CALCULI INTEGRALIS

LIBER PRIOR.

PARS SECUNDA,

SEU

METHODUS INVENIENDI FUNCTIONES UNIUS VARIABILIS EX DATA RELATIONE DIFFERENTIALIUM SECUNDI ALTIORISVE GRADUS.

SECTIO POSTERIOR,

DE

RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM TERTII ALTIORUMQUE GRADUUM QUAE DUAS TANTUM VARIABILES INVOLVUNT.

CAPUT I.

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM TERTII ALTIORISVE GRADUS SIMPLICIUM.

Problema 140.

1100.

Sumto elemento ∂x constante, invenire integrale completum harum formularum $\partial^3 y = 0$, $\partial^4 y = 0$, $\partial^5 y = 0$ etc. atque in genere hujus $\partial^n y = 0$.

Solutio.

Cum ∂x sit constans, aequatio $\partial^3 y = 0$ per integrationem dat $\partial \partial y = \alpha \partial x^2$, hincque porro integrando $\partial y = \alpha x \partial x + \beta \partial x$, et tandem $y = \frac{1}{2} \alpha x^2 + \beta x + \gamma$.

Simili modo ex aequatione $\partial^4 y = 0$, per quadruplicem integrationem reperitur

1) $\partial^3 y = \alpha \partial x^3$,

2) $\partial \partial y = \alpha x \partial x^2 + \beta \partial x^2$,

3) $\partial y = \frac{1}{2} \alpha x x \partial x + \beta x \partial x + \gamma \partial x$,

et tandem

4) $y = \frac{1}{6} \alpha x^3 + \frac{1}{2} \beta x^2 + \gamma x + \delta$.

Ex aequatione autem $\partial^5 y = 0$, integratio quinquies repetita dat

$$y = \tfrac{1}{24} \alpha x^4 + \tfrac{1}{6} \beta x^3 + \tfrac{1}{2} \gamma x^2 + \delta x + \varepsilon.$$

$$\int t\,\partial p = qs - \tfrac{1}{2}rr, \text{ hinc } \int p\,\partial t = pt - qs + \tfrac{1}{2}rr,$$
$$\int u\,\partial q = rt - \tfrac{1}{2}ss, \text{ hinc } \int q\,\partial u = qu - rt + \tfrac{1}{2}ss.$$

Hinc porro definitur $\int y\,\partial u = yu - \int u\,\partial y$, at $\frac{\partial y}{p} = \frac{\partial t}{u}$, unde

$$\int y\,\partial u = yu - \int p\,\partial t = yu - pt + qs - \tfrac{1}{2}rr.$$

Quare si differentialia iterum introducamus, obtinebimus sequent formulas integrales

$$\int y\,\partial y = \tfrac{1}{2}xy$$
$$\int y\,\partial^3 y = y\,\partial\partial y - \tfrac{1}{2}\partial y^2$$
$$\int y\,\partial^5 y = y\,\partial^4 y - \partial y\,\partial^3 y + \tfrac{1}{2}\partial\partial y^2$$
$$\int y\,\partial^7 y = y\,\partial^6 y - \partial y\,\partial^5 y + \partial\partial y\,\partial^4 y - \tfrac{1}{2}\partial^3 y^2$$
etc.

ita ut formula $\int y\,\partial^n y$ sit integrabilis, quoties n est numerus impa

Scholion 2.

1105. In aequationibus differentialibus secundi gradus forma simpliciores ita constituimus, ut q aequetur functioni vel ipsius tantum, vel ipsius y, vel ipsius p, quas litteras majusculas pr functionibus minuscularum scribendo ita repraesentare licet, ut sit

vel $q = X$, vel $q = Y$, vel $q = P$.

Hinc simili modo pro aequationibus differentialibus tertii gradus fo mas simpliciores constituere possumus

$$r = X;\ r = Y,\ r = P,\ r = Q,$$

ita ut tantum binas quantitates variabiles involvant.

Pro quarto autem gradu essent formae simpliciores

$$s = X,\ s = Y,\ s = P,\ s = Q,\ s = R,$$

et pro quinto

$$t = X,\ t = Y,\ t = P,\ t = Q,\ t = R,\ t = S,$$

atque ita porro pro superioribus.

Verum hae formae non omnes aeque integrationem admittunt, dum aliae ne semel quidem, aliae semel tantum, aliae per omnes integrationes usque ad relationem inter x et y perduci possunt, cujusmodi sunt primae quaeque in quovis gradu. Semper autem id est propositum, ut relatio inter binas variabiles principales x et y eliciatur.

Problema 141.

1106. Posito $\partial y = p \partial x$, $\partial p = q \partial x$, $\partial q = r \partial x$, $\partial r = s \partial x$, $\partial s = t \partial x$, etc. pro quovis differentialium gradu, si litterarum p, q, r, s, t, etc. quaepiam aequetur functioni ipsius x, quae sit X, invenire relationem inter x et y.

Solutio.

Si primo sit $p = X$, per ∂x multiplicando erit $p \partial x = \partial y = X \partial x$ hincque $y = \int X \partial x$, qui est casus formularum differentialium primi gradus simplicium.

Sit secundo $q = X$, erit $q \partial x = \partial p = X \partial x$, hinc $p = \int X \partial x$, et $p \partial x = \partial y = \partial x \int X \partial x$, ergo $y = \int \partial x \int X \partial x$, seu per simplicia integralia $y = x \int X \partial x - \int X x \partial x$.

Sit tertio $r = X$, ob $\partial q = r \partial x$ erit $q = \int X \partial x$, hincque

$$p = \int q \partial x = \int \partial x \int X \partial x = x \int X \partial x - \int X x \partial x,$$

ac tandem

$$y = \int p \partial x = \int \partial x \int \partial x \int X \partial x = \tfrac{1}{2} x x \int X \partial x - x \int X x \partial x + \tfrac{1}{2} \int X x x \partial x.$$

Sit quarto $s = X$, ac reperitur $y = \int \partial x \int \partial x \int \partial x \int X \partial x$, quae expressio evolvitur in hanc

$$y = \tfrac{1}{6} x^3 \int X \partial x - \tfrac{1}{2} x x \int X x \partial x + \tfrac{1}{2} x \int X x x \partial x - \tfrac{1}{6} \int X x^3 \partial x.$$

Sit quinto $t = X$, erit $y = \int \partial x \int \partial x \int \partial x \int \partial x \int X \partial x$, seu

$$y = \tfrac{1}{24} x^4 \int X \partial x - \tfrac{1}{6} x^3 \int X x \partial x + \tfrac{1}{4} x^2 \int X x x \partial x - \tfrac{1}{6} x \int X x^3 \partial x + \tfrac{1}{24} \int X x^4 \partial x;$$

unde lex ulterius progrediendi est manifesta.

Corollarium 1.

1107. Tot ergo habentur formulae integrales, quoti gradus aequatio fuerit differentialis, et quia quaelibet constantem arbitrariam assumit, totidem constantes in integrale ingrediuntur, quibus id completum redditur; quod idem ex priori forma, ubi totidem signa integralia implicantur, intelligitur.

Corollarium 2.

1108. Sumto elemento ∂x constante, sequentium formularum more consueto expressarum integralia completa ita se habebunt

I. Si $\partial y = X \partial x$ est $y = \int X \partial x$,

II. Si $\partial \partial y = X \partial x^2$ est $1 y = x \int X \partial x - \int X x \partial x$,

III. Si $\partial^3 y = X \partial x^3$ est
$$2 y = x^2 \int X \partial x - 2 x \int X x \partial x + \int X x^2 \partial x,$$

IV. Si $\partial^4 y = X \partial x^4$ est
$$6 y = x^3 \int X \partial x - 3 x^2 \int X x \partial x + 3 x \int X x^2 \partial x - \int X x^3 \partial x,$$

V. Si $\partial^5 y = X \partial x^5$ est
$$24 y = x^4 \int X \partial x - 4 x^3 \int X x \partial x + 6 x^2 \int X x^2 \partial x - 4 x \int X x^3 \partial x + \int X x^4 \partial x,$$

etc.

Scholion.

1109. Formulas autem, quas supra secundo loco constituimus, functionem Y complectentes, post secundum gradum integrare non licet. Ex tertio enim ordine formula $r = Y$ etsi novimus esse

$$r = \frac{p\,\partial q}{\partial y} = \frac{q\,\partial q}{\partial p} = \frac{\partial q}{\partial x},$$

nullo modo integrari potest; neque etiam hinc q per y determinari potest. Nam sumta forma $p\,\partial q = Y\,\partial y$, existente $p\,\partial p = q\,\partial y$ ob $p = \frac{Y\,\partial y}{\partial q}$, erit

$$\partial p = \frac{\partial y\,\partial Y}{\partial q} + Y\,\partial \,.\, \frac{\partial y}{\partial q},$$

hincque p elidendo

$$\frac{Y\,\partial y^2\,\partial Y}{\partial q^2} + \frac{YY\,\partial y}{\partial q}\,\partial \,.\, \frac{\partial y}{\partial q} = q\,\partial y,$$

quae quidem aequatio est secundi gradus, sed neutiquam in genere resolutionem admittit. Ex quarto genere formula $s = Y$, ob $\int s\,\partial y = p\,r - \frac{1}{2}\,q\,q = \int Y\,\partial y$, semel integrari potest, sed hinc ulterius progredi non licet. Quas autem supra pro quovis gradu formulas simpliciores ultimo loco constituimus itemque penultimo, eae tractabiles deprehenduntur: earum ergo integrationem investigemus.

Problema 142.

1110. Posito ut hactenus $\partial y = p\,\partial x$, $\partial p = q\,\partial x$, $\partial q = r\,\partial x$, etc. litterae Y, P, Q, R, etc. denotent functiones cujusque litterae minusculae cognominis, investigare integralia harum formularum simplicium

$$p = Y,\; q = P,\; r = Q,\; s = R,\; t = S, \text{ etc.}$$

Solutio.

Aequatio prima ob $p = \frac{\partial y}{\partial x}$ statim dat $\partial x = \frac{\partial y}{Y}$, ideoque $x = \int \frac{\partial y}{Y}$.

Aequatio secunda $q = P$ ob $q = \frac{\partial p}{\partial x}$ praebet $\partial x = \frac{\partial p}{P}$, et $\partial y = \frac{p\,\partial p}{P}$, unde cum P sit functio ipsius p, utraque variabilis x

et y per p determinatur hoc modo

$$x = \int \frac{\partial p}{P} \text{ et } y = \int \frac{p\,\partial p}{P}.$$

Aequatio tertia $r = Q$, ob $r = \frac{\partial q}{\partial x}$, dat $\partial x = \frac{\partial q}{Q}$, hin $q\,\partial x = \partial p = \frac{q\,\partial q}{Q}$, ita ut sit $x = \int \frac{\partial q}{Q}$ et $p = \int \frac{q\,\partial q}{Q}$ unde coll gimus $p\,\partial x = \partial y = \frac{\partial q}{Q} \int \frac{q\,\partial q}{Q}$, ergo $y = \int \frac{\partial q}{Q} \int \frac{q\,\partial q}{Q}$. Quare pe eandem variabilem q utraque variabilis x et y ita determinatu ut sit

$$x = \int \frac{\partial q}{Q} \text{ et } y = \int \frac{\partial q}{Q} \int \frac{q\,\partial q}{Q}.$$

Aequatio quarta $s = R = \frac{\partial r}{\partial x}$ dat $\partial x = \frac{\partial r}{R}$, unde collig mus $r\,\partial x = \partial q = \frac{r\,\partial r}{R}$, ita ut sit $q = \int \frac{r\,\partial r}{R}$. Porro $q\,\partial x = \partial$ dat $\partial p = \frac{\partial r}{R} \int \frac{r\,\partial r}{R}$, hincque $p = \int \frac{\partial r}{R} \int \frac{r\,\partial r}{R}$, et quia $p\,\partial x = \partial$ habebimus $\partial y = \frac{\partial r}{R} \int \frac{\partial r}{R} \int \frac{r\,\partial r}{R}$; quare per r ambae variabiles pri cipales x et y ita definiuntur

$$x = \int \frac{\partial r}{R} \text{ et } y = \int \frac{\partial r}{R} \int \frac{\partial r}{R} \int \frac{r\,\partial r}{R}.$$

Aequatio quinta $t = S$ simili modo tractata praebet

$$x = \int \frac{\partial s}{S} \text{ et } y = \int \frac{\partial s}{S} \int \frac{\partial s}{S} \int \frac{\partial s}{S} \int \frac{s\,\partial s}{S}$$

sicque facile ulterius progredi licet.

Corollarium 1.

1111. Ex formula secunda intelligitur, si x aequetur functio ipsius p, ut sit $x = P$, fore $y = \int p\,\partial P = Pp - \int P\,\partial p$, qu quidem per se est manifestum.

Corollarium 2.

1112. Sin autem sit $x = Q$, ob $\partial x = \partial Q$ erit

$$q\,\partial x = \partial p = q\,\partial Q, \text{ et } p = \int q\,\partial Q,$$

hincque

$$y = \int \partial Q \int q\,\partial Q, \text{ seu } y = Q \int q\,\partial Q - \int q\,Q\,\partial Q.$$

Vel etiam cum sit

$$y=\int\partial Q(qQ-\int Q\partial q),\ \text{erit}\ y=\tfrac{1}{2}qQQ+\tfrac{1}{2}\int QQ\partial q-Q\int Q\partial q,$$

sive hoc modo

$$2y=QQq-2Q\int Q\partial q-\int QQ\partial q.$$

Corollarium 3.

1113. Simili modo si $x=R$, erit

$$q=\int r\partial x=\int r\partial R\ \text{et}\ p=\int q\partial x=\int\partial R\int r\partial R,$$

atque

$$y=\int p\partial x=\int\partial R\int\partial R\int r\partial R,\ \text{seu}$$
$$2y=\int\partial R(RRr-2R\int R\partial r+\int RR\partial r),$$

per praecedens Corollarium. Ergo per similes reductiones

$$6y=R^3r-3R^2\int R\partial r+3R\int RR\partial r-\int R^3\partial r.$$

Corollarium 4.

1114. At si fuerit $x=S$, reperietur per similes reductiones

$$24y=S^4s-4S^3\int S\partial s+6S^2\int SS\partial s-4S\int S^3\partial s+\int S^4\partial s:$$

hinc ergo per differentiationes retrogrediendo

$$24p\partial S=4S^3s\partial S-12SS\partial S\int S\partial s+12S\partial S\int SS\partial s-4\partial S\int S^3\partial s,$$

seu

$$6p=S^3s-3SS\int S\partial s+3S\int SS\partial s-\int S^3\partial s,\ \text{et}$$
$$2q=S^2s-2S\int S\partial s+\int SS\partial s,$$

tum $r=Ss-\int S\partial s$ et $s=s$.

Problema 143.

1115. Iisdem manentibus denominationibus, quibus hactenus sumus usi, investigare integralia harum formularum simpliciorum

$$q=Y,\ r=P,\ s=Q,\ t=R,\ \text{etc.}$$

Solutio.

Pro formula prima $q = Y$, cum sit $q = \frac{p\,\partial p}{\partial y}$; erit $p\,\partial p = Y\,\partial y$ et $pp = 2\int Y\,\partial y$, hinc $p = \sqrt{2\int Y\,\partial y} = \frac{\partial y}{\partial x}$, unde colligitur $x = \int \frac{\partial y}{\sqrt{2\int Y\,\partial y}}$, sicque x per y determinatur.

Pro formula secunda $r = P$ ob $r = \frac{q\,\partial q}{\partial p}$, habebimus $q\,\partial q = P\,\partial p$ et $q = \sqrt{2\int P\,\partial p} = \frac{p\,\partial p}{\partial y} = \frac{\partial p}{\partial x}$, unde concludimus

$$x = \int \frac{\partial p}{\sqrt{2\int P\,\partial p}}, \text{ et } y = \int \frac{p\,\partial p}{\sqrt{2\int P\,\partial p}}.$$

Pro formula tertia $s = Q$ ob $s = \frac{r\,\partial r}{\partial q}$, fiet $r = \sqrt{2\int Q\,\partial q} = \frac{q\,\partial q}{\partial p}$, unde sequitur $p = \int \frac{q\,\partial q}{\sqrt{2\int Q\,\partial q}}$. Cum vero sit $r = \frac{\partial q}{\partial x}$, erit $\partial x = \frac{\partial q}{\sqrt{2\int Q\,\partial q}}$, et ob $p\,\partial x = \partial y$ habebimus

$$x = \int \frac{\partial q}{\sqrt{2\int Q\,\partial q}}, \text{ et } y = \int \frac{\partial q}{\sqrt{2\int Q\,\partial q}} \int \frac{q\,\partial q}{\sqrt{2\int Q\,\partial q}}.$$

Pro formula quarta $t = R$ ob $t = \frac{s\,\partial s}{\partial r}$, nanciscimur $s = \sqrt{2\int R\,\partial r}$. At est $s = \frac{\partial r}{\partial x}$, unde fit $\partial x = \frac{\partial r}{\sqrt{2\int R\,\partial r}}$. Est vero etiam $s = \frac{r\,\partial r}{\partial q}$, ideoque $q = \int \frac{r\,\partial r}{\sqrt{2\int R\,\partial r}}$, sed quoniam $p = \int q\,\partial x$, fit $p = \int \frac{\partial r}{\sqrt{2\int R\,\partial r}} \int \frac{r\,\partial r}{\sqrt{2\int R\,\partial r}}$, ex quo prodit $y = \int p\,\partial x$. Quocirca x et y ita per r determinantur, ut sit

$$x = \int \frac{\partial r}{\sqrt{2\int R\,\partial r}}, \text{ et } y = \int \frac{\partial r}{\sqrt{2\int R\,\partial r}} \int \frac{\partial r}{\sqrt{2\int R\,\partial r}} \int \frac{r\,\partial r}{\sqrt{2\int R\,\partial r}}.$$

Pro formula quinta $u = S$, ob $u = \frac{t\,\partial t}{\partial s}$, adipiscimur $t = \sqrt{2\int S\,\partial s} = \frac{\partial s}{\partial x}$, ut sit $\partial x = \frac{\partial s}{\sqrt{2\int S\,\partial s}}$. Est vero etiam $t = \frac{s\,\partial s}{\partial r}$, ergo $r = \int \frac{s\,\partial s}{\sqrt{2\int S\,\partial s}}$. Tum $q = \int r\,\partial x$, $p = \int q\,\partial x$ et $y = \int p\,\partial x$, ex quibus conficitur

$$x = \int \frac{\partial s}{\sqrt{2\int S\,\partial s}} \text{ et } y = \int \frac{\partial s}{\sqrt{2\int S\,\partial s}} \int \frac{\partial s}{\sqrt{2\int S\,\partial s}} \int \frac{\partial s}{\sqrt{2\int S\,\partial s}} \int \frac{s\,\partial s}{\sqrt{2\int S\,\partial s}}.$$

Scholion.

1116. Hi sunt casus, quibus formulas illas simpliciores supra recensitas resolvere licet, neque methodus patet, qua reliquae tractari queant. Multo pauciores occurrunt casus tractabiles in formis magis compositis, ubi $\frac{\partial^n y}{\partial x^n}$ aequatur functioni binarum pluriumve quantitatum variabilium, ob quam penuriam parum admodum suppetit, quod in hac sectione exponere queamus. Aequationum autem, quae per methodos adhuc inventas tractari possunt, haec fere est forma generalis

$$A y + B . \frac{\partial y}{\partial x} + C . \frac{\partial \partial y}{\partial x^2} + D . \frac{\partial^3 y}{\partial x^3} + \text{etc.} = 0,$$

sumto elemento ∂x constante, quae etiam ponendo

$$\partial y = p \partial x, \ \partial p = q \partial x, \ \partial q = r \partial x, \text{ etc.}$$

ita repraesentari potest

$$A y + B p + C q + D r + E s + \text{etc.} = 0.$$

Deinde vero etiam aequationes hac forma latius patente contentae resolutionem admittunt

$$A y + B p + C q + D r + E s + \text{etc.} = X,$$

denotante X functionem quamcunque ipsius x. Porro quoque sequentes formae, quae quidem ad illas reduci possunt

$$A y + \frac{B p}{x} + \frac{C q}{x x} + \frac{D r}{x^3} + \frac{E s}{x^4} + \text{etc.} = 0 \text{ et}$$

$$A y + \frac{B p}{x} + \frac{C q}{x x} + \frac{D r}{x^3} + \frac{E s}{x^4} + \text{etc.} = X,$$

quarum resolutio adeo succedit, ad quantumvis gradum etiam differentialitas assurgat; in harum ergo aequationum evolutione tractatio nostra versabitur.