CAPUT II.

DE RESOLUTIONE AEQUATIONUM HUJUS FORMAE

$$A\,y + B\,.\frac{\partial y}{\partial x} + C\,.\frac{\partial\partial y}{\partial x^2} + D\,.\frac{\partial^3 y}{\partial x^3} + E\,.\frac{\partial^4 y}{\partial x^4} + \text{etc.} = 0,$$

SUMTO ELEMENTO ∂x CONSTANTE.

Problema 144.

1117.

Aequationis differentialis tertii gradus

$$A\,y + \frac{B\,\partial y}{\partial x} + \frac{C\,\partial\partial y}{\partial x^2} + \frac{D\,\partial^3 y}{\partial x^3} = 0,$$

sumto elemento ∂x constante, integrale completum invenire.

Solutio

Cum A, B, C, D sint quantitates constantes, levi attentio adhibita patet, isti aequationi hujusmodi formam $y = e^{\lambda x}$ satisfacer cum enim hinc sit

$$\frac{\partial y}{\partial x} = \lambda\, e^{\lambda x}, \quad \frac{\partial\partial y}{\partial x^2} = \lambda^2 e^{\lambda x}, \quad \frac{\partial^3 y}{\partial x^3} = \lambda^3 e^{\lambda x},$$

his substitutis et aequatione per $e^{\lambda x}$ divisa, fit

$$A + \lambda B + \lambda^2 C + \lambda^3 D = 0,$$

unde exponens λ determinatur, qui cum tres valores sortiatur, q sint α, β, γ, habebimus tres formulas satisfacientes $y = e^{\alpha}$ $y = e^{\beta x}$, $y = e^{\gamma x}$. Verum ex natura aequationis propositae pe spicuum est, si ei satisfaciant valores $y = P$, $y = Q$, $y =$ etiam his utcunque conjungendis satisfacturam

$$y = \mathfrak{A}\,P + \mathfrak{B}\,Q + \mathfrak{C}\,R.$$

Quare ex ternis formulis inventis nanciscimur hanc latissime patentem aeque satisfacientem

$$y = \mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x},$$

quae forma cum tres constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ complectatur, revera erit integrale completum aequationis nostrae propositae.

Corollarium 1.

1118. Integrale ergo completum tot constat partibus, quot radices habeat seu factores aequatio

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0,$$

cujus si factor fuerit $a + \lambda$, pars integralis erit e^{-ax}.

Corollarium 2.

1119. Haec scilicet pars erit integrale hujus aequationis $ay + \frac{\partial x}{\partial y} = 0$. Unde si sit

$$A + B\lambda + C\lambda^2 + D\lambda^3 = (a + \lambda)(b + \lambda)(c + \lambda),$$

quaerantur valores ipsius y ex his aequationibus simplicibus

$$ay + \frac{\partial y}{\partial x} = 0, \quad by + \frac{\partial y}{\partial x} = 0, \quad cy + \frac{\partial y}{\partial x} = 0,$$

qui si sint $y = P$, $y = Q$, $y = R$, integrale aequationis propositae erit

$$y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R.$$

Corollarium 3.

1120. Si binae radices sint aequales, puta $\beta = \alpha$, consideretur differentia ut evanescens vel $\beta = \alpha + \omega$, et cum sit

$$e^{\beta x} = e^{\alpha x} \cdot e^{\omega x} = e^{\alpha x}(1 + \omega x),$$

evidens est loco $\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x}$ scribi debere

$$\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\alpha x} x = e^{\alpha x}(\mathfrak{A} + \mathfrak{B} x).$$

Ac si omnes tres radices fuerint aequales $\alpha = \beta = \gamma$, ut aequatio sit

$$y + \frac{3\partial y}{a\partial x} + \frac{3\partial\partial y}{a^2\partial x^2} + \frac{\partial^3 y}{a^3\partial x^3} = 0,$$

ob $\alpha = \beta = \gamma = -a$, integrale completum erit

$$e^{-ax}(\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx).$$

Corollarium 4.

1121. Si binae radices fiant imaginariae, ut fit

$$\alpha = \mu + \nu\sqrt{-1} \text{ et } \beta = \mu - \nu\sqrt{-1},$$

loco $\mathfrak{A}e^{\alpha x} + \mathfrak{B}e^{\beta x}$ scribi debet

$$e^{\mu x}(\mathfrak{A}e^{\nu x\sqrt{-1}} + \mathfrak{B}e^{-\nu x\sqrt{-1}}),$$

quae reducitur ad hanc formam

$$e^{\mu x}(\mathfrak{A}\cos.\nu x + \mathfrak{B}\sin.\nu x).$$

Scholion 1.

1122. Quamquam aequatio proposita triplicem integrationem postulat, antequam ad relationem finitam inter x et y perveniatur, hic tamen una operatione, quae ne integrationi quidem est affinis, eo pertigimus. Conjectura scilicet formam collegimus aequationi particulariter satisfacientem, simulque tres hujusmodi formas sumus consecuti. Deinde ex ipsa aequationis indole intelleximus, si valores singuli $y = P$, $y = Q$, $y = R$ satisfaciant, etiam formam ex iis compositam $y = \mathfrak{A}P + \mathfrak{B}Q + \mathfrak{C}R$ satisfacere debere, quod nisi commode evenisset, ex illis ternis valoribus nihil amplius concludi potuisset. Ex eodem ergo principio in genere hujusmodi aequationes differentiales, quotcunque fuerint gradus, uno quasi actu ita resolvi poterunt, ut adeo integrale completum assignetur.

Scholion 2.

1123. Quoniam aequationem differentialem tertii gradus

$$A y + B \frac{\partial y}{\partial x} + C \frac{\partial \partial y}{\partial x^2} + D \frac{\partial^3 y}{\partial x^3} = 0,$$

in genere resolvere licuit, ut integrale completum esset

$$y = \mathfrak{A}\, e^{\alpha x} + \mathfrak{B}\, e^{\beta x} + \mathfrak{C}\, e^{\gamma x},$$

existentibus α, β, γ radicibus hujus aequationis cubicae

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0,$$

hinc usum non contemnendum pro aliis aequationibus, in quas illam transformare licet, percipiemus. Primo autem illam aequationem ad differentialem secundi gradus revocare licet ope substitutionis $y = e^{\int u \partial x}$; unde fit

$$\frac{\partial y}{\partial x} = e^{\int u \partial x}\, u, \quad \frac{\partial \partial y}{\partial x^2} = e^{\int u \partial x} \left(\frac{\partial u}{\partial x} + u u\right), \text{ et}$$

$$\frac{\partial^3 y}{\partial x^3} = e^{\int u \partial x} \left(\frac{\partial \partial u}{\partial x^2} + \frac{3 u \partial u}{\partial x} + u^3\right),$$

ut aequatio transformata sit, divisione per $e^{\int u \partial x}$ facta,

$$A + B u + C u u + D u^3 + C \frac{\partial u}{\partial x} + \frac{3 D u \partial u}{\partial x} + \frac{D \partial \partial u}{\partial x^2} = 0;$$

cujus ergo ob $u = \frac{\partial y}{y \partial x}$ integrale est

$$u = \frac{\alpha \mathfrak{A}\, e^{\alpha x} + \beta \mathfrak{B}\, e^{\beta x} + \gamma \mathfrak{C}\, e^{\gamma x}}{\mathfrak{A}\, e^{\alpha x} + \mathfrak{B}\, e^{\beta x} + \mathfrak{C}\, e^{\gamma x}}.$$

Haec autem aequatio ulterius ad gradum primum reducitur ponendo $\partial x = \frac{\partial u}{t}$, cum enim elementum ∂x sumtum sit constans, erit $t \partial \partial u - \partial t \partial u = 0$, seu $\partial \partial u = \frac{\partial t \partial u}{t}$, unde fit $\frac{\partial u}{\partial x} = t$, et $\frac{\partial \partial u}{\partial x^2} = \frac{t \partial t}{\partial u}$, ita ut prodeat haec aequatio differentialis primi gradus

$$A + B u + C u u + D u^3 + t (C + 3 D u) + \frac{D t \partial t}{\partial u} = 0,$$

cujus ergo resolutio quoque est in potestate: utriusque scilicet variabilis u et t valor per eandem variabilem x exprimi potest.

Cum enim y per x detur, erit primo $u = \frac{\partial y}{y \partial x}$; tum vero $t + uu = \frac{\partial \partial y}{y \partial x^2}$, ob $\frac{\partial u}{\partial x} = t$. Loco y ergo substituto valore supra invento erit

$$u = \frac{\alpha \mathfrak{A} e^{\alpha x} + \beta \mathfrak{B} e^{\beta x} + \gamma \mathfrak{C} e^{\gamma x}}{\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x}}, \text{ et}$$

$$t + uu = \frac{\alpha \alpha \mathfrak{A} e^{\alpha x} + \beta \beta \mathfrak{B} e^{\beta x} + \gamma \gamma \mathfrak{C} e^{\gamma x}}{\mathfrak{A} e^{\alpha x} + \mathfrak{B} e^{\beta x} + \mathfrak{C} e^{\gamma x}},$$

dummodo α, β, γ sint radices ex hac aequatione

$$A + B\lambda + C\lambda^2 + D\lambda^3 = 0.$$

Observari autem convenit, illam aequationem, posito $t + uu = z$, abire in hanc formam

$$A + Bu + z(C + Du) + \frac{D \partial z}{\partial u}(z - uu) = 0,$$

quae latius patere videtur, quam illae ejusdem generis aequationes, quas supra tractavimus; cujus, quia ratio per methodos cognitas integrandi non constat, resolutio facillime instituitur ponendo

$$u = \frac{\partial y}{y \partial x} \text{ et } z = \frac{\partial \partial y}{y \partial x^2},$$

unde fit

$$\partial u = \frac{\partial \partial y}{y \partial x} - \frac{\partial y^2}{yy \partial x} \text{ et } \partial z = \frac{\partial^3 y}{y \partial x^2} - \frac{\partial y \partial \partial y}{yy \partial x^2},$$

ideoque

$$\frac{\partial z}{\partial u} = \frac{y \partial^3 z - \partial y \partial \partial y}{\partial x (y \partial \partial y - \partial y^2)} \text{ et } z - uu = \frac{y \partial \partial y - \partial y^2}{yy \partial x^2},$$

sicque resultat haec aequatio

$$A + \frac{B \partial y}{y \partial x} + \frac{C \partial \partial y}{y \partial x^2} + \frac{D \partial y \partial \partial y}{yy \partial x^3} + \frac{D y \partial^3 y - D \partial y \partial \partial y}{yy \partial x^3} = 0, \text{ seu}$$

$$Ay + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} = 0,$$

cujus resolutio est ostensa.

Scholion 3.

1124. Aequatio illa differentialis primi gradus

$$D t \partial t + t \partial u (C + 3 D u) + \partial u (A + B u + C u^2 + D u^3) = 0,$$

cujus integrale invenimus, diligentiore evolutione est digna. Ac primo quidem observo, eam integrabilem reddi, si dividatur per hanc formam

$$DDt^3 - Dtt(B + 2Cu + 3Duu)$$
$$+ t(C + 3Du)(A + Bu + Cu^2 + Du^3) + (A + Bu + Cu^2 + Du^3)^2,$$

unde concludimus et hanc aequationem

$$D z \partial z + D u u \partial z + z \partial u (C + D u) + \partial u (A + B u) = 0$$

integrabilem fieri, si dividatur per hanc formam

$$DDz^3 + Dzz(B + 2Cu) + z[AC + (3AD + BC)u + (BD + CC)uu]$$
$$+ AA + 2ABu + (AC + BB)uu + (AD + BC)u^3.$$

Utrinque autem divisor iste nihilo aequatus praebet integrale particulare, unde cum t vel z ternos obtineant valores, singuli exhibebunt integralia particularia. Hinc operae pretium erit in genere aequationem

$$y \partial y + y P \partial x + Q \partial x = 0,$$

investigare, quae per formam

$$y^3 + Lyy + My + N,$$

divisa integrabilis evadat. Per operationem autem supra explicatam invenitur

$$\partial L = 2 P \partial x, \quad \partial M = P L \partial x + 3 Q \partial x,$$
$$\partial N = 2 Q L \partial x, \text{ et } PN - QM = 0;$$

unde colligitur

$$P \partial x = \tfrac{1}{2} \partial L, \quad Q \partial x = \frac{\partial N}{2L},$$
$$\partial M = \tfrac{1}{2} L \partial L + \frac{3 \partial N}{2N}, \text{ et } N \partial L = \frac{M \partial N}{L}, \text{ seu } M = \frac{N L \partial L}{\partial N},$$

qui valor ibi substitutus sumto ∂N constante dat

$$3\partial N^2 = LL\partial L\partial N + 2NLL\partial\partial L + 2NL\partial L^2,$$

quae per ∂L multiplicata, transit in

$$3\partial L\partial N^3 = \partial . NL^2\partial L^2.$$

Verum commodius, ac singulari quidem modo, illae aequation resolvuntur, statuendo

$$N = \alpha Z^2 \text{ et } L = \frac{\partial Z}{\partial z},$$

unde sumto elemento ∂z constante deducitur $M = \frac{Z\partial\partial Z}{2\partial z^2}$, hincqu

$$\partial M = \frac{Z\partial^3 Z + \partial Z\partial\partial Z}{2\partial z^2} \text{ et}$$

$$\tfrac{1}{2}L\partial L + \frac{3\partial N}{2L} = \frac{\partial Z\partial\partial Z}{2\partial z^2} + 3\alpha Z\partial z.$$

Ergo $\partial^3 Z = 6\alpha\partial z^3$, ideoque

$$Z = \alpha z^3 + \beta zz + \gamma z + \delta,$$

$$P\partial x = \frac{\partial\partial Z}{2\partial z} \text{ et } Q\partial x = \alpha Z\partial z.$$

Quocirca sumto $Z = \alpha z^3 + \beta z^2 + \gamma z + \delta$, haec aequatio

$$y\partial y + y . \frac{\partial\partial Z}{2\partial z} + \alpha Z\partial z = 0$$

integrabilis redditur, divisa per hanc formam

$$y^3 + y^2 . \frac{\partial Z}{\partial z} + y . \frac{Z\partial\partial Z}{2\partial z^2} + \alpha ZZ.$$

Praeterea si Z habeat factores, ut proposita sit haec aequatio

$$y\partial y + y\partial z(\alpha + \beta + \gamma + 3z) + \partial z(\alpha + z)(\beta + z)(\gamma + z) = 0$$

divisor eam integrabilem reddens erit

$$[y + (\alpha + z)(\beta + z)][y + (\alpha + z)(\gamma + z)][y + (\beta + z)(\gamma + z)],$$

cujus singuli factores nihilo aequati praebent integrale particular Ex unoquoque autem, more magis consueto, integrale completu ita elicitur. Ponatur

$$y = v - (\alpha + z)(\beta + z),$$

ac reperitur

$$v\,\partial v + v\,\partial z(\gamma + z) - \partial v(\alpha + z)(\beta + z) = 0,$$

sit porro $\partial v = p\,\partial z$, eritque $v = \frac{p(\alpha + z)(\beta + z)}{p + \gamma + z}$, et differentiando locoque ∂v ponendo $p\,\partial z$, orietur haec aequatio

$$\partial p(\alpha+z)(\beta+z)(\gamma+z) = \partial z[p^3 + (2\gamma - \alpha - \beta)p^2 + (\gamma - \alpha)(\gamma - \beta)p]$$

quae dat hanc separatam

$$\frac{\partial z}{(\alpha + z)(\beta + z)(\gamma + z)} = \frac{\partial p}{p(p + \gamma - \alpha)(p + \gamma - \beta)}.$$

Problema 145.

1125. Aequationis differentialis cujuscunque gradus

$$A\,y + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + E\frac{\partial^4 y}{\partial x^4} + \text{etc.} = 0$$

sumto elemento ∂x constante, integrale completum invenire.

Solutio.

Et huic aequationi evidens est satisfacere formulam $y = e^{\lambda x}$, cum enim hinc sit

$$\frac{\partial y}{\partial x} = \lambda\, e^{\lambda x}, \quad \frac{\partial \partial y}{\partial x^2} = \lambda^2 e^{\lambda x}, \quad \frac{\partial^3 y}{\partial x^3} = \lambda^3 e^{\lambda x},$$

et in genere $\frac{\partial^n y}{\partial x^n} = \lambda^n e^{\lambda x}$, facta substitutione pervenietur ad hanc aequationem, postquam scilicet per $e^{\lambda x}$ diviserimus,

$$A + B\lambda + C\lambda^2 + D\lambda^3 + E\lambda^4 + \text{etc.} = 0$$

ex qua valorem ipsius λ definiri oportet. Hinc littera λ totidem valores obtinebit, quoti fuerit ordinis aequatio differentialis proposita, quorum singuli aequationi aeque satisfacient. Qui valores si sint α, β, γ, δ, etc. integralia quidem particularia erunt

$$y = \mathfrak{A}\, e^{\alpha x}, \quad y = \mathfrak{B}\, e^{\beta x}, \quad y = \mathfrak{C}\, e^{\gamma x}, \quad \text{etc.}$$

Verum ex ipsa aequationis natura perspicuum est, aggregata quotcunque horum valorum, ideoque etiam omnium, perinde satisfacere. Cum igitur aggregatum omnium

$$y = \mathfrak{A}\, e^{\alpha x} + \mathfrak{B}\, e^{\beta x} + \mathfrak{C}\, e^{\gamma x} + \mathfrak{D}\, e^{\delta x} + \text{etc.}$$

tot complectatur constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. quot ordinis differentialis est aequatio proposita, quin haec forma ejus sit integrale completum, dubitari nequit. Ascendat aequatio differentialis ad gradum n, ut sit

$$A\, y + B\frac{\partial y}{\partial x} + C\frac{\partial \partial y}{\partial x^2} + D\frac{\partial^3 y}{\partial x^3} + \ldots\ldots + N\frac{\partial^n y}{\partial x^n} = 0,$$

atque integrale completum ex n partibus constabit, quas ex resolutione hujus aequationis algebraicae ordinis n, scilicet

$$A + B\lambda + C\lambda^2 + D\lambda^3 + \ldots\ldots + N\lambda^n = 0,$$

definire oportet. Singuli nimirum ejus factores simplices partes illa patefacient, ita si factor sit $\alpha - \lambda$, ex eo integralis pars nascitur $\mathfrak{A}\, e^{\alpha x}$, quae, uti manifestum est, ex integratione aequationis differentialis simplicis $\alpha y - \frac{\partial y}{\partial x} = 0$ nascitur. Simili modo duo factores conjunctim

$$(\alpha - \lambda)(\beta - \lambda) = \alpha\beta - (\alpha + \beta)\lambda + \lambda\lambda$$

integralis portionem $\mathfrak{A}\, e^{\alpha x} + \mathfrak{B}\, e^{\beta x}$ suppeditant, quae simul est integrale hujus aequationis differentialis secundi gradus

$$\alpha\beta y - (\alpha + \beta)\frac{\partial y}{\partial x} + \frac{\partial \partial y}{\partial x^2} = 0.$$

Atque in genere si aequationis illius algebraicae factor sit

$$a + b\lambda + c\lambda^2 + f\lambda^3 + \text{etc.} = 0,$$

ex hoc vicissim formetur aequatio differentialis

$$a y + b\frac{\partial y}{\partial x} + c\frac{\partial \partial y}{\partial x^2} + f\frac{\partial^3 y}{\partial x^3} + \text{etc.} = 0;$$

cujus integrale completum si sit $y = P$, id simul erit pars integralis aequationis propositae. Atque hoc modo ex singulis factoribus aequationis algebraicae

$$A + B\lambda + C\lambda^2 + D\lambda^3 + \ldots\ldots + N\lambda^n = 0$$

derivabuntur singulae partes integralis quaesiti, quae junctae ejus integrale completum constituent, ita ut praecipuum negotium resolutioni hujus aequationis innitatur.

Corollarium 1.

1126. Si igitur istius aequationis algebraicae omnes factores simplices fuerint reales simulque inaequales, integratio nullam habet difficultatem. Si enim factor simplex sit $f + g\lambda$, integralis pars inde oriunda est $\mathfrak{A} e^{\frac{-fx}{g}}$.

Corollarium 2.

1127. Si bini factores simplices sint aequales, seu factor fuerit $(f + g\lambda)^2$, pars integralis inde oriunda est $e^{\frac{-fx}{g}} (\mathfrak{A} + \mathfrak{B}x)$. Si factor sit cubus $(f + g\lambda)^3$, inde oritur pars integralis

$$e^{\frac{-fx}{g}} (\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2),$$

et ex factore biquadrato $(f + g\lambda)^4$ hujusmodi pars

$$e^{\frac{-fx}{g}} (\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3),$$

et ita porro pro quotcunque factoribus aequalibus, uti ex §. 1120. colligere licet.

Corollarium 3.

1128. Si factores occurrant imaginarii, bini conjuncti exhibent factorem trinomium realem, cujus forma ita repraesentatur

$$ff + 2fg\lambda \cos.\zeta + gg\lambda\lambda,$$

unde deducitur

$$\lambda = -\tfrac{f}{g}(\cos.\zeta \pm \sqrt{-1}\,.\sin.\zeta),$$

quo cum §. 1121. collato, fit $\mu = \frac{-f\cos.\zeta}{g}$ et $\nu = \frac{f\sin.\zeta}{g}$: Ex quo pars integralis ex tali factore oriunda erit

$$e^{\frac{-fx\cos.\zeta}{g}}\left(\mathfrak{A}\cos.\frac{fx\sin.\zeta}{g} + \mathfrak{B}\sin.\frac{fx\sin.\zeta}{g}\right).$$

Corollarium 4.

1129. Si hujusmodi formae quadratum inter factores occurrat

$$(ff + 2fg\lambda\cos.\zeta + gg\lambda\lambda)^2,$$

seu duo hujusmodi factores sint aequales, considerentur quasi infinite parum discrepantes, ut in altero loco $\frac{f}{g}$ sit $\frac{f}{g}(1+\omega)$, et ob

$$e^{\frac{-fx\cos.\zeta}{g}(1+\omega)} = e^{\frac{-fx\cos.\zeta}{g}}\left(1 - \frac{\omega fx}{g}\cos.\zeta\right),$$

$$\cos.\frac{fx\sin.\zeta}{g}(1+\omega) = \cos.\frac{fx\sin.\zeta}{g} - \frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}, \text{ et}$$

$$\sin.\frac{fx\sin.\zeta}{g}(1+\omega) = \sin.\frac{fx\sin.\zeta}{g} + \frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}:$$

ex hoc factore colligitur pars integralis

$$e^{\frac{-fx\cos.\zeta}{g}}\left\{\begin{array}{l}\mathfrak{A}'\cos.\frac{fx\sin.\zeta}{g} - \mathfrak{A}'\frac{\omega fx\cos.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g} - \mathfrak{A}'\frac{\omega fx\sin.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g}\\ + \mathfrak{B}'\sin.\frac{fx\sin.\zeta}{g} - \mathfrak{B}'\frac{\omega fx\cos.\zeta}{g}\sin.\frac{fx\sin.\zeta}{g} + \mathfrak{B}'\frac{\omega fx\sin.\zeta}{g}\cos.\frac{fx\sin.\zeta}{g}\end{array}\right\}$$

cui prior addi debet. Hunc in finem constantes ita contrahamus ponendo

$$\mathfrak{A} + \mathfrak{A}' = \mathfrak{E}, \quad \frac{-\mathfrak{A}'\omega f\cos.\zeta}{g} + \frac{\mathfrak{B}'\omega f\sin.\zeta}{g} = \mathfrak{G},$$

$$\mathfrak{B} + \mathfrak{B}' = \mathfrak{F}, \quad \frac{-\mathfrak{A}'\omega f\sin.\zeta}{g} - \frac{\mathfrak{B}'\omega f\cos.\zeta}{g} = \mathfrak{H},$$

unde illae constantes utique determinantur, eritque pars integralis respondens

$$e^{\frac{-fx\cos.\zeta}{g}}\left[(\mathfrak{E} + \mathfrak{G}x)\cos.\frac{fx\sin.\zeta}{g} + (\mathfrak{F} + \mathfrak{H}x)\sin.\frac{fx\sin.\zeta}{g}\right],$$

Scholion.

1130. En ergo universam methodum hujusmodi aequationum differentialium integralia inveniendi

$$A y + B \frac{\partial y}{\partial x} + C \frac{\partial \partial y}{\partial x^2} + D \frac{\partial^3 y}{\partial x^3} + \ldots + N \frac{\partial^n y}{\partial x^n} = 0$$

ita in compendium contractam. Procedatur ut iste laterculus indicat

loco	y	$\frac{\partial y}{\partial x}$	$\frac{\partial \partial y}{\partial x^2}$	$\frac{\partial^3 y}{\partial x^3}$	$\frac{\partial^4 y}{\partial x^4}$	$\ldots\ldots$	$\frac{\partial^n y}{\partial x^n}$
scribatur	1	z	z^2	z^3	z^4	$\ldots\ldots$	z^n

ut oriatur haec aequatio algebraica

$$A + B z + C z^2 + D z^3 + E z^4 + \ldots\ldots + N z^n = 0,$$

cujus singuli factores reales, sive simplices sive duplicati, notentur, atque insuper casus quibus duo pluresve sunt inter se aequales, probe observentur. Tum cujusmodi partes pro integrali quaesito ex singulis factoribus oriantur, ex sequente tabella intelligere licet:

Factores	Partes integralis
$f + g z$	$\mathfrak{A} e^{\frac{-f x}{g}}$
$(f + g z)^2$	$(\mathfrak{A} + \mathfrak{B} x) e^{\frac{-f x}{g}}$
$(f + g z)^3$	$(\mathfrak{A} + \mathfrak{B} x + \mathfrak{C} x^2) e^{\frac{-f x}{g}}$
$(f + g z)^4$	$(\mathfrak{A} + \mathfrak{B} x + \mathfrak{C} x^2 + \mathfrak{D} x^3) e^{\frac{-f x}{g}}$
$ff + 2 f g z \cos.\zeta + g g z z$	$e^{\frac{-f x \cos.\zeta}{g}} (\mathfrak{A} \cos.\frac{f x \sin.\zeta}{g} + \mathfrak{B} \sin.\frac{f x \sin.\zeta}{g})$
$(ff + 2 f g z \cos.\zeta + g g z z)^2$	$e^{\frac{-f x \cos.\zeta}{g}} \left\{ \begin{array}{l} (\mathfrak{A} + \mathfrak{B} x) \cos.\frac{f x \sin.\zeta}{g} \\ (\mathfrak{a} + \mathfrak{b} x) \sin.\frac{f x \sin.\zeta}{g} \end{array} \right\}$
$(ff + 2 f g z \cos.\zeta + g g z z)^3$	$e^{\frac{-f x \cos.\zeta}{g}} \left\{ \begin{array}{l} (\mathfrak{A} + \mathfrak{B} x + \mathfrak{C} x x) \cos.\frac{f x \sin.\zeta}{g} \\ (\mathfrak{a} + \mathfrak{b} x + \mathfrak{c} x x) \sin.\frac{f x \sin.\zeta}{g} \end{array} \right\}$

etc.

Pro singulis autem factoribus diversae litterae constantes scribi debent, ut integrale omnibus numeris completum obtineatur.

Exemplum 1.

1131. *Aequationis differentialis quarti gradus*

$$y - \frac{2 \partial y}{\partial x} + \frac{2 \partial \partial y}{\partial x^2} - \frac{2 \partial^3 y}{\partial x^3} + \frac{\partial^4 y}{\partial x^4} = 0$$

integrale completum assignare.

Hinc oritur aequatio algebraica

$$1 - 2z + 2zz - 2z^3 + z^4 = 0.$$

quae in hos factores resolvitur $(1 - z)^2 (1 + zz)$, quorum prior ob $f = 1$ et $g = -1$ praebet hanc partem integralis $(\mathfrak{A} + \mathfrak{B} x) e^x$ alter vero factor ob $f = 1$, $\cos. \zeta = 0$, $g = 1$, et $\sin. \zeta = 1$ dat $\mathfrak{A} \cos. x + \mathfrak{B} \sin. x$. Quare integrale completum, quod quaeritur, erit

$$y = (\mathfrak{A} + \mathfrak{B} x) e^x + \mathfrak{C} \cos. x + \mathfrak{D} \sin. x$$

continens quatuor constantes arbitrarias. Quod si velimus, ut posito $x = 0$ fiat $y = 0$, fieri oportet $\mathfrak{A} + \mathfrak{C} = 0$, si etiam $\frac{\partial y}{\partial x}$ eodem casu evanescere debeat, ob

$$\frac{\partial y}{\partial x} = (\mathfrak{A} + \mathfrak{B} + \mathfrak{B} x) e^x - \mathfrak{C} \sin. x + \mathfrak{D} \cos. x$$

fieri debet $\mathfrak{A} + \mathfrak{B} + \mathfrak{D} = 0$. Si praeterea $\frac{\partial \partial y}{\partial x^2}$ evanescere debeat ob

$$\frac{\partial \partial y}{\partial x^2} = (\mathfrak{A} + 2 \mathfrak{B} + \mathfrak{B} x) e^x - \mathfrak{C} \cos. x - \mathfrak{D} \sin. x$$

fieri debet $\mathfrak{A} + 2 \mathfrak{B} - \mathfrak{C} = 0$. Quare his tribus conditionibus satisfaciemus sumendo $\mathfrak{C} = -\mathfrak{A}$, $\mathfrak{B} = -\mathfrak{A}$ et $\mathfrak{D} = 0$, ita ut sit integrale

$$y = \mathfrak{A} (1 - x) e^x - \mathfrak{A} \cos. x.$$

Exemplum 2.

1132. *Aequationem differentialem quarti ordinis*

$$Ay + C\frac{\partial\partial y}{\partial x^2} + E\frac{\partial^4 y}{\partial x^4} = 0,$$

sumto elemento ∂x constante, integrare.

Aequatio algebraica ad integrationem perducens est

$$A + Czz + Ez^4 = 0,$$

quae semper duos factores duplicatos reales habet, quorum forma duplex esse potest

$$\text{vel } (aa + 2maz + nzz)(aa - 2maz + nzz)$$
$$\text{vel } (aa + mzz)(aa + nzz).$$

Ex priori est

$$A = a^4,\ C = 2naa - 4mmaa,\ E = nn,$$

ex posteriore vero

$$A = a^4,\ C = (m+n)aa,\ E = mn:$$

semper autem terminum primum A biquadrato a^4 repraesentare licet, et prior resolutio locum habet, si E sit numerus positivus, et $2naa - C$ seu $2\sqrt{AE} - C$ quoque positivus, ideoque $4AE > CC$. Posterior vero si $CC > 4AE$. Tum igitur videndum est, ad quamnam classem singuli factores pertineant, unde sequentes casus occurrent.

I. Si omnes quatuor factores simplices sunt reales, erit

$$A + Czz + Ez^4 = (a+z)(a-z)(b+z)b-z),$$

et haec habebitur aequatio

$$aabby - (aa+bb)\frac{\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0,$$

cujus integrale completum est

$$y = \mathfrak{A}e^{ax} + \mathfrak{B}e^{-ax} + \mathfrak{C}e^{bx} + \mathfrak{D}e^{-bx}.$$

Ac si sit $b = a$, hujus aequationis

$$a^4 y - \frac{2aa\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0,$$

integrale completum erit

$$y = (\mathfrak{A} + \mathfrak{B} x) e^{ax} + (\mathfrak{C} + \mathfrak{D} x) e^{-ax}.$$

II. Si duo factores simplices sint reales, duo vero imaginar erit

$$A + Czz + Ez^4 = (a + z)(a - z)(bb + zz),$$

et haec habebitur aequatio

$$aabby + (aa - bb)\frac{\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0,$$

cujus integrale completum est

$$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{-ax} + \mathfrak{C} \cos. bx + \mathfrak{D} \sin. bx.$$

III. Si omnes factores simplices sint imaginarii, duo casu sunt evolvendi:

1) si $A + Czz + Ez^4 = (aa + zz)(bb + zz)$,

unde hujus aequationis

$$aabby + (aa + bb)\frac{\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0$$

integrale completum erit

$$y = \mathfrak{A} \cos. ax + \mathfrak{B} \sin. ax + \mathfrak{C} \cos. bx + \mathfrak{D} \sin. bx:$$

2) si $A + Czz + Ez^4 = (aa + 2az\cos.\zeta + zz)(aa - 2az\cos.\zeta + zz$

unde hujus aequationis

$$a^4 y - \frac{2aa\partial\partial y}{\partial x^2}\cos. 2\zeta + \frac{\partial^4 y}{\partial x^4} = 0,$$

integrale completum est

$$y = e^{+ax\cos.\zeta}(\mathfrak{A}\cos. ax \sin.\zeta + \mathfrak{B}\sin. ax \sin.\zeta)$$
$$+ e^{-ax\cos.\zeta}(\mathfrak{C}\cos. ax \sin.\zeta + \mathfrak{D}\sin. ax \sin.\zeta).$$

At si sit priori casu $b = a$, seu posteriori $\cos.\zeta = 0$, huju aequationis

$$a^4 y + \frac{2aa\partial\partial y}{\partial x^2} + \frac{\partial^4 y}{\partial x^4} = 0$$

integrale completum est

$$y = (\mathfrak{A} + \mathfrak{B}x)\cos. ax + (\mathfrak{C} + \mathfrak{D}x)\sin. ax.$$

Scholion 1.

1133. Cum igitur aequationis

$$Ay + \frac{C\partial\partial y}{\partial x^2} + \frac{E\partial^4 y}{\partial x^4} = 0$$

integrale assignari possit, omnes aequationes quas inde derivare licet, integrari poterunt. At haec aequatio per $2\partial y$ multiplicata per integrationem ad differentialem tertii ordinis reducitur

$$Ayy + \frac{C\partial y^2}{\partial x^2} + \frac{2E\partial y\partial^3 y - E\partial\partial y^2}{\partial x^4} = \text{Const.}$$

In integrali autem ante invento constantes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ ita definire licet, ut haec Const. evanescat, ideoque hujus aequationis

$$Ayy + \frac{C\partial y^2}{\partial x^2} + \frac{2E\partial y\partial^3 y - E\partial\partial y^2}{\partial x^4} = 0$$

integrale completum erit in nostra potestate. Nunc ponatur $y = e^{\int v\partial x}$, ut sit $v = \frac{\partial y}{y\partial x}$, et ob $\frac{\partial y}{\partial x} = e^{\int v\partial x}v$,

$$\frac{\partial\partial y}{\partial x^2} = e^{\int x\partial x}\left(\frac{\partial v}{\partial x} + vv\right), \text{ atque}$$

$$\frac{\partial^3 y}{\partial x^3} = e^{\int v\partial x}\left(\frac{\partial\partial v}{\partial x^2} + \frac{3v\partial v}{\partial x} + v^3\right),$$

aequatio nostra hanc induit formam

$$A + Cvv + E\left(\frac{2v\partial\partial v}{\partial x^2} + \frac{4vv\partial v}{\partial x} + v^4 - \frac{\partial v^2}{\partial x^2}\right) = 0.$$

Sit porro $\partial x = \frac{\partial v}{s}$, ut sit

$$s = \frac{\partial v}{\partial x} = \frac{\partial\partial y}{y\partial x^2} - \frac{\partial y^2}{yy\partial x^2}, \text{ erit}$$

$$\frac{\partial\partial v}{\partial x} = \partial s \text{ et } \frac{\partial\partial v}{\partial x^2} = \frac{s\partial s}{\partial v}:$$

unde resultat haec aequatio differentialis primi gradus

$$A + Cxv + E\left(\frac{2vs\partial s}{\partial v} - ss + 4vvs + v^4\right) = 0,$$

cujus relatio inter v et s ita ex relatione inter x et y inventa definitur, ut sit

$$v = \frac{\partial y}{y\,\partial x} \text{ et } s = \frac{y\,\partial\partial y - \partial y^2}{yy\,\partial x^2}.$$

Scholion 2.

1134. Retenta autem illa constante per integrationem ingressa, ut habeatur

$$A\,yy + \frac{C\,\partial y^2}{\partial x^2} + \frac{2E\,\partial y\,\partial^3 y - E\,\partial\partial y^2}{\partial x^4} = G,$$

in integrali completo, quo y per x exprimitur, constantes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, quantitati huic G conformiter determinari poterunt. Nunc igitur ponatur $\partial x = \frac{\partial y}{u}$, ut sit $\frac{\partial y}{\partial x} = u$, erit

$$\frac{\partial\partial y}{\partial x^2} = \frac{u\,\partial u}{\partial y}, \text{ et } \frac{\partial^3 y}{\partial x^2} = \partial\,.\,\frac{u\,\partial u}{\partial y},$$

ideoque $\frac{\partial^3 y}{\partial x^3} = \frac{u}{\partial y}\partial\,.\,\frac{u\,\partial u}{\partial y}$. Unde obtinetur haec aequatio differentialis secundi gradus

$$A\,yy + C\,uu + E\left(\frac{2\,uu}{\partial y}\partial\,.\,\frac{u\,\partial u}{\partial y} - \frac{uu\,\partial u^2}{\partial y^2}\right) = G,$$

ubi consideratio elementi pro constante assumti est exuta. Nihil ergo impedit, quo minus sumamus ∂y pro constante, fietque

$$A\,yy + C\,uu + E\left(\frac{2\,u^3\,\partial\partial u}{\partial y^2} + \frac{uu\,\partial u^2}{\partial y^2}\right) = G,$$

quae ergo aequatio etiam integrari potest.

Vel si ponamus $yy = p$ et $uu = q$, sumto elemento ∂p constante, prodibit haec aequatio

$$A\,p + C\,q + E\left(\frac{4\,pq\,\partial\partial q + 2\,q\,\partial p\,\partial q - p\,\partial q^2}{\partial p^2}\right) = G.$$

Vel si in illa aequatione ponatur $u = r^{\frac{2}{3}}$, erit

$$A\,yy + C\,r^{\frac{4}{3}} + \tfrac{4}{3}\,E\,r^{\frac{5}{3}}\,.\,\frac{\partial\partial r}{\partial y^2} = G.$$

Quarum formarum integratio sine hoc subsidio maxime ardua videtur.

Problema 146.

1135. Proposita aequatione differentiali ordinis cujuscunque $a^n y \pm \frac{\partial^n y}{\partial x^n} = 0$, ubi elementum ∂x constans est assumtum, ejus integrale completum investigare.

Solutio.

Aequatio algebraica solutioni inserviens est $a^n \pm z^n = 0$, pro cujus resolutione duos casus considerari convenit, prout signum vel superius vel inferius valeat.

I. Valeat superius, ut haec proposita sit aequatio

$$a^n y + \frac{\partial^n y}{\partial x^n} = 0,$$

et formulae $a^n + z^n$ factores reales sunt

$$aa - 2az\cos.\tfrac{\pi}{n} + zz,\quad aa - 2az\cos.\tfrac{3\pi}{n} + zz,$$
$$aa - 2az\cos.\tfrac{5\pi}{n} + zz,\ \text{etc.}$$

quorum ultimus est vel $aa - 2az\cos.\frac{n\pi}{n} + zz$ vel

$$aa - 2az\cos.\tfrac{(n-1)\pi}{n} + zz,$$

prout vel n vel $n-1$ fuerit numerus impar, atque illo quidem casu loco factoris quadrati $aa + 2az + zz$ ejus radix $a + z$ sumi debet.

Hinc istius aequationis integrale completum est

$$y = + e^{ax\cos.\frac{\pi}{n}}\left(\mathfrak{A}\cos.ax\sin.\tfrac{\pi}{n} + \mathfrak{B}\sin.ax\sin.\tfrac{\pi}{n}\right)$$
$$+ e^{ax\cos.\frac{3\pi}{n}}\left(\mathfrak{C}\cos.ax\sin.\tfrac{3\pi}{n} + \mathfrak{D}\sin.ax\sin.\tfrac{3\pi}{n}\right)$$
$$+ e^{ax\cos.\frac{5\pi}{n}}\left(\mathfrak{E}\cos.ax\sin.\tfrac{5\pi}{n} + \mathfrak{F}\sin.ax\sin.\tfrac{5\pi}{n}\right)$$
etc.

cujus expressionis, si n sit numerus impar, ultima pars fit $\mathfrak{N}\, e^{-ax}$. Quod integrale etiam ita potest exhiberi

$$y = \mathfrak{A}\, e^{ax\cos.\frac{\pi}{n}} \cos.(ax\sin.\tfrac{\pi}{n} + \mathfrak{a}) + \mathfrak{B}\, e^{ax\cos.\frac{3\pi}{n}} \cos.(ax\sin.\tfrac{3\pi}{n} + \mathfrak{b})$$
$$+ \mathfrak{C}\, e^{ax\cos.\frac{5\pi}{n}} \cos.(ax\sin.\tfrac{5\pi}{n} + \mathfrak{c}) + \mathfrak{D}\, e^{ax\cos.\frac{7\pi}{n}} \cos.(ax\sin.\tfrac{7\pi}{n} + \mathfrak{d})$$
etc.

quae forma eousque continuari debet, quoad similes termini recurrant.

II. Si valeat signum inferius, propositaque sit haec aequatio

$$a^n y - \frac{\partial^n y}{\partial x^n} = 0,$$

formulae $a^n - z^n$ factores reales sunt

$$a - z,\ aa - 2z\cos.\tfrac{2\pi}{n} + zz,\ aa - 2az\cos.\tfrac{4\pi}{n} + zz$$
$$aa - 2az\cos.\tfrac{6\pi}{n} + zz, \text{ etc.}$$

quorum si n numerus par, ultimus est $a + z$, sin autem impar

$$aa - 2az\cos.\frac{(n-1)\pi}{n} + zz.$$

Quare aequationis hujus integrale completum est

$$y = \mathfrak{A}\, e^{ax} + e^{ax\cos.\frac{2\pi}{n}} (\mathfrak{B}\cos.ax\sin.\tfrac{2\pi}{n} + \mathfrak{C}\sin.ax\sin.\tfrac{2\pi}{n})$$
$$+ e^{ax\cos.\frac{4\pi}{n}} (\mathfrak{D}\cos.ax\sin.\tfrac{4\pi}{n} + \mathfrak{E}\sin.ax\sin.\tfrac{4\pi}{n})$$
$$+ e^{ax\cos.\frac{6\pi}{n}} (\mathfrak{F}\cos.ax\sin.\tfrac{6\pi}{n} + \mathfrak{G}\sin.ax\sin.\tfrac{6\pi}{n});$$
etc.

quod integrale etiam ita exprimi potest

$$y = \mathfrak{A}\, e^{ax} + \mathfrak{B}\, e^{ax\cos.\frac{2\pi}{n}} \cos.(ax\sin.\tfrac{2\pi}{n} + \mathfrak{b})$$
$$+ \mathfrak{C}\, e^{ax\cos.\frac{4\pi}{n}} \cos.(ax\sin.\tfrac{4\pi}{n} + \mathfrak{c}) + \mathfrak{D}\, e^{ax\cos.\frac{6\pi}{n}} \cos.(ax\sin.\tfrac{6\pi}{n} + \mathfrak{d})$$
etc.

quae forma eousque est continuanda, quamdiu termini a prioribus diversi prodeunt.

Scholion 1.

1136. Pro variis ergo exponentis n valoribus integralia sequenti modo se habebunt, ac primo quidem pro aequatione $a^n y + \frac{\partial^n y}{\partial x^n} = 0$.

I. Aequationis $a y + \frac{\partial y}{\partial x} = 0$ integrale est
$y = \mathfrak{A}\, e^{-ax}$.

II. Aequationis $a^2 y + \frac{\partial \partial y}{\partial x^2} = 0$ integrale est
$y = \mathfrak{A} \cos.(a x + \mathfrak{a})$.

III. Aequationis $a^3 y + \frac{\partial^3 y}{\partial x^3} = 0$ integrale est

$$y = \mathfrak{A}\, e^{\frac{1}{2} a x} \cos.\left(\frac{a x \sqrt{3}}{2} + \mathfrak{a}\right) + \mathfrak{B}\, e^{-ax}.$$

IV. Aequationis $a^4 y + \frac{\partial^4 y}{\partial x^4} = 0$ integrale est

$$y = \mathfrak{A}\, e^{\frac{a x}{\sqrt{2}}} \cos.\left(\frac{a x}{\sqrt{2}} + \mathfrak{a}\right) + \mathfrak{B}\, e^{\frac{-a x}{\sqrt{2}}} \cos.\left(\frac{a x}{\sqrt{2}} + \mathfrak{b}\right).$$

V. Aequationis $a^5 y + \frac{\partial^5 y}{\partial x^5} = 0$ integrale est

$$y = \mathfrak{A}\, e^{a x \cos. 36^0} \cos.(a x \sin. 36^0 + \mathfrak{a})$$
$$+ \mathfrak{B}\, e^{-a x \cos. 72^0} \cos.(a x \sin. 72^0 + \mathfrak{b}) + \mathfrak{C}\, e^{-ax}.$$

VI. Aequationis $a^6 y + \frac{\partial^6 y}{\partial x^6} = 0$ integrale est

$$y = \mathfrak{A}\, e^{\frac{a x \sqrt{3}}{2}} \cos.\left(\tfrac{1}{2} a x + \mathfrak{a}\right)$$
$$+ \mathfrak{B} \cos.(a x + \mathfrak{b}) + \mathfrak{C}\, e^{\frac{-a x \sqrt{3}}{2}} \cos.\left(\tfrac{1}{2} a x + \mathfrak{c}\right),$$

etc.

Simili autem modo pro altera forma

$$a^n y - \frac{\partial^n y}{\partial x^n} = 0,$$

integrationes ad valores simpliciores exponentis n accommodatae ita se habebunt

I. Aequationis $a y - \frac{\partial y}{\partial x} = 0$ integrale est
$y = \mathfrak{A} e^{ax}$.

II. Aequationis $a^2 y - \frac{\partial \partial y}{\partial x^2} = 0$ integrale est
$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{-ax}$.

III. Aequationis $a^3 y - \frac{\partial^3 y}{\partial x^3} = 0$ integrale est

$$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{-\frac{1}{2} a x} \cos. \left(\frac{a x \sqrt{3}}{2} + \mathfrak{b}\right).$$

IV. Aequationis $a^4 y - \frac{\partial^4 y}{\partial x^4} = 0$ integrale est
$y = \mathfrak{A} e^{ax} + \mathfrak{B} \cos. (a x + \mathfrak{b}) + \mathfrak{C} e^{-ax}$.

V. Aequationis $a^5 y - \frac{\partial^5 y}{\partial x^5} = 0$ integrale est
$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{a x \cos. 72^0} \cos. (a x \sin. 72^0 + \mathfrak{b})$
$+ \mathfrak{C} e^{- a x \cos. 36^0} \cos. (a x \sin. 36^0 + \mathfrak{c})$.

VI. Aequationis $a^6 y - \frac{\partial^6 y}{\partial x^6} = 0$ integrale est

$$y = \mathfrak{A} e^{ax} + \mathfrak{B} e^{\frac{1}{2} a x} \cos. \left(\frac{a x \sqrt{3}}{2} + \mathfrak{b}\right)$$

$$+ \mathfrak{C} e^{-\frac{1}{2} a x} \cos. \left(\frac{a x \sqrt{3}}{2} + \mathfrak{c}\right) + \mathfrak{D} e^{-ax},$$

sicque quousque libuerit progredi licet

Scholion 2.

1137. Quamvis methodus, qua hic sum usus, expedite integralia aequationum in proposita forma contentarum suppeditet, a principiis tamen integrationis omnino abhorret. Cum enim aequatio differentialis est altioris gradus, leges integrationis postulant, ut toties seorsim integretur, antequam ad relationem finitam inter binas variabiles perveniatur, et dum singulae integrationes constantem arbitrariam recipiunt, hoc demum modo integrale completum obtinetur. Hactenus autem una quasi operatione integrale postremum eruimus, cum omnibus constantibus, quibus id completum redditur; revera scilicet sola conjectura utentes plura integralia particularia sumus adepti, atque natura aequationis commode permisit, ut ex iis integrale completum formare liceret. Verum si leges integrandi stricte observare velimus, proposita verbi gratia aequatione differentiali quarti gradus, quadruplici integratione opus erit, quarum prima ea reducatur ad aequationem differentialem tertii gradus, tum vero ista per novam integrationem ad aequationem differentialem secundi gradus, quae denuo integrata ad gradum primum perducatur, haecque tandem iterum integrata relationem quaesitam inter binas variabiles patefaciat. Atque hoc modo etiam aequationum hic tractatarum formam resolvere licet, ut per continuas integrationes ad gradus simpliciores redigatur, quibus tandem eadem integralia, quae hic elicuimus, inveniantur. Cum autem haec methodus latius pateat, quam ad formas hic consideratas, ejusque ope haec aequatio generalior integrari queat

$$X = Ay + \frac{B\partial y}{\partial x} + \frac{C\partial\partial y}{\partial x^2} + \frac{D\partial^3 y}{\partial x^3} + \text{etc.}$$

denotante X functionem quamcunque ipsius x, cui resolvendae praecedens methodus minime sufficit, novam methodum statim ad hanc formam generaliorem accommodabo.