CAPUT IX.

DE TRANSFORMATIONE AEQUATIONUM DIFFERENTIO-DIFFERENTIALIUM.

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0.$$

Problema 125.

993.

Aequationem differentio - differentialem

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0,$$

in qua L, M, N sunt functiones quaecunque ipsius x, sumto elemento ∂x constante, ope substitutionis $y = e^{\int P\partial x} z$ in aliam formam transmutare.

Solutio.

Cum hinc sit $\frac{\partial y}{y} = P\partial x + \frac{\partial z}{z}$, erit differentiando

$$\frac{\partial\partial y}{y} - \frac{\partial y^2}{y^2} = \partial x\partial P + \frac{\partial\partial z}{z} - \frac{\partial z^2}{zz}, \text{ ergo}$$

$$\frac{\partial\partial y}{y} = \frac{\partial\partial z}{z} + \frac{2P\partial x\partial z}{z} + \partial x\partial P + PP\partial x^2.$$

Quare cum aequatio nostra sit

$$\frac{L\partial\partial y}{y} + \frac{M\partial x\partial y}{y} + N\partial x^2 = 0,$$

erit facta substitutione

$$\frac{L\partial\partial z}{z} + \frac{2LP\partial x\partial z}{z} + L\partial x\partial P + LPP\partial x^2$$
$$+ \frac{M\partial x\partial z}{z} + MP\partial x^2 + N\partial x^2 = 0,$$

seu per x multiplicando

$$L\partial\partial z + (2LP + M)\partial x\partial z$$
$$+ z\partial x(L\partial P + LPP\partial x + MP\partial x + N\partial x) =$$

ubi pro P functionem quamcunque ipsius x accipere licet, un innumerabiles aequationes inter binas variabiles x et z obtinentur

C o r o l l a r i u m 1.

994. Quodsi ergo hanc aequationem transformatam integra vel per seriem resolvere liceat, ex invento valore ipsius z hab bitur $y = e^{\int P\partial x} z$.

C o r o l l a r i u m 2.

995. Aequatio transformata similis est propositae, propter quod in ea variabilis z cum suis differentialibus ∂z et $\partial\partial z$ ul que unicam dimensionem occupat, perinde ac y in aequatione pr posita.

C o r o l l a r i u m 3.

996. Si eveniat, ut ambae aequationes, proposita ac trar formata, aeque commode per series resolvi possint, hoc mo plures resolutiones ejusdem aequationis exhiberi possunt.

S c h o l i o n 1.

997. Cum aequationes commode per series resolubiles hac forma contineantur

$$xx(a + bx^n)\partial\partial x + x(c + ex^n)\partial x\partial y + (f + gx^n)y\partial x^2 =$$

ubi est

$$L = xx(a + bx^n),\ M = x(c + ex^n),\ N = f + g$$

ut transformata similem obtineat formam, fieri oportet $LP = x(\mu + \nu x$

ideoque $P = \frac{\mu + \nu x^n}{x(a + bx^n)}$. Hinc erit

$$\partial P = \frac{-\mu a - \mu(n+1) b x^n + \nu(n-1) a x^n - \nu b x^{2n}}{x x (a + b x^n)^2} \partial x$$

ideoque

$$L \partial P + L P P \partial x + M P \partial x =$$

$$\left\{\begin{array}{ll} -\mu a - (n+1)\mu b x^n + (n-1)\nu a x^n & -\nu b x^{2n} \\ +\mu\mu + 2\mu\nu x^n & +\nu\nu x^{2n} \\ +\mu c + \mu e x^n + \nu c x^n & +\nu e x^{2n} \end{array}\right\} : a + b x^n$$

ubi divisio per $a + b x^n$ succedere debet. Statuatur quotus $= \mu h + \nu k x^n$, fietque

$$\mu = a - c + a h, \ \nu = b - e + a k,$$

ac praeterea

$$2\mu\nu - (n+1)\mu b + (n-1)\nu a + \mu e + \nu c = \mu b h + \nu a k,$$

ubi priores valores substituti praebent

$$(h-k+n)(bc - ae) = n a b (h-k) + a b (h-k)^2;$$

unde fit vel

$$h - k = \frac{be + ae}{ab}, \text{ vel } h - k = -n.$$

Litterarum ergo b et k altera arbitrio nostro relinquitur, fitque aequatio transformata

$$x x (a + b x^n) \partial\partial z + x [2\mu + c + (2\nu + e) x^n] \partial x \partial z$$
$$+ [f + \mu h + (g + \nu k) x^n] z \partial x^2 = 0.$$

Hujus autem resolutio tam per series ascendentes, quam descendentes similes ipsius x postulat potestates: Substitutio autem ipsa fit

$$y = x^{\frac{a-c}{a} + h} (a + b x^n)^{\frac{bc - ae}{nab} - \frac{(b-k)}{n}} z,$$

ubi ne sola potestas ipsius x ingrediatur, sumi debet $h - k = -n$. Nihil interest quomodo hic h accipiatur, sumto ergo $h = 0$, fit $k = n$, et substitutio

$$y = x^{\frac{a-c}{a}} (a + b x^n)^{\frac{bc-ae}{nab} + 1} z,$$

quae ducit ad hanc aequationem

$$x x (a + b x^n) \partial \partial z + x [2 a - c + (2 (n+1) b - e) x^n] \partial x \partial z$$
$$+ [f + (n (n+1) b - ne + g) x^n] z \partial x^2 = 0.$$

Scholion 2.

998. Supra §. 970. vidimus, aequationem propositam inter x et y algebraicum admittere integrale, si fuerit

$$\frac{c}{2a} - \frac{e}{2b} \,..\, \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} \,..\, \frac{\sqrt{[(b-e)^2 - 4bg]}}{2b} = i n,$$

quae si transformata simili modo tractetur, integrale algebraicum assignari poterit, si fuerit

$$-\frac{c}{2a} + \frac{e}{2b} \,..\, \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} \,..\, \frac{\sqrt{[(b-e)^2 - 4bg]}}{2b} - n = i n,$$

quibus conditionibus conjunctis concludere licet, integrale algebraicum satisfacere, dummodo haec formula

$$\frac{bc - aa}{2ab} \,..\, \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} \,..\, \frac{\sqrt{[(b-e)^2 - 4bg]}}{2b}:$$

divisibilis extiterit per exponentem n, hic signum .. ad ambiguitatem positivi ac negativi designandam adhibui. Quare si ponamus

$$f = \frac{(a-c)^2 - bb}{4a} \text{ et } g = \frac{(b-e)^2 - kk}{4b},$$

integrabilitas locum habet, quoties haec expressio $\frac{bc - ae \pm bb \pm ak}{2nab}$ fuerit numerus integer, sive positivus sive negativus.

Exemplum.

999. *Proposita aequatione*

$$x x (1 - x x) \partial \partial y + x (1 + 2 m x x) \partial x \partial y$$
$$- m (m+1) x x y \partial x^2 = 0,$$

invenire casus, quibus integrale algebraicum saltem particulare assignari potest.

Hic est $a = 1$, $b = -1$, $c = 1$, $e = 2m$, $f = 0$, $g = -m(m+1)$ et $n = 2$. Hinc deducimus

$$h = \sqrt{[(a-c)^2 - 4af]} = 0, \text{ et}$$
$$k = \sqrt{[(b-e)^2 - 4bg]} = \sqrt{[(2m+1)^2 - 4m(m+1)]}$$

hoc est $k = \pm 1$. Formula ergo numero integro aequalis est, $\frac{-1-2m\pm 1}{-4}$, unde geminos pro m casus nanciscimur

$$\text{vel } 2m+2 = \pm 4i, \text{ vel } 2m = \pm 4i, \text{ hoc est}$$
$$\text{vel } m = \pm 2i - 1, \text{ vel } m = \pm 2i,$$

dummodo ergo m sit numerus integer sive positivus sive negativus, integrale particulare algebraicum exhiberi potest. Substitutio autem aequationem transformatam praebens est

$$y = (1-xx)^{\frac{-1-2m}{-2}+1} z = (1-xx)^{\frac{2m+3}{2}} z,$$

ipsa vero aequatio transformata

$$xx(1-xx)\partial\partial z + x[1-2(m+3)xx]\partial x\partial z$$
$$-(m+2)(m+3)xxz\partial x^2 = 0,$$

quam ex illa oriri manifestum est, si loco m scribatur $-m-3$. Ipsa autem haec integralia reperiuntur, ob $\lambda\lambda = 0$, ponendo

$$y = A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + \text{etc.}$$

unde fit

$$\left.\begin{array}{lllll}
2Bxx & +12Cx^4 & +30Dx^6 & +56Ex^8 & +\text{etc.} \\
 & -2B & -12C & -30D & \\
+2B & +4C & +6D & +8E & \\
 & +4mB & +8mC & +12mD & \\
-m(m+1)A & -m(m+1)B & -m(m+1)C & -m(m+1)D &
\end{array}\right\} = 0.$$

Ergo determinatio coëfficientium ita se habet

$B = \frac{m(m+1)}{4}A$, $C = \frac{(m-1)(m-2)}{16}B$, $D = \frac{(m-3)(m-4)}{36}C$, etc.

Ac si ponatur

$$z = \mathfrak{A} + \mathfrak{B}x^2 + \mathfrak{C}x^4 + \mathfrak{D}x^6 + \mathfrak{E}x^8 + \text{etc. erit}$$

$$\mathfrak{B} = \frac{(m+2)(m+3)}{4}\mathfrak{A},\ \mathfrak{C} = \frac{(m+4)(m+5)}{16}\mathfrak{B},\ \mathfrak{D} = \frac{(m+6)(m+7)}{36}\mathfrak{C},\ \text{etc.}$$

Problema 126.

1000. Aequationem differentio-differentialem

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0,$$

ope substitutionis $\frac{\partial y}{y} = \frac{Pz\partial x^2}{\partial z}$, in aliam ejusdem formae transmutare.

Solutio.

Hic scilicet quaeritur, qualem functionem ipsius x pro P accipi oporteat, ut facta substitutione variabilis z cum suis differentialibus ∂z et $\partial\partial z$ ubique unicam dimensionem obtineat. Cum igitur sit $\frac{\partial y}{y} = \frac{Pz\partial x^2}{\partial z}$ erit differentiando

$$\frac{\partial\partial y}{y} - \frac{\partial y^2}{y^2} = \frac{-Pz\partial x^2\partial\partial z}{\partial y^2} + \frac{z\partial x^2\partial P}{\partial z} + P\partial x^2 \text{ et}$$

$$\frac{\partial\partial y}{y} = \frac{-Pz\partial x^2\partial\partial z}{\partial z^2} + \frac{z\partial x^2\partial P}{\partial z} + \frac{PPzz\partial x^4}{\partial z^2} + P\partial x^2,$$

quibus valoribus substitutis fit

$$\frac{LPz\partial x^2\partial\partial z}{\partial z^2} + \frac{Lz\partial x^2\partial P}{\partial z} + LP\partial x^2 + \frac{LPPzz\partial x^4}{\partial z^2}$$
$$+ \frac{MPz\partial x^3}{\partial z} + N\partial x^2 = 0.$$

Sumamus ergo $LP + N = 0$, seu $P = \frac{-N}{L}$, et multiplicando per $\frac{-\partial z^2}{Pz\partial x^2}$, nanciscemur

$$-L\partial\partial z - \frac{L\partial P\partial z}{P} - LPz\partial x^2 - M\partial x\partial z = 0, \text{ seu}$$

$$L\partial\partial z - M\partial x\partial z - \frac{L\partial N\partial z}{N} + \partial L\partial z + Nz\partial x^2 = 0.$$

Aequatio ergo proposita ope substitutionis $\frac{\partial y}{y} = \frac{-Nz\partial x^2}{L\partial x}$ transformatur in hanc

$$L\partial\partial z + \left(\tfrac{\partial L}{\partial x} - M - \tfrac{L\partial N}{N\partial x}\right)\partial x\partial z + Nz\partial x^2 = 0.$$

Quodsi ergo hinc valor ipsius z erui possit, habebitur etiam valor ipsius y per x expressus.

Corollarium 1.

1001. Si in hac aequatione transformata vicissim ponatur $\frac{\partial z}{z} = \frac{-Nz\partial x^2}{L\partial y}$, ipsa aequatio proposita exoritur, unde hae duae aequationes ita inter se cohaerent, ut altera ex altera per similem substitutionem producatur.

Corollarium 2.

1002. Si in aequatione transformata secundum substitutionem priorem ponatur $\frac{\partial z}{z} = Q\partial x + \frac{\partial v}{v}$, obtinebitur haec nova transformata

$$L\partial\partial v + \left(2LQ + \tfrac{\partial L}{\partial x} - M - \tfrac{L\partial N}{N\partial x}\right)\partial x\partial v$$
$$+ v\partial x\left(L\partial Q + LQQ\partial x + Q\partial L - MQ\partial x - \tfrac{LQ\partial N}{N} + N\partial x\right) = 0,$$

quae ergo ex ipsa proposita deducitur ponendo

$$\frac{\partial y}{y} = \frac{-Nv\partial x^2}{L(\partial v + Qv\partial x)}.$$

Scholion 1.

1003. Hinc combinando ambas substitutiones, quibus in binis praecedentibus problematibus sumus usi, substitutionem hujusmodi generalem adipiscimur

$$\frac{\partial y}{y} = \frac{P\partial z + Qz\partial x}{R\partial z + Sz\partial x}\cdot\partial x,$$

quae si in aequatione proposita

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0$$

substituatur, functiones P, Q, R, S ita definiri debent, ut in aequatione resultante variabilis z cum suis differentialibus nusquam plus una dimensione teneat. Oriuntur autem termini quadrato ∂z^2 affecti ad quos destruendos fieri oportet

$$L\partial x(PP + QR - PS) + L(R\partial P - P\partial R) + MPR\partial x + NRR\partial x = 0$$

seu $Q = \frac{PS}{R} - \frac{PP}{R} - \frac{\partial P}{\partial x} + \frac{P\partial R}{R\partial x} - \frac{MP}{L} - \frac{NR}{L}$;

tum vero pervenitur ad hanc aequationem

$$\left.\begin{aligned} &L\partial\partial z(PS - QR) + L\partial z(R\partial Q - Q\partial R + S\partial P - P\partial S) \\ &\quad + \partial x\partial z[2LPQ + M(QR + PS) + 2NRS] \\ &\quad + Lz\partial x(S\partial Q - Q\partial S + QQ\partial x) + Sz\partial x^2(MQ + NS) \end{aligned}\right\} = 0$$

Verum facilius ad hanc aequationem generalem pervenitur, si ambae substitutiones alternatim in usum vocentur.

Scholion 2.

1004. Transformatio autem hic exposita eo magis est notatu digna, quod etiamsi aequatio transformata resolutionem admittat inde tamen non nisi difficulter ipsa proposita resolvatur. Cum enim reperta fuerit functio ipsius x, quae loco z substituta aequationi transformatae satisfaciat, pro valore ipsius y inveniendo, insuper integrale hujus aequationis $\frac{\partial y}{y} = \frac{-Nz\partial x}{L\partial z}$ investigari oportet, ubi etsi variabiles x et y a se invicem sunt separatae, tamen difficultates insignes in ipsa integratione se exerere possunt. Fieri ergo poterit, ut ope hujus substitutionis, ejusmodi aequationum integralia exhiberi queant, quae directa via vix investigare liceat. Scilicet si eveniat, ut integrale aequationis transformatae vel ope methodi cujusdam supra expositae inveniri, vel per seriem abruptam exprimi possit, tum etiam ipsius aequationis propositae integrale habebitur. Etsi enim casu posteriori integrale tantum particulare innotescit, tamen ex eo semper in hoc aequationum genere integrale completum elici potest. Namque si aequationi

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0$$

particulariter satisfaciat valor $y = X$, ponatur $y = Xv$, fietque

$$\left.\begin{array}{r} L X \partial\partial v + 2 L \partial X \partial v + L v \partial\partial X \\ + M X \partial x \partial v + M v \partial x \partial X \\ + N X v \partial x^2 \end{array}\right\} = 0.$$

At quia $X = y$ per hypothesin aequationi satisfacit, erit

$$L \partial\partial X + M \partial x \partial X + N X \partial x^2 = 0 \text{ et}$$

$$L X \partial\partial v + (2 L \partial X + M X \partial x) \partial v = 0, \text{ seu}$$

$$\frac{\partial\partial v}{\partial v} + \frac{2 \partial X}{X} + \frac{M \partial x}{L} = 0,$$

unde integrando oritur

$$X X \partial v = C e^{-\int \frac{M \partial x}{L}} \partial x, \text{ porroque}$$

$$v = \int \frac{C \partial x}{X X} e^{-\int \frac{M \partial x}{L}};$$

ita ut integrale completum sit,

$$y = C X \int \frac{\partial x}{X X} e^{-\int \frac{M \partial x}{L}},$$

quod ergo ex quolibet integrali particulari $y = X$ elici potest.

Exemplum.

1005. *Aequationem differentio-differentialem*

$$x x (a + b x^n) \partial\partial y + x (c + e x^n) \partial x \partial y + f y \partial x^2 = 0$$

transformare ac per seriem integrare.

Cum hic sit $L = x x (a + b x^n)$, $M = x (c + e x^n)$ et $N = f$, utendum est hac substitutione

$$\frac{\partial y}{y} = \frac{-f z \partial x^2}{x x (a + b x^n) \partial z},$$

qua nostra aequatio reducitur ad hanc formam

$$x x (a + b x^n) \partial\partial z + x (2 a - c + [(n+2) b - e] x^n) \partial x \partial z + f z \partial x^2 = 0,$$

pro cujus resolutione si ponatur

$$z = A x^{\lambda} + B x^{\lambda + n} + C x^{\lambda + 2n} + \text{etc.}$$

fieri debet

$$\lambda (\lambda - 1) a + \lambda (2a - c) + f = 0, \text{ seu}$$

$$\lambda \lambda a + \lambda (a - c) + f = 0, \text{ ergo } \lambda = \frac{-a + c \pm \sqrt{[(a-c)^2 - 4af]}}{2a}$$

Series autem abrumpetur per 970, si haec expressio

$$-\frac{c}{2a} + \frac{e}{2b} - \frac{n}{2} \pm \left(\frac{e}{2b} - \frac{n}{2} - \frac{1}{2}\right) \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = i n,$$

denotante i numerum integrum positivum, hoc est

$$\text{vel } -\frac{c}{2a} + \frac{e}{b} - \frac{1}{2} - n \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = i n,$$

$$\text{vel } -\frac{c}{2a} + \frac{1}{2} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = i n.$$

Sin autem ipsa aequatio proposita hoc modo in seriem resolvatur haec abrumpetur, si fuerit

$$\frac{c}{2a} - \frac{e}{2b} \pm \frac{(b-e)}{2b} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = i n,$$

hoc est

$$\text{vel } \frac{e}{2a} - \frac{e}{b} + \frac{1}{2} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = i n$$

$$\text{vel } \frac{c}{2a} - \frac{1}{2} \pm \frac{\sqrt{[(a-c)^2 - 4af]}}{2a} = i n.$$

Unde intelligitur, integrale finitum exhiberi posse, sive numerus integer i sit positivus sive negativus. Ad hanc vero duplicitatem jam prior substitutio perduxerat (998.), ita ut haec nova substitutio nullos novos casus integrabiles suppeditet.

Scholion 1.

1006. Ut tamen pateat, quomodo ex valore finito ipsius z valor finitus ipsius y elici queat, contemplemur casum

$$x x (a + b x^2) \partial \partial y + x (3a + e x^2) \partial x \partial y - 24 a y \partial x^2 = 0,$$

ubi $n = 2$, $c = 3a$ et $f = -24a$, quae facta substitutione $\frac{\partial y}{y} = \frac{24 a z \partial x^2}{x x (a + b x^2) \partial z}$ abit in hanc

$$x x (a + b x^2) \partial\partial z + x [- a + (4 b - e) x x] \partial x \partial z$$
$$- 24 a z \partial x^2 = 0,$$

ubi pro serie ascendente fit

$$\lambda\lambda - 2\lambda - 24 = 0 \text{ vel } (\lambda - 6)(\lambda + 4) = 0.$$

Statuatur

$$z = A x^{-4} + B x^{-2} + C + D x^2 + \text{etc.}$$

erit

$$\left.\begin{array}{lllll} 20 A a x^{-4} + & 6 B a x^{-2} & * & + 2 D a x^2 + \text{etc.} \\ 0 & + 20 A b & + 6 B b & * \\ + 4 A a & + 2 B a & * & - 2 D a \\ & - 4 A (4 b - e) & - 2 B (4 b - e) & * \\ - 24 A a & - 24 B a & - 24 C a & - 24 D a \end{array}\right\} = 0.$$

Cum ergo sit $D = 0$, sequentes termini omnes tolluntur. Tum vero est

$$16 B a = 4 A (b + e), \quad 24 C a = - 2 B b + 2 B e,$$

ergo

$$B = \frac{b + e}{4 a}, \quad C = \frac{e - b}{12 a} B = \frac{e e - b b}{24 a a} A,$$

hincque

$$z = A \left(\frac{1}{x^4} + \frac{b + e}{4 a x^2} + \frac{e e - b b}{48 a a}\right) = \frac{A [48 a a + 12 a (b + e) x x + (e e - b b) x^4]}{48 a a x^4},$$

unde sequitur

$$\partial z = A \partial x \left(\frac{-4}{x^5} - \frac{b - e}{2 a x^3}\right) = \frac{- A \partial x}{2 a x^5} [8 a + (b + e) x x].$$

Ergo

$$\frac{\partial y}{y} = \frac{- [48 a a + 12 a (b + e) x x + (e e - b b) x^4]}{x (a + b x^2) [8 a + (b + e) x x]} \partial x,$$

seu resolvendo

$$\frac{\partial y}{y} = \frac{- 6 \partial x}{x} + \frac{(5 b - e) x \partial x}{a + b x x} + \frac{2 (b + e) x \partial x}{8 a + (b + e) x x},$$

hincque integrando

$$y = \frac{A}{x^6} (a + b x x)^{\frac{5 b - e}{2 b}} [8 a + (b + e) x x].$$

Scholion 2.

1007. Quod hic casu fortuito evenisse videtur, ut ex valore ipsius z invento quantitas y commode definiri potuerit, idem perpetuo evenire oportere, sequenti modo in genere ostendi potest. Cum enim aequatio proposita

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0$$

ope substitutionis $\frac{\partial y}{y} = -\frac{Nz\partial x^2}{L\partial z}$ in hanc sit transformata

$$L\partial\partial z - M\partial x\partial z - \frac{L\partial N\partial z}{N} + \partial L\partial z + Nz\partial x^2 = 0,$$

si haec per $L\partial z$ dividatur, prodit

$$\frac{\partial\partial z}{\partial z} - \frac{M\partial x}{L} - \frac{\partial N}{N} + \frac{\partial L}{L} = -\frac{Nz\partial x^2}{L\partial z} = \frac{\partial y}{y},$$

ex qua integrando elicitur

$$y = \frac{aL\partial z}{N\partial x} e^{-\int\frac{M\partial x}{L}},$$

quae invento valore ipsius z, statim sine ulteriori integratione praebet valorem ipsius y.

Cum porro sit

$$\partial y = \frac{-Nyz\partial x^2}{L\partial z}, \text{ erit } \partial y = -az\partial x \,.\, e^{-\int\frac{M\partial x}{L}},$$

hincque

$$y\partial y = \frac{-aaLz\partial z}{N} e^{-2\int\frac{M\partial x}{L}},$$

atque hae relationes eo magis sunt notatu dignae, quod ex iis aequatio proposita nonnisi per plures ambages ad transformatam reduci possit. Ipsa enim formula pro y substituta perducit ad aequationem differentialem tertii gradus, quae autem manifesto integrationem admittit, ipsamque aequationem hic inventam suppeditat. Hinc igitur occasionem adipiscimur ejusmodi substitutiones inves

[..]ndi, quae quidem ad differentialia tertii gradus ascendant, verum [..]men per integrationem ad differentialia secunda redigi se patiantur.

Problema 127.

1008. Aequationem differentio-differentialem

$$L \partial \partial y + M \partial x \partial y + N y \partial x^2 = 0$$

[..]pe hujusmodi substitutionis $y = \frac{P \partial z}{\partial x}$ in aliam aequationem pariter [d]ifferentio-differentialem transformare.

Solutio.

Ob $y = \frac{P \partial z}{\partial x}$, fit

$$\partial y = \frac{P \partial \partial z + \partial P \partial z}{\partial x} \text{ et } \partial \partial y = \frac{P \partial^3 z + 2 \partial P \partial \partial z + \partial z \partial \partial P}{\partial x},$$

[q]uibus formulis substitutis oritur haec aequatio differentialis tertii [g]radus

$$L P \partial^3 z + 2 L \partial P \partial \partial z + L \partial z \partial \partial P + M P \partial x \partial \partial z$$
$$+ M \partial x \partial P \partial z + N P \partial x^2 \partial z = 0,$$

[q]uam ita comparatam assumamus, ut per functionem ipsius x, quae [s]it Q, multiplicata integrabilis evadat. Integrabilis ergo sit haec [f]orma

$$L P Q \partial^3 z + 2 L Q \partial P \partial \partial z + M P Q \partial x \partial \partial z + L Q \partial z \partial \partial P$$
$$+ M Q \partial x \partial P \partial z + N P Q \partial x^2 \partial z = 0,$$

[c]ujus integrale sit

$$L P Q \partial \partial z + S \partial x \partial z + T z \partial x^2 = C \partial x^2,$$

[u]nde colligitur

$$\partial \partial z (2 L Q \partial P + M P Q \partial x) = \partial \partial z (\partial . L P Q + S \partial x),$$
$$\partial z (L Q \partial \partial P + M Q \partial x \partial P + N P Q \partial x^2) = \partial z (\partial x \partial S + T \partial x^2),$$

et $z \partial x^2 \partial T = 0$, ideoque T quantitas constans.

Inde autem fit

$$S \partial x = L Q \partial P - L P \partial Q - P Q \partial L + M P Q \partial x,$$

ex quo per alteram conditionem elicitur

$$T\partial x^2 = LQ\partial\partial P + MQ\partial x\partial P + NPQ\partial x^2 - LQ\partial\partial P - L\partial P\partial Q - Q\partial P\partial L$$
$$+ LP\partial\partial Q + L\partial P\partial Q + P\partial Q\partial L + PQ\partial\partial L + P\partial Q\partial L + Q\partial P\partial L$$
$$- MP\partial x\partial Q - MQ\partial x\partial P - PQ\partial x\partial M, \text{ sive}$$
$$T\partial x^2 = P\partial\partial . LQ - P\partial x\partial . MQ + PNQ\partial x^2.$$

Quare cum T sit quantitas constans, ponatur $T = a$, atque hinc commode functio P definitur, scilicet

$$P = \frac{a\partial x^2}{\partial\partial . LQ - \partial x\partial . MQ + NQ\partial x^2},$$

hocque valore pro P assumto, aequatio proposita ope substitutionis $y = \frac{P\partial z}{\partial x}$ transformatur in hanc

$$LPQ\partial\partial z + \partial z(LQ\partial P - LP\partial Q - PQ\partial L + MPQ\partial x)$$
$$+ az\partial x^2 = C\partial x^2,$$

ubi cum z constante quantitate augere liceat, constans C omitti potest. Dividatur ergo haec aequatio per PQ et prodibit

$$L\partial\partial z + \partial z\left(\frac{L\partial P}{P} - \frac{L\partial Q}{Q} - \partial L + M\partial x\right) + \frac{az\partial x^2}{PQ} = 0,$$

seu in postremo termino valorem ipsius P substituendo

$$L\partial\partial z + \partial z\left(\frac{L\partial P}{P} - \frac{\partial . LQ}{Q} + M\partial x\right)$$
$$+ \frac{z}{Q}(\partial\partial . LQ - \partial x\partial . MQ + NQ\partial x^2) = 0,$$

atque hic pro Q functionem quamcunque ipsius x accipere licet.

Corollarium 1.

1009. Hinc praecedens substitutio derivatur ponendo

$$\partial\partial . LQ - \partial x\partial . MQ = 0, \text{ ideoque}$$
$$\partial . LQ - MQ\partial x = C\partial x, \text{ seu}$$
$$e^{-\int\frac{M\partial x}{L}} LQ = C\int e^{-\int\frac{M\partial x}{L}}\partial x + D.$$

Namque si hic capiatur $C = 0$, erit

$$Q = \frac{D}{L} e^{+\int \frac{M \partial x}{L}}, \text{ et } P = \frac{\alpha \partial x^2}{N Q \partial x^2} = \frac{\alpha}{N Q}, \text{ seu}$$

$$P = \frac{\alpha L}{N} e^{-\int \frac{M \partial x}{L}}, \text{ ut ante.}$$

Corollarium 2.

1010. Sin autem ponamus

$$\partial \partial . L Q - \partial x \partial . M Q = \partial X \partial x, \text{ ut sit}$$

$$P = \frac{\alpha \partial x}{\partial X + N Q \partial x}, \text{ erit}$$

$$\partial . L Q - M Q \partial x = X \partial x + A \partial x,$$

porroque integrando

$$e^{-\int \frac{M \partial x}{L}} L Q = \int e^{-\int \frac{M \partial x}{L}} \partial x (X + A) + B \text{ et}$$

$$Q = \frac{1}{L} e^{\int \frac{M \partial x}{L}} \int e^{-\int \frac{M \partial x}{L}} \partial x (X + A) + \frac{B}{L} e^{\int \frac{M \partial x}{L}}.$$

Corollarium 3.

1011. Ponatur $\int e^{-\int \frac{M \partial x}{L}} X \partial x = e^{-\int \frac{M \partial x}{L}} V$, et $A = 0$, $B = 0$, erit $X = \frac{\partial V}{\partial x} - \frac{M V}{L}$ et $Q = \frac{V}{L}$, ideoque

$$P = \frac{\alpha \partial x}{\frac{\partial \partial V}{\partial x} - \frac{M}{L} \partial V - V \partial . \frac{M}{L} + \frac{N}{L} V \partial x}.$$

Si igitur sit $V = \alpha$, erit $Q = \frac{\alpha}{L}$,

$$P = \frac{L L \partial x}{L N \partial x - L \partial M + M \partial L},$$

et aequatio resultans

$$L \partial \partial z + \partial z \left(\frac{L \partial P}{P} + M \partial x\right) + \frac{z \partial x (L N \partial x - L \partial M + M \partial L)}{L} = 0.$$

Scholion.

1012. Haec autem nimis sunt generalia, quam ut inde quicquam ad usum communem concludi possit. Utcunque autem transformatio instituatur, et aequatio transformata in seriem resolv haec nullis aliis casibus abrumpi videtur, nisi iis quibus ipsa ae tio proposita, et inde per primam substitutionem transformata, seriem alicubi terminatam reducitur. Ex quo perspicuum est hujusmodi transformationum vix unquam novos casus integrabiles posse. Verum dum hactenus loco variabilis y aliam z per su tutionem introduximus, altera x, ex cujus potestatibus series fo bantur, retenta, nunc etiam paucis exploremus, quomodo loco ipsi aliam variabilem t introducendo, transformationem perfici oport ubi imprimis notetur necesse est, cum ante elementum ∂x ass tum fuerit constans, jam in transformata elementum ∂t cons accipi debere. Hic igitur t scribetur loco certae cujuspiam fun nis ipsius x, quam autem ita comparatam esse decet, ut aequ resultans ne nimis fiat complicata.

Problema 128.

1013. Proposita aequatione differentio-differentiali

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0,$$

loco quantitatis x aliam t introducere, quae functioni cuipiam ipsi aequetur.

Solutio.

Divisa aequatione per ∂x, reqraesentetur aequatio ita

$$L\partial . \frac{\partial y}{\partial x} + M\partial y + Ny\partial x = 0,$$

ut jam consideratio elementi ∂x, quod constans erat assum sit exclusa. Cum t aequetur functioni cuipiam ipsius x, fiat $\partial t = P\partial x$, seu $\partial x = \frac{\partial t}{P}$, unde nanciscimur

$$L\partial . \frac{P\partial y}{\partial t} + M\partial y + \frac{Ny\partial t}{P} = 0,$$

ac sumto elemento ∂t constante

$$LP\partial\partial y + L\partial P\partial y + M\partial t\partial y + \frac{Ny\partial t^2}{P} = 0,$$

tantum superest, ut in quantitatibus finitis, quae adhuc variabilem x complectuntur, ejus loco altera t introducatur.

Exemplum.

1014. *Proposita sit haec aequatio*

$$xx(a+bx^n)\partial\partial y+x(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0,$$

in quam loco formulae $h+kx^n$ introducatur t.

Cum ergo sit

$t=h+kx^n$, erit $\partial t=nkx^{n-1}\partial x$,

ideoque

$P=nkx^{n-1}$ et $\partial P=n(n-1)kx^{n-2}\partial x=\frac{(n-1)\partial t}{x}$.

Quare habebimus

$$nkx^{n+1}(a+bx^n)\partial\partial y+(n-1)x\partial t\partial y(a+bx^n)+x(c+ex^n)\partial t\partial y$$
$$+\frac{(f+gx^n)y\partial t^2}{nkx^{n-1}}=0$$

sive

$$nk(a+bx^n)\partial\partial y+\frac{(n-1)\partial t\partial y(a+bx^n)+\partial t\partial y(c+ex^n)}{x^n}$$
$$+\frac{(f+gx^n)y\partial t^2}{nkx^{2n}}=0.$$

Nunc vero est $x^n=\frac{t-h}{k}$, qui valor substitutus praebet

$$n(ak-bh+bt)\partial\partial y+\frac{(n-1)\partial t\partial y(ak-bh+bt)+\partial t\partial y(ck-eh+et)}{t-h}$$
$$+\frac{(fk-gh+gt)y\partial t^2}{n(t-h)^2}=0.$$

Verum hic ita ubique $t-h$ occurrit, ut aequatio simplicior evadat loco $t-h$ scribendo u, tum autem perinde est, ac si loco

potestatis x^n scripsissemus quantitatem u: neque ergo hinc quicqu
lucri pro novis seriebus eruendis redundat.

Corollarium.

1015. Si in aequatione generali loco x^m scribere velimus
erit

$$\partial t = m x^{m-1} \partial x \text{ et } P = m x^{m-1},$$

et aequatio resultabit, ob

$$\partial P = m(m-1) x^{m-2} \partial x = \frac{(m-1)\partial t}{x},$$

ista

$$m L x^{m-1} \partial \partial y + \frac{(m-1) L \partial t \partial y}{x} + M \partial t \partial y + \frac{N y \partial t^2}{m x^{m-1}} =$$

seu

$$m L \partial \partial y + \frac{(m-1) L \partial t \partial y}{t} + \frac{M x \partial t \partial y}{t} + \frac{N x x y \partial t^2}{m t t} = 0$$

Scholion.

1016. Plura de hujusmodi aequationum transformationibu
tradere haud necesse videtur, cum ex his fontibus haud difficulte
omnes transformationes ad usum idoneae derivari queant. Datu
autem alia methodus prorsus singularis hujusmodi aequationum diffe
rentio-differentialium integralia exprimendi, quae per formulas inte
grales binas variabiles involventes expeditur, dum altera in integra
tione ut constans tractatur. Ita si P fuerit functio quaecunque bina
rum variabilium x et u, ac ponatur $y = \int P \partial x$, considerando
in integratione ut constantem, integrale hoc $\int P \partial x$ erit functi
ipsarum x et u, quod ita determinatum, ut evanescat posito $x = 0$
si deinceps statuatur $x = a$, obtinebitur functio ipsius u ipsi
aequalis, quae si satisfaciat aequationi cuipiam differentiali inter
et y propositae, haec aequatio resolvetur formula $y = \int P \partial x$
quae ut ejus integrale spectari poterit. Atque hoc modo innume

bilium aequationum differentio-differentialium integralia exhiberi possint, quae aliis methodis prorsus intractabiles videntur. Quamquam autem formula $\int P \partial x$, spectata quantitate u ut constante, actu integrari nequit, tamen ejus integrale in hoc negotio pro cognito accipi potest, quia ejus valor saltem per approximationes assignari potest. Scilicet dum sumta x pro abscissa, si P denotet applicatam orthogonalem ei convenientem, formula $\int P \partial x$ exprimet aream ejusdem curvae abcissae x insistentem, ac posito $x = a$, area habetur determinata valori $y = \int P \partial x$, prouti cum modo definivimus, aequationis, quae ergo, uti loqui solent, per quadraturas curuarum assignari potest, ex quo haec integrandi ratio commode appellatur constructio per quadraturas. Hic autem imprimis ad eam rationem, qua integralia in particularia et completa distinximus, attendi conveniet; unde sollicite est cavendum, ne integralia hoc modo inventa pro completis habeantur, nisi quatenus binas constantes arbitrarias involvant. Cum igitur eidem aequationi differentiali infinita integralia particularia conveniant, mirandum non est, si hoc modo pro eadem aequatione proposita plura integralia diversa inveniamus. Hoc autem argumentum fere prorsus est novum, neque a quoquam adhuc pertractatum, si quidem nonnulla specimina, quae equidem jam dudum dedi, excipiantur; ex quo dubitare non licet, quin ista methodus, si diligentius excolatur, aliquando forte praeclara incrementa in Analysin sit allatura.