CAPUT VII.

DE RESOLUTIONE AEQUATIONIS

$\partial y y + a x^n y \partial x^2 = 0$

PER SERIES INFINITAS.

Problema 117.

929.

Sumto elemento ∂x constante, aequationem differentio-differentialem $\partial \partial y + a x^n y \partial x^2 = 0$ per seriem infinitam integrare.

Solutio.

Quaerimus hic seriem secundum potestates ipsius x progredientem, quae valorem ipsius y exprimat; et quia in altero aequationis nostrae termino quantitas x cum suo differentiali ∂x nullam, in altero vero $n + 2$ dimensiones occupat, evidens est exponentes potestatum ipsius x differentia $n + 2$ ascendere vel descendere debere

I. Ascendant primo exponentes, et fingatur series

$$y = A x^\lambda + B x^{\lambda + n + 2} + C x^{\lambda + 2n + 4} + \text{etc.}$$

eritque

$$\frac{\partial \partial y}{2} = \lambda(\lambda - 1) A x^{\lambda - 2} + (\lambda + n + 2)(\lambda + n + 1) B x^{\lambda + n} + \text{etc.}$$

$$a x^n y = \quad . \quad . \quad . \quad . \quad a A x^{\lambda + n}$$

unde patet primum terminum solitarium evanescere debere, ut sit $\lambda(\lambda - 1) = 0$. Quare capi oportet vel $\lambda = 0$ vel $\lambda = 1$, sicque duplex series obtinetur

$$y = A + B x^{n+2} + C x^{2n+4} + D x^{3n+6} + E x^{4n+8} + \text{etc.}$$
$$+ \mathfrak{A} x + \mathfrak{B} x^{n+3} + \mathfrak{C} x^{2n+5} + \mathfrak{D} x^{3n+7} + \mathfrak{E} x^{4n+9} + \text{etc.}$$

substitutione ergo facta fieri oportet

$$0 = (n+2)(n+1)Bx^n + (2n+4)(2n+3)Cx^{2n+2} + (3n+6)(3n+5)Dx^{3n+4} + \text{etc.}$$
$$+ aA \quad + \quad aB \quad + aC$$
$$0 = (n+3)(n+2)\mathfrak{B}x^{n+1} + (2n+5)(2n+4)\mathfrak{C}x^{2n+3} + (3n+7)(3n+6)\mathfrak{D}x^{3n+5} + \text{etc.}$$
$$+ a\mathfrak{A} \quad + a\mathfrak{B} \quad + a\mathfrak{C}$$

unde litteris A et $\mathfrak{A}$ arbitrio nostro relictis, reliquae per eas ita determinantur

$$B = \frac{-aA}{(n+1)(n+2)};\ C = \frac{-aB}{2(2n+3)(n+2)};\ D = \frac{-aC}{3(3n+5)(n+2)}\ \text{etc.}$$
$$\mathfrak{B} = \frac{-a\mathfrak{A}}{(n+3)(n+2)};\ \mathfrak{C} = \frac{-a\mathfrak{B}}{2(2n+5)(n+2)};\ \mathfrak{D} = \frac{-a\mathfrak{C}}{3(3n+7)(n+2)}\ \text{etc.}$$

sicque habebitur integrale completum ita expressum

$$y = A - \frac{aAx^{n+2}}{1(n+1).(n+2)} + \frac{a^2 A x^{2n+4}}{1.2(n+1)(2n+3).(n+2)^2}$$
$$- \frac{a^2 A x^{3n+6}}{1.2.3(n+1)(2n+3)(3n+5).(n+2)^3} + \text{etc.}$$
$$+ \mathfrak{A} x - \frac{a\mathfrak{A} x^{n+3}}{1(n+3).(n+2)} + \frac{a^2 \mathfrak{A} x^{2n+5}}{1.2(n+3)(2n+5).(n+2)^2}$$
$$- \frac{a^3 \mathfrak{A} x^{3n+7}}{1.2.3(n+3)(2n+5)(3n+7).(n+2)^3} + \text{etc.}$$

II. Descendant jam exponentes, et ficta serie

$$y = A x^{\lambda} + B x^{\lambda-n-2} + C x^{\lambda-2n-4} + \text{etc.}$$

habebitur

$$\frac{\partial\partial y}{\partial x^2} = \lambda(\lambda-1) A x^{\lambda-2} + (\lambda-n-2)(\lambda-n-3) B x^{\lambda-n-4} + \text{etc.}$$
$$a x^n y = a A x^{\lambda+n} + a B x^{\lambda-2} + \text{etc.}$$

ubi cum terminus $x^{\lambda+n}$ sui similem non habeat, tolli nequit, ita ut hinc nulla aequationis resolutio obtineatur.

Corollarium 1.

930. Geminata series pro y inventa, quoniam litterae A et $\mathfrak{A}$ arbitrio nostro relinquuntur, integrale completum aequationis differentio-differentialis $\partial\partial y + ax^n y\partial x^2 = 0$ exhibet; tribuendo autem litteris A et $\mathfrak{A}$ datos valores, integralia particularia nascentur.

Corollarium 2.

931. Si ponamus $n+2=m$, seu $n=m-2$, hujus aequationis $\partial\partial y + ax^{m-2}y\partial x^2 = 0$ integrale completum ita commodius exprimetur

$$y = \begin{cases} A - \dfrac{a A x^m}{1(m-1).m} + \dfrac{a^2 A x^{2m}}{1.2(m-1)(2m-1).m^2} \\ - \dfrac{a^3 A x^{3m}}{1.2.3(m-1)(2m-1)(3m-1).m^3} + \text{etc.} \\ \mathfrak{A}x - \dfrac{a\mathfrak{A}x^{m+1}}{1(m+1).m} + \dfrac{a^2\mathfrak{A}x^{2m+1}}{1.2(m+1)(2m+1).m^2} \\ - \dfrac{a^3\mathfrak{A}x^{3m+1}}{1.2.3(m+1)(2m+1)(3m+1).m^3} + \text{etc.} \end{cases}$$

Corollarium. 3.

932. Si exponens m fuerit positivus et unitate major, hae series eo magis convergunt, quo minor valor quantitati x tribuatur: aliis vero casibus in praxi hae series adhiberi nequeunt, nisi forte eae ipsae in alias convergentes transformari possint.

Scholion 1.

933. Dantur tamen casus, quibus hae series omni plane usu destituuntur, quod evenit, si quispiam factorum denominatores con-

stituentium evanescat, sicque omnes termini sequentes in infinitum excrescant, quibus casibus series in alias formas transmutari convenit. Hic primo occurrit casus $m = 0$ seu $n = -2$, quo utriusque seriei omnes termini praeter primos fiunt infiniti, hoc vero casu aequatio, quae est $\partial\partial y + \frac{a y \partial x^2}{x^2} = 0$, cum sit homogenea, singularem integrationem admittit: inveniri enim potest potestas ipsius x, quae pro y substituta aequationi satisfacit. Ponatur scilicet $y = x^\lambda$, prodibitque

$$\lambda(\lambda - 1) x^{\lambda - 2} + a x^{\lambda - 2} = 0, \text{ seu } \lambda\lambda - \lambda + a = 0$$

unde colligitur $\lambda = \frac{1}{2} \pm \sqrt{(\frac{1}{4} - a)}$, ob quem duplicem valorem est integrale completum

$$y = A x^{\frac{1}{2} + \sqrt{(\frac{1}{4} - a)}} + B x^{\frac{1}{2} - \sqrt{(\frac{1}{4} - a)}},$$

quae aequatio casu $a > \frac{1}{4}$ abit in hanc formam

$$y = A x^{\frac{1}{2}} \sin. [(a - \tfrac{1}{4})^{\frac{1}{2}} l x + \alpha];$$

unde patet, casu $a = \frac{1}{4}$ fore

$$y = (A + B\, l x) \sqrt{x}.$$

Scholion 2.

934. Reliqui casus ad incommodum ducentes sunt, si vel $m = \frac{1}{i}$ vel $m = -\frac{1}{i}$ denotante i numerum quemcunque integrum. Casu $m = \frac{1}{i}$ prior tantum series fit incongrua, casu vero $m = -\frac{1}{i}$ posterior tantum. Quare illo casu ponendo $A = 0$ hoc vero $\mathfrak{A} = 0$, series saltem una idonea habetur, integrale particulare exhibens. Verum cognito integrali particulari quod sit $y = P$, inde aequationis

$$\partial\partial y + a x^{m-2} y \partial x^2 = 0$$

integrale completum eruitur ponendo $y = P z$, unde fit

$$P\partial\partial z + 2\partial P\partial z + z\partial\partial P + a x^{m-2} P z \partial x^2 = 0,$$

at per hypothesin est

$$\partial\partial P + a x^{m-2} P \partial x^2 = 0,$$

ergo prodit

$$P\partial\partial z + 2\partial P\partial z = 0, \text{ seu}$$

$$PP\partial z = C\partial x \text{ et } z = C\int\frac{\partial x}{PP}.$$

Cum autem P sit series infinita, hinc valorem ipsius z cognoscere haud licet. At casibus illis memoratis pars integralis logarithmum ipsius x involvit, quod vel inde intelligitur quod $\frac{x^0}{0}$ aequivaleat ipsi lx. Quare in aequatione

$$\partial\partial y + a x^{m-2} y \partial x^2 = 0,$$

ponendo $y = p + q\,lx$, ob $\partial y = \partial p + \frac{q\partial x}{x} + \partial q\,lx$, erit

$$\partial\partial p + \frac{2\partial x\partial q}{x} - \frac{q\partial x^2}{xx} + \partial\partial q\,lx + a p x^{m-2}\partial x^2$$
$$+ a q x^{m-2}\partial x^2\,lx = 0;$$

in qua partes lx involventes seorsim destruantur necesse est; ita ut hae binae habeantur aequationes

$$\partial\partial q + a q x^{m-2}\partial x^2 = 0 \text{ et}$$
$$\partial\partial p + \frac{2\partial x\partial q}{x} - \frac{q\partial x^2}{xx} + a p x^{m-2}\partial x^2 = 0,$$

ubi pro q ea binarum superiorum serierum accipi debet, quae casu oblato incommodo caret, eaque constituta ex posteriori aequatione facile quantitas p per seriem exprimetur. Hujusmodi casus in sequentibus exemplis evolvamus; tantum notemus illam operationem perinde se habere, etiamsi loco lx sumatur $lx + \alpha$, ita ut inventis p et q futurum sit

$$y = \alpha q + p + q\,lx, \text{ seu } y = p + q\,l\beta x.$$

Exemplum 1.

935. *Posito* $m = 1$, *hanc aequationem*

$$\partial \partial y + \frac{a y \partial x^2}{x} = 0$$

per series resolvere.

Posito $y = p + q\, l x$, capi oportet

$$q = \mathfrak{A} x - \frac{a \mathfrak{A} x^2}{1.2} + \frac{a^2 \mathfrak{A} x^2}{1.2.2.3} - \frac{a^3 \mathfrak{A} x^4}{1.2.2.3.3.4} + \text{etc.}$$

pro qua ponamus brevitatis gratia

$$q = \mathfrak{A} x + \mathfrak{B} x^2 + \mathfrak{C} x^3 + \mathfrak{D} x^4 + \text{etc.}$$

tum vero quaeratur p ex hac aequatione

$$\frac{\partial \partial p}{\partial x^2} + \frac{2 \partial q}{x \partial x} - \frac{q}{x x} + \frac{a p}{x} = 0;$$

fingamus ergo

$$p = \mathrm{A} + \mathrm{B} x + \mathrm{C} x^2 + \mathrm{D} x^3 + \mathrm{E} x^4 + \text{etc.}$$

eritque facta substitutione

$$\left.\begin{array}{llllll} & 2\mathrm{C} & + 6 \mathrm{D} x & + 12 \mathrm{E} x x & + 20 \mathrm{F} x^3 & + 30 \mathrm{G} x^4 \\ + 2 \mathfrak{A} \frac{1}{x} & + 4 \mathfrak{B} & + 6 \mathfrak{C} & + 8 \mathfrak{D} & + 10 \mathfrak{E} & + 12 \mathfrak{F} \\ - \mathfrak{A} & - \mathfrak{B} & - \mathfrak{C} & - \mathfrak{D} & - \mathfrak{E} & - \mathfrak{F} \\ + a \mathrm{A} & + a \mathrm{B} & + a \mathrm{C} & + a \mathrm{D} & + a \mathrm{E} & + a \mathrm{F} \end{array}\right\} = 0$$

Cum jam dentur coëfficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc. erit $\mathrm{A} = -\frac{\mathfrak{A}}{a}$ quantitas B non determinatur, tum vero

$$\mathrm{C} = \frac{-3 \mathfrak{B}}{2} - \frac{a \mathrm{B}}{1.2} = \frac{+3 a \mathfrak{A}}{1^3.2^2} - \frac{a \mathrm{B}}{1.2},$$

$$\mathrm{D} = \frac{-5 \mathfrak{C}}{6} - \frac{a \mathrm{C}}{2.3} = \frac{-5 a^2 \mathfrak{A}}{1^2.2^3.3^2} - \frac{a \mathrm{C}}{2.3},$$

$$\mathrm{E} = \frac{-7 \mathfrak{D}}{12} - \frac{a \mathrm{D}}{3.4} = \frac{+7 a^3 \mathfrak{A}}{1^2.2^2.3^3.4^2} - \frac{a \mathrm{D}}{3.4},$$

$$\mathrm{F} = \frac{-9 \mathfrak{E}}{20} - \frac{a \mathrm{E}}{4.5} = \frac{-9 a^4 \mathfrak{A}}{1^2.2^2.3^2.4^3.5^2} - \frac{a \mathrm{E}}{4.5}, \text{ etc.}$$

ubi pro B scribere licet 0, quandoquidem in integrali $y = p + q\, l x$ addimus partem $a q$, quae ex littera B oritur, ita ut sit

$$p = \mathrm{A} + \mathrm{C} x^2 + \mathrm{D} x^3 + \mathrm{E} x^4 + \mathrm{F} x^5 + \text{etc.}$$

Hinc erit

$$C = \frac{3\,\alpha\,\mathfrak{A}}{1^3 . 2^2},\quad D = \frac{-14\,\alpha^2\,\mathfrak{A}}{1^3 . 2^3 . 3^2},\quad E = \frac{+70\,\alpha^3\,\mathfrak{A}}{1^3 . 2^3 . 3^3 . 4^2},\quad F = \frac{-404\,\alpha^4\,\mathfrak{A}}{1^3 . 2^3 . 3^3 . 4^3 . 5^2};\ \text{etc.}$$

ubi notetur esse

$$14 = 3.3 + 5.1,\quad 70 = 4.14 + 7.1.2,\quad 404 = 5.70 + 9.1.2.3,$$

et pro sequente $2688 = 6.404 + 11.1.2.3.4$; estque

$$y = p + \alpha q + q l x.$$

Exemplum 2.

936. *Posito* $m = -1$ *hanc aequationem*

$$\partial \partial y + \frac{\alpha y \partial x^2}{x^3} = 0,$$

per series resolvere.

Posito $y = p + \alpha q + q l x$, capi oportet

$$q = A - \frac{\alpha A}{1.2x} + \frac{\alpha^2 A}{1.2^2.3x^2} - \frac{\alpha^3 A}{1.2^2.3^2.4x^3} + \text{etc.}$$

pro qua ponatur brevitatis gratia

$$q = A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \frac{E}{x^4} + \text{etc.}$$

tum vero quantitas p ex hac aequatione definiri debet

$$\partial \partial p + \frac{2\partial q \partial x}{x} - \frac{q \partial x^2}{xx} + \frac{\alpha p \partial x^2}{x^3} = 0.$$

Fingamus ergo

$$p = \mathfrak{A} x + \mathfrak{B} + \frac{\mathfrak{C}}{x} + \frac{\mathfrak{D}}{xx} + \frac{\mathfrak{E}}{x^3} + \frac{\mathfrak{F}}{x^4} + \text{etc.}$$

unde facta substitutione prodit

$$\left.\begin{array}{cccccc} \frac{\alpha\mathfrak{A}}{xx} + & \frac{\alpha\mathfrak{B}}{x^3} + & \frac{\alpha\mathfrak{C}}{x^4} + & \frac{\alpha\mathfrak{D}}{x^5} + & \frac{\alpha\mathfrak{E}}{x^6} + & \frac{\alpha\mathfrak{F}}{x^7} + \text{etc.} \\ -A - & B - & C - & D - & E - & F \\ & -2B - & 4C - & 6D - & 8E - & 10F \\ & +2\mathfrak{C} + & 6\mathfrak{D} + & 12\mathfrak{E} + & 20\mathfrak{F} + & 30\mathfrak{G} \end{array}\right\} = 0;$$

et coëfficientes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, etc. ita determinantur, ut sit

$\mathfrak{A} = \frac{A}{a}$; secundus $\mathfrak{B}$ non definitur, tum vero est

$$\mathfrak{C} = \frac{3\,B}{1 \cdot 2} - \frac{a\mathfrak{B}}{1 \cdot 2} = \frac{-3\,a\,A}{1^3 \cdot 2^2} - \frac{a\,\mathfrak{B}}{1 \cdot 2},$$

$$\mathfrak{D} = \frac{5\,C}{2 \cdot 3} - \frac{a\,\mathfrak{C}}{2 \cdot 3} = \frac{5\,a^2\,A}{2^3 \cdot 3^2} - \frac{a\,\mathfrak{C}}{2 \cdot 3},$$

$$\mathfrak{E} = \frac{7\,D}{3 \cdot 4} - \frac{a\,\mathfrak{D}}{3 \cdot 4} = \frac{-\,a^3\,A}{2^2 \cdot 3^3 \cdot 4^2} - \frac{a\,\mathfrak{D}}{3 \cdot 4},$$

$$\mathfrak{F} = \frac{9\,E}{4 \cdot 5} - \frac{a\,\mathfrak{E}}{4 \cdot 5} = \frac{9\,a^4\,A}{2^2 \cdot 3^2 \cdot 4^3 \cdot 5^2} - \frac{a\,\mathfrak{E}}{4 \cdot 5}, \text{ etc.}$$

si sumatur, id quod sine detrimento generalitatis fieri licet, $\mathfrak{B} = 0$, ita ut sit

$$p = \mathfrak{A}\,x + \frac{\mathfrak{C}}{x} + \frac{\mathfrak{D}}{xx} + \frac{\mathfrak{E}}{x^3} + \frac{\mathfrak{F}}{x^4} + \text{etc. erit}$$

$$\mathfrak{C} = \frac{-3\,a\,A}{1^3 \cdot 2^2},\; \mathfrak{D} = \frac{+14\,a^2\,A}{1^3 \cdot 2^3 \cdot 3^2},\; \mathfrak{E} = \frac{-70\,a^3\,A}{1^3 \cdot 2^3 \cdot 3^3 \cdot 4^2},$$

$$\mathfrak{F} = \frac{+404\,a^4\,A}{1^3 \cdot 2^3 \cdot 3^3 \cdot 4^3 \cdot 5^2}, \text{ etc.}$$

qui valores similes sunt praecedentibus.

Exemplum 3.

937. *Posito* $m = \frac{1}{2}$, *hanc aequationem*

$$\partial\partial y + \frac{a y \partial x^2}{x \sqrt{x}} = 0$$

per series resolvere.

Posito $y = p + \alpha q + q\,lx$, capi oportet

$$q = \mathfrak{A}x - \frac{4\,a\,\mathfrak{A}}{1 \cdot 3}\,x^{\frac{3}{2}} + \frac{16\,a^2\,\mathfrak{A}}{1 \cdot 2 \cdot 3 \cdot 4}\,x^2 - \frac{64\,a^3\,\mathfrak{A}}{1 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 5}\,x^{\frac{5}{2}}$$

$$+ \frac{256\,a^4\,\mathfrak{A}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 3 \cdot 4 \cdot 5 \cdot 6}\,x^3 - \text{etc.}$$

pro qua brevitatis gratia scribatur

$$q = \mathfrak{A}x + \mathfrak{B}x^{\frac{3}{2}} + \mathfrak{C}x^2 + \mathfrak{D}x^{\frac{5}{2}} + \mathfrak{E}x^3 + \mathfrak{F}x^{\frac{7}{2}} + \mathfrak{G}x^4 + \text{etc.}$$

tum vero quantitas p ex hac aequatione definiri debet:

$$\partial\partial p + \frac{2\partial x \partial q}{x} - \frac{q \partial x^2}{xx} + \frac{\alpha p \partial x^2}{x \sqrt{x}} = 0.$$

Fingamus ergo

$$p = \Delta + A x^{\frac{1}{2}} + B x + C x^{\frac{3}{2}} + D x^{2} + E x^{\frac{5}{2}} + F x^{3} + \text{etc.}$$

prodibitque facta substitutione

$$\left.\begin{array}{llllllll} \frac{a\Delta}{x\sqrt{x}} & +\frac{aA}{x} & +\frac{aB}{\sqrt{x}} & +aC & +aD\sqrt{x} & +aEx & +aFx\sqrt{x} \\ & -\mathfrak{A} & -\mathfrak{B} & -\mathfrak{C} & -\mathfrak{D} & -\mathfrak{E} & -\mathfrak{F} \\ & +2\mathfrak{A} & +3\mathfrak{B} & +4\mathfrak{C} & +5\mathfrak{D} & +6\mathfrak{E} & +7\mathfrak{F} \\ & -\frac{A}{4} & +0 & +\frac{3}{4}C & +2D & +\frac{15}{4}E & +6F & +\frac{35}{4}G \end{array}\right\} = 0.$$

Hinc colligitur fore

$$A = \frac{-\mathfrak{A}}{a}, \quad \Delta = \frac{-\mathfrak{A}}{4aa},$$

et B non determinatur, porro

$$C = \frac{-4aB}{1.3} - \frac{8\mathfrak{B}}{1.3} = \frac{-4aB}{1.3} + \frac{32a\mathfrak{A}}{1^2.3^2},$$

$$D = \frac{-4aC}{2.4} - \frac{12\mathfrak{C}}{2.4} = \frac{-4aC}{2.4} - \frac{12.16.a^2\mathfrak{A}}{1.3.2^2.4^2},$$

$$E = \frac{-4aD}{3.5} - \frac{16\mathfrak{D}}{3.5} = \frac{-4aD}{3.5} + \frac{16.64.a^3\mathfrak{A}}{1.3.2.4.3^2.5^2},$$

$$F = \frac{-4aE}{4.6} - \frac{20\mathfrak{E}}{4.6} = \frac{-4aE}{4.6} - \frac{20.256.a^4\mathfrak{A}}{1.3.2.4.3.5.4^2.6^2}, \text{ etc.}$$

Quodsi jam ponatur $B = 0$ ut sit

$$p = \frac{-\mathfrak{A}}{4aa} - \frac{\mathfrak{A}}{a}\sqrt{x} + * + Cx\sqrt{x} + Dx^2 + Ex^2\sqrt{x} + Fx^3 + \text{etc.}$$

erit

$$C = \frac{8.4a\mathfrak{A}}{1^2.3^2}, \quad D = \frac{-100.16.a^2\mathfrak{A}}{1^2.3^2.2^2.4^2}, \quad E = \frac{1884.64.a^3\mathfrak{A}}{1^2.3^2.2^2.4^2.3^2.5^2},$$

$$F = \frac{-52416.256a^4\mathfrak{A}}{1^2.3^2.2^2.4^2.3^2.5^2.4^2.6^2}, \text{ etc.}$$

ubi notetur esse

$$100 = 2.4.8 + 1.3.12, \quad 1884 = 3.5.100 + 1.3.2.4.16,$$
$$52416 = 4.6.1884 + 1.3.2.4.3.5.20.$$

Exemplum 4.

938. *Posito* $m = -\frac{1}{2}$, *hanc aequationem*

$$\partial\partial y + \frac{a y \partial x^2}{x x \sqrt{x}} = 0$$

per series resolvere.

Posito $y = p + \alpha q + q l x$, capi oportet

$$q = A - \frac{4 a A}{1.3} x^{-\frac{1}{2}} + \frac{16\, a^2 A}{1.3.2.4} x^{-1} - \frac{64\, a^3 A}{1.3.2.4.3.5} x^{-\frac{3}{2}} + \text{etc.}$$

pro qua brevitatis causa scribamus

$$q = A + B x^{-\frac{1}{2}} + C x^{-1} + D x^{-\frac{3}{2}} + E x^{-2} + F x^{-\frac{5}{2}} + \text{etc}$$

et littera p ex hac aequatione definiri debet

$$\partial\partial p + \frac{2 \partial x \partial q}{x} - \frac{q \partial x^2}{x x} + \frac{a p \partial x^2}{x x \sqrt{x}} = 0.$$

Fingamus

$$p = \Delta x + \mathfrak{A}\sqrt{x} + \mathfrak{B} + \mathfrak{C} x^{-\frac{1}{2}} + \mathfrak{D} x^{-1} + \mathfrak{E} x^{-\frac{3}{2}}$$
$$+ \mathfrak{F} x^{-2} + \mathfrak{G} x^{-\frac{5}{2}} + \text{etc.}$$

et facta substitutione prodit

$$\left.\begin{array}{ccccccc}
\frac{\mathfrak{A}\Delta}{x\sqrt{x}} & + \frac{a\mathfrak{A}}{xx} & + \frac{a\mathfrak{B}}{x x \sqrt{x}} & + \frac{a\mathfrak{C}}{x^3} & + \frac{a\mathfrak{D}}{x^3\sqrt{x}} & + \frac{a\mathfrak{E}}{x^4} & + \frac{a\mathfrak{F}}{x^4\sqrt{x}} \\
 & - A & - B & - C & - D & - E & - F \\
 & & - B & - 2C & - 3D & - 4E & - 5F \\
- \frac{1}{4}\mathfrak{A} & & + \frac{3}{4}\mathfrak{C} & + 2\mathfrak{D} & + \frac{15}{4}\mathfrak{E} & + 6\mathfrak{F} & + \frac{35}{4}\mathfrak{G}
\end{array}\right\} = 0,$$

unde sequentes determinationes colliguntur

$$\mathfrak{A} = \frac{A}{a} \text{ et } \Delta = \frac{\mathfrak{A}}{4a} = \frac{A}{4aa},$$

at $\mathfrak{B}$ non determinatur: porro

$$\mathfrak{C} = \frac{-4a\mathfrak{B}}{1.3} + \frac{8\,B}{1.3} = \frac{-4a\mathfrak{B}}{1.3} - \frac{8.4\,a A}{1^2.3^2},$$

$$\mathfrak{D} = \frac{-4a\mathfrak{C}}{2.4} + \frac{12\,C}{2.4} = \frac{-4a\mathfrak{C}}{2.4} + \frac{12.16\,a^2 A}{1.3.2^2.4^2},$$

$$\mathfrak{E} = \frac{-4a\mathfrak{D}}{3.5} + \frac{16\,D}{3.5} = \frac{-4a\mathfrak{D}}{3.5} - \frac{16.64\,a^3 A}{1.3.2.4.3^2.5^2},$$

$$\mathfrak{F} = \frac{-4a\mathfrak{E}}{4.6} + \frac{20\,E}{4.6} = \frac{-4a\mathfrak{E}}{4.6} + \frac{20.256\,a^4 A}{1.3.2.4.3.5.4^2 6^2}, \text{ etc.}$$

Quodsi jam sumatur $\mathfrak{B} = 0$, erit

$$\mathfrak{C} = \frac{-8.4\,a\,A}{1^2.3^2},\quad \mathfrak{D} = \frac{+100.16\,a^2 A}{1^2.3^2.2^2.4^2},\quad \mathfrak{E} = \frac{-1884.64\,a^3 A}{1^2.3^2.2^2.4^2.3^2.6^2},\ \text{etc.}$$

qui numeri ut ante progrediuntur.

Scholion.

939. Ex his exemplis perspicitur, quomodo series aequationem

$$\partial\partial y + a\,x^{m-2}\,y\,\partial x^2 = 0$$

resolventes in reliquis casibus, quibus $m = \pm\frac{1}{i}$, inveniri oporteat; ubi observetur, si sit $m = +\frac{1}{i}$, pro q hanc seriem accipi debere

$$q = \mathfrak{A}\,x + \mathfrak{B}\,x^{1+\frac{1}{i}} + \mathfrak{C}\,x^{1+\frac{2}{i}} + \mathfrak{D}\,x^{1+\frac{3}{i}} + \text{etc.}$$

tum vero formam ipsius p tali serie exprimi

$$p = A + B\,x^{\frac{1}{i}} + C\,x^{\frac{2}{i}} + D\,x^{\frac{3}{i}} + \text{etc.}$$

cujus coëfficientes ex superioribus ut ante definiantur. Sin autem sit $m = -\frac{1}{i}$, pro q sumatur series

$$q = A + B\,x^{-\frac{1}{i}} + C\,x^{-\frac{2}{i}} + D\,x^{-\frac{3}{i}} + \text{etc.}$$

at pro p hujusmodi formam accipi conveniet

$$p = \mathfrak{A}\,x + \mathfrak{B}\,x^{1-\frac{1}{i}} + \mathfrak{C}\,x^{1-\frac{2}{i}} + \mathfrak{D}\,x^{1-\frac{3}{i}} + \text{etc.}$$

unde pariter singulos coëfficientes uno excepto determinare licebit. Atque hoc artificium in genere est tenendum, quoties in resolutione aequationis generalis ad series pervenitur, cujus coëfficientes certis casibus in infinitum excrescunt, quod plerumque indicio est, logarithmos esse introducendos. Verum etiam eadem aequatio $\partial\partial y + a\,x^n\,y\,\partial x^2 = 0$ aliis modis per series resolvi potest, dum ea ante resolutionem in aliam formam transmutatur, ubi cum evenire possit, ut series certis casibus abrumpatur, quibus adeo integrale revera assignari potest, talem transformationem maxime notabilem hic explicemus.

Problema 118.

940. Aequationem differentio-differentialem

$$\partial \partial y + a x^n y \partial x^2 = 0$$

in aliam formam transfundere, cujus resolutio per series infinit commode institui possit.

Solutio.

Utamur substitutione $y = e^{\int p \partial x} z$, ubi p sit certa functio i sius x aequationem commode resolubilem suppeditans. Erit ergo

$$\partial y = e^{\int p \partial x} (\partial z + p z \partial x) \text{ et}$$

$$\partial \partial y = e^{\int p \partial x} (\partial \partial z + 2 p \partial x \partial z + z \partial x \partial p + p p z \partial x^2),$$

unde aequatio proposita abit in

$$\partial \partial z + 2 p \partial x \partial z + z \partial x \partial p + p p z \partial x^2 + a x^n z \partial x^2 = 0$$

ubi p ita capiatur, ut fiat

$$p p + a x^n = 0, \text{ seu } p = x^{\frac{n}{2}} \sqrt{-a}.$$

Ponamus ideo $a = -c c$ et $n = 2m$, ut proposita sit haec a quatio

$$\partial \partial y - c c x^{2m} y \partial x^2 = 0,$$

quae posito

$$p = c x^m \text{ et } y = e^{\int p \partial x} z = e^{\frac{c}{m+1} x^{m+1}} z,$$

induet hanc formam

$$\partial \partial z + 2 c x^m \partial x \partial z + m c x^{m-1} z \partial x^2 = 0,$$

in qua, cum x occupet vel nullam, vel $m + 1$ dimensiones, fi gamus ipsius z valorem

$$z = A x^\lambda + B x^{\lambda + m + 1} + C x^{\lambda + 2m + 2} + \text{etc.}$$

quo substituto fit

$$\left.\begin{array}{l} \lambda(\lambda - 1)\mathrm{A}x^{\lambda-2} + (\lambda + m + 1)(\lambda + m)\mathrm{B}x^{\lambda+m-1} \\ \qquad\qquad\qquad\;\; + 2\lambda\mathrm{A}c \\ \qquad\qquad\qquad\;\; + m\mathrm{A}c \end{array}\right\} = 0,$$

unde perspicuum est sumi debere vel $\lambda = 0$ vel $\lambda = 1$. Consequimur ergo seriem duplicatam hujusmodi

$$\begin{array}{l} z = \mathrm{A} + \mathrm{B}x^{m+1} + \mathrm{C}x^{2m+2} + \mathrm{D}x^{3m+3} + \mathrm{E}x^{4m+4} + \text{etc.} \\ \quad\; \mathfrak{A}x + \mathfrak{B}x^{m+2} + \mathfrak{C}x^{2m+3} + \mathfrak{D}x^{3m+4} + \mathfrak{E}x^{4m+5} + \text{etc.} \end{array}$$

qua substituta fit

$$\left.\begin{array}{llll} (m+1)m\mathrm{B}x^{m-1} & + 2(m+1)(2m+1)\mathrm{C}x^{2m} & + 3(m+1)(3m+2)\mathrm{D}x^{3m+1} \\ + m\mathrm{A}c & + 2(m+1)\mathrm{B}c & + 4(m+1)\mathrm{C}c \\ & + m\mathrm{B}c & + m\mathrm{C}c \end{array}\right\} = 0$$

$$\left.\begin{array}{llll} + (m+1)(m+2)\mathfrak{B}x^{m} & + 2(m+1)(2m+3)\mathfrak{C}x^{2m+1} & + 3(m+1)(3m+4)\mathfrak{D}x^{3m+2} \\ + 2\mathfrak{A}c & + 2(m+2)\mathfrak{B}c & + 2(2m+3)\mathfrak{C}c \\ + m\mathfrak{A}c & + m\mathfrak{B}c & + m\mathfrak{C}c \end{array}\right\} = 0$$

unde utrique coëfficientes sequenti modo determinantur

$$\begin{array}{l|l} \mathrm{B} = \dfrac{-m\mathrm{A}c}{m(m+1)} & \mathfrak{B} = \dfrac{-(m+2)\mathfrak{A}c}{(m+2)(m+1)} \\ \mathrm{C} = \dfrac{-(3m+2)\mathrm{B}c}{2(2m+1)(m+1)} & \mathfrak{C} = \dfrac{-(3m+4)\mathfrak{B}c}{2(2m+3)(m+1)} \\ \mathrm{D} = \dfrac{-(5m+4)\mathrm{C}c}{3(3m+2)(m+1)} & \mathfrak{D} = \dfrac{-(5m+6)\mathfrak{C}c}{3(3m+4)(m+1)} \\ \mathrm{E} = \dfrac{-(7m+6)\mathrm{D}c}{4(4m+3)(m+1)} & \mathfrak{E} = \dfrac{-(7m+8)\mathfrak{D}c}{4(4m+5)(m+1)} \\ \qquad\text{etc.} & \qquad\text{, etc.} \end{array}$$

ubi bini coëfficientes A et $\mathfrak{A}$ manent indeterminati, ita ut hoc integrale completum sit censendum.

Aliter. Sumta serie, in qua exponentes ipsius x decrescant, fieri debet

$$2\lambda + m = 0 \quad \text{seu} \quad m = -2\lambda,$$

ut aequatio nostra sit

$$\partial\partial y - cc\,x^{-4\lambda}y\,\partial x^2 = 0,$$

quae posito

$$p = c\,x^{-2\lambda} \text{ et } y = e^{\int p\,\partial x}\,z = e^{\frac{-c}{2\lambda-1}x^{-2\lambda+1}}\,z,$$

abit in

$$\partial\partial z + 2\,c\,x^{-2\lambda}\,\partial x\,\partial z - 2\,\lambda\,c\,x^{-2\lambda-1}\,z\,\partial x^2 = 0.$$

Ponamus ergo

$$z = A\,x^{\lambda} + B\,x^{3\lambda-1} + C\,x^{5\lambda-2} + D\,x^{7\lambda-3} + \text{etc.}$$

et substitutione facta prodit

$$\left.\begin{array}{llll} \lambda(\lambda-1)Ax^{\lambda-2} + (3\lambda-1)(3\lambda-2)Bx^{3\lambda-3} & + (5\lambda-2)(5\lambda-3)Cx^{5\lambda-4} & & \\ +2\lambda Acx^{-\lambda-1} + 2(3\lambda-1)Bc & + 2(5\lambda-2)Cc & + 2(7\lambda-3)Dc & \\ -2\lambda Ac \qquad -2\lambda Bc & -2\lambda Cc & -2\lambda Dc & \end{array}\right\} = 0$$

unde coëfficientes ita determinantur

$$B = \frac{-\lambda(\lambda-1)A}{(4\lambda-2)c} = \frac{-\lambda(\lambda-1)A}{2(2\lambda-1)c}$$

$$C = \frac{-(3\lambda-1)(3\lambda-2)B}{(8\lambda-4)c} = \frac{-(3\lambda-1)(3\lambda-2)B}{4(2\lambda-1)c}$$

$$D = \frac{-(5\lambda-2)(5\lambda-3)C}{(12\lambda-6)c} = \frac{-(5\lambda-2)(5\lambda-3)C}{6(2\lambda-1)c}$$

$$E = \frac{-(7\lambda-3)(7\lambda-4)D}{(16\lambda-8)c} = \frac{-(7\lambda-3)(7\lambda-4)D}{8(2\lambda-1)c}$$

etc.

Hic unius tantum litterae A valor arbitrio nostro relinquitur, ex quo haec series tantum integrale particulare exhibet.

Corollarium 1.

941. Ex solutione priori patet, alteram seriem terminari quoties

$$(2\,i + 1)\,m + 2\,i = 0, \text{ seu } m = \frac{-2i}{2i+1}$$

alteram vero quoties

$$(2\,i - 1)\,m + 2\,i = 0, \text{ seu } m = \frac{-2i}{2i-1}$$

denotante i numerum integrum quemcunque. His ergo casibus integrale saltem particulare finite exprimi potest.

Corollarium 2.

942. Altera solutio praebet seriem finitam, quoties fuerit vel
$$(2i+1)\lambda - i = 0 \text{ vel } (2i-1)\lambda - i = 0,$$
hoc est
$$\lambda = \frac{i}{2i\pm 1} \text{ et } m = \frac{-2i}{2i\pm 1},$$
ut ante. Reliquis vero casibus haec series in infinitum excurrit.

Corollarium 3.

943. Casus ergo, quibus haec aequatio
$$\partial\partial y - cc x^n \partial x^2 = 0$$
atque adeo posito $y = e^{\int u \partial x}$ etiam haec
$$\partial u + uu\partial x = ccx^n \partial x$$
integrationem saltem particularem admittit, sunt $n = \frac{-4i}{2i\pm 1}$ sumendo pro i numerum integrum quemcunque.

Scholion.

944. Sufficit autem integrale particulare invenisse, cum ex eo facile integrale completum erui possit. Cum enim in integrali insit littera c, dum aequatio differentialis tantum quadratum cc continet, perinde est sive in integrali sumatur $+c$ sive $-c$. Hinc si integrale particulare sit $y = P + cQ$, erit etiam $y = P - cQ$ integrale particulare, unde integrale completum erit

$$y = \alpha(P + cQ) + \beta(P - cQ), \text{ seu } y = \alpha P \mp \beta c Q.$$

Quo haec clarius explicentur, ad solutionem alteram aequationis $\partial\partial y - ccx^{-4\lambda} y \partial x^2 = 0$ accommodentur, pro qua ponendo brevitatis gratia $\frac{1}{1-2\lambda} x^{1-2\lambda} = t$, fecimus $y = e^{ct} z$, et invenimus:
$$z = Ax^{\lambda} - \frac{\lambda(\lambda-1)A}{2(2\lambda-1)c} x^{3\lambda-1} + \frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)A}{2.4(2\lambda-1)^2 cc} x^{5\lambda-2}$$
$$- \frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)(5\lambda-2)(5\lambda-3)A}{2.4.6(2\lambda-1)^3 c^3} x^{7\lambda-3} + \text{etc.}$$

Pro qua expressione distinguendo terminos per potestates pares i
sius c divisos ab iis, qui per potestates impares sunt divisi, scrib
mus $z = P - c\,Q$, ita ut jam P et Q tantum potestates pares i
sius c contineant, eritque integrale particulare unum

$$y = e^{ct}(P - c\,Q)$$

et alterum

$$y = e^{-ct}(P + c\,Q),$$

unde completum erit

$$y = \tfrac{1}{2}P(\alpha e^{ct} + \beta e^{-ct}) - \tfrac{1}{2}c\,Q(\alpha e^{ct} - \beta e^{-ct}),$$

Hinc si c sit numerus imaginarius seu $cc = -bb$, ut aequatio

$$\partial\partial y + bb\,x^{-4\lambda}\,y\,\partial x^2 = 0, \text{ erit}$$

$$z = P - b\,Q\sqrt{-1} \text{ et}$$

$$e^{ct} = e^{bt\sqrt{-1}} = \cos. bt + \sqrt{-1}.\sin. bt, \text{ ergo}$$

$$y = P\left(\tfrac{\alpha+\beta}{2}\cos. bt + \tfrac{\alpha-\beta}{2}\sqrt{-1}.\sin. bt\right)$$
$$- b\,Q\left(\tfrac{\alpha-\beta}{2}\cos. bt + \tfrac{\alpha+\beta}{2}\sqrt{-1}.\sin. bt\right)\sqrt{-1}.$$

Sit $\frac{\alpha+\beta}{2} = \gamma$ et $\frac{\alpha-\beta}{2}\sqrt{-1} = \delta$,

atque integrale completum hoc casu ita exprimetur

$$y = P(\gamma\cos. bt + \delta\sin. bt) - b\,Q(\delta\cos. bt - \gamma\sin. bt), \text{ se}$$

$$y = (\gamma P - \delta b\,Q)\cos. bt + (\delta P + \gamma b\,Q)\sin. bt.$$

Casus ergo hoc modo integrabiles evolvamus.

Exemplum 1.

945. *Integrale aequationis* $\partial\partial y - cc\,y\,\partial x^2 = 0$ *inveni.*

Hic est $\lambda = 0$ et $z = A$, atque $t = x$, unde ob $P =$
et $Q = 0$, erit integrale completum

$$y = \alpha e^{cx} + \beta e^{-cx}.$$

Casu autem $cc = -bb$, aequationis $\partial\partial y + bb\,y\,\partial x^2 = 0$ i
tegrale completum erit

$$y = \gamma \cos. b x + \delta \sin. b x.$$

Exemplum 2.

946. *Integrale aequationis* $\partial \partial y - c c x^{-4} y \partial x^2 = 0$ *invenire.*

Hic ob $\lambda = 1$ est $z = A x$, et $t = -\frac{1}{x}$, unde ob $P = x$ et $Q = 0$, fit

$$y = (\alpha e^{c t} + \beta e^{-c t}) x.$$

Casu autem $cc = -bb$ aequationis

$$\partial \partial y + b b x^{-4} y \partial x^2 = 0$$

integrale est

$$y = (\alpha \cos. b t + \beta \sin. b t) x, \text{ existente } t = -\frac{1}{x}.$$

Exemplum 3.

947. *Integrale aequationis* $\partial \partial y - c c x^{-\frac{4}{3}} y \partial x^2 = 0$ *invenire.*

Ob $\lambda = \frac{1}{3}$, fit $B = -\frac{A}{3c}$, et $z = A x^{\frac{1}{3}} - \frac{A}{3c}$, atque $t = 3 x^{\frac{1}{3}}$, unde $P = x^{\frac{1}{3}}$ et $Q = \frac{1}{3cc}$. Integrale ergo erit

$$y = (\alpha e^{ct} + \beta e^{-ct}) x^{\frac{1}{3}} - (\alpha e^{ct} - \beta e^{-ct}) \frac{1}{8c}.$$

Casu autem $cc = -bb$, aequationis

$$\partial \partial y + b b x^{-\frac{4}{3}} y \partial x^2 = 0$$

integrale est

$$y = (\alpha \cos. b t + \beta \sin. b t) x^{\frac{1}{3}} + \frac{1}{3b} (\beta \cos. b t - \alpha \sin. b t).$$

Exemplum 4.

948. *Integrale aequationis* $\partial\partial y - ccx^{-\frac{8}{3}} y \partial x^2 = 0$ *invenire.*

Ob $\lambda = \frac{2}{3}$, fit $B = \frac{A}{3c}$ et $z = Ax^{\frac{2}{3}} + \frac{A}{3c}x$, ita ut sit $P = x^{\frac{2}{3}}$ et $Q = \frac{-x}{3cc}$. Posito ergo $t = -3x^{-\frac{1}{3}}$, integrale ita exprimitur

$$y = x^{\frac{2}{3}}(\alpha e^{ct} + \beta e^{-ct}) + \frac{x}{3c}(\alpha e^{ct} - \beta e^{-ct}).$$

Casu autem $cc = -bb$ aequationis

$$\partial\partial y + bbx^{-\frac{8}{3}} y \partial x^2 = 0$$

integrale est

$$y = x^{\frac{2}{3}}(\alpha \cos. bt + \beta \sin. bt) - \frac{x}{3b}(\beta \cos. bt - \alpha \sin. bt).$$

Exemplum 5.

949. *Integrale aequationis* $\partial\partial y - ccx^{-\frac{8}{5}} y \partial x^2 = 0$ *invenire.*

Ob $\lambda = \frac{2}{5}$, est $B = \frac{-3A}{5c}$ et $C = \frac{+3A}{5^2cc}$, hinc

$$z = Ax^{\frac{2}{5}} - \frac{3A}{5c}x^{\frac{1}{5}} + \frac{3A}{5^2cc}, \text{ ideoque}$$

$$P = x^{\frac{2}{5}} + \frac{3}{5^2cc} \text{ et } Q = \frac{3}{5cc}x^{\frac{1}{5}}.$$

Posito ergo $t = 5x^{\frac{1}{5}}$, integrale erit

$$y = (x^{\frac{2}{5}} + \frac{3}{5^2cc})(\alpha e^{ct} + \beta e^{-ct}) - \frac{3}{5c}x^{\frac{1}{5}}(\alpha e^{ct} - \beta e^{-ct})$$

casu autem $cc = -bb$, aequationis

$$\partial\partial y + bbx^{-\frac{8}{5}} y\partial x^2 = 0$$

integrale est

$$y = (x^{\frac{2}{5}} - \frac{3}{5^2 bb})(\alpha \cos. bt + \beta \sin. bt) + \frac{3}{5b} x^{\frac{1}{5}} (\beta \cos. bt - \alpha \sin. bt).$$

Problema 119.

950. Aequationis differentio-differentialis

$$\partial\partial y + cc x^{\frac{-4i}{2i-1}} y \partial x^2 = 0$$

integrale completum assignare, denotante i numerum integrum quemcunque.

Solutio.

Sit brevitatis gratia $t = -(2i-1) x^{\frac{-1}{2i-1}}$ unde fit $x^{\frac{1}{2i-1}} = -\frac{(2i-1)}{t}$, ac posito $y = e^{ct} z$, pro valore ipsius z per seriem invento

$$z = A x^{\frac{i}{2i-1}} + B x^{\frac{i+1}{2i-1}} + C x^{\frac{i+2}{2i-1}} + D x^{\frac{i+3}{2i-1}} + \text{etc.}$$

ob $\lambda = \frac{+i}{2i-1}$, hi coëfficientes ita determinantur

$$B = \frac{i(i-1)A}{2(2i-1)c}, \quad C = \frac{(i+1)(i-2)B}{4(2i-1)c}, \quad D = \frac{(i+2)(i-3)C}{6(2i-1)c} \text{ etc.}$$

quibus substitutis, et introducto valore $x^{\frac{1}{2i-1}} = \frac{-(2i-1)}{t}$, erit

$$z = A x^{\frac{i}{2i-1}} (1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2.4cctt} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6c^3t^3} + \text{etc.}$$

sive hoc modo

$$z = \frac{A}{t^i} (1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2.4c.tt} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6.c^3t^3} + \text{etc.}).$$

Atque hinc aequationis propositae integrale completum ita exprimetur

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$$y = t^{-i}\left(1 + \frac{i(ii-1)(i-2)}{2.4\,cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,c^4t^4} + \text{etc.}\right)(\alpha e^{ct} + \beta e^{-ct})$$
$$- t^{-i}\left(\frac{i(i-1)}{2ct} + \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,c^3t^3} + \text{etc.}\right)(\alpha e^{ct} - \beta e^{-ct})$$

ubi in utraque progressione lex formationis singulorum terminorum est manifesta.

Corollarium 1.

951. Hinc quoque illius aequationis

$$\partial\partial y + bb\,x^{\frac{-4i}{2i-1}}\,y\,\partial x^2 = 0$$

integrale completum, manente $t = -(2i-1)\,x^{\frac{-1}{2i-1}}$, est

$$y = t^{-i}\left(1 - \frac{i(ii-1)(i-2)}{2.4\,bbtt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,b^4t^4} - \text{etc}\right)(\alpha\cos.bt + \beta\sin.bt)$$
$$+ t^{-i}\left(\frac{i(i-1)}{2bt} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,b^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8.10\,b^5t^5} - \text{etc.}\right)(\beta\cos.bt - \alpha\sin.bt)$$

Corollarium 2.

952. Si i sit numerus negativus, haec integratio perinde succedit, aequationis enim

$$\partial\partial y - ce\,x^{\frac{-4i}{2i+1}}\,y\,\partial x^2 = 0$$

posito $t = (2i+1)\,x^{\frac{1}{2i+1}}$, integrale erit

$$y = t^{i}\left(1 + \frac{i(ii-1)(i+2)}{2.4\,cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8\,c^4t^4} + \text{etc.}\right)(\alpha e^{ct} + \beta e^{-ct})$$
$$- t^{i}\left(\frac{i(i+1)}{2ct} + \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\,c^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\,c^5t^5} + \text{etc.}\right)(\alpha e^{ct} - \beta e^{-ct}).$$

Corollarium 3.

953. Simili modo hujus aequationis

$$\partial\partial y + bb\,x\,b^{\frac{-4i}{2i+1}}\,y\,\partial x^2 = 0,$$

posito $t = (2i + 1)\, x^{\frac{1}{2i+1}}$, integrale completum erit

$$y = t^i \left(1 - \frac{i(ii-1)(i+2)}{2.4\, bb\, tt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8.\, b^4 t^4} - \text{etc}\right)(\alpha \cos. bt + \beta \sin. bt)$$

$$+ t^i \left(\frac{i(i+1)}{2bt} - \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\, b^3 t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\, b^5 t^5} - \text{etc.}\right)(\beta \cos. bt - \alpha \sin. bt).$$

Corollarium 4.

954. In formulis sinus et cosinus continentibus si ponatur

$$\alpha = C \sin. \zeta \text{ et } \beta = C \cos. \zeta,$$

expressiones nostrae ita contrahuntur, ut fiat

$$\alpha \cos. bt + \beta \sin. bt = C \sin. (bt + \zeta) \text{ et}$$

$$\beta \cos. bt - \alpha \sin. bt = C \cos. (bt + \zeta),$$

ut jam hic C et ζ sint constantes arbitrariae integrale completum reddentes.

Scholion.

955. Hinc egregium adipiscimur adminiculum ad casus integrabilitatis hujus aequationis differentialis primi gradus

$$\partial u + uu \partial x + a x^n \partial x = 0$$

agnoscendos, simulque integralia completa definienda; nascitur enim haec aequatio ex ista

$$\partial \partial y + a x^n y \partial x^2 = 0, \text{ ponendo } z = e^{\int u \partial x},$$

unde ex illa vicissim haec oritur, ponendo $u = \frac{\partial y}{y \partial x}$. Cum igitur istius integrale assignare licuerit casibus quibus exponens $n = \frac{-4i}{2i \pm 1}$ iisdem casibus integrale aequationis differentialis primi gradus assignare licebit, ubi quidem duos casus evolvi convenit, prout a fuerit vel numerus negativus $a = -cc$ vel positivus $a = +bb$. Hos igitur duos casus pertractasse operae erit pretium.

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Problema 120.

956. Denotante i numerum integrum sive positivum sive negativum quemcunque, invenire integrale hujus aequationis

$$\partial u + u u \partial x - c c x^{\frac{-4i}{2i+1}} \partial x = 0.$$

Solutio.

Posito $u = \frac{\partial y}{y \partial x}$, haec aequatio transformatur in istam

$$\partial \partial y - c c x^{\frac{-4i}{2i+1}} y \partial x^2 = 0,$$

sumto elemento ∂x constante, cujus integrale assignavimus.

Posito scilicet $t = (2i+1) x^{\frac{1}{2i+1}}$, est

$$y = (\alpha e^{ct} + \beta e^{-ct}) \left(t^i + \frac{i(ii-1)(i+2)}{2.4cc} t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8c^4} t^{i-4} + \text{etc.}\right)$$
$$- (\alpha e^{ct} - \beta e^{-ct}) \left(\frac{i(i+1)}{2c} t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2.4.6c^3} t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10c^5} t^{i-5} + \text{etc.}\right)$$

Ponamus brevitatis gratia

$$y = (\alpha e^{ct} + \beta e^{-ct}) P - (\alpha e^{ct} - \beta e^{-ct}) Q,$$

et cum sit

$$\partial t = x^{\frac{-2i}{2i+1}} \partial x, \text{ seu } \partial x = x^{\frac{2i}{2i+1}} \partial t,$$

erit

$$\frac{\partial y}{\partial x} = \frac{(\alpha e^{ct} + \beta e^{-ct})(\partial P - cQ\partial t) + (\alpha e^{ct} - \beta e^{-ct})(cP\partial t - \partial Q)}{x^{\frac{2i}{2i+1}} \partial t},$$

sive

$$\frac{\partial y}{\partial x} = \frac{\alpha e^{ct}(\partial P + cP\partial t - \partial Q - cQ\partial t) + \beta e^{-ct}(\partial P - cP\partial t + \partial Q - cQ\partial t)}{x^{\frac{2i}{2i+1}} \partial t}.$$

At vero est

$$\frac{\partial P}{\partial t} = i\,t^{i-1} + \frac{i(ii-1)(ii-4)}{2.4\,c^2}\,t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)}{2.4.6.8.\,c^4}\,t^{i-5} + \text{etc.}$$

$$cQ = \frac{i(i+1)}{2}\,t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\,c^2}\,t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\,c^4}\,t^{i-5} + \text{etc.}$$

$$cP = c\,t^i + \frac{i(ii-1)(i+2)}{2.4\,c}\,t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8\,c^3}\,t^{i-4} + \text{etc.}$$

$$\frac{\partial Q}{\partial t} = \frac{i(ii-1)}{2\,c}\,t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)}{2.4.6\,c^3}\,t^{i-4} + \text{etc.}$$

unde colligitur

$$\frac{\partial P - cQ\partial t}{\partial t} = -\frac{i(i-1)}{2}\,t^{i-1} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,c^2}\,t^{i-3} - \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8.10\,c^4}\,t^{i-5} - \text{etc.}$$

$$\frac{cP\partial t - \partial Q}{\partial t} = c\,t^i + \frac{i(ii-1)(i-2)}{2.4\,c}\,t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,c^3}\,t^{i-4} + \text{etc.}$$

Ponamus ad abbreviandum

$$P = t^i + \frac{i(ii-1)(i+2)}{2.4\,cc}\,t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8\,c^4}\,t^{i-4} + \text{etc.}$$

$$Q = \frac{i(i+1)}{2c}\,t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2.4.6\,c^3}\,t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10\,c^5}\,t^{i-5} + \text{etc.}$$

$$R = t^i + \frac{i(ii-1)(i-2)}{2.4\,cc}\,t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8\,c^4}\,t^{i-4} + \text{etc.}$$

$$S = \frac{i(i-1)}{2c}\,t^{i-1} + \frac{i(ii-1)(ii-4)(i-3)}{2.4.6\,c^3}\,t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8.10\,c^5}\,t^{i-5} + \text{etc.}$$

ut sit

$$\frac{\partial y}{\partial x} = \frac{(\alpha\,e^{ct} + \beta\,e^{-ct})(-c\,S) + (\alpha\,e^{ct} - \beta\,e^{-ct})(c\,R)}{x^{\frac{2i}{2i+1}}}.$$

Quare cum sit $u = \frac{\partial y}{y\,\partial x}$, erit nostrae aequationis integrale completum

$$\tfrac{1}{c}\,x^{\frac{2i}{2i+1}}\,u = \frac{(\alpha\,e^{ct} - \beta\,e^{-ct})\,R - (\alpha\,e^{ct} + \beta\,e^{-ct})\,S}{(\alpha\,e^{ct} + \beta\,e^{-ct})\,P - (\alpha\,e^{ct} - \beta\,e^{-ct})\,Q}, \text{ sive}$$

$$\tfrac{1}{c}\,x^{\frac{2i}{2i+1}}\,u = \frac{\alpha\,e^{ct}\,(R - S) - \beta\,e^{-ct}\,(R + S)}{\alpha\,e^{ct}\,(P - Q) + \beta\,e^{-ct}\,(P + Q)},$$

quod ob rationem constantium $\alpha : \beta$ arbitrariam, est completum.

Corollarium 1.

957. Quaternae formulae P, Q, R, S, quae singulae casibus, quibus i est numerus integer, abrumpuntur, ita a se invicem pendent, ut sit primo

$$R = P - \frac{\partial Q}{c \partial t} \text{ et } S = Q - \frac{\partial P}{c \partial t};$$

tum vero

$$\partial P + \partial R = \frac{+2iR\partial t}{t} \text{ et } \partial Q + \partial S = \frac{2iS\partial t}{t}.$$

Corollarium 2.

958. Posito ergo vel $\alpha = 0$ vel $\beta = 0$, integralia particularia algebraica aequationis

$$\partial u + uu \partial x - cc\, x^{\frac{-4i}{2i+1}} \partial x = 0$$

exhiberi possunt, quae sunt

$$\frac{1}{c} x^{\frac{2i}{2i+1}} u = \frac{R - S}{P - Q} \text{ et } \frac{1}{c} x^{\frac{2i}{2i+1}} u = \frac{-S - R}{P + Q},$$

ideoque hac una formula comprehendi possunt

$$\frac{1}{c} x^{\frac{2i}{2i+1}} u = \frac{-S \pm R}{P \mp Q}.$$

Scholion 1.

959. Pro variis ergo valoribus numeri i tam quantitas t quam litterae P, Q, R, S sequenti modo se habebunt: Primo scilicet si $i = 0$, erit $t = x$, atque $P = 1$, $Q = 0$, $R = 1$ et $S = 0$; reliquos casus in sequenti tabella repraesentemus

$i = -1,\ t = -\frac{1}{x}$	$i = 1,\ t = 3x^{\frac{1}{3}}$
$P = \frac{1}{t},\quad Q = 0$	$P = t,\ Q = \frac{1}{c}$
$R = \frac{1}{t},\quad S = \frac{1}{ctt}$	$R = t,\ S = 0$

$i = -2,\ t = -\frac{3}{x^{\frac{1}{3}}}$ $P = \frac{1}{t^2},\ Q = \frac{1}{c\,t^3}$ $R = \frac{1}{t^2} + \frac{3}{c\,c\,t^4},\ S = \frac{3}{c\,t^3},$	$i = 2,\ t = 5\,x^{\frac{1}{5}}$ $P = t^2 + \frac{3}{c\,c},\ Q = \frac{3}{c}\,t$ $R = t^2,\ S = \frac{1}{c}\,t.$
$i = -3,\ t = \frac{-5}{x^{\frac{1}{5}}}$ $P = \frac{1}{t^3} + \frac{1.3}{c\,c\,t^5}$ $Q = \frac{3}{c\,t^4}$ $R = \frac{1}{t^3} + \frac{3.5}{c\,c\,t^5}$ $S = \frac{6}{c\,t^4} + \frac{3.5}{c^3\,t^6}$	$i = 3,\ t = 7\,x^{\frac{1}{7}}$ $P = t^3 + \frac{3.5}{c\,c}\,t$ $Q = \frac{6}{c}\,t\,t + \frac{3.5}{c^3}$ $R = t^3 + \frac{1.3}{c\,c}\,t$ $S = \frac{3}{c}\,t\,t$
$i = -4,\ t = -\frac{7}{x^{\frac{1}{7}}}$ $P = \frac{1}{t^4} + \frac{3.5}{c\,c\,t^6}$ $Q = \frac{6}{c\,t^5} + \frac{3.5}{c^3\,t^7}$ $R = \frac{1}{t^4} + \frac{3.3.5}{c\,c\,t^6} + \frac{3.5.7}{c^4\,t^8}$ $S = \frac{10}{c\,t^5} + \frac{3.5.7}{c^3\,t^7}$	$i = 4,\ t = 9\,x^{\frac{1}{9}}$ $P = t^4 + \frac{3.3.5}{c\,c}\,t^2 + \frac{3.5.7}{c^4}$ $Q = \frac{10}{c}\,t^3 + \frac{3.5.7}{c^3}\,t$ $R = t^4 + \frac{3.5}{c\,c}\,t^2$ $S = \frac{6}{c}\,t^3 + \frac{3.5}{c^3}\,t$
$i = -5,\ t = \frac{-9}{x^{\frac{1}{9}}}$ $P = \frac{1}{t^5} + \frac{3.3.5}{c\,c\,t^7} + \frac{3.5.7}{c^4\,t^9}$ $Q = \frac{10}{c\,t^6} + \frac{3.5.7}{c^3\,t^8}$ $R = \frac{1}{t^5} + \frac{3.5.7}{c\,c\,t^7} + \frac{3.5.7.9}{c^4\,t^9}$ $S = \frac{15}{c\,t^6} + \frac{4.3.5.7}{c^3\,t^8} + \frac{3.5.7.9}{c^8\,t^{10}}$	$i = 5,\ t = 11\,x^{\frac{1}{11}}$ $P = t^5 + \frac{3.5.7}{c\,c}\,t^3 + \frac{3.5.7.9}{c^4}\,t$ $Q = \frac{15}{c}\,t^4 + \frac{4.3.5.7}{c^3}\,t^2 + \frac{3.5.7.9}{c^5}$ $R = t^5 + \frac{3.3.5}{c\,c}\,t^3 + \frac{3.5.7}{c^4}\,t$ $S = \frac{10}{c}\,t^4 + \frac{3.5.7}{c^3}\,t^2$

$$i = -6, \quad t = \frac{-11}{x^{\frac{1}{11}}}$$

$$P = \frac{1}{t^6} + \frac{3.5.7}{cct^8} + \frac{3.5.7.9}{c^4t^{10}}$$

$$Q = \frac{15}{ct^7} + \frac{4.3.5.7}{c^3t^9} + \frac{3.5.7.9}{c^5t^{11}}$$

$$R = \frac{1}{t^6} + \frac{2.3.5.7}{cct^8} + \frac{5.3.5.7.9}{c^4t^{10}} + \frac{3.5.7.9.11}{c^6t^{12}}$$

$$S = \frac{21}{ct^7} + \frac{4.5.7.9}{c^3t^9} + \frac{3.5.7.9.11}{c^5t^{11}}$$

$$i = 6, \quad t = 13\,x^{\frac{1}{13}}$$

$$P = t^6 + \frac{2.3.5.7}{cc}t^4 + \frac{5.3.5.7.9}{c^4}t^2 + \frac{3.5.7.9.11}{c^6}$$

$$Q = \frac{21}{c}t^5 + \frac{4.5.7.9}{c^3}t^3 + \frac{3.5.7.9.11}{c^4}t$$

$$R = t^6 + \frac{3.5.7}{cc}t^4 + \frac{3.5.7.9}{c^4}t^2$$

$$S = \frac{15}{c}t^5 + \frac{4.3.5.7}{c^3}t^3 + \frac{3.5.7.9}{c^5}t.$$

Scholion. 2.

960. Dum hae formulae diligentius considerantur, nova se prodit relatio inter valores litterarum P, Q, R, S, quae in hoc consistit, ut perpetuo sit $PR - QS = t^{2i}$, cujus veritas primo quidem per inductionem deprehenditur, tum vero etiam per relationes supra datas demonstrari potest. Si enim valores

$$R = P - \frac{\partial Q}{c\,\partial t} \quad \text{et} \quad S = Q - \frac{\partial P}{c\,\partial t}$$

in aequationibus

$$\partial P + \partial R = \frac{2iR\,\partial t}{t} \quad \text{et} \quad \partial Q + \partial S = \frac{2iS\,\partial t}{t}$$

substituantur, oriuntur hae duae aequationes

$$2\partial P - \frac{\partial\partial Q}{c\,\partial t} = \frac{2iP\,\partial t}{t} - \frac{2i\,\partial Q}{ct} \quad \text{et}$$

$$2\partial Q - \frac{\partial\partial P}{c\,\partial t} = \frac{2iQ\,\partial t}{t} - \frac{2i\,\partial P}{ct},$$

quarum illa per P, haec vero per $-$ Q multiplicata junctim dant

$$2\,P\,\partial P - 2\,Q\,\partial Q + \frac{Q\,\partial\partial P - P\,\partial\partial Q}{c\,\partial t} = \frac{2\,i\,\partial t}{t}(P\,P - Q\,Q)$$
$$+ \frac{2\,i}{c\,t}(Q\,\partial P - P\,\partial Q).$$

Ponatur

$$P\,P - Q\,Q = M \text{ et } \frac{Q\,\partial P - P\,\partial Q}{c\,\partial t} = N, \text{ erit}$$
$$\partial M + \partial N = \frac{2\,i\,\partial t}{t}(M + N), \text{ seu } \frac{\partial M + \partial N}{M + N} = \frac{2\,i\,\partial t}{t}$$

hincque integrando $M + N = C\,t^{2\,i}$. At est

$$M + N = P\left(P - \frac{\partial Q}{c\,\partial t}\right) - Q\left(Q - \frac{\partial P}{c\,\partial t}\right) = P\,R - Q\,S;$$

evidens autem est pro constante C unitatem accipi debere.

Problema 121.

961. Denotante i numerum integrum sive positivum sive negativum quemcunque, invenire integrale completum hujus aequationis

$$\partial u + u\,u\,\partial x + b\,b\,x^{\frac{-4\,i}{2\,i+1}}\,\partial x = 0.$$

Solutio.

Posito $u = \frac{\partial y}{y\,\partial x}$, haec aequatio transformatur in istam

$$\partial\partial y + b\,b\,x^{\frac{-4\,i}{2\,i+1}}\,y\,\partial x^2 = 0,$$

sumto elemento ∂x constante, cujus integrale supra est assignatum. Scilicet posito $t = (2\,i + 1)\,x^{\frac{1}{2\,i+1}}$, invenimus (953, 954)

$$y = C\left(t^i - \frac{i(ii-1)(i+2)}{2.4\,b\,b}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8\,b^4}t^{i-4} - \text{etc.}\right)\sin.(b\,t + \zeta)$$
$$+ C\left(\frac{i(i+1)}{2\,b}t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2.4.6.b^3}t^{i-3} + \text{etc.}\right)\cos.(b\,t + \zeta),$$

cujus loco brevitatis gratia scribamus

$$y = C\,P\,\sin.(b\,t + \zeta) + C\,Q\,\cos.(b\,t + \zeta).$$

Hinc ob

$$\partial t = x^{\frac{-2i}{2i+1}} \partial x \text{ seu } \partial x = x^{\frac{2i}{2i+1}} \partial t,$$

erit

$$\frac{\partial y}{\partial x} = \frac{C(\partial P - bQ\partial t)\sin.(bt+\zeta) + C(\partial Q + bP\partial t)\cos.(bt+\zeta)}{x^{\frac{2i}{2i+1}}\partial t}:$$

unde cum sit $u = \frac{\partial y}{y \partial x}$, erit

$$u = \frac{(\partial P - bQ\partial t)\sin.(bt+\zeta) + (\partial Q + bP\partial t)\cos.(bt+\zeta)}{x^{\frac{2i}{2i+1}}\partial t\,[P\sin.(bt+\zeta) + Q\cos.(bt+\zeta)]}.$$

Ponamus

$$P + \frac{\partial Q}{b\partial t} = R \text{ et } Q - \frac{\partial P}{b\partial t} = S,$$

ut sit

$$\frac{1}{b} x^{\frac{2i}{2i+1}} u = \frac{R\cos.(bt+\zeta) - S\sin.(bt+\zeta)}{P\sin.(bt+\zeta) + Q\cos.(bt+\zeta)},$$

erit

$$P = t^i - \frac{i(ii-1)(i+2)}{2.4bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2.4.6.8.10b^4}t^{i-4} - \text{etc.}$$

$$Q = \frac{i(i+1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2.4.6b^3}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2.4.6.8.10b^5}t^{i-5} - \text{etc.}$$

$$R = t^i - \frac{i(ii-1)(i-2)}{2.4.bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2.4.6.8b^4}t^{i-4} - \text{etc.}$$

$$S = \frac{i(i-1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i-3)}{2.4.6b^3}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2.4.6.8.10b^5}t^{i-5} - \text{etc.}$$

atque ob angulum ζ introductum hoc integrale erit completum.

Corollarium 1.

962. Quaternarum ergo litterarum P, Q, R, S valores ita a se invicem pendent, ut sit primo

$$R = P + \frac{\partial Q}{b\partial t} \text{ et } S = Q - \frac{\partial P}{b\partial t}$$

tum vero etiam patet fore

$$\partial P + \partial R = \frac{2iR\partial t}{t} \text{ et } \partial Q + \partial S = \frac{2iS\partial t}{t}.$$

Corollarium 2.

963. Deinde etiam colligitur fore $PR + QS = t^{2i}$ quae aequalitas ex praecedentis problematis formulis deducitur, sumto $cc = -bb$, ubi Q et S abeunt in $Q\sqrt{-1}$ et $S\sqrt{-1}$.

Corollarium 3.

964. Hic casus a praecedente etiam hoc differt, quod hic nulla dentur integralia particularia algebraica. Quicunque enim valor angulo constanti ζ tribuatur, integrale semper sinum et cosinum cujusdam anguli involvit.

Scholion 1.

965. Cum igitur aequationis

$$\partial u + uu\partial x + bb\,x^{\frac{-4i}{2i+1}}\partial x = 0$$

integrale completum, posito $t = (2i+1)\,x^{\frac{1}{2i+1}}$, sit

$$\frac{1}{b}\,x^{\frac{2i}{2i+1}}\,u = \frac{R\cos.(bt+\zeta) - S\sin.(bt+\zeta)}{P\sin.(bt+\zeta) + Q\cos.(bt+\zeta)};$$

pro singulis valoribus numeri i quantitas t cum litteris P, Q, R, S ita se habebit: Primo si $i = 0$, erit $P = 1$, $Q = 0$, $R = 1$ et $S = 0$, item $t = x$, ita ut integrale sit $\frac{1}{b}u = \frac{\cos.(bx+\zeta)}{\sin.(bx+\zeta)}$; reliquos casus sequens tabella exhibet:

$i = -1,\ t = -\frac{1}{x}$	$i = 1,\ t = 3x^{\frac{1}{3}}$
$P = \frac{1}{t},\quad Q = 0$	$P = t,\ Q = \frac{1}{b}$
$R = \frac{1}{t},\quad S = \frac{1}{btt}$	$R = t,\ S = 0$

$i = -2,\ t = -\frac{3}{x^{\frac{1}{3}}}$ $P = \frac{1}{t^2},\ Q = \frac{1}{b\,t^3}$ $R = \frac{1}{t^2} + \frac{3}{b\,b\,t^4},\ S = \frac{3}{b\,t^3},$	$i = 2,\ t = 5\,x^{\frac{1}{5}}$ $P = t^2 + \frac{3}{b\,b},\ Q = \frac{3}{b}\,t$ $R = t^2,\ S = \frac{1}{b}\,t.$
$i = -3,\ t = \frac{-5}{x^{\frac{1}{5}}}$ $P = \frac{1}{t^3} + \frac{1.3}{b\,b\,t^5}$ $Q = \frac{3}{b\,t^4}$ $R = \frac{1}{t^3} + \frac{3.5}{b\,b\,t^5}$ $S = \frac{6}{b\,t^4} + \frac{3.5}{b^3\,t^6}$	$i = 3,\ t = 7\,x^{\frac{1}{7}}$ $P = t^3 + \frac{3.5}{b\,b}\,t$ $Q = \frac{6}{b}\,t\,t + \frac{3.5}{b^3}$ $R = t^3 + \frac{1.3}{b\,b}\,t$ $S = \frac{3}{b}\,t\,t$
$i = -4,\ t = -\frac{7}{x^{\frac{1}{7}}}$ $P = \frac{1}{t^4} + \frac{3.5}{b\,b\,t^6}$ $Q = \frac{6}{b\,t^5} + \frac{3.5}{b^3\,t^7}$ $R = \frac{1}{t^4} + \frac{3.3.5}{b\,b\,t^6} + \frac{3.5.7}{b^4\,t^8}$ $S = \frac{10}{b\,t^5} + \frac{3.5.7}{b^3\,t^7}$	$i = 4,\ t = 9\,x^{\frac{1}{9}}$ $P = t^4 + \frac{3.3.5}{b\,b}\,t^2 + \frac{3.5.7}{b^4}$ $Q = \frac{10}{b}\,t^3 + \frac{3.5.7}{b^3}\,t$ $R = t^4 + \frac{3.5}{b\,b}\,t^2$ $S = \frac{6}{b}\,t^3 + \frac{3.5}{b^3}\,t$
$i = -5,\ t = \frac{-9}{x^{\frac{1}{9}}}$ $P = \frac{1}{t^5} + \frac{3.3.5}{b\,b\,t^7} + \frac{3.5.7}{b^4\,t^9}$ $Q = \frac{10}{b\,t^6} + \frac{3.5.7}{b^3\,t^8}$ $R = \frac{1}{t^5} + \frac{3.5.7}{b\,b\,t^7} + \frac{3.5.7.9}{b^4\,t^9}$ $S = \frac{15}{b\,t^6} + \frac{4.3.5.7}{b^3\,t^8} + \frac{3.5.7.9}{b^5\,t^{10}}$	$i = 5,\ t = 11\,x^{\frac{1}{11}}$ $P = t^5 + \frac{3.5.7}{b\,b}\,t^3 + \frac{3.5.7.9}{b^4}\,t$ $Q = \frac{15}{b}\,t^4 + \frac{4.3.5.7}{b^3}\,t^2 + \frac{3.5.7.9}{b^5}$ $R = t^5 + \frac{3.3.5}{b\,b}\,t^3 + \frac{3.5.7}{b^4}\,t$ $S = \frac{10}{b}\,t^4 + \frac{3.5.7}{b^3}\,t^2$

$$i = -6,\ t = \frac{-11}{x^{\frac{1}{11}}}$$

$$P = \frac{1}{t^6} + \frac{3.5.7}{bbt^8} + \frac{3.5.7.9}{b^4 t^{10}}$$

$$Q = \frac{15}{bt^7} + \frac{4.3.5.7}{b^3 t^9} + \frac{3.5.7.9}{b^5 t^{11}}$$

$$R = \frac{1}{t^6} + \frac{2.3.5.7}{bbt^8} + \frac{5.3.5.7.9}{b^4 t^{10}} + \frac{3.5.7.9.11}{b^6 t^{12}}$$

$$S = \frac{21}{bt^7} + \frac{4.5.7.9}{b^3 t^9} + \frac{3.5.7.9.11}{b^5 t^{11}}$$

$$i = 6,\ t = 13\, x^{\frac{1}{13}}$$

$$P = t^6 + \frac{2.3.5.7}{bb} t^4 + \frac{5.3.5.7.9}{b^4} t^2 + \frac{3.5.7.9.11}{b^6}$$

$$Q = \frac{21}{b} t^5 + \frac{4.5.7.9}{b^3} t^3 + \frac{3.5.7.9.11}{b^5} t$$

$$R = t^6 + \frac{3.5.7}{bb} t^4 + \frac{3.5.7.9}{b^4} t^2$$

$$S = \frac{15}{b} t^5 + \frac{4.3.5.7}{b^3} t^3 + \frac{3.5.7.9}{b^5} t.$$

Scholion 2.

966. Forma integralis inventa modum suppeditat, aequationem propositam

$$\partial u + uu\,\partial x + A\,x^{\frac{-4i}{2i+1}}\,\partial x = 0$$

in speciem simpliciorem transformandi. Primo enim ponatur

$$x^{\frac{2i}{2i+1}}\,u = v,\ \text{seu}\ u = x^{\frac{-2i}{2i+1}}\,v,$$

ac prodibit

$$x^{\frac{-2i}{2i+1}}\,\partial v - \frac{2i}{2i+1}\,x^{\frac{-4i-1}{2i+1}}\,v\,\partial x + x^{\frac{-4i}{2i+1}}\,vv\,\partial x$$

$$+ A\,x^{\frac{-4i}{2i+1}}\,\partial x = 0,\ \text{seu}$$

$$\partial v - \frac{2i}{2i+1}\cdot\frac{v\,\partial x}{x} + x^{\frac{-2i}{2i+1}}\,vv\,\partial x + A\,x^{\frac{-2i}{2i+1}}\,\partial x = 0.$$

Ponatur porro $t = (2i+1)\, x^{\frac{1}{2i+1}}$, erit

$$\partial t = x^{\frac{-2i}{2i+1}} \partial x, \text{ et } \frac{\partial t}{t} = \frac{1}{2i+1} \cdot \frac{\partial x}{x},$$

unde fit

$$\partial v - \frac{2iv\partial t}{t} + vv\partial t + A\partial t = 0.$$

Sit insuper $v = \frac{i}{t} + z$, ut prodeat

$$-\frac{i\partial t}{tt} + \partial z - \frac{2ii\partial t}{tt} - \frac{2iz\partial t}{t} + \frac{ii\partial t}{tt} + \frac{2iz\partial t}{t} + zz\partial t + A\partial t = 0$$

seu

$$\partial z + zz\partial t - \frac{i(i+1)\partial t}{tt} + A\partial t = 0,$$

quae ergo quoties i est numerus integer, est integrabilis. Simili modo haec aequatio

$$\partial u + uu\partial x + A x^n \partial x = 0$$

generalius ita transformari potest: posito $u = x^\lambda v$ et $v = z - \frac{1}{2}\lambda x^{-\lambda-1}$, obtinetur

$$\partial z + x^\lambda zz\partial x + \tfrac{1}{4}\lambda(\lambda+2) x^{-\lambda-2}\partial x + A x^{n-\lambda}\partial x = 0,$$

quae porro posito $x^\lambda \partial x = \partial t$, seu $x^{\lambda+1} = (\lambda+1)t$, abit in

$$\partial z + zz\partial t + \frac{\lambda(\lambda+2)\partial t}{4(\lambda+1)^2 tt} + A(\lambda+1)^{\frac{n-2\lambda}{\lambda+1}}\, t^{\frac{n-2\lambda}{\lambda+1}}\, \partial t = 0,$$

quae aequatio est integrabilis, quoties $n = \frac{-4i}{2i+1}$, unde numerum λ pro lubitu assumendo, innumerabiles formae exhiberi possunt. Si capiatur $\lambda = -1$, fit $t = lx$ et

$$\partial z + zz\partial t - \tfrac{1}{4}\partial t + A e^{(n+2)t}\partial t = 0.$$