CAPUT VIII.

DE ALIARUM AEQUATIONUM DIFFERENTIO - DIFFERENTIALIUM RESOLUTIONE PER SERIES INFINITAS.

Problema 122.

967.

Formam generalem aequationum differentio-differentialium, quas commode per series resolvere licet, exhibere, earumque integralia investigare.

Solutio.

Primo alias aequationes commode per series resolvere non licet, nisi in quibus altera variabilis y cum suis differentialibus ∂y et $\partial \partial y$ nusquam plus una dimensione obtinet; quoniam pro y seriem infinitam substituendo in calculos nimis molestos incideremus, si usquam plures dimensiones ingrederentur. Hujusmodi ergo aequationes in hac forma

$$\partial \partial y + M \partial x \partial y + N y \partial x^2 = X \partial x^2$$

continentur. Tum vero ut seriei pro y assumtae quilibet terminus per solum praecedentem determinetur, qui est casus resolutionis maxime notabilis, duplicis tantum generis terminos ratione alterius variabilis x inesse oportet, siquidem ad dimensiones, quas ipsa x cum suo differentiali ∂x constituit, respiciamus. Unde primo quidem, rejecto termino $X \partial x^2$, aequationes hoc modo resolubiles in hac forma continentur

$$xx(a+bx^n)\partial\partial y+x(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0.$$

Pro cujus resolutione fingamus

$$y=\mathrm{A}x^\lambda+\mathrm{B}x^{\lambda+n}+\mathrm{C}x^{\lambda+2n}+\mathrm{D}x^{\lambda+3n}+\text{etc.}$$

et facta substitutione, sequens serierum summa ad nihilum redigi debet

$$\begin{array}{lll}
\lambda(\lambda-1)\mathrm{A}ax^\lambda+ & (\lambda+n)(\lambda+n-1)\mathrm{B}ax^{\lambda+n}+ & (\lambda+2n)(\lambda+2n-1)\mathrm{C}ax^{\lambda+2n}+\text{etc}\\
 & +\lambda(\lambda-1)\mathrm{A}b & +(\lambda+n)(\lambda+n-1)\mathrm{B}b\\
+\lambda\mathrm{A}c & +(\lambda+n)\mathrm{B}c & +(\lambda+2n)\mathrm{C}c\\
 & +\lambda\mathrm{A}e & +(\lambda+n)\mathrm{B}e\\
+\mathrm{A}f & +\mathrm{B}f & +\mathrm{C}f\\
 & +\mathrm{A}g & +\mathrm{B}g.
\end{array}$$

Hic ergo primo exponens λ ita accipi debet, ut sit

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

tum vero pro reliquis fieri oportet

$$[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f]\mathrm{B}=-[\lambda(\lambda-1)b+\lambda e+g]\mathrm{A},$$

$$[(\lambda+2n)(\lambda+2n-1)a+(\lambda+2n)c+f]\mathrm{C}=$$
$$-[(\lambda+n)(\lambda+n-1)b+(\lambda+n)e+g]\mathrm{B},$$

$$[(\lambda+3n)(\lambda+3n-1)a+(\lambda+3n)c+f]\mathrm{D}=$$
$$-[(\lambda+2n)(\lambda+2n-1)b+(\lambda+2n)e+g)\mathrm{C}$$

etc.

Cum igitur sit

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

si ponamus brevitatis causa

$$(\lambda(\lambda-1)b+\lambda e+g=h,$$

erit

$$[n(n+2\lambda-1)a+nc]\mathrm{B}=-h\mathrm{A}$$
$$[2n(2n+2\lambda-1)a+2nc]\mathrm{C}=-[n(n+2\lambda-1)b+ne+h]\mathrm{B}$$
$$[3n(3n+2\lambda-1)a+3nc]\mathrm{D}=-[2n(2n+2\lambda-1)b+2ne+h]\mathrm{C}.$$
etc.

Quia ergo nisi $a=0$, pro λ gemini inveniuntur valores, scilicet

$$\lambda=\frac{a-c\pm\sqrt{[(a-c)^2-4af]}}{2a},$$

binae series pro y inveniuntur, quae utcunque combinatae integrale completum aequationis propositae praebent.

Aliter

Proposita aequatione hac

$$xx(a+bx^n)\partial\partial y+x(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2=0,$$

series quoque ordine retrogrado fingi potest

$$y=\mathrm{A}x^{\lambda}+\mathrm{B}x^{\lambda-n}+\mathrm{C}x^{\lambda-2n}+\mathrm{D}x^{\lambda-3n}+\text{etc.}$$

unde oritur ad nihilum reducendum

$$\begin{array}{lll}
+\lambda(\lambda-1)\mathrm{A}bx^{\lambda+n} & +(\lambda-n)(\lambda-n-1)\mathrm{B}bx^{\lambda} & +(\lambda-2n)(\lambda-2n-1)\mathrm{C}bx^{\lambda-n}+\text{etc}\\
 & +\lambda(\lambda-1)\mathrm{A}a & +(\lambda-n)(\lambda-n-1)\mathrm{B}a\\
+\lambda\mathrm{A}e & +(\lambda-n)\mathrm{B}e & +(\lambda-2n)\mathrm{C}e\\
 & +\lambda\mathrm{A}c & +(\lambda-n)\mathrm{B}c\\
+\mathrm{A}g & +\mathrm{B}g & +\mathrm{C}g\\
 & +\mathrm{A}f & +\mathrm{B}f
\end{array}$$

Hic ergo exponentem λ ita accipi oportet, ut fiat

$$\lambda(\lambda-1)b+\lambda e+g=0.$$

Tum vero si ponamus

$$\lambda(\lambda-1)a+\lambda c+f=h,$$

determinatio coëfficientium ita se habebit

$$n[(n-2\lambda+1)b-e]B=-h\ A,$$
$$2n[(2n-2\lambda+1)b-e]C=-[n(n-2\lambda+1)a-nc+h]B,$$
$$3n[(3n-2\lambda+1)b-e]D=-[2n(2n-2\lambda+1)a-2nc+h]C,$$
etc.

Corollarium 1.

968. Ex priore solutione, si i denotet numerum integrum positivum, series assumta alicubi abrumpetur, si fuerit

$$in(in+2\lambda-1)b+ine+h=0, \text{ vel}$$
$$(\lambda+in)(\lambda+in-1)b+(\lambda+in)e+g=0,$$

hoc est

$$\left.\begin{array}{l}\lambda\lambda b+\lambda(2in-1)b+in(in-1)b\\ \quad+\lambda e \qquad\qquad\quad +ine+g\end{array}\right\}=0.$$

Corollarium 2.

969. Aequatio ergo nostra integrationem admittit, si litterae f et g ita fuerint comparatae, ut sit

$$f=-\lambda(\lambda-1)a-\lambda c \text{ et}$$
$$g=-(\lambda+in)(\lambda+in-1)b-(\lambda+in)e.$$

Vel sumtis duobus numeris μ et ν, ut sit $\nu-\mu$ divisibile per exponentem n, si fuerit

$$f=-\mu(\mu-1)a-\mu c \text{ et } g=-\nu)\nu-1)b-\nu e.$$

Corollarium. 3.

970. Cum hinc sit

$$\mu=\frac{a-c\pm\sqrt{[(a-c)^2-4af]}}{2a} \text{ et } \nu=\frac{b-e\pm\sqrt{[(b-e)^2-4bg]}}{2b},$$

aequatio habebit integrale algebraicum, si fuerit $\nu-\mu=in$, denotante i numerum integrum positivum: hoc est si sit

$$in=\frac{c}{2a}-\frac{e}{2b}\pm\frac{\sqrt{[(b-e)^2-4bg]}}{2b}\mp\frac{\sqrt{[(a-c)^2-4af)}}{2a}.$$

Corollarium 4.

971. Pro serie autem invenienda si eveniat, ut exponens λ fiat imaginarius, notari convenit esse

$$x^{\alpha+\beta\sqrt{-1}} = x^{\alpha} \cdot e^{\beta\sqrt{-1} \cdot lx} = x^{\alpha}(\cos.\beta lx + \sqrt{-1} \cdot \sin.\beta lx),$$

unde binae series ita combinari poterunt, ut integrale consequatur formam realem.

Scholion.

972. Utraque solutio generatim spectata duplicem seriem pro variabili y suppeditat, pro gemino exponentis λ valore, quarum combinatio integrale completum exhibet. Solutio scilicet prior pro exponente λ hos duos praebet valores

$$\lambda = \frac{a - c \pm \sqrt{[(a - c)^2 - 4af]}}{2a},$$

solutio vero posterior

$$\lambda = \frac{b - e \pm \sqrt{[(b - e)^2 - 4bg]}}{2b},$$

ita ut hoc modo integrale completum duplici modo exprimi possit; quae binae formae etiamsi maxime diversae, atque adeo interdum altera per exponentes imaginarios progrediatur, dum altera habet reales, tamen sibi aequipollentes esse debent. Quin etiam evenire potest, ut altera solutio vel etiam utraque ad integrale completum exhibendum sit inepta, dum unicam seriem suppeditet. Incommodum hoc pro utraque solutione duplici modo accidere potest, pro priori nempe solutione, ubi exponentem λ ex hac aequatione

$$\lambda(\lambda - 1)a + \lambda c + f = 0,$$

definiri oportet, unicus inde pro λ eruitur valor, si fuerit vel $a = 0$, vel $4af = (a - c)^2$, priori casu tantum fit $\lambda = -\frac{f}{c}$, altero ipsius λ valore quasi in infinitum abeunte. Posteriori casu vero ambo ipsius λ valores fiunt inter se aequales, scilicet $\lambda = \frac{a - c}{2a}$ Idem incommodum in altera solutione locum habet, si fuerit vel $b = 0$, vel $4bg = (b-e)^2$: unde patet fieri posse, ut altera solutio h jusmodi incommodo laboret, dum altera eo careat, quin etiam utraque eodem inquinetur. Quocirca ostendi conveniet, quemadm dum etiam his casibus integrale completum investigari debeat; quc sum etiam casum referamus, quo ambo ipsius λ valores fiunt im ginarii, quandoquidem ad imaginariam speciem tollendam singul artificio est opus. Denique vero etiam binae series pro y exhibe dae difficultate premuntur, quoties bini valores ipsius λ differentia habent per exponentem n divisibilem, quorum casuum evolutio etia explicari meretur.

Problema 123.

973. Proposita aequatione differentio-differentiali

$$xx(a+bx^n)\partial\partial y + x(c+ex^n)\partial x\partial y + (f+gx^n)y\partial x^2 = 0$$

si eveniat, ut binae series ascendentes pro y assumtae vel in una coalescant, vel altera fiat impossibilis, integrale completum per s ries exprimere.

Solutio.

Assumta serie

$$y = Ax^\lambda + Bx^{\lambda+n} + Cx^{\lambda+2n} + Dx^{\lambda+3n} + \text{etc.}$$

si eveniat ut bini valores ipsius λ ex aequatione

$$\lambda(\lambda-1)a + \lambda c + f = 0$$

vel fiant aequales, vel differentiam per n divisibilem obtineant, va lor ipsius y praeter potestates ipsius x etiam logarithmum ipsius involvet. Quare pro aequationis resolutione statim ponamus $y =$ $+ v\,lkx$, ut sit $y = u + v\,lx + \alpha v$, denotante α quantitatem constantem quamcunque. Hinc erit

$$\partial y = \partial u + \frac{v\partial x}{x} + \partial v\,lkx,$$

$$\partial\partial y = \partial\partial u + \frac{2\partial x\partial v}{x} - \frac{v\partial x^2}{xx} + \partial\partial v\,lkx,$$

quibus valoribus substitutis, aequatio nostra hanc induet formam

$$\left.\begin{array}{r} xx(a+bx^n)\partial\partial u+2x(a+bx^n)\partial x\partial v-(a+bx^n)v\partial x^2 \\ +x(c+ex^n)\partial x\partial u+(c+ex^n)v\partial x^2 \\ +(f+gx^n)u\partial x^2 \\ +[xx(a+bx^n)\partial\partial v+x(c+ex^n)\partial x\partial v+(f+gx^n)v\partial x^2]\,lkx \end{array}\right\}=0,$$

ubi partem postremam logarithmo affectam seorsim nihilo aequari oportet. Quare posito

$$v=\mathrm{A}\,x^{\lambda}+\mathrm{B}\,x^{\lambda+n}+\mathrm{C}\,x^{\lambda+2n}+\mathrm{D}\,x^{\lambda+3n}+\text{etc.}$$

exponenti λ ex aequatione

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

is valor tribuatur, qui nulli incommodo est obnoxius; eritque pro reliquis coëfficientibus,

$$\text{ponendo } \lambda(\lambda-1)b+\lambda e+g=nh,$$

ut sequitur

$$[(n+2\lambda-1)a+c]\mathrm{B}+h\mathrm{A}=0,$$

$$2[(2n+2\lambda-1)a+c]\mathrm{C}+[(n+2\lambda-1)b+e]\mathrm{B}+h\mathrm{B}=0,$$

$$3[(3n+2\lambda-1)a+c]\mathrm{D}+2[(2n+2\lambda-1)b+e]\mathrm{C}+h\mathrm{C}=0,$$

$$4[(4n+2\lambda-1)a+c]\mathrm{E}+3[(3n+2\lambda-1)b+e]\mathrm{D}+h\mathrm{D}=0,$$

etc.

His coëfficientibus ita definitis, quorum primus A arbitrio nostro relinquitur, ponamus

$$u=\Delta+\mathfrak{A}\,x^{\lambda}+\mathfrak{B}\,x^{\lambda+n}+\mathfrak{C}\,x^{\lambda+2n}+\mathfrak{D}\,x^{\lambda+3n}+\text{etc.}$$

qui valor si in priori aequatione cum serie pro v inventa substituatur, sequentes series ad nihilum reduci oporteat

$$\begin{array}{llll}
xx(a+bx^n)\frac{\partial\partial\Delta}{\partial x^2} & +\lambda(\lambda-1)\mathfrak{A}ax^\lambda & +(\lambda+n)(\lambda+n-1)\mathfrak{B}ax^{\lambda+n} & +(\lambda+2n)(\lambda+2n-1)\mathfrak{C}ax^{\lambda+2n} \\
+x(c+ex^n)\frac{\partial\Delta}{\partial x} & & +\lambda(\lambda-1)\mathfrak{A}b & +(\lambda+n)(\lambda+n-1)\mathfrak{B}b \\
 & +\lambda\mathfrak{A}c & +(\lambda+n)\mathfrak{B}c & +(\lambda+2n)\mathfrak{C}c \\
+(f+gx^n)\Delta & & +\lambda\mathfrak{A}e & +(\lambda+n)\mathfrak{B}e \\
 & +\mathfrak{A}f & +\mathfrak{B}f & +\mathfrak{C}f \\
 & & +\mathfrak{A}g & +\mathfrak{B}g \\
 & +2\lambda\mathrm{A}a & +2(\lambda+n)\mathrm{B}a & +2(\lambda+2n)\mathrm{C}a \\
 & & +2\lambda\mathrm{A}b & +2(\lambda+n)\mathrm{B}b \\
 & +\mathrm{A}(c-a) & +\mathrm{B}(c-a) & +\mathrm{C}(c-a) \\
 & & +\mathrm{A}(e-b) & +\mathrm{B}(e-b)
\end{array}$$

Cum autem sit

$$\lambda(\lambda-1)a+\lambda c+f=0 \text{ et } \lambda(\lambda-1)b+\lambda e+g=nh,$$

expressio haec transmutabitur in hanc formam

$$xx(a+bx^n)\frac{\partial\partial\Delta}{\partial x^2}+x(c+ex^n)\frac{\partial\Delta}{\partial x}+(f+gx^n)\Delta$$

$$\begin{array}{lll}
[(2\lambda-1)a+c]\mathrm{A}x^\lambda & +[(2n+2\lambda-1)a+c]\mathrm{B}x^{\lambda+n} & +[(4n+2\lambda-1)a+c]\mathrm{C}x^{\lambda+2n} \\
 & +[(2\lambda-1)b+e]\mathrm{A} & +[(2n+2\lambda-1)b+e]\mathrm{B} \\
 & +n[(n+2\lambda-1)a+c]\mathfrak{B} & +2n[(2n+2\lambda-1)a+c]\mathfrak{C} \\
 & & +n[(n+2\lambda-1)b+e]\mathfrak{B} \\
 & +nh\mathfrak{A} & +nh\mathfrak{B}
\end{array}$$

ubi Δ denotat quosdam terminos seriei

$$\mathfrak{A}x^\lambda+\mathfrak{B}x^{\lambda+n}+\text{etc.}$$

praemittendos, ita ut ordine retrogrado sit

$$\Delta=\mathfrak{a}x^{\lambda-n}+\mathfrak{b}x^{\lambda-2n}+\mathfrak{c}x^{\lambda-3n}+\ldots+\mathfrak{j}x^{\lambda-in}.$$

Quod principium quomodo quovis casu sit constituendum, sequentia sunt observanda.

I. Principium hoc locum habere nequit, nisi fuerit

$$(\lambda - in)(\lambda - in - 1)a + (\lambda - in)c + f = 0,$$

cum igitur sit

$$\lambda(\lambda - 1)a + \lambda c + f = 0,$$

inde erit

$$\lambda = in + \frac{a - c - \sqrt{[(a-c)^2 - 4af]}}{2a},$$

hinc vero

$$\lambda = \frac{a - c + \sqrt{[(a-c)^2 - 4af]}}{2a},$$

quandoquidem hi duo valores convenire nequeunt, nisi ibi signum radicale negative, hic vero positive accipiatur. Aequatis autem his valoribus fit

$$in = \tfrac{1}{a}\sqrt{[(a-c)^2 - 4af]}, \text{ seu}$$

$$iinnaa = (a-c)^2 - 4af, \text{ hincque}$$

$$f = \frac{(a-c)^2}{4a} - \tfrac{1}{4}iinna,$$

unde fit

$$\lambda = \frac{a-c}{2a} + \tfrac{1}{2}in$$

Quare si aequatio proposita ita fuerit comparata, ut sit

$$ina = \sqrt{[(a-c)^2 - 4af]},$$

tum sumto

$$\lambda = \frac{a-c}{2a} + \tfrac{1}{2}in,$$

ac pro v sumta serie

$$v = \mathrm{A}x^{\lambda} + \mathrm{B}x^{\lambda+n} + \text{etc.}$$

alteram seriem u ita constitui convenit

$$u = \mathfrak{j}x^{\lambda - in} + \ldots\ldots + \mathfrak{a}x^{\lambda - n} + \mathfrak{A}x^{\lambda} + \mathfrak{B}x^{\lambda+n}$$
$$+ \mathfrak{C}x^{\lambda+2n} + \text{etc.}$$

Hic est casus, quo bini valores ipsius λ ex aequatione

$$\lambda(\lambda-1)a+\lambda c+f=0,$$

differentiam habent per n divisibilem, ubi notandum, seriem v a majore valore ipsius λ, seriem vero u a minore inchoari debere

II. Principium Δ omitti nequit, nisi fuerit

$$(2\lambda-1)a+c=0,$$

quo casu fit $\lambda=\frac{a-c}{2}$: atque hic est casus, quo aequationis

$$\lambda(\lambda-1)a+\lambda c+f=0$$

binae radices fiunt inter se aequales, ideoque $f=\frac{(a-c)^2}{4a}$. Continetur ergo hic casus in praecedente, sumendo ibi $i=0$. Quare hoc modo resolventur casus, quibus bini valores ipsius λ vel sunt inter se aequales, vel differentiam habent per exponentem n divisibilem. Sicque reperitur integrale completum per duas series ascendentes v et u expressum, quarum illa v per lx multiplicatur.

Corollarium 1.

974. Quando ergo in aequatione proposita coëfficientes a, c et f ita sunt comparati, ut aequationis

$$\lambda(\lambda-1)a+\lambda c+f=0$$

radices sint $\lambda=\mu$ et $\lambda=\mu-in$, denotante i numerum integrum positivum, integrale completum hujusmodi habebit formam $y=u+\alpha v+vlx$.

Corollarium 2,

975. Hic autem binae quantitates v et u ex his aequationibus

$$\text{I. } xx(a+bx^n)\partial\partial v + x(c+ex^n)\partial x\partial v + (f+gx^n)v\partial x^2 = 0,$$

$$\text{II. } \left.\begin{array}{l} xx(a+bx^n)\partial\partial u + x(c+ex^n)\partial x\partial u + (f+gx^n)u\partial x^2 \\ +2x(a+bx^n)\partial x\partial v - (a+bx^n)v\partial x^2 \\ +(c+ex^n)v\partial x^2 \end{array}\right\} = 0,$$

ita determinari poterunt, ut ponatur

$$v = A x^\mu + B x^{\mu+n} + C x^{\mu+2n} + D x^{\mu+3n} + \text{etc.}$$

$$u = \mathfrak{A} x^{\mu-in} + \mathfrak{B} x^{\mu-in+n} + \mathfrak{C} x^{\mu-in+2n} + \mathfrak{D} x^{\mu-in+3n} + \text{etc.}$$

Has scilicet series substituendo omnes coëfficientes ex uno definire licebit.

Scholion.

976. Logarithmo ergo ipsius x in subsidium vocato, his casibus, quos commemoravimus, integrale completum aequationis propositae per series ascendentes exhiberi potest, dum sine hoc artificio integrale tantum particulare invenitur. Quando enim aequatio $\lambda(\lambda-1)a + \lambda c + f = 0$ duas radices habet, quarum differentia per exponentem n est divisibiles, puta $\lambda = \mu$ et $\lambda = \mu - in$, priore methodo sola series, quae incipit a potestate x^μ, determinari potest; si enim altera a potestate $x^{\mu-in}$ incipiens pro y assumeretur, coëfficiens cujusdam termini reperiretur infinitus, unde sequentes omnes forent quoque infiniti, quod incommodum introducendo logarithmus ipsius x feliciter tollitur. Hunc igitur usum istius resolutionis aliquot exemplis illustrasse juvabit.

Exemplum 1.

977. *Aequationis differentio - differentialis*

$$x\partial\partial y + \partial x\partial y + gx^{n-1}y\partial x^2 = 0$$

integrale completum per series ascendentes exhibere.

Hanc aequationem ad nostram formam reducendo habebimus

$$xx\partial\partial y + x\partial x\partial y + gx^n y\partial x^2 = 0,$$

ubi ergo est $a = 1$, $b = 0$, $c = 1$, $e = 0$ et $f = 0$. Hinc $\lambda(\lambda - 1) + \lambda = 0$, seu $\lambda\lambda = 0$, ita ut bini valores ipsius λ sint aequales et $= 0$. Quare posito $y = u + \alpha v + v\,lx$, resolvi oportet has aequationes:

$$\text{I. } xx\partial\partial v + x\partial x\partial v + gx^n v\partial x^2 = 0 \text{ et}$$

$$\text{II. } \left.\begin{array}{l} xx\partial\partial u + x\partial x\partial u + gx^n u\partial x^2 \\ \quad + 2x\partial x\partial v. \end{array}\right\} = 0$$

Statuamus ergo

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc. et}$$

$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

ac prior aequatio praebet

$$\left.\begin{array}{llll} n(n-1)Bx^n & + 2n(2n-1)Cx^{2n} & + 3n(3n-1)Dx^{3n} & + \text{etc.} \\ + nB & + 2nC & + 3nD & \\ + Ag & + Bg & + Cg & \end{array}\right\} = 0$$

unde fit

$$B = \frac{-Ag}{nn}, \quad C = \frac{-Bg}{4nn}, \quad D = \frac{-Cg}{9nn}, \quad E = \frac{-Dg}{16\alpha n}, \text{ etc.}$$

Tum vero altera aequatio dat

$$\left.\begin{array}{llll} nn\mathfrak{B}x^n & + 4nn\mathfrak{C}x^{2n} & + 9nn\mathfrak{D}x^{3n} & + \text{etc.} \\ + \mathfrak{A}g & + \mathfrak{B}g & + \mathfrak{C}g & \\ + 2nB & + 4nC & + 6nD & \end{array}\right\} = 0,$$

unde colligitur

$$\mathfrak{B} = \frac{-\mathfrak{A}g}{nn} - \frac{2B}{n}, \quad \mathfrak{C} = \frac{-\mathfrak{B}g}{4nn} - \frac{2C}{2n}, \quad \mathfrak{D} = \frac{-\mathfrak{C}g}{9nn} - \frac{2D}{3n}, \text{ etc.}$$

Hic autem tuto assumere licet $\mathfrak{A} = 0$, quoniam termini ex $\mathfrak{A}$ orundi, continentur in membro αv. Cum igitur sit

$$B = \frac{-Ag}{n^2}, \quad C = \frac{+Ag^2}{1.4\,n^4}, \quad D = \frac{-Ag^3}{1.4.9\,n^6}, \quad E = \frac{+Ag^4}{1.4.9.16\,n^8}, \text{ etc.}$$

erit ut sequitur

$$\mathfrak{B} = \frac{2Ag}{n^3}, \quad \mathfrak{C} = \frac{-2Agg}{4n^5} - \frac{2Agg}{2.1.4n^5} = \frac{-6Agg}{2.1.4n^5},$$

$$\mathfrak{D} = \frac{6Ag^3}{2.1.4.9n^7} + \frac{2Ag^3}{3.1.4.9n^7} = \frac{22Ag^3}{2.3.1.4.9n^7},$$

$$\mathfrak{E} = \frac{-22Ag^4}{2.3.1.4.9.16n^9} - \frac{2Ag^4}{4.1.4.9.16n^9} = \frac{-100Ag^4}{2.3.4.1.4.9.16n^9},$$

$$\mathfrak{F} = \frac{100Ag^5}{2.3.4.1.4.9.16.25n^{11}} + \frac{2Ag^5}{5.1.4.9.16.25n^{11}} = \frac{548Ag^5}{2.3.4.5.1.4.9.16.25n^{11}}$$

sicque obtinentur sequentes valores

$$\mathfrak{B} = \frac{2Ag}{n^3}, \quad \mathfrak{C} = \frac{-6Ag^2}{1.8n^5}, \quad \mathfrak{D} = \frac{22Ag^3}{1.8.27n^7}, \quad \mathfrak{E} = \frac{-100Ag^4}{1.8.27.64n^9},$$

$$\mathfrak{F} = \frac{548Ag^5}{1.8.27.64.125n^{11}}, \quad \mathfrak{G} = \frac{-3528Ag^6}{1.8.27.64.125.216n^{13}}, \text{ etc.}$$

ubi numeratores 2, 6, 22, 100, 548, 3528, etc. singuli ita per binos praecedentes definiuntur

$$6 = 3.2 - 1.0, \quad 22 = 5.6 - 4.2, \quad 100 = 7.22 - 9.6,$$

$$548 = 9.100 - 16.22, \quad 3528 = 11.548 - 25.100 \text{ etc.}$$

Consequenter integrale ita exprimetur

$$y = \frac{2Ag}{n^3}x^n - \frac{6Ag^2}{1.8n^5}x^{2n} + \frac{22Ag^3}{1.8.27n^7}x^{3n} - \frac{100Ag^4}{1.8.27.64n^9}x^{4n} + \text{etc.}$$

$$+ A\left(1 - \frac{g}{nn}x^n + \frac{g^2}{1.4n^4}x^{2n} - \frac{g^3}{1.4.9n^6}x^{3n} + \frac{g^4}{1.4.9.16n^8}x^{4n} - \text{etc.}\right) lx$$

$$+ \alpha - \frac{\alpha g}{nn}x^n + \frac{\alpha g^2}{1.4n^4}x^{2n} - \frac{\alpha g^3}{1.4.9n^6}x^{3n} + \frac{\alpha g^4}{1.4.9.16n^8}x^{4n} - \text{etc.}$$

ubi A et α sunt binae constantes arbitrariae.

Exemplum 2.

978. *Aequationis differentio-differentialis*

$$x(1 - xx)\partial\partial y - (1 + xx)\partial x\partial y + xy\partial x^2 = 0,$$

integrale completum per series ascendentes assignare.

Ad formam nostram reducta est

$$xx(1 - xx)\partial\partial y - x(1 + xx)\partial x\partial y + xxy\partial x^2 = 0,$$

ita ut sit $n = 2$, $a = 1$, $b = -1$, $c = -1$, $e = -1$, $f = 0$ et $g = 1$, unde aequationis $\lambda(\lambda - 1) - \lambda = 0$ radices sint $\lambda = 0$ et $\lambda = 2$, quarum differentia per $n = 2$ divisa dat 1.

Posito ergo $y = u + \alpha v + v\,l\,x$, statui debet

$$v = \mathrm{A}\,x^2 + \mathrm{B}\,x^4 + \mathrm{C}\,x^6 + \mathrm{D}\,x^8 + \text{etc. et}$$

$$u = \mathfrak{A} + \mathfrak{B}\,x^2 + \mathfrak{C}\,x^4 + \mathfrak{D}\,x^6 + \mathfrak{E}\,x^8 + \text{etc.}$$

quae series ex sequentibus aequationibus determinari debent

$$\text{I.}\quad x\,x\,(1 - x\,x)\,\partial\,\partial\,v - x\,(1 + x\,x)\,\partial\,x\,\partial\,v + x\,x\,v\,\partial\,x^2 = 0,$$

$$\text{II.}\quad \left.\begin{array}{r} x\,x\,(1 - x\,x)\,\partial\,\partial\,u - x\,(1 + x\,x)\,\partial\,x\,\partial\,u + x\,x\,u\,\partial\,x^2 \\ + 2\,x\,(1 - x\,x)\,\partial\,x\,\partial\,v - 2\,v\,\partial\,x^2 \end{array}\right\} = 0.$$

Hinc pro prioris determinatione fit

$$\left.\begin{array}{lllll} & 2\,\mathrm{A}\,x^2 & + 12\,\mathrm{B}\,x^4 & + 30\,\mathrm{C}\,x^6 & + 56\,\mathrm{D}\,x^8 + \text{etc.} \\ & & - 2\,\mathrm{A} & - 12\,\mathrm{B} & - 30\,\mathrm{C} \\ & - 2\,\mathrm{A} & - 4\,\mathrm{B} & - 6\,\mathrm{C} & - 8\,\mathrm{D} \\ & & - 2\,\mathrm{A} & - 4\,\mathrm{B} & - 6\,\mathrm{C} \\ & & + \mathrm{A} & + \mathrm{B} & + \mathrm{C} \end{array}\right\} = 0,$$

ideoque

$$2.4\,\mathrm{B} = 1.3\,\mathrm{A},\quad 4.6\,\mathrm{C} = 3.5\,\mathrm{B},\quad 6.8\,\mathrm{D} = 5.7\,\mathrm{C},\ \text{etc.}$$

seu

$$\mathrm{C} = \frac{1.3}{2.4}\mathrm{A},\quad \mathrm{C} = \frac{1.3.3.5}{2.4.4.6}\mathrm{A},\quad \mathrm{D} = \frac{1.3.3.5.5.7}{2.4.4.6.6.8}\mathrm{A},\ \text{etc.}$$

Ex altera vero aequatione reperitur

$$\left.\begin{array}{lllll} & 2\,\mathfrak{B}\,x^2 & + 12\,\mathfrak{C}\,x^4 & + 30\,\mathfrak{D}\,x^6 & + 56\,\mathfrak{E}\,x^8 + \text{etc.} \\ & & - 2\,\mathfrak{B} & - 12\,\mathfrak{C} & - 30\,\mathfrak{D} \\ & - 2\,\mathfrak{B} & - 4\,\mathfrak{C} & - 6\,\mathfrak{D} & - 8\,\mathfrak{E} \\ & & - 2\,\mathfrak{B} & - 4\,\mathfrak{C} & - 6\,\mathfrak{D} \\ & + \mathfrak{A} & + \mathfrak{B} & + \mathfrak{C} & + \mathfrak{D} \\ & + 4\,\mathrm{A} & + 8\,\mathrm{B} & + 12\,\mathrm{C} & + 16\,\mathrm{D} \\ & & - 4\,\mathrm{A} & - 8\,\mathrm{B} & - 12\,\mathrm{C} \\ & - 2\,\mathrm{A} & - 2\,\mathrm{B} & - 2\,\mathrm{C} & - 2\,\mathrm{D} \end{array}\right\} = 0$$

unde fieri oportet

$$\mathfrak{A} + 2A = 0, \quad 2.4\,\mathfrak{C} - 1.3\,\mathfrak{B} + 6\,B - 4\,A = 0,$$
$$4.6\,\mathfrak{D} - 3.5\,\mathfrak{C} + 10\,C - 8\,B = 0,$$
$$6.8\,\mathfrak{E} - 5.7\,\mathfrak{D} + 14\,D - 12\,C = 0, \text{ etc.}$$

seu cum sit

$$B = \frac{1.3}{2.4}A, \quad C = \frac{3.5}{4.6}B, \quad D = \frac{5.7}{6.8}C, \text{ etc.}$$

erit $\mathfrak{A} = -2A$, tum vero

$$2.\,4\,\mathfrak{C} - 1.\,3\,\mathfrak{B} - \frac{2.7}{2.4}A = 0, \quad \mathfrak{C} = \frac{1.3}{2.4}\mathfrak{B} + \frac{2.7}{2^2.4^2}A,$$
$$4.\,6\,\mathfrak{D} - 3.\,5\,\mathfrak{C} - \frac{2.21}{4.6}B = 0, \quad \mathfrak{D} = \frac{3.5}{4.6}\mathfrak{C} + \frac{2.21}{4^2.6^2}B,$$
$$6.\,8\,\mathfrak{E} - 5.\,7\,\mathfrak{D} - \frac{2.4^3}{6.8}C = 0, \quad \mathfrak{E} = \frac{5.7}{6.8}\mathfrak{D} + \frac{2.4^3}{6^2.8^2}C,$$
$$8.\,10\,\mathfrak{F} - 7.\,9\,\mathfrak{E} - \frac{2.7^3}{8.10}D = 0, \quad \mathfrak{F} = \frac{7.9}{8.10}\mathfrak{E} + \frac{2.7^3}{8^2.10^2}D.$$

Dum ergo capiatur $\mathfrak{A} = -2A$, littera $\mathfrak{B}$ pro lubitu accipi potest; nihilque impedit, quo minus nihilo aequalis statuatur, siquidem constans α supra est inducta.

Exemplum 3.

979. *Aequationis differentio-differentialis*

$$xx(1+bxx)\partial\partial y + x(-5+exx)\partial x\partial y + (5+gxx)y\partial x^2 = 0$$

integrale completum per series ascendentes exhibere.

Quia hic est $a=1$, $c=-5$ et $f=5$, aequatio $\lambda(\lambda-1)-5\lambda+5=0$, seu $\lambda\lambda-6\lambda+5=0$, radices habet $\lambda=1$ et $\lambda=5$, quarum differentia 4 per $n=2$ dividi potest. Posito ergo $y=u+\alpha v+vlx$ statuatur

$$v = A\,x^5 + B\,x^7 + C\,x^9 + D\,x^{11} + E\,x^{13} + \text{etc.}$$
$$u = \mathfrak{A}\,x + \mathfrak{B}x^3 + \mathfrak{C}\,x^5 + \mathfrak{D}\,x^7 + \text{etc. et}$$

aequationes resolvendae erunt

$$\text{I.} \quad xx(1+bxx)\partial\partial v + x(-5+exx)\partial x\partial v + (5+gxx)v\partial x^2 = 0 \quad \text{et}$$

$$\text{II.} \quad \left.\begin{aligned} & xx(1+bxx)\partial\partial u + x(-5+exx)\partial x\partial u + (5+gxx)u\partial x^2 \\ & + 2x(1+bxx)\partial x\partial v - (1+bxx)v\partial x^2 \\ & + (-5+exx)v\partial x^2 \end{aligned}\right\} = 0.$$

ubi prior ducit ad

$$\left.\begin{array}{lllll}
 & 5.4\,A x^5 & +7.6\,B x^7 & +9.8\,C x^9 & +11.10\,D x^{11} + \text{etc.} \\
 & -5.5\,A & -5.7\,B & -5.9\,C & -5.11\,D \\
 & +5\,A & +5\,B & +5\,C & +5\,D \\
 & & +5.4\,A b & +7.6\,B b & +9.8\,C b \\
 & & +5\,A e & +7\,B e & +9\,C e \\
 & & +A g & +B g & +C g
\end{array}\right\} = 0,$$

posterior vero ad

$$\left.\begin{array}{llllll}
 & & +2.3\mathfrak{B}x^3 & +4.5\mathfrak{C}x^5 & +6.7\mathfrak{D}x^7 & +8.9\mathfrak{E}x^9 + \text{etc.} \\
 & -5\mathfrak{A}x & -5.3\mathfrak{B} & -5.5\mathfrak{C} & -5.7\mathfrak{D} & -5.9\mathfrak{E} \\
 & +5\mathfrak{A} & +5\mathfrak{B} & +5\mathfrak{C} & +5\mathfrak{D} & +5\mathfrak{E} \\
 & & & +2.3\mathfrak{B}b & +4.5\mathfrak{C}b & +6.7\mathfrak{D}b \\
 & & +\mathfrak{A}e & +3\mathfrak{B}e & +5\mathfrak{C}e & +7\mathfrak{D}e \\
 & & +\mathfrak{A}g & +\mathfrak{B}g & +\mathfrak{C}g & +\mathfrak{D}g \\
 & & & +2.5A & +2.7B & +2.9C \\
 & & & -6A & -6B & -6C \\
 & & & & +2.5Ab & +2.7Bb \\
 & & & & -Ab & -Bb \\
 & & & & +Ae & +Be
\end{array}\right\} = 0$$

Inde fit	seu
$12\,B + A(20\,b + 5\,e + g) = 0$	$2.\ 6\,B + A(4.5\,b + 5e + g) = 0,$
$32\,C + B(42\,b + 7\,e + g) = 0$	$4.\ 8\,C + B(6.7\,b + 7e + g) = 0,$
$60\,D + C(72\,b + 9\,e + g) = 0$	$6.\ 10\,D + C(8.9\,b + 9e + g) = 0,$
etc.	etc.

Hinc autem

$$-4\mathfrak{B}+\mathfrak{A}(e+g)=0,$$

$$0\mathfrak{C}+\mathfrak{B}(2.3b+3e+g)+\ 4A=0,$$

$$2.6\mathfrak{D}+\mathfrak{C}(4.5b+5e+g)+\ 8B+A(\ 9b+e)=0,$$

$$4.8\mathfrak{E}+\mathfrak{D}(6.7b+7e+g)+12C+B(13b+e)=0,$$

etc.

Ex prioribus formulis litterae B, C, D, etc. per A determinantur, ex posteriorum vero secunda fit $\mathfrak{B}=\frac{-4A}{2.3b+3e+g}$, ex prima autem $\mathfrak{A}=\frac{4\mathfrak{B}}{e+g}$, tum vero $\mathfrak{C}$ pro lubitu assumi potest, indeque reliqui coëfficientes $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, etc. definiuntur.

Scholion.

980. Exemplum hoc occasionem nobis suppeditat phaenomena quaedam singularia observandi. Scilicet etiamsi integrale completum in genere lx involvat, tamen id a logarithmo liberum prodit certis casibus. Primo nempe si sit $g=-e$, fit $\mathfrak{B}=0$, manente $\mathfrak{A}$ indefinito, tum vero ob $\mathfrak{B}=0$ capi oportet $A=0$, $B=0$, $C=0$ etc. ideoque $v=0$. Porro vero erit

$$2.\ 6\mathfrak{D}+4\mathfrak{C}(5b+e)=0,$$

$$4.\ 8\mathfrak{E}+6\mathfrak{D}(7b+e)=0,$$

$$6.\ 10\mathfrak{F}+8\mathfrak{E}(9b+e)=0,$$

etc.

ubi $\mathfrak{C}$ altera est constans arbitraria, eritque aequationis

$$xx(1+bxx)\partial\partial y+x(-5+exx)\partial x\partial y+(5-exx)y\partial x^2=0,$$

integrale completum

$$y=\mathfrak{A}x-*+\mathfrak{C}x^5+\mathfrak{D}x^7+\mathfrak{E}x^9+\mathfrak{F}x^{11}+\text{etc.}$$

quod adeo finite exprimitur si $e=-(2i+5)b$, pro i sumendo numeros 0, 1, 2, 3, 4, etc.

Secundo si sit

$$2.\ 3b + 3e + g = 0, \text{ seu } g = -6b - 3e,$$

fit $\mathfrak{B} = -\frac{1}{2}\mathfrak{A}(3b+e)$, tum vero $A = 0$, $B = 0$, $C = 0$, etc. $v = 0$. Porro vero reperitur

$$\mathfrak{D} = -\tfrac{1}{6}\mathfrak{C}(7b+e),\ \mathfrak{E} = -\tfrac{1}{8}\mathfrak{D}(9b+e),\ \mathfrak{F} = -\tfrac{1}{10}\mathfrak{E}(11b+e) \text{ etc.}$$

hincque

$$y = \mathfrak{A}x - \tfrac{1}{2}\mathfrak{A}(3b+e)x^3 + \mathfrak{C}x^5 + \mathfrak{D}x^7 + \mathfrak{E}x^9 + \text{etc.}$$

ubi $\mathfrak{A}$ et $\mathfrak{C}$ arbitrio nostro relinquuntur.

Tertio si sit

$$4.\ 5b + 5e + g = 0, \text{ seu } g = -20b - 5e,$$

primo fit $B=0$, $C=0$, $D=0$, etc. ideoque $v = Ax^5$, tum vero $\mathfrak{B} = -\mathfrak{A}(5b+e)$, $-\mathfrak{B}(14b+2e) + 4A = 0$, seu $\mathfrak{B} = \frac{2A}{7b+e}$, que $A = -\frac{1}{2}\mathfrak{A}(5b+e)(7b+e)$, porro

$$2.\ 6\mathfrak{D} + A(9b+e) = 0,$$
$$4.\ 8\mathfrak{E} + 2\mathfrak{D}(11b+e) = 0,$$
$$6.\ 10\mathfrak{F} + 2\mathfrak{E}(13b+e) = 0,$$
etc.

Per $\mathfrak{A}$ ergo definiuntur coëfficientes $\mathfrak{B}$, A, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, etc. quoque arbitrio nostro relinquitur, unde integrale completum casu erit

$$y = Ax^5 lx + \mathfrak{C}x^5 + \mathfrak{A}x + \mathfrak{B}x^3 + * + \mathfrak{D}x^7 + \mathfrak{E}x^9 + \text{ete}$$

quae expressio fit finita quoties $(2i+5)b + e = 0$.

Exemplum 4.

981. *Si in priori exemplo sit* $e = -7b$ et $g = 1$ *aequationis*

$$xx(1+bxx)\partial\partial y - x(5+7bxx)\partial x\partial y + 6(1+3bxx)y\partial x^2 = 0$$

integrale completum algebraice exhibere.

Erit ergo $\mathfrak{B} = + 2\mathfrak{A} b$, $A = 0$, $\mathfrak{D} = 0$, $\mathfrak{E} = 0$, ideoque $v = 0$ et $u = \mathfrak{A} x + 2 \mathfrak{A} b x^3 + \mathfrak{C} x^5$, unde pro $\mathfrak{A}$ et $\mathfrak{C}$ sumendo constantes quascunque, erit integrale completum

$$y = \mathfrak{A} x (1 + 2 b x x) + \mathfrak{C} x^5.$$

Integralia ergo particularia erunt

$$y = a x (1 + 2 b x x),\ y = a x^5,\ y = a x (1 + b x x)^2.$$

Corollarium 1.

982. Posito $y = e^{\int z \partial x}$, ut sit $z = \frac{\partial y}{y \partial x}$, aequationis hujus differentialis primi gradus

$$x x (1 + b x x) \partial z + x x (1 + b x x) z z \partial x - x (5 + 7 b x x) z \partial x + 5 (1 + 3 b x x) \partial x = 0,$$

integrale completum est $z = \frac{\mathfrak{A} (1 + 6 b x x) + 5 \mathfrak{C} x^4}{\mathfrak{A} x (1 + 2 b x x) + \mathfrak{C} x^5}$.

Corollarium 2.

983. Aequatio autem differentio-differentialis integrabilis redditur, si dividatur per $x x (1 + b x x)^2$, eritque integrale

$$\frac{x \partial y - 5 y \partial x}{x (1 + b x x)} C \partial x, \text{ seu } \partial y - \frac{5 y \partial x}{x} = C \partial x (1 + b x x),$$

quae per x^5 divisa integrale praebet

$$\frac{y}{x^5} = \frac{-C}{4 x^4} - \frac{b C}{2 x^2} + D \text{ seu}$$
$$y = -\tfrac{1}{4} C x \left(1 + 2 b x x\right) + D x^5,$$

ut ante.

Scholion.

984. Deficit autem adhuc integratio completa nostrae aequationis generalis per series ascendentes, casu quo $a = 0$, ideoque $\lambda c + f = 0$, unde unicus pro exponente λ valor definitur $\lambda = \frac{-f}{c}$, qui tantum integrale particulare suppeditat, atque hoc etiam tollitur, si fuerit $c = 0$. Quia autem his casibus $a = 0$, coëfficiens b

certo adsit necesse est, ex quo integrale completum per descendentes exhiberi poterit, cum aequatio $\lambda(\lambda-1)b+\lambda e+$ duas semper contineat radices, ex quibus duplex series obti Simile autem hic incommodum usu venire potest, quando radices ipsius λ vel prodeunt aequales, vel differentiam haber exponentem n divisibilem. Verum huic incommodo, seriem p multiplicatam introducendo, simili methodo medela affertur, c hoc problemate sumus usi, ac superfluum foret istam evolut hic repetere. Quodsi autem binae radices ipsius λ tam pro bus ascendentibus quam descendentibus fiant imaginariae, oste dum restat, quomodo integrale completum per series infinitas e oporteat.

Problema 124.

985. Proposita aequatione differentio-differentiali

$$xx(a+bx^n)\partial\partial y+x(c+ex^n)\partial x\partial y+(f+gx^n)y\partial x^2$$

si eveniat ut aequatio

$$\lambda(\lambda-1)a+\lambda c+f=0$$

radices habeat imaginarias, ejus integrale completum per ascendentes exhibere.

Solutio.

Ex supra allatis (971) colligitur hoc casu statui deber

$$y=v\sin.\mu lx+u\cos.\mu lx \text{ unde fit}$$

$$\partial y=\left(\partial v-\frac{\mu u\partial x}{x}\right)\sin.\mu lx+\left(\frac{\mu v\partial x}{x}+\partial u\right)\cos.\mu lx$$

$$\partial\partial y=\left(\partial\partial v-\frac{2\mu\partial x\partial u}{x}+\frac{\mu u\partial x^2}{xx}-\frac{\mu\mu v\partial x^2}{xx}\right)\sin.\mu lx$$

$$+\left(\partial\partial u+\frac{2\mu\partial x\partial v}{x}-\frac{\mu v\partial x^2}{xx}-\frac{\mu\mu u\partial x^2}{xx}\right)\cos.\mu lx,$$

qua facta substitutione, si terminos tam $\sin.\mu lx$ quam co affectos seorsim ad nihilum redigamus, obtinebimus duas se aequationes

$$\text{I.}\quad \left.\begin{aligned} xx(a+bx^n)\,\partial\partial v + x\,(c+ex^n)\,\partial x\partial v + (f+gx^n)\,v\partial x^2 \\ -2\mu x(a+bx^n)\partial x\partial u - \mu\mu(a+bx^n)v\partial x^2 \\ +\mu\,(a+bx^n)\,u\partial x^2 \\ -\mu\,(c+ex^n)\,u\partial x^2 \end{aligned}\right\} = 0$$

$$\text{II.}\quad \left.\begin{aligned} xx(a+bx^n)\,\partial\partial u + x\,(c+ex^n)\,\partial x\partial u + (f+gx^n)\,u\partial x^2 \\ +2\mu x(a+bx^n)\partial x\partial v - \mu\mu(a+bx^n)u\partial x^2 \\ -\mu\,(a+bx^n)\,v\partial x^2 \\ +\mu\,(c+ex^n)\,v\partial x^2 \end{aligned}\right\} = 0$$

Jam pro v et u assumamus has series ascendentes

$$v = \mathrm{A}\,x^\lambda + \mathrm{B}\,x^{\lambda+n} + \mathrm{C}\,x^{\lambda+2n} + \mathrm{D}\,x^{\lambda+3n} + \text{etc.}$$
$$u = \mathfrak{A}\,x^\lambda + \mathfrak{B}x^{\lambda+n} + \mathfrak{C}\,x^{\lambda+2n} + \mathfrak{D}x^{\lambda+3n} + \text{etc.}$$

hisque substitutis, prior aequatio abit in hanc

$\lambda(\lambda-1)\mathrm{A}ax^\lambda$	$+(\lambda+n)(\lambda+n-1)\mathrm{B}ax^{\lambda+n}$	$+(\lambda+2n)(\lambda+2n-1)\mathrm{C}ax^{\lambda+2n}$
	$+\lambda(\lambda-1)\mathrm{A}b$	$+(\lambda+n)(\lambda+n-1)\mathrm{B}b$
$+\lambda\mathrm{A}c$	$+(\lambda+n)\mathrm{B}c$	$+(\lambda+2n)\mathrm{C}c$
	$+\lambda\mathrm{A}e$	$+(\lambda+n)\mathrm{B}e$
$+\mathrm{A}f$	$+\mathrm{B}f$	$+\mathrm{C}f$
	$+\mathrm{A}g$	$+\mathrm{B}g$
$-2\mu\lambda\mathfrak{A}a$	$-2\mu(\lambda+n)\mathfrak{B}a$	$-2\mu(\lambda+2n)\mathfrak{C}a$
	$-2\mu\lambda\mathfrak{A}b$	$-2\mu(\lambda+n)\mathfrak{B}b$
$-\mu\mu\mathrm{A}a$	$+\mu\mu\mathrm{B}a$	$-\mu\mu\mathrm{C}a$
	$-\mu\mu\mathrm{A}b$	$-\mu\mu\mathrm{B}b$
$+\mu\mathfrak{A}a$	$+\mu\mathfrak{B}a$	$+\mu\mathfrak{C}a$
	$+\mu\mathfrak{A}b$	$+\mu\mathfrak{B}b$
$-\mu\mathfrak{A}c$	$-\mu\mathfrak{B}c$	$-\mu\mathfrak{C}c$
	$-\mu\mathfrak{A}e$	$-\mu\mathfrak{B}e$

Hinc altera aequatio facile formatur permutandis litteris latinis germanicis atque insuper signum numeri μ mutando.

Utrinque ergo potestas prima x^λ exigit has aequationes

$$A[\lambda(\lambda-1)a+\lambda c+f-\mu\mu a]-\mu\mathfrak{A}(2\lambda a-a+c)=0$$

$$\mathfrak{A}[\lambda(\lambda-1)a+\lambda c+f-\mu\mu a]+\mu A(2\lambda a-a+c)=0,$$

unde necesse est ut sit

$$\text{tam } 2\lambda a-a+c=0$$

$$\text{quam } \lambda(\lambda-1)a+\lambda c+f-\mu\mu a=0.$$

Inde fit $\lambda=\frac{1}{2}-\frac{c}{2a}$, qui valor hic substitutus dat

$$-a\left(\tfrac{1}{6}-\frac{cc}{4aa}\right)+\frac{c}{2}-\frac{cc}{2a}+f=\mu\mu a=-\frac{a}{4}+\frac{c}{2}-\frac{cc}{4a}+f, \text{ s}$$

$$\mu\mu a=\frac{4af-(a-c)^2}{4a}, \text{ ideoque}$$

$$\mu=\frac{\sqrt{[4af-(a-c)^2]}}{2a} \text{ et } \lambda=\frac{a-c}{2a}.$$

Unde patet hanc solutionem locum habere si $4af>(a-c$ quo ipso casu praecedens solutio fiebat imaginaria. Hic aute quantitates A et $\mathfrak{A}$ arbitrio nostro relinquuntur.

Terminus vero $x^{\lambda+n}$ utrinque postulat has aequationes

$$B[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a]+A[\lambda(\lambda-1)b+\lambda e+g-\mu\mu b$$
$$-\mu\mathfrak{B}[2(\lambda+n)a-a+c]-\mu\mathfrak{A}(2\lambda b-b+e)=0$$

$$\mathfrak{B}[(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a]+\mathfrak{A}[\lambda(\lambda-1)b+\lambda e+g-\mu\mu b$$
$$+\mu B[2(\lambda+n)a-a+c]+\mu A(2\lambda b-b+e)=0$$

Sit brevitatis gratia

$$(\lambda+n)(\lambda+n-1)a+(\lambda+n)c+f-\mu\mu a=nna=$$

$$\lambda(\lambda-1)b+\lambda e+g-\mu\mu b=\beta$$

$$2(\lambda+n)a-a+c=2na=\gamma$$

$$2\lambda b-b+e=\delta,$$

habeamus

$$B\alpha + A\beta - \mu\mathfrak{B}\gamma - \mu\mathfrak{A}\delta = 0 \text{ et}$$

$$\mathfrak{B}\alpha + \mathfrak{A}\beta + \mu B\gamma + \mu A\delta = 0,$$

unde colligitur

$$B = \frac{-A(\alpha\beta + \mu\mu\gamma\delta) + \mu\mathfrak{A}(\alpha\delta - \beta\gamma)}{\alpha\alpha + \mu\mu\gamma\gamma} \text{ et}$$

$$\mathfrak{B} = \frac{-\mathfrak{A}(\alpha\beta + \mu\mu\gamma\delta) - \mu A(\alpha\delta - \beta\gamma)}{\alpha\alpha + \mu\mu\gamma\gamma}.$$

At vero ex valoribus assumtis est

$$\alpha = nna,\ \beta = \frac{(ae - bc)(a - c)}{2aa} - \frac{bf}{a} + g,\ \gamma = 2na,\ \delta = \frac{ae - bc}{a},$$

unde ex assumtis A et $\mathfrak{A}$ definiuntur B et $\mathfrak{B}$, hincque porro C, $\mathfrak{C}$; D, $\mathfrak{D}$ etc.

Exemplum 1.

986. *Sit $c = a$ et $f = a$, ut fiat $\mu = 1$, et investigetur integrale hujus aequationis*

$$xx(a + bx^n)\partial\partial y + x(a + ex^n)\partial x\partial y + (a + gx^n)y\partial x^2 = 0.$$

Hic ergo erit $\lambda = 0$ et $\mu = 1$, unde posito

$$y = v \sin. lx + u \cos. lx,$$

ac pro v et u sumtis seriebus

$$v = A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.}$$

$$u = \mathfrak{A} + \mathfrak{B}x^n + \mathfrak{C}x^{2n} + \mathfrak{D}x^{3n} + \text{etc.}$$

coëfficientes A et $\mathfrak{A}$ pro lubitu accipi possunt. Ex iis primo, ob $\alpha = nna$, $\beta = g - b$, $\gamma = 2na$ et $\delta = e - b$, erit

$$B = \frac{-A[nna(g-b) + 2na(e-b)] + \mathfrak{A}[nna(e-b) - 2na(g-b)]}{n^4aa + 4nnaa} \text{ seu}$$

$$B = \frac{-A[n(g-b) + 2(e-b)] + \mathfrak{A}[n(e-b) - 2(g-b)]}{na(nn+4)} \text{ et}$$

$$\mathfrak{B} = \frac{-\mathfrak{A}[n(g-b) + 2(e-b)] - A[n(e-b) - 2(g-b)]}{na(nn+4)}.$$

Pro sequentibus coëfficientibus habebimus

$$C\,[2n(2n-1)a+2na+a-a]+B\,[n(n-1)b+ne+g-b]$$
$$-\mathfrak{C}(4na-a+a)-\mathfrak{B}(2nb-b+e)=0, \text{ seu}$$

$$4nnCa+B\,[(nn-n-1)b+ne+g]-4n\mathfrak{C}a$$
$$-\mathfrak{B}\,[(2n-1)b+e]=0, \text{ et}$$

$$4nn\mathfrak{C}a+\mathfrak{B}\,[(nn-n-1)b+ne+g]+4nCa$$
$$+B\,[(2n-1)b+e]=0,$$

quarum illa per n multiplicata huic addatur, ut prodeat

$$4n(nn+1)Ca+B\,[(n^3-nn+n-1)b+(nn+1)e+ng]$$
$$+\mathfrak{B}\,[-(nn+1)b+g]=0, \text{ hinc}$$

$$C=\frac{-B\,[(n-1)(nn+1)b+(nn+1)e+ng]+\mathfrak{B}\,[(nn+1)b-g]}{4na(nn+1)} \text{ et}$$

$$\mathfrak{C}=\frac{-\mathfrak{B}\,[(n-1)(nn+1)b+(nn+1)e+ng]-B\,[(nn+1)b-g]}{4na(nn+1)}.$$

Porro erit

$$9nnDa+C\,[(4nn-2n-1)b+2ne+g]+6n\mathfrak{D}a$$
$$-\mathfrak{C}\,[(4n-1)b+e]=0$$

$$9nn\mathfrak{D}a+\mathfrak{C}\,[(4nn-2n-1)b+2ne+g]+6nDa$$
$$+C\,[(4n-1)b+e]=0,$$

quarum illa per $3n$, haec vero per 2 multiplicata junctim dant

$$3n(9nn+4)Da+C\,[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]$$
$$+\mathfrak{C}\,[(-4nn-n-2)b+ne+2g]=0;$$

unde sequitur

$$D=\frac{-C\,[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]+\mathfrak{C}\,[(4nn+n+2)b-ne-2g]}{3n(9nn+4)a}$$

$$\mathfrak{D}=\frac{-\mathfrak{C}\,[(12n^3-6nn+5n-2)b+2(3nn+1)e+3ng]-C\,[(4nn+n+2)b-ne-2g]}{3n(9nn+4)a}.$$

In genere autem ex coëfficientibus quibuscunque M et $\mathfrak{M}$ sequentes N et $\mathfrak{N}$ definiuntur per has formulas

$$in(iinn+4)\mathrm{N}a$$
$$+\mathrm{M}[[i(i-1)^2n^3-i(i-1)nn+(3i-4)n-2]b+i(i-1)nne+2e+ing]$$
$$-\mathfrak{M}[[2(i-1)nn+(i-2)n+2]b-(i-2)ne-2g]=0$$
$$in(iinn+4)\mathfrak{N}a$$
$$+\mathfrak{M}[[i(i-1)^2n^3-i(i-1)nn+(3i-4)n-2]b+i(i-1)nne+2e+ing]$$
$$+\mathrm{M}[[2(i-1)nn+(i-2)n+2]b-(i-2)ne-2g]=0$$

Corollarium 1.

987. Si quantitates b, e, g ita sint comparatae, ut binae litterae sibi respondentes N et $\mathfrak{N}$ evanescant, sequentes omnes evanescent, et integrale completum forma finita exprimetur. Ita ut B et $\mathfrak{B}$ evanescant, fieri debet

$$2(g-b)=n(e-b) \text{ et } n(g-b)=-2(e-b);$$

unde fit $g=e=b$, et ipsa aequatio proposita factorem habebit $a+bx^n$.

Corollarium 2.

988. In genere autem integrale finite exprimetur, si denotante i numerum integrum quemcunque positivum sit

$$g=[(i-1)nn+\tfrac{1}{2}(i-2)n+1]b-\tfrac{1}{2}(i-2)ne,$$

tum vero

$$e=-[2(i-1)n-1]b,$$

unde fit

$$g=[(i-1)^2nn+1]b.$$

Exemplum 2.

989. *Sumto* $n=1$, *si sit* $e=-b$ et $g=2b$, *hujus aequationis*

$$xx(a+bx)\partial\partial y+x(a-bx)\partial x\partial y+(a+2bx)y\partial x^2=0$$

integrale completum assignare.

Ex formulis modo inventis colligimus

$$B = \frac{-A(g+2e-3b)+\mathfrak{A}(e+b-2g)}{5a} = \frac{3Ab-4\mathfrak{A}b}{5a} \text{ et}$$

$$\mathfrak{B} = \frac{5\mathfrak{A}b+4Ab}{5a};$$

tum vero

$$C = \frac{-B(2e+g)+\mathfrak{B}(2b-g)}{5a} = 0 \text{ et } \mathfrak{C} = 0.$$

Quocirca habebimus

$$v = A + \frac{(3A-4\mathfrak{A})b}{5a}x, \text{ et } u = \mathfrak{A} + \frac{(3\mathfrak{A}+4A)b}{5a}x;$$

hincque integrale completum elicitur

$$y = A\sin.lx + \mathfrak{A}\cos.lx + \tfrac{bx}{5a}[(3A-4\mathfrak{A})\sin.lx + (3\mathfrak{A}+4A)\cos.lx].$$

Corollarium 1.

990. Sumto $\mathfrak{A} = 0$, habebitur integrale particulare

$$y = A\ (\sin.lx + \frac{3bx}{5a}\sin.lx + \frac{4bx}{5a}\cos.lx).$$

Sin autem sit $A = 0$, aliud habebitur

$$y = \mathfrak{A}\ (\cos.lx - \frac{4bx}{5a}\sin.lx + \frac{3bx}{5a}\cos.lx).$$

Corollarium 2.

991. Posito $y = e^{\int s\partial x}$, aequatio nostra reducitur ad hanc

$$xx(a+bx)\partial s + xx(a+bx)ss\partial x + x(a-bx)s\partial x + (a+2bx)\partial x = 0,$$

cujus integrale habetur $s = \frac{\partial y}{y\partial x}$ inde definiendum, quae aequatio in plures alias formas transfundi potest.

Scholion.

992. Simili modo integratio per series descendentes instituitur, si exponentes singulorum terminorum prodeant imaginarii; quod

seorsim exposuisse ne opus quidem erit. Atque haec sufficiunt, ut pateat, quibusnam cautelis in resolutione aequationum per series infinitas sit utendum. Summus autem usus istarum evolutionum in hoc consistit, ut aequationes differentio-differentiales exhiberi queant, quarum saltem integrale particulare algebraicum assignare liceat, quos casus supra §. 969 indicavimus. Similis porro integratio per series infinitas pari modo extendi potest ad hujusmodi aequationes

$$x\,x\,(a + b\,x^n + \beta\,x^{2n})\,\partial\partial y + x\,(c + e\,x^n + \varepsilon\,x^{2n})\,\partial x\,\partial y$$
$$+ (f + g\,x^n + \gamma\,x^{2n})\,y\,\partial x^2 = 0$$

tum autem seriei quaesitae quilibet terminus per duos praecedentes determinatur, ita ut si bini contigui evanescant, sequentes omnes in nihilum sint abituri. Quodsi autem terminus ab y vacuus affuerit, resolutio in series fit facilior, cui propterea non immorandum censeo. Veluti si proponatur haec aequatio

$$x\,x\,\partial\partial y - x\,\partial x\,\partial y + a\,x^n\,y\,\partial x^2 = b\,x^m\,\partial x^2,$$

series a potestate x^m est inchoanda, ponendo

$$y = \mathrm{A}\,x^m + \mathrm{B}\,x^{m+n} + \mathrm{C}\,x^{m+2n} + \mathrm{D}\,x^{m+3n} + \text{etc}$$

unde fit

$$\begin{array}{llll} m(m-1)\mathrm{A}x^m & +(m+n)(m+n-1)\mathrm{B}x^{m+n} & +(m+2n)(m+2n-1)\mathrm{C}x^{m+2n} & + \text{etc.} \\ -m\mathrm{A} & -(m+n)\mathrm{B} & -(m+n)\mathrm{C} & \\ -b & +\mathrm{A}a & +\mathrm{B}a & \end{array}$$

hincque

$$\mathrm{A} = \frac{b}{m\,(m-2)},\quad \mathrm{B} = \frac{-\mathrm{A}\,a}{(m+n)\,(m+n-2)},\quad \mathrm{C} = \frac{-\mathrm{B}\,a}{(m+2n)\,(m+2n-2)},\ \text{etc.}$$

ubi quidem multa observanda occurrunt, quae per praecepta supra data expedire licet. Imprimis autem in hoc negotio juvat, aequationem propositam ope substitutionis in alias transformasse, quarum resolutio per series fiat simplicior, quod cum pluribus modis fieri possit, hoc argumentum sequenti capite diligentius pertractare visum est, idque pro forma aequationum

$$L\partial\partial y + M\partial x\partial y + Ny\partial x^2 = 0,$$

quandoquidem pro aliis formis hujusmodi transformatio raro locum invenit.