus

$$X = \mathfrak{A} v + \mathfrak{B} \frac{\partial v}{\partial x} + \mathfrak{C} \frac{\partial \partial v}{\partial x^2} + \ldots\ldots + \mathfrak{N} \frac{\partial^m v}{\partial x^m},$$

indeque valorem ipsius v per x et X definiverimus, valorem ipsius y pro aequatione proposita ex hac aequatione erutum iri

$$v = \mathfrak{a}\, y + \mathfrak{b}\frac{\partial y}{\partial x} + \mathfrak{c}\frac{\partial \partial y}{\partial x^2} + \cdot\cdot\cdot\cdot\cdot + \mathfrak{n}\frac{\partial^n y}{\partial x^n},$$

cujus ratio adeo in promtu est posita; dum ex hac aequatione valores pro v ejusque differentialibus substituantur. Prodibit enim

$$\begin{array}{llll} X = \mathfrak{A}\,\mathfrak{a}\, y + \mathfrak{A}\,\mathfrak{b}\,.\,\frac{\partial y}{\partial x} & + \mathfrak{A}\,\mathfrak{c}\,.\,\frac{\partial \partial y}{\partial x^2} & + \mathfrak{A}\,\mathfrak{d}\,.\,\frac{\partial^3 y}{\partial x^3} & + \text{etc.} \\ \qquad\qquad\quad + \mathfrak{B}\,\mathfrak{a} & + \mathfrak{B}\,\mathfrak{b} & + \mathfrak{B}\,\mathfrak{c} & \\ & + \mathfrak{C}\,\mathfrak{a} & + \mathfrak{C}\,\mathfrak{b} & \\ & & + \mathfrak{D}\,\mathfrak{a} & \end{array}$$

Cum autem per hypothesin sit $P = Q\,R$, seriebus Q et R in se multiplicatis, necesse est fieri

$$A = \mathfrak{A}\,\mathfrak{a},\; B = \mathfrak{A}\,\mathfrak{b} + \mathfrak{B}\,\mathfrak{a},\; C = \mathfrak{A}\,\mathfrak{c} + \mathfrak{B}\,\mathfrak{b} + \mathfrak{C}\,\mathfrak{a},\; \text{etc.}$$

sicque haec postrema aequatio ad ipsam propositam reducitur.

Corollarium 1.

1186. Si tantum ad factores simplices respiciamus, prioris aequationis integrale per hujusmodi terminos exprimitur

$$v = \Gamma\, e^{-\alpha x} \int e^{\alpha x}\, X\, \partial x \;\; \text{etc.}$$

posterioris vero aequationis integrale per hujusmodi

$$y = \Delta\, e^{-\beta x} \int e^{\beta x}\, v\, \partial x \;\; \text{etc.}$$

Corollarium 2.

1187. Quodsi jam in singulis terminis posterioris integralis substituamus singulos prioris, fiet

$$y = \Gamma \Delta\, e^{-\beta x} \int e^{(\beta - \alpha)x}\, \partial x \int e^{\alpha x}\, X\, \partial x,$$

quae forma ad hanc reducitur

$$y = \frac{\Gamma \Delta}{\beta - \alpha}\left(e^{-\alpha x} \int e^{\alpha x}\, X\, \partial x - e^{-\beta x} \int e^{\beta x}\, X\, \partial x\right),$$

cujusmodi termini per integrationem aequationis propositae immediate inveniuntur.

Corollarium 3.

1188. Si hic fuisset $\beta = \alpha$, sine ulla reductione statim prodiisset forma

$$y = \Gamma \Delta e^{-\alpha x} \int \partial x \int e^{\alpha x} X \partial x$$

supra pro casu duorum factorum simplicium aequalium inventa. Interim cum totum negotium ad resolutionem in factores vel simplices vel duplices reales redeat, ipsa aequatio proposita modo exposito facillime expeditur.