CAPUT VI.

DE INTEGRATIONE ALIARUM AEQUATIONUM DIFFERENTIO-DIFFERENTIALIUM PER IDONEOS MULTIPLICATORES INSTITUENDA.

Problema 112.

906.

Posito elemento ∂x constante, si proposita sit hujusmodi aequatio

$$\partial \partial y + \frac{A y \partial x^2}{(Byy + C + 2Dx + Exx)^2} = 0;$$

invenire multiplicatorem, quo ea integrabilis reddatur.

Solutio.

Tentetur talis multiplicatoris forma $2 P \partial y + 2 Q y \partial x$, ubi P et Q sint functiones ipsius x, et producti

$$2 \partial \partial y (P \partial y + Q y \partial x) + \frac{2 A y \partial x^2 (P \partial y + Q y \partial x)}{(Byy + C + 2Dx + Exx)^2} = 0,$$

integrale statuatur

$$P \partial y^2 + 2 Q y \partial x \partial y + V \partial x^2 = \text{Const.} \partial x^2,$$

ubi V sit functio binas variabiles x et y complectens. Erit ergo facta aequalitate

$$\begin{matrix} + \partial P \partial y^2 & + 2 y \partial x \partial Q \partial y \\ + 2 Q \partial x \partial y^2 & + \partial x^2 \partial V \end{matrix} - \frac{2 A y \partial x^2 (P \partial y + Q y \partial x)}{(Byy + C + 2Dx + Exx)^2} = 0,$$

quae per integrationem valorem ipsius V suppeditare nequit, nisi sit $\partial P + 2 Q \partial x = 0$, eritque tum

$$\partial V = \frac{Ay(2P\partial y - y\partial P)}{(Byy + C + 2Dx + Exx)^2} - 2y\partial y \cdot \frac{\partial Q}{\partial x},$$

cujus formulae, siquidem integrationem admittat, integrale ex variabilitate ipsius y erit

$$\frac{-AP}{B(Byy + C + 2Dx + Exx)} - yy \cdot \frac{\partial Q}{\partial x} = V.$$

Sumto ergo y constante, necesse est sit

$$-\frac{A\partial P(Byy + C + 2Dx + Exx) + 2AP\partial x(D + Ex)}{B(Byy + C + 2Dx + Exx)^2} - yy \cdot \frac{\partial\partial Q}{\partial x} = \frac{-Ayy\partial P}{(Byy + C + 2Dx + Exx)^2}$$

cui satisfit, si $\frac{\partial\partial Q}{\partial x^2} = 0$ et

$$-\partial P(C + 2Dx + Exx) + 2P\partial x(D + Ex) = 0,$$

quae duae conditiones an simul consistere possint, videndum est. Posterior autem dat

$$\frac{\partial P}{P} = \frac{2D\partial x + 2Ex\partial x}{C + 2Dx + Exx}, \text{ ideoque } P = C + 2Dx + Exx,$$

unde fit

$$Q = -\frac{\partial P}{2\partial x} = -D - Ex, \text{ hinc } \frac{\partial Q}{\partial x} = -E \text{ et } \frac{\partial\partial Q}{\partial x^2} = 0.$$

Multiplicator ergo quaesitus est

$$2\partial y(C + 2Dx + Exx) - 2y\partial x(D + Ex),$$

hincque obtinetur integrale

$$\frac{\partial y^2}{\partial x^2}(C + 2Dx + Exx) - \frac{2y\partial y}{\partial x}(D + Ex) - \frac{A(C + 2Dx + Exx)}{B(Byy + C + 2Dx + Exx)} + Eyy = \text{Const.}$$

seu utrinque addendo $\frac{A}{B}$,

$$\frac{\partial y^2}{\partial x^2}(C + 2Dx + Exx) - \frac{2y\partial y}{\partial x}(D + Ex) + \frac{Ayy}{Byy + C + 2Dx + Exx} + Eyy = \text{Const.}$$

Corollarium 1.

907. Haec ergo aequatio $\partial\partial y + \frac{aay\partial x^2}{(yy + xx)^2} = 0$, ubi $A = aa$, $B = 1$, $C = 0$, $D = 0$, et $E = 1$, integrabilis redditur multipli

...atore $2xx\partial y - 2yx\partial x$, ejusque integrale erit

$$\frac{xx\partial y^2}{\partial x^2} - \frac{2xy\partial y}{\partial x} + yy + \frac{aayy}{yy+xx} = bb.$$

Corollarium 2.

908. Si hic ponatur $y = ux$, ob $\partial y = u\partial x + x\partial u$, habebimus

$$\begin{aligned} &xxuu + \frac{2ux^3\partial u}{\partial x} + \frac{x^4\partial u^2}{\partial x^2} + \frac{aauu}{1+uu} = bb \\ &- 2xxuu - \frac{2ux^3\partial u}{\partial x} \\ &+ xxuu, \end{aligned}$$ sive

$$\frac{x^4\partial u^2}{\partial x^2} = \frac{bb+(bb-aa)uu}{1+uu}, \text{ ergo } \frac{\partial x}{xx} = \frac{\partial u\sqrt{(1+uu)}}{\sqrt{(bb+(bb-aa)uu)}},$$

unde tam x quam y per u determinatur.

Corollarium 4.

909. Simili etiam modo integratio in genere perfici potest. Sit enim brevitatis gratia

$$C + 2Dx + Exx = Bzz, \text{ erit } D + Ex = \frac{Bz\partial z}{\partial x},$$

et aequatio nostra fiet

$$\frac{Bzz\partial y^2}{\partial x^2} - \frac{2Byz\partial y\partial z}{\partial x^2} + Eyy + \frac{Ayy}{B(yy+zz)} = \frac{K}{B};$$

quae posito $y = uz$, abit in

$$\frac{Bz^4\partial u^2}{\partial x^2} - \frac{Buuzz\partial z^2}{\partial x^2} + Euuzz + \frac{Auu}{B(1+uu)} = \frac{K}{B}.$$

At $\frac{zz\partial z^2}{\partial x^2} = \frac{(D+Ex)^2}{BB}$, unde oritur

$$\frac{Bz^4\partial u^2}{\partial x^2} + \frac{CE-DD}{B}uu = \frac{K+(K-A)uu}{B(1+uu)}, \text{ seu}$$

$$\frac{BBz^4\partial u^2}{\partial x^2} = \frac{K+(K-A+DD-CE)uu+(DD-CE)u^4}{1+uu},$$

ita ut sit, restituto valore ipsius z,

$$\frac{\partial x}{C+2Dx+Exx} = \frac{\partial u\sqrt{(1+uu)}}{\sqrt{[K+(K-A+DD-CE)uu+(DD-CE)u^4]}}.$$

sicque x definitur per u, indeque etiam

$$y = u z = u \sqrt{\frac{C + 2 D x + E x x}{B}}.$$

Scholion.

910. Hinc patet substitutio, qua tam ipsa aequatio differentio-differentialis proposita, quam multiplicator ad formam commodiorem reduci debet. Posito enim ad abbreviandum $C + 2 D x + E x x = B z z$, aequatio nostra

$$\partial \partial y - \frac{A y \partial x^2}{B^2 (y y + z z)^2} = 0,$$

ope substitutionis $y = u z$ transformatur

$$z \partial \partial u + 2 \partial z \partial u + u \partial \partial z + \frac{A u \partial x^2}{B B z^3 (1 + u u)^2} = 0,$$

cujus multiplicator est

$$2 B (z z \partial y - y z \partial z), \text{ seu } 2 B z^3 \partial u,$$

vel simpliciter $z^3 \partial u$. Cum autem sit $\partial z = \frac{\partial z (D + E x)}{B z}$, erit $\partial \partial z = \frac{E \partial x^2}{B z} - \frac{\partial x \partial z (D + E x)}{B z z} = \frac{E \partial x^2}{B z} - \frac{\partial x^2 (D + E x)^2}{B B z^3} = \frac{(CE - DD) \partial x^2}{B B z^3}$ ita ut sit $z^3 \partial \partial z = \frac{CE - DD}{BB} \partial x^2$, unde aequatio nostra per $z^3 \partial u$ multiplicata induit hanc formam

$$z^4 \partial u \partial \partial u + 2 z^3 \partial z \partial u^2 + \frac{CE - DD}{BB} u \partial u \partial x^2 + \frac{A u \partial u \partial x^2}{B B (1 + u u)^2} = 0,$$

manifesto integrabilem, integrali existente

$$\tfrac{1}{2} z^4 \partial u^2 + \frac{CE - DD}{2 B B} u u \partial x^2 - \frac{A \partial x^2}{2 B B (1 + u u)} = \tfrac{1}{2} \text{Const.} \partial x^2,$$

cujus adeo nova integratio ob z functionem ipsius x mox in oculos incurrit, cum sit

$$z z \partial u = \partial x \sqrt{\left(\text{Const.} + \frac{DD - CE}{BB} u u + \frac{A}{B B (1 + u u)}\right)},$$

ubi variabiles u et x sponte separantur. Caeterum hic notetur, functionem pro z assumtam satisfacere aequationi $z^3 \partial \partial z = \alpha \partial x^2$, cum tamen ejus ratio non sit manifesta. Multiplicando autem hanc aequationem per $\frac{2 \partial x}{z^3}$ prodit, $2 \partial z \partial \partial z = \frac{2 \alpha \partial x^2 \partial z}{z^3}$ cujus integrale est $\partial z^2 = \beta \partial x^2 - \frac{\alpha \partial x^2}{zz}$, seu $\partial x = \frac{z \partial z}{\sqrt{(\beta zz - \alpha)}}$ unde porro fit

$$\beta zz = \alpha + \gamma\gamma + 2\beta\gamma x + \beta\beta xx,$$

quae est ipsa nostra forma.

Problema 113.

911. Sumto elemento ∂x constante, invenire formam generaliorem aequationum differentio-differentialium quae ope hujusmodi multiplicatoris $My\partial x + N\partial y$ integrabiles reddantur.

Solutio.

Quia multiplicator ope substitutionis $y = Ru$ in formam simplicissimam $S\partial u$ transmutari potest, hac substitutione ipsa aequatio differentio-differentialis induat hanc formam

$$\partial\partial u + P\partial x \partial u + \frac{U\partial x^2}{S} = 0,$$

cujus postremum membrum per $S\partial u$ multiplicatum sponte est integrabile, si quidem U denotet functionem quamcunque ipsius u, dum R, S et P sint functiones ipsius x. Cum ergo aequatio

$$S\partial u \partial\partial u + PS\partial x \partial u^2 + U\partial x^2 \partial u = 0,$$

debeat esse integrabilis, posito integrali

$$\tfrac{1}{2} S\partial u^2 + \partial x^2 \int U\partial u = \tfrac{1}{2} C\partial x^2,$$

necesse est sit

$$\tfrac{1}{2}\partial S\partial u^2 = PS\partial x\partial u^2, \text{ seu } P\partial x = \frac{\partial S}{2S}.$$

Quocirca haec forma generalis

$$\partial\partial u + \frac{\partial S\partial u}{2S} + \frac{U\partial x^2}{S} = 0,$$

por $S\partial u$ multiplicata dabit integrale

$$S\partial u^2 = \partial x^2 (C - 2\int U\partial u),$$

quod denuo integratum praebet

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$$\int\frac{\partial x}{\sqrt{S}}=\int\frac{\partial u}{\sqrt{(C-2\int C\partial u)}}.$$

Cum igitur haec sint manifesta, ponendo $u=\frac{y}{R}$ ad formas magis intricatas regrediamur, ite ut jam sit $U=$ funct. $\frac{y}{R}$. Nunc vero est

$$\partial u=\frac{\partial y}{R}-\frac{y\partial R}{RR} \text{ et } \partial\partial u=\frac{\partial\partial y}{R}-\frac{2\partial R\partial y}{RR}-\frac{y\partial\partial R}{RR}+\frac{2y\partial R^2}{R^3},$$

unde aequatio nostra fit

$$\frac{\partial\partial y}{R}-\frac{2\partial R\partial y}{RR}-\frac{y\partial\partial R}{RR}+\frac{2y\partial R^2}{R^3}+\frac{U\partial x^2}{S}=0,$$
$$+\frac{\partial S\partial y}{2RS} \qquad -\frac{y\partial R\partial S}{2RRS},$$

quae per $\frac{S}{RR}(R\partial y-y\partial R)$ multiplicata integrabilis redditur. Ut igitur ad formam supra propositam accedamus, statuamus $S=\alpha R^4$, et aequatio

$$\frac{\partial\partial y}{R}-\frac{y\partial\partial R}{RR}+\frac{U\partial x^2}{\alpha R^4}=0$$

per $\alpha RR(R\partial y-y\partial R)$ multiplicata integrabilis redditur. Seu haec aequatio

$$R\partial\partial y-y\partial\partial R+\frac{\partial x^2}{RR}f:\frac{y}{R}=0$$

per $R\partial y-y\partial R$ multiplicata fit integrabilis.

Ut via ad integrationem perveniendi magis occultetur, ponatur $f:\frac{y}{R}=\frac{\alpha y}{R}+V$, ut V sit functio homogenea nullius dimensionis ipsarum y et R, ac ponatur $y\partial\partial R=\frac{\alpha y\partial x^2}{R^3}$, ut fiat

$$R\partial\partial y+\frac{V\partial x^2}{RR}=0, \text{ seu } \partial\partial y+\frac{V\partial x^2}{R^3}=0,$$

quae multiplicatore $R(R\partial y-y\partial R)$ redditur integrabilis. At cum sit $\partial\partial R=\frac{\alpha\partial x^2}{R^3}$, erit ut supra vidimus

$$R=\sqrt{(\alpha+2\beta x+\gamma xx)},$$

unde dum V sit functio homogenea nullius dimensionis ipsarum y et $R=\sqrt{(\alpha+2\beta x+\gamma xx)}$, aequatio

$$\partial\partial y + \frac{V\partial x^2}{(\alpha + 2\beta x + \gamma xx)^{\frac{3}{2}}} = 0$$

ope multiplicatoris

$$(\alpha + 2\beta x + \gamma xx)\partial y - (\beta + \gamma x) y \partial x,$$

integrabilis evadit.

Corollarium 1.

912. Posito autem $R = \sqrt{(\alpha + 2\beta x + \gamma xx)}$, aequatio nostra per $RR\partial y - Ry\partial R$ multiplicata fit

$$RR\partial y\partial\partial y - Ry\partial R\partial\partial y + \frac{V\partial x^2 (R\partial y - y\partial R)}{RR} = 0,$$

cujus integrale est

$$\tfrac{1}{2}RR\partial y^2 - Ry\partial R\partial y + \int y\partial y (R\partial\partial R + \partial R^2)$$
$$+ \partial x^2 \int V\partial\frac{y}{R} = \text{Const.}\, \partial x^2,$$

ubi est

$$R\partial\partial R + \partial R^2 = \partial.R\partial R = \partial.(\beta + \gamma x)\partial x = V\partial x^2,$$

sicque integrale est

$$RR\partial y^2 - 2Ry\partial R\partial y + \gamma yy\partial x^2 \int V\partial.\frac{y}{R} = \text{Const.}\, \partial x^2.$$

Corollarium 2.

913. Quia V est functio ipsius $\frac{y}{R}$, formulae $\int V\partial.\frac{y}{R}$ integrale habetur. Pro ulteriore vero integratione posito $y = Ru$ et $\int V\partial u = U$, habebitur

$$R^4\partial u^2 - RRuu\partial R^2 + \gamma RRuu\partial x^2 + 2U\partial x^2 = G\partial x^2,$$

seu $R^4\partial u^2 = \partial x^2 [G - 2U + (\beta\beta - \alpha\gamma)uu]$,

hincque

$$\frac{\partial x}{\alpha + 2\beta x + \gamma xx} = \frac{\partial u}{\sqrt{[G - 2U + (\beta\beta - \alpha\gamma)uu]}},$$

ac porro $y = u\sqrt{(\alpha + 2\beta x + \gamma xx)}$.

Scholion.

914. Haec ergo aequatio $\partial\partial y + \frac{V\partial x^2}{R^3} = 0$, existente $R = \sqrt{(\alpha + 2\beta x + \gamma xx)}$, multo latius patet ea quam in praecedente problemate tractavimus, propterea quod hic pro V accipere licet functionem quamcunque homogeneam nullius dimensionis ipsarum y et R. Si enim sumatur $V = \frac{A R^3 y}{(myy+RR)^2}$, ipsa aequatio primum tractata oritur. Caeterum ex methodo, qua illam aequationem elicuimus apparet, eam per restrictionem ad hanc formam occultam esse perductam, cum ea aequatio, unde est nata

$$R\partial\partial y - y\partial\partial R + \frac{\partial x^2}{RR} f : \frac{y}{R} = 0$$

perspicue integrationem admittat, si per $R\partial y - y\partial R$ multiplicetur. Est enim

$$R\partial\partial y - y\partial\partial R = \partial . (R\partial y - y\partial R) \text{ et } \frac{R\partial y - y\partial R}{RR} = \partial . \frac{y}{R},$$

unde facta multiplicatione habebimus

$$(R\partial y - y\partial R)\,\partial . (R\partial y - y\partial R) + \partial x^2 f : \frac{y}{R}\,\partial . \frac{y}{R} = 0,$$

cujus aequationis utrumque membrum per se est integrabile. In aequatione autem inde eruta integrabilitas minus perspicitur, multo magis integratio est abscondita in aequationibus sequentibus.

Problema 114.

915. Sumto elemento ∂x constante, integrationem hujus aequationis

$$yy\partial\partial y + y\partial y^2 + Ax\partial x^2 = 0$$

ope multiplicatoris eam integrabilem reddentis perficere.

Solutio.

Hic frustra tentatur multiplicator hujus formae $L\partial y + M\partial x$; tentetur ergo haec forma

$$3L\partial y^2 + 2M\partial x\partial y + N\partial x^2$$

ac ponatur producti integrale

$$Lyy\partial y^3 + Myy\partial x\partial y^2 + Nyy\partial x^2\partial y + V\partial x^3 = C\partial x^3$$

cujus differentiatio perducit ad hanc aequationem

$$\begin{aligned}\partial x^3\partial V = &3Ly\partial y^4 + 2My\partial x\partial y^3 + Ny\partial x^2\partial y^2 + 2AMx\partial x^3\partial y + ANx\partial x^4\\ &-2Ly\partial y^4 - yy\partial x\partial y^3\left(\tfrac{\partial L}{\partial x}\right) + 3ALx\partial x^2\partial y^2 - yy\partial x^3\partial y\left(\tfrac{\partial N}{\partial x}\right)\\ &-yy\partial y^4\left(\tfrac{\partial L}{\partial y}\right) + 2My\partial x\partial y^3 - yy\partial x^2\partial y^2\left(\tfrac{\partial M}{\partial x}\right)\\ &-yy\partial x\partial y^3\left(\tfrac{\partial M}{\partial y}\right) - 2Ny\partial x^2\partial y^2\\ &-yy\partial x^2\partial y^2\left(\tfrac{\partial N}{\partial y}\right)\end{aligned}$$

quae formula ut integrationem admittat, membra, quae ∂y^4, ∂y^3 et ∂y^2 continent, evanescere debent: unde primo colligitur $L - y\left(\frac{\partial L}{\partial y}\right) = 0$, ubi $\left(\frac{\partial L}{\partial y}\right)$ nascitur ex differentiatione ipsius L posito x constante. Consideretur ergo x ut quantitas constans, eritque $\frac{\partial L}{L} = \frac{\partial y}{y}$, ideoque $L = y f : x$. Negligamus autem hanc functionem ipsius x, seu ejus loco unitatem sumamus, ut sit $L = y$ et $\left(\frac{\partial L}{\partial x}\right) = 0$: secundo ergo esse debet $\left(\frac{\partial M}{\partial y}\right) = 0$. Sumamus igitur $M = 0$, etiamsi M denotare possit functionem quamcunque ipsius x, quandoquidem videbimus hoc modo negotium confici posse. Tertio itaque habebimus

$$-Ny + 3Axy - yy\left(\frac{\partial N}{\partial y}\right) = 0;$$

sumto ergo x constante, erit $3Ax\partial y = N\partial y + y\partial N$, ideoque $Ny = 3Axy$ seu $N = 3Ax$, ubi iterum functionem ipsius x, quae loco constantis ingrederetur, negligimus. Cum igitur hactenus invenimus $L = y$, $M = 0$ et $N = 3Ax$, erit $\partial V = -3Ayy\partial y + 3AAxx\partial x$, quae formula cum sponte sit integrabilis scilicet $V = -Ay^3 + AAx^3$, multiplicator nostram aequationem integrabilem reddens erit $= 3y\partial y^2 + 3Ax\partial x^2$, et producti integrale habebitur

$$y^3\partial y^3 + 3Axyy\partial x^2\partial y - Ay^3\partial x^3 + AAx^3\partial x^3 = C\partial x^3$$

quod ob constantem C est integrale completum.

Corollarium 1.

916. Hujus integralis membrum primùm commode in tres factores resolvi potest. Si ponantur formulae $z^3 - A$ factores $(z-\alpha)(z-\beta)(z-\gamma)$, ut sit

$$\alpha = \sqrt[3]{A},\quad \beta = \frac{-1+\sqrt{-3}}{2}\sqrt[3]{A} \text{ et } \gamma = \frac{-1-\sqrt{-3}}{2}\sqrt[3]{A},$$

erit integrale inventum

$$\left(\frac{y\partial y}{\partial x} - \alpha y + \alpha\alpha x\right)\left(\frac{y\partial y}{\partial x} - \beta y + \beta\beta x\right)\left(\frac{y\partial y}{\partial x} - \gamma y + \gamma\gamma x\right) = C,$$

existente

$$\alpha + \beta + \gamma = 0,\ \alpha\beta + \alpha\gamma + \beta\gamma = 0 \text{ et } \alpha\beta\gamma = 1.$$

Posito enim $\frac{y\partial y}{\partial x} = z$, habetur haec forma

$$z^3 + 3Axyz - Ay^3 + AAx^3,$$

cujus factor si ponatur $z - p - q$, fit

$$z^3 - 3pqz - p^3 - q^3 = 0,$$

ideoque

$$p = y\sqrt[3]{A} \text{ et } q = -x\sqrt[3]{A^2}.$$

Corollarium 2.

917. Sumto ergo constante $C = 0$, tria obtinentur integralia particularia

$$y\partial y - \alpha y\partial x + \alpha\alpha x\partial x = 0,$$

et loco α scribendo β et γ,

$$y\partial y - \beta y\partial x + \beta\beta x\partial x = 0 \text{ et}$$
$$y\partial y - \gamma y\partial x + \gamma\gamma x\partial x = 0,$$

quae posito $y = ux$ dant $\frac{\partial x}{x} = \frac{-u\partial u}{uu - \alpha u + \alpha\alpha}$, et porro integrando

$$lx = l\frac{\alpha}{\sqrt{(\alpha\alpha - \alpha u + uu)}} - \frac{1}{\sqrt{3}}\text{Ang. tang.}\frac{u\sqrt{3}}{2\alpha - u} + \text{Const.}$$

Scholion 1.

918. Aequationem autem differentialem primi ordinis inventam difficile est denuo integrare. A potestatibus quidem differentia-

lium, ponendo $\partial y = p \partial x$ et $y = ux$, unde fit $\frac{\partial x}{x} = \frac{\partial u}{p-u}$, liberari potest, prodit enim

$$x^3 (u^3 p^3 + 3 A u u p - A u^3 + A A) = C,$$

quae sumtis logarithmis, differentiata dat

$$\frac{\partial x}{x} + \frac{u u \partial p (u p p + A) + u \partial u (u p^3 + 2 A p - A u)}{u^3 p^3 + 3 A u u p - A u^3 + A A} = 0,$$

quae loco $\frac{\partial x}{x}$ scripto $\frac{\partial u}{p-u}$, abit in

$$\partial u (u p p + A)^2 + u u (p - u) \partial p (u p p + A) = 0,$$

ac per $u p p + A$ dividendo, oritur

$$A \partial u + u p p \partial u + p u u \partial p - u^3 \partial p = 0,$$

quae ponendo $p = \frac{q}{u}$ aliquanto fit simplicior, scilicet

$$A \partial u + q \partial q + q u \partial u - u u \partial q = 0,$$

cui autem posito $A = m^3$, etsi particulariter satisfacit $q = mu - mm$, tamen inde integratio completa erui vix posse videtur. Caeterum eadem haec aequatio inter p et u immediate elicitur ex aequatione differentio-differentiali proposita, quoniam in ea binae variabiles x et y ubique eundem dimensionum numerum constituunt. Posito enim $\partial y = p \partial x$ et $y = u x$, abit ea in

$$u u x \partial p + u p p \partial x + A \partial x = 0, \text{ seu } \frac{\partial x}{x} = \frac{-u u \partial p}{A + u p p} = \frac{\partial u}{p-u},$$

quae est ipsa praecedens aequatio.

Scholion 2.

919. Interim tamen aequatio proposita complete integrari potest, indeque etiam eae, quas ex ea elicuimus. Hoc autem prorsus singulari ratione praestatur, aequationem illam adeo ad differentialia tertii ordinis evehendo. Cum enim sit

$$y \partial . \frac{y \partial y}{\partial x^2} + A x \partial x = 0,$$

statuatur $\frac{\partial x}{y} = \partial v$, ut fiat

$$y\,\partial . \frac{\partial y}{\partial v} + A\,x\,\partial x = 0, \text{ seu } \partial . \frac{\partial y}{\partial v} + A\,x\,\partial v = 0,$$

quae sumto elemento ∂v constante, denuo differentiata praebet

$$\frac{\partial^3 y}{\partial v} + A\,\partial x\,\partial v = 0, \text{ seu } \partial^3 y + A\,y\,\partial v^3 = 0:$$

quae forma ita est comparata, ut si ei particulariter satisfaciant $y = P$, $y = Q$, $y = R$, etiam satisfaciat $y = DP + EQ + FR$. Jam vero illi satisfacit $y = e^{-\alpha v}$, si fuerit $\alpha^3 = A$, cum igitur in Coroll. 1. ternae litterae α, β, γ eadem conditione sint praeditae, habebitur integrale completum

$$y = D\,e^{-\alpha v} + E\,e^{-\beta v} + F\,e^{-\gamma v};$$

unde ob $A\,x = -\frac{\partial\partial y}{\partial v^2}$, erit

$$x = \frac{-D\,\alpha\,\alpha\,e^{-\alpha v} - E\,\beta\,\beta\,e^{-\beta v} - F\,\gamma\,\gamma\,e^{-\gamma v}}{A}:$$

seu mutatis constantibus, ob $A = \alpha^3 = \beta^3 = \gamma^3$,

$$x = +\mathfrak{A}\,e^{-\alpha v} + \mathfrak{B}\,e^{-\beta v} + \mathfrak{C}\,e^{-\gamma v}$$

$$y = -\mathfrak{A}\,\alpha\,e^{-\alpha v} - \mathfrak{B}\,\beta\,e^{-\beta v} - \mathfrak{C}\,\gamma\,e^{-\gamma v}.$$

Hinc ergo aequationis

$$A\,\partial u + q\,\partial q + q\,u\,\partial u - u\,u\,\partial q = 0$$

integrale completum his formulis continetur

$$u = \frac{-\mathfrak{A}\,\alpha\,e^{-\alpha v} - \mathfrak{B}\,\beta\,e^{-\beta v} - \mathfrak{C}\,\gamma\,e^{-\gamma v}}{\mathfrak{A}\,e^{-\alpha v} + \mathfrak{B}\,e^{-\beta v} + \mathfrak{C}\,e^{-\gamma v}} \text{ et}$$

$$q = \frac{\mathfrak{A}\,\alpha\,\alpha\,e^{-\alpha v} + \mathfrak{B}\,\beta\,\beta\,e^{-\beta v} + \mathfrak{C}\,\gamma\,\gamma\,e^{-\gamma v}}{\mathfrak{A}\,e^{-\alpha v} + \mathfrak{B}\,e^{-\beta v} + \mathfrak{C}\,e^{-\gamma v}},$$

ob $q = pu = \frac{y\,\partial y}{x\,\partial x} = \frac{\partial y}{x\,\partial v}$, quod insigne est specimen integrationis methodo directa vix perficiendae.

Problema 115.

920. Sumto elemento ∂x constante, si proponatur haec ae-

quatio $2y^3\partial\partial y + yy\partial y^2 + X\partial x^2 = 0$, existente $X = \alpha + \beta x + \gamma xx$, invenire multiplicatorem, qui eam integrabilem reddat.

Solutio.

Hic frustra tentantur multiplicatores formae

$$L\partial y + M\partial x \text{ et } L\partial y^2 + M\partial x\partial y + N\partial x^2;$$

sumamus ergo multiplicatorem hujus formae

$$2L\partial y^3 + M\partial x^2\partial y + N\partial x^3,$$

et integrale statuatur

$$Ly^3\partial y^4 + My^3\partial x^2\partial y^2 + 2Ny^3\partial x^3\partial y + S\partial x^4 = 0,$$

unde per differentiationem colligitur

$$\begin{aligned}\partial x^4\partial S = {} & 2Lyy\partial y^5 + Myy\partial x^2\partial y^3 + Nyy\partial x^3\partial y^2 + MX\partial x^4\partial y + NX\partial x^5 \\ & - 3Lyy\partial y^5 + 2LX\partial x^2\partial y^3 + y^3\partial x^3\partial y^2\left(\tfrac{\partial M}{\partial x}\right) - 2y^3\partial x^4\partial y\left(\tfrac{\partial N}{\partial x}\right) \\ & - y^3\partial y^5\left(\tfrac{\partial L}{\partial y}\right) - 3Myy\partial x^2\partial y^3 - 6Nyy\partial x^3\partial y^2 \\ & - y^3\partial x^2\partial y^3\left(\tfrac{\partial M}{\partial y}\right) - 2y^3\partial x^3\partial y^2\left(\tfrac{\partial N}{\partial y}\right)\end{aligned}$$

ubi sumimus L esse functionem ipsius y tantum. Ut ergo termini ∂y^5 continentes destruantur, erit

$$-L - \frac{y\partial L}{\partial y} = 0 \text{ et } L = \frac{1}{y}.$$

Deinde pro destructione terminorum per ∂y^3 affectorum erit

$$-2Myy + \frac{2X}{y} - y^3\left(\frac{\partial M}{\partial y}\right) = 0,$$

et sumto x constante

$$\partial M + \frac{2M\partial y}{y} = \frac{2X\partial y}{y^4},$$

quae per yy multiplicata et integrata praebet

$$Myy = P - \frac{2X}{y} \text{ et } M = \frac{P}{yy} - \frac{2X}{y^3}$$

denotante P functionem quamcunque ipsius x. Jam ad terminos

∂y^2 tollendos, erit

$$-5\,N y y - y\frac{\partial P}{\partial x} + \frac{2\,\partial X}{\partial x} - 2\,y^3\left(\frac{\partial N}{\partial y}\right) = 0,$$

et sumto x constante

$$2\,y^3\,\partial N + 5\,N y y\,\partial y = \frac{2\,\partial X}{\partial x}\,\partial y - \frac{\partial P}{\partial x}\cdot y\,\partial y,$$

quae per $\sqrt{y}$ divisa et integrata dat

$$2\,N\,y^{\frac{5}{2}} = \frac{4\,\partial X}{\partial x}\sqrt{y} - \frac{2\,\partial P}{3\,\partial x}\,y\sqrt{y},$$

neglecta functione ipsius x addenda, quoniam irrationalitas $\sqrt{y}$ in calculum non ingreditur. Erit ergo $N = \frac{2\,\partial X}{y y\,\partial x} - \frac{\partial P}{3 y\,\partial x}$, ac propterea

$$\partial S = \partial y\left(\frac{PX}{yy} - \frac{2XX}{y^3} - \frac{4y\,\partial\partial X}{\partial x^2} + \frac{2yy\,\partial\partial P}{3\,\partial x^2}\right) + \frac{2X\,\partial X}{yy} - \frac{X\,\partial P}{3y},$$

unde fit integrando

$$S = \frac{XX}{yy} - \frac{PX}{y} - \frac{2yy\,\partial\partial X}{\partial x^2} + \frac{2y^3\,\partial\partial P}{9\,\partial x^2}$$
$$+\int\left(\frac{P\,\partial X}{y} + \frac{2X\,\partial P}{3y} + \frac{2yy\,\partial^3 X}{\partial x^2} - \frac{2y^3\,\partial^3 P}{9\,\partial x^2}\right),$$

quae finite exprimetur si $P = 0$, cum ob $X = \alpha + \beta x + \gamma x x$ fit $\partial^3 X = 0$. Quocirca habemus

$$L = \frac{1}{y},\quad M = -\frac{2X}{y^3},\ \text{et}\ N = \frac{2\,\partial X}{yy\,\partial x},$$

atque

$$S = \frac{XX}{yy} - \frac{2yy\,\partial\partial X}{\partial x^2} + \text{Const.}$$

unde aequatio integralis est

$$y^2\partial y^4 - 2X\partial x^2\partial y^2 + 4y\partial X\partial x^2\partial y + \frac{XX\,\partial x^4}{yy} - 2\,yy\,\partial x^2\,\partial\partial X = C\,\partial x^4.$$

Aequatio ergo proposita

$$2\,y^3\,\partial\partial y + yy\,\partial y^2 + \partial x^2(\alpha + \beta x + \gamma x x) = 0,$$

integrabilis redditur multiplicata per

$$\frac{2\,\partial y^3}{y} - \frac{2(\alpha + \beta x + \gamma x x)\,\partial x^2\,\partial y}{y^3} + \frac{2\,\partial x^3(\beta + 2\gamma x)}{yy}$$

cujus vero est integrale

$$y^2\partial y^4 - 2\partial x^2\partial y^2(\alpha+\beta x+\gamma xx) + 4y\partial x^3\partial y(\beta+2\gamma x)$$
$$-4\gamma yy\partial x^4 + \frac{(\alpha+\beta x+\gamma xx)^2\partial x^4}{\gamma y} = C\partial x^4,$$

seu

$$[yy\partial y^2 - (\alpha+\beta x+\gamma xx)\partial x^2]^2 + 4y^3\partial x^3\partial y(\beta+2\gamma x)$$
$$-4\gamma y^4\partial x^4 = C yy\partial x^4.$$

Scholion 1.

921. Integrale hoc ita est intricatum, ut alia methodo vix inveniri potuisse videatur, verum etiam ita est comparatum, ut nulla pateat methodus id porro integrandi, unde prima integratio parum lucri attulisse est judicanda. Quemadmodum autem in praecedente problemate integrale completum ex alio fonte hausimus, ita hic simili modo integrale eruere licet, quod eo magis est notatu dignum, cum aequatio proposita in se spectata solutu sit difficillima. Ponamus scilicet itidem $\partial x = y\partial v$, et cum sit

$$\partial\partial y = \partial x\partial . \frac{\partial y}{\partial x} = y\partial v\partial . \frac{\partial y}{y\partial v},$$

erit sumendo jam elementum ∂v constans

$$\partial\partial y = y\partial v\left(\frac{\partial\partial y}{y\partial v} - \frac{\partial y^2}{yy\partial v}\right) = \partial\partial y - \frac{\partial y^2}{y}.$$

Hinc nostra aequatio induit hanc formam

$$2y^3\partial\partial y - yy\partial y^2 + yy\partial v^2(\alpha+\beta x+\gamma xx) = 0, \text{ seu}$$
$$2y\partial\partial y - \partial y^2 + \partial v^2(\alpha+\beta x+\gamma xx) = 0,$$

quae denuo differentiata praebet

$$2y\partial^3 y + y\partial v^3(\beta+2\gamma x) = 0, \text{ seu}$$
$$2\partial^3 y + \partial v^3(\beta+2\gamma x) = 0,$$

differentietur iterum, prodibitque

$$2\partial^4 y + 2\gamma y\partial v^4 = 0, \text{ seu } \partial^4 y + \gamma y\partial v^4 = 0,$$

quam aequationem si aliunde resolvere, valoremque ipsius y per v exprimere liceat, erit $x = \int y\partial v$, seu sine integratione $x = -\frac{\partial^3 y}{\gamma\partial v^3}$

$-\frac{\beta}{2\gamma}$. At manifestum est, isti aequationi differentiali quarti ordinis satisfacere $y = e^{\lambda v}$, si sit $\lambda^4 + \gamma = 0$. Ponamus ergo $\gamma = -n^4$, et quatuor ipsius λ habebuntur valores $\lambda = \pm n$ et $\lambda = \pm n\sqrt{-1}$, unde ejus integrale completum est

$$y = A e^{nv} + B e^{-nv} + C \sin.(nv + \zeta),$$

hincque

$$x = + \frac{A}{n} e^{nv} - \frac{B}{n} e^{-nv} - \frac{C}{n} \cos.(nv + \zeta) + \frac{\beta}{2n^4},$$

qui ergo valores quoque satisfacient aequationi inter x et y propositae, dummodo constantes A, B, C, et ζ ita a se pendentes capiantur, ut quantitati quoque α conveniant. His nempe valoribus substitutis fieri debet

$$\alpha + \beta x - n^4 xx + \frac{2y\partial\partial y - \partial y^2}{\partial v^2} = 0,$$

ubi tantum terminos constantes considerasse sufficit, quibus accenseri debent ii, qui quadratum sinus cosinusve anguli $nv + \zeta$ continent, quippe ex quorum combinatione quantitas constans exsurgit. Cum ergo sit

$$2y = 2 A e^{nv} + 2 B e^{-nv} + 2 C \sin.(nv + \zeta),$$
$$\frac{\partial\partial y}{\partial v^2} = nn A e^{nv} + nn B e^{-nv} - nn C \sin.(nv + \zeta),$$
$$\frac{\partial y}{\partial v} = n A e^{nv} - n B e^{-nv} + n C \cos.(nv + \zeta),$$
$$x = \frac{A}{n} e^{nv} - \frac{B}{n} e^{-nv} - \frac{C}{n} \sin.(nv + \zeta) + \frac{\beta}{2n^4},$$

erit sumtis terminis memoratis

$$\beta x = \frac{\beta\beta}{2n^4}$$
$$-n^4 xx = 2nn AB - nn CC \cos.(nv + \zeta)^2 - \frac{\beta\beta}{4n^4},$$
$$\frac{2y\partial\partial y}{\partial v^2} = 4nn AB - 2nn CC \sin.(nv + \zeta)^2,$$
$$-\frac{\partial y^2}{\partial v^2} = 2nn AB - nn CC \cos.(nv + \zeta)^2,$$

ergo

$$\alpha + 8nn AB - 2nn CC + \frac{\beta\beta}{4n^4} = 0, \text{ ideoque}$$
$$C = \sqrt{\left(\frac{\alpha}{2nn} + \frac{\beta\beta}{8n^6} + 4\alpha B\right)}, \text{ vel}$$

$$\alpha = 2nn(CC - 4AB) - \frac{\beta\beta}{4n^4} \text{ et}$$

$$\alpha + \beta x + \gamma xx = 2nn(CC - AB) - \left(\frac{\beta}{2nn} - nnx\right)^2.$$

Manent ergo tres constantes A, B, et ζ indeterminatae, ita ut nullum sit dubium, quin formulae pro x et y datae integrale completum exhibeant.

Scholion 2.

922. Aequationes differentio-differentiales, quas in his duobus problematibus tractavimus, ad similem formam reduci possunt. Prior enim

$$y(y\partial\partial y + \partial y^2) + X\partial x^2 = 0,$$

existente $X = Ax$ vel $X = \alpha + \beta x$, si ponatur $y\partial y = \frac{1}{2}\partial z$ seu $yy = z$, induit hanc formam

$$\tfrac{1}{2}\partial\partial z\sqrt{z} + X\partial x^2 = 0,$$

quae ope multiplicatoris $\frac{3\partial z^2}{4\sqrt{z}} + 3X\partial x^2$ integrabilis redditur. Altera vero aequatio

$$yy(2y\partial\partial y + \partial y^2) + X\partial x^2 = 0,$$

existente $X = \alpha + \beta x + \gamma xx$, posito $y = z^{\frac{2}{3}}$, fit

$$\partial y = \tfrac{2}{3}z^{-\frac{1}{3}}\partial z \text{ et } \partial\partial y = \tfrac{2}{3}z^{-\frac{1}{3}}\partial\partial z - \tfrac{2}{9}z^{-\frac{4}{3}}\partial z^2,$$

hinc $2y\partial\partial y + \partial y^2 = \frac{4}{3}z^{\frac{1}{3}}\partial\partial z$, sicque aequatio hanc induit formam $\frac{4}{3}z^{\frac{5}{3}}\partial\partial z + X\partial x^2 = 0$, quae integrabilis redditur ope hujus multiplicatoris:

$$\frac{16\partial z^3}{27z^{\frac{5}{3}}} - \frac{4X\partial x^2\partial z}{3z^{\frac{7}{3}}} + \frac{2\partial X\partial x^2}{z^{\frac{4}{3}}}.$$

Hinc colligimus, pro aequatione $\partial\partial z + \frac{2X\partial x^2}{\sqrt{z}} = 0$ fore multiplicatorem $\partial z^2 + X\partial x^2 \sqrt{z}$, pro aequatione autem

$$\partial\partial z + \frac{3X\partial x^2}{4z\sqrt[3]{zz}} = 0$$

multiplicatorem fore

$$\partial z^3 - \frac{9X\partial x^2\partial z}{4\sqrt[3]{zz}} + \tfrac{27}{8}\partial X\partial x^2 \sqrt[3]{z},$$

seu sub uno conspectu

pro aequatione	multiplicator erit
$\partial\partial z + \frac{X\partial x^2}{\sqrt{z}} = 0$	$\partial z^2 + 2X\partial x^2\sqrt{z}$,
$\partial\partial z + \frac{X\partial x^2}{z\sqrt[3]{zz}} = 0$	$\partial z^3 + \frac{3X\partial x^4\partial z}{\sqrt[3]{zz}} + \frac{9}{2}\partial X\partial x^2\sqrt[3]{z}$.

Caeterum hae integrationes maxime sunt notatu dignae, cum ex aequationibus differentialibus altioribus perfici queant. Ita cum ex hac aequatione, ubi ∂v constans

$$\partial^3 y + A\partial v\partial\partial y + B\partial v^2\partial y + Cy\partial v^3 = 0 \text{ sit}$$

$$y = \mathfrak{A}\, e^{\alpha v} + \mathfrak{B}\, e^{\beta v} + \mathfrak{C}\, e^{\gamma v},$$

si fuerint α, β, γ, radices hujus aequationis

$$r^3 + Ar^2 + Br + C = 0,$$

ponamus $\partial v = \frac{\partial x}{y}$, et cum sit

$$\partial\partial y = \partial v\partial\frac{\partial y}{\partial v} = \frac{\partial x}{y}\partial.\frac{y\partial y}{\partial x} \text{ et}$$

$$\partial^3 y = \partial v^2\partial.\frac{\partial\partial y}{\partial v^2} = \partial v^2\partial.\left(\frac{1}{\partial v}\partial.\frac{\partial y}{\partial v}\right) = \frac{\partial x^2}{yy}\,\partial.\left(\frac{y}{\partial x}\partial.\frac{y\partial y}{\partial x}\right),$$

si jam ∂x constans sumamus, erit

$$\partial\partial y = \partial\partial y + \frac{\partial y^2}{y} \text{ et}$$

$$\partial^3 y = \frac{1}{yy}\partial \cdot y\,(y\,\partial\partial y + \partial y^2) = \partial^3 y + \frac{4\,\partial y\,\partial\partial y}{y} + \frac{\partial y^3}{yy},$$

hincque per $y\,y$ multiplicando

$$y\,y\,\partial^3 y + 4\,y\,\partial y\,\partial\partial y + \partial y^3 + \mathrm{A}\,\partial x\,(y\,\partial\partial y + \partial y^2) + \mathrm{B}\,\partial x^2\,\partial y + \mathrm{C}\,\partial x^3 = 0,$$

quae integrata dat

$$y\,y\,\partial\partial y + y\,\partial y^2 + \mathrm{A}\,y\,\partial x\,\partial y + \mathrm{B}\,y\,\partial x^2 + (\mathrm{C}x + \mathrm{D})\,\partial x^2 = 0,$$

quae ergo per superiora integrari potest.

Problema 116.

923. Definire conditiones functionum P, Q, R et L, M, N, ut haec aequatio differentio-differentialis

$$\partial\partial y + \mathrm{P}\,\partial y^2 + \mathrm{Q}\,\partial x\,\partial y + \mathrm{R}\,\partial x^2 = 0$$

integrabilis reddatur multiplicatore

$$3\,\mathrm{L}\,\partial y^2 + 2\,\mathrm{M}\,\partial x\,\partial y + \mathrm{N}\,\partial x^2.$$

Solutio.

Facta multiplicatione integratio terminorum per $\partial\partial y$ affectorum dat

$$\mathrm{L}\,\partial y^3 + \mathrm{M}\,\partial x\,\partial y^2 + \mathrm{N}\,\partial x^2\,\partial y,$$

quare ponatur integrale

$$\mathrm{L}\,\partial y^3 + \mathrm{M}\,\partial x\,\partial y^2 + \mathrm{N}\,\partial x^2\,\partial y + \mathrm{V}\,\partial x^3 = \mathrm{C}\,\partial x^3,$$

cujus differentiale aequari debet formulae propositae in multiplicatorem ductae, unde oritur

$$\begin{array}{lllll}
\partial x^3\partial \mathrm{V} = 3\,\mathrm{LP}.\partial y^4 & + 3\,\mathrm{LQ}.\partial x\partial y^3 & + 3\,\mathrm{LR}.\partial x^2\partial y^2 & & \\
 & + 2\,\mathrm{MP} & + 2\,\mathrm{MQ} & + 2\,\mathrm{MR}\partial x^3\partial y & \\
\quad -\left(\frac{\partial \mathrm{L}}{\partial y}\right) & -\left(\frac{\partial \mathrm{L}}{\partial x}\right) & + \mathrm{NP} & + \mathrm{NQ} & + \mathrm{NR}\partial x^4 \\
 & -\left(\frac{\partial \mathrm{M}}{\partial y}\right) & + \left(\frac{\partial \mathrm{M}}{\partial x}\right) & -\left(\frac{\partial \mathrm{N}}{\partial x}\right) & \\
 & & -\left(\frac{\partial \mathrm{N}}{\partial y}\right). & &
\end{array}$$

Hic ergo fieri oportet

$$3LP - \left(\frac{\partial L}{\partial y}\right) = 0,$$
$$3LQ + 2MP - \left(\frac{\partial L}{\partial x}\right) - \left(\frac{\partial M}{\partial y}\right) = 0,$$
$$3LR + 2MQ + NP - \left(\frac{\partial M}{\partial x}\right) - \left(\frac{\partial N}{\partial y}\right) = 0.$$

Tum vero erit

$$\partial V = \left[2MR + NQ - \left(\frac{\partial N}{\partial x}\right)\right]\partial y + NR\,\partial x,$$

quae formula integrabilis esse debet. Ex illis autem aequationibus colligitur

$$P = \frac{1}{3L}\left(\frac{\partial L}{\partial y}\right),\quad Q = \frac{1}{3L}\left(\frac{\partial L}{\partial x}\right) + \frac{1}{3L}\left(\frac{\partial M}{\partial y}\right) - \frac{2M}{9LL}\left(\frac{\partial L}{\partial y}\right) \text{ et}$$
$$R = \frac{1}{3L}\left(\frac{\partial M}{\partial x}\right) + \frac{1}{3L}\left(\frac{\partial N}{\partial y}\right) - \frac{N}{9LL}\left(\frac{\partial L}{\partial y}\right) - \frac{2M}{9LL}\left(\frac{\partial L}{\partial x}\right) - \frac{2M}{9LL}\left(\frac{\partial M}{\partial y}\right) + \frac{4MM}{27L^3}\left(\frac{\partial L}{\partial y}\right).$$

Corollarium 1.

924. Si L, M, et N fuerint functiones ipsius x tantum, erit $P = 0$, $Q = \frac{\partial L}{3L\partial x}$ et $R = \frac{\partial M}{3L\partial x} - \frac{2M\partial L}{9LL\partial x}$, hinc

$$\partial V = \left(\frac{2M\partial M}{3L\partial x} - \frac{4MM\partial L}{9LL\partial x} + \frac{N\partial L}{3L\partial x} - \frac{\partial N}{\partial x}\right)\partial y + \frac{N\partial M}{3L} - \frac{2MN\partial L}{9LL},$$

ac coëfficiens ipsius ∂y debet esse constans. Quare per $L^{\frac{1}{3}}$ dividendo habebitur

$$\frac{C\,\partial x}{\sqrt[3]{L}} = \frac{2M\partial M}{3L\sqrt[3]{L}} + \frac{N\partial L}{3L\sqrt[3]{L}} - \frac{\partial N}{\sqrt[3]{L}},$$

et integrando

$$C\int\frac{\partial x}{\sqrt[3]{L}} = \frac{MM}{3L\sqrt[3]{L}} - \frac{N}{\sqrt[3]{L}},\ \text{seu}\ N = \frac{MM}{\partial L} - C\,L^{\frac{1}{3}}\int\frac{\partial x}{\sqrt[3]{L}},$$

ergo

$$V = Cy + \int\left(\frac{\partial M}{3L} - \frac{2M\partial L}{9LL}\right)\left(\frac{MM}{3L} - C\,L^{\frac{1}{3}}\int\frac{\partial x}{\sqrt[3]{L}}\right).$$

Corollarium 2.

925. Sit $M = S\sqrt[3]{L^2}$, erit

$$\partial M = \partial S \sqrt[3]{L^2} + \frac{2 S \partial L}{3\sqrt[3]{L}}, \text{ et}$$

$$V = C y + \tfrac{1}{3}\int \frac{\partial S}{\sqrt[3]{L}} \left(\tfrac{1}{3} S S \sqrt[3]{L} - C L^{\frac{1}{3}} \int \frac{\partial x}{\sqrt[3]{L}}\right), \text{ seu}$$

$$V = C y + \tfrac{1}{27} S^3 - \tfrac{1}{3} C \int \partial S \int \frac{\partial x}{\sqrt[3]{L}},$$

tum vero

$$N = \tfrac{1}{3} S S \sqrt[3]{L} - C L^{\frac{1}{3}} \int \frac{\partial x}{\sqrt[3]{L}} = \left(\tfrac{1}{3} S S - C \int \frac{\partial x}{\sqrt[3]{L}}\right) \sqrt[3]{L},$$

atque $P = 0$, $Q = \frac{\partial L}{3 L \partial x}$ et $R = \frac{\partial S}{3 \partial x \sqrt[3]{L}}$. Quare haec aequatio

$$\partial \partial y + \frac{\partial L \partial y}{3L} + \frac{\partial S \partial x}{3\sqrt[3]{L}} = 0$$

integrabilis redditur multiplicatore

$$3 L \partial y^2 + 2 S \partial x \partial y \sqrt[3]{L^2} + \partial x^2 \left(\tfrac{1}{2} S S - C \int \frac{\partial x}{\sqrt[3]{L}}\right) \sqrt[3]{L},$$

et integrale est

$$L \partial y^3 + S \partial x \partial y^2 \sqrt[3]{L L} + \partial x^2 \partial y \left(\tfrac{1}{2} S S - C \int \frac{\partial x}{\sqrt[3]{L}}\right) \sqrt[3]{L} + C y \partial x^3$$
$$+ \tfrac{1}{27} S^3 \partial x^3 - \tfrac{1}{3} C \partial x^3 \int \partial S \int \frac{\partial x}{\sqrt[3]{L}} = 0.$$

Corollarium 3.

926. Hic quidquid pro constante C assumatur, idem integrale prodire debet. Hinc si $C = 0$, aequationis

$$\partial \partial y + \frac{\partial L \partial y}{3 L} + \frac{\partial S \partial x}{3\sqrt[3]{L}} = 0$$

multiplicator erit

$$3\,L\,\partial y^2 + 2\,S\,\partial x\,\partial y\,\sqrt[3]{L^2} + \tfrac{1}{2}\,S\,S\,\partial x^2\,\sqrt[3]{L},$$

et integrale

$$L\,\partial y^3 + S\,\partial x\,\partial y^2\,\sqrt[3]{L\,L} + \tfrac{1}{3}\,S\,S\,\partial x^2\,\partial y\,\sqrt[3]{L} + \tfrac{1}{27}\,S^3\,\partial x^3 = D\,\partial x^3,$$

seu $(\partial y\,\sqrt[3]{L} + \tfrac{1}{3}\,S\,\partial x)^3 = D\,\partial x^3$.

Scholion 1.

927. Ex iisdem quoque conditionibus, si dentur functiones P, Q et R, definiri poterunt functiones L, M, N, quatenus quidem postrema conditio integrabilitatis patitur. Veluti si sit $P = \frac{n}{y}$, $Q = 0$ et R functio ipsius x tantum, puta $R = X$, ut habeatur haec aequatio

$$\partial\,\partial y + \frac{n\,\partial y^2}{y} + X\,\partial x^2 = 0,$$

cujus multiplicator si sumatur

$$3\,L\,\partial y^2 + 2\,M\,\partial x\,\partial y + N\,\partial x^2,$$

ut integrale sit

$$L\,\partial y^3 + M\,\partial x\,\partial y^2 + N\,\partial x^2\,\partial y + V\,\partial x^3 = C\,\partial x^3,$$

erit primo $\frac{3nL}{y} - \left(\frac{\partial L}{\partial y}\right) = 0$, et sumta x constante $\frac{\partial L}{L} = \frac{3n\partial y}{y}$, hinc $L = S\,y^{3n}$, denotante S functionem ipsius x. Deinde est

$$\frac{2nM}{y} - y^{3n}\frac{\partial S}{\partial x} - \left(\frac{\partial M}{\partial y}\right) = 0,$$

et sumta x constante

$$\partial M - \frac{2nM\,\partial y}{y} + \frac{\partial S}{\partial x}\cdot y^{3n}\,\partial y = 0,$$

quae per y^{-2n} multiplicata et integrata dat

$$y^{-2n}M + \frac{\partial S}{(n+1)\,\partial x}\,y^{n+1} = T \text{ funct. ipsius } x.$$

Ergo

$$M = T\,y^{2n} - \frac{\partial S}{(n+1)\,\partial x}\,y^{3n+1}.$$

Tertio fieri debet

$$3SXy^{3n}+\frac{nN}{y}-\frac{\partial T}{\partial x}\cdot y^{2n}+\frac{\partial\partial S}{(n+1)\partial x^2}\cdot y^{3n+1}-\frac{\partial N}{\partial y}=0,$$

unde sumta x constante

$$\partial N+\frac{nN\partial y}{y}+\frac{\partial T}{\partial x}\cdot y^{2n}\partial y-\frac{\partial\partial S}{(n+1)\partial x^2}y^{3n+1}\partial y-3SXy^{3n}\partial y=0,$$

quae per y^{-n} multiplicata et integrata dat

$$y^{-n}N+\frac{\partial T}{(n+1)\partial x}y^{n+1}-\frac{\partial\partial S}{2(n+1)^2\partial x^2}y^{2n+2}-\frac{3SX}{2n+1}y^{2n+1}=Uf:x,$$

seu

$$N=Uy^n-\frac{\partial T}{(n+1)\partial x}y^{2n+1}+\frac{\partial\partial S}{2(n+1)^2\partial x^2}y^{3n+2}+\frac{3SX}{2n+1}y^{3n+1}.$$

Ex his autem fit

$$\partial V=\partial y\left\{\begin{array}{l}2TXy^{2n}-\frac{2X\partial S}{(n+1)\partial x}y^{3n+1}-\frac{\partial U}{\partial x}\cdot y^n+\frac{\partial\partial T}{(n+1)\partial x^2}y^{2n+1}\\ \qquad -\frac{\partial^3 S}{2(n+1)^2\partial x^3}y^{3n+2}-\frac{3\partial.SX}{(2n+1)\partial x}y^{3n+1}\end{array}\right\}$$

$$+X\partial x\left(Uy^n-\frac{\partial T}{(n+1)\partial x}y^{2n+1}+\frac{\partial\partial S}{2(n+1)^2\partial x^2}y^{3n+2}+\frac{3SX}{2n+1}y^{3n+1}\right),$$

quae formula ut integrationem admittat, esse oportet

$$2y^{2n}\partial.TX-2y^{3n+1}\cdot\frac{\partial.X\partial S}{(n+1)\partial x}-y^n\cdot\frac{\partial\partial U}{\partial x}+y^{2n+1}\cdot\frac{\partial^3 T}{(n+1)\partial x^2}$$
$$-y^{3n+2}\cdot\frac{\partial^4 S}{2(n+1)^2\partial x^3}-3y^{3n+1}\frac{\partial\partial.SX}{(2n+1)\partial x}-nUXy^{n-1}\partial x$$
$$+\frac{(2n+1)X\partial T}{u+1)}y^{2n}-\frac{(3n+2)X\partial\partial S}{2(n+1)^2\partial x}y^{3n+1}-\frac{3(3n+1)SXX\partial x}{(2n+1)}y^{3n}=0,$$

hic ergo singulae potestates ipsius y, quatenus sunt inaequales, seorsim destrui debent. Quare potestas y^{n-1} dat $U=0$; unde etiam potestas y^n ad nihilum redigitur. Potestas y^{2n} dat

$$(2n+2)T\partial X+(2n+2)X\partial T+(2n+1)X\partial T=0,$$

seu $X^{2n+2}T^{4n+3}=A$; at potestas y^{2n+1} praebet $\partial^3 T=0$, seu $T=\alpha+\beta x+\gamma xx$. Potestas vero y^{3n} postulat $S=0$, nisi sit $n=-\frac{1}{3}$; quo casu etiam potestates y^{3n+1} et y^{3n+2} sponte evanescunt. Cum ergo sit $U=0$, $S=0$ et $T=\alpha+\beta x+\gamma xx$, hincque $X=B(\alpha+\beta x+\gamma xx)^{\frac{-4n-3}{2n+2}}$ haec aequatio

$$\partial\partial y + \frac{n\,\partial y^2}{y} + B(\alpha + \beta x + \gamma x x)^{\frac{-4n-3}{2n+2}}\,\partial x^2 = 0\,;$$

integrabilis redditur ope multiplicatoris

$$2(\alpha + \beta x + \gamma x x)\,y^{2n}\,\partial y - \frac{\partial x(\beta + 2\gamma x)}{n+1}\,y^{2n+1}.$$

Scholion 2.

928. Quanquam plurimum abest, quominus haec methodus satis adhuc sit culta, tamen specimina in hoc capite tradita abunde declarant, quanta incrementa inde expectare queamus, unde ejus cultura maxime Geometris commendanda videtur. Quoniam igitur methodi, quibus in resolutione aequationum differentio-differentialium uti convenit, satis luculenter sunt expositae, ad sequens caput progrediamur, ubi integrationem hujusmodi aequationum, quatenus quidem id commode fieri potest, per series infinitas ostendemus.