CAPUT V.

DE

INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM SECUNDI GRADUS, IN QUIBUS ALTERA VARIABILIS UNAM DIMENSIONEM NON SUPERAT, PER FACTORES.

Problema 107.

865.

Sumto elemento ∂x constante, si proponatur haec aequatio

$$\partial \partial y + A \partial x \partial y + B y \partial x^2 = X \partial x^2,$$

ubi X denotat functionem quamcunque ipsius x, invenire functionem ipsius x, per quam haec aequatio multiplicata fiat integrabilis.

Solutio.

Ponatur $\partial y = p \partial x$, ut habeatur forma differentialis primi gradus

$$\partial p + A p \partial x + B y \partial x = X \partial x,$$

quae multiplicata per V, functionem quandam ipsius x, fiat integrabilis; scilicet

$$V \partial p + A V p \partial x + B V y \partial x = V X \partial x,$$

ubi cum posterius membrum $V X \partial x$ sit integrabile, idem in priori eveniat, necesse est. At primo perspicuum est ejus integralis partem fore Vp, unde id ponatur $V p + S$ ut sit $V p + S = \int V X \partial x$, fietque

$$\partial y + g y \partial x = e^{-fx} \partial x \int e^{fx} X \partial x,$$

sicque per integrationem ad aequationem differentialem primi gradus reducitur, quae denuo integrabilis redditur si per e^{gx} multiplicetur.

Scholion.

868. Multiplicatorem V ita determinari oportebat, ut formula $\partial y (AV - \frac{\partial V}{\partial x}) + BVy\partial x$ fieret per se integrabilis. Tum autem cum V sit functio ipsius x, integrale erit $y(AV - \frac{\partial V}{\partial x})$, unde fiat necesse est

$$A\partial V - \frac{\partial\partial V}{\partial x} = BV\partial x, \text{ seu } \partial\partial V - A\partial x\partial V + BV\partial x^2 = 0,$$

a cujus aequationis integratione pendet inventio factoris quaesiti V. Sufficit autem ejus integrale particulare sumsisse, dummodo enim aequatio proposita integrabilis reddatur, constans arbitraria pro integrali completo reddendo ipsa integratione introducitur.

Problema 108.

869. Sumto elemento ∂x constante, si proponatur haec aequatio

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = X\partial x^2,$$

ubi P, Q et X sint functiones quaecunque ipsius x, invenire multiplicatorem V, qui sit functio ipsius x, quo illa aequatio integrabilis reddatur.

Solutio.

Quia aequatio per V multiplicata

$$V\partial\partial y + VP\partial y\partial x + VQy\partial x^2 = VX\partial x^2,$$

integrabilis existit, prioris partis integrale ponatur $V\partial y + Sy\partial x$, aliam enim formam habere nequit, ac fieri oportet

$$VP\partial y\partial x + VQy\partial x^2 = \partial y\partial V + S\partial y\partial x + y\partial S\partial x,$$

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ubi cum S sit necessario functio ipsius x, erit

$$VP\partial x = \partial V + S\partial x \text{ et } VQ\partial x = \partial S.$$

Inde autem est $S = VP - \frac{\partial V}{\partial x}$, quare multiplicator V definiri debet ex hac aequatione

$$VQ\partial x = V\partial P + P\partial V - \frac{\partial\partial V}{\partial x}, \text{ seu}$$

$$\partial\partial V - P\partial V\partial x + V\partial x(Q\partial x - \partial P) = 0,$$

quae ergo si resolvi potuerit, vel si saltem ejus integrale quodpiam particulare innotescat, ut habeatur multiplicator V, aequationis propositae integrale erit

$$V\partial y + y + y(VP\partial x - \partial V) = \partial x \int VX\partial x,$$

quae porro integrabilis redditur, si ducatur in $\frac{1}{VV}e^{\int P\partial x}$, obtinebitur enim integrale

$$\frac{y}{V}e^{\int P\partial x} = \int\frac{\partial x}{VV}e^{\int P\partial x}\int VX\partial x, \text{ seu}$$

$$y = e^{-\int P\partial x}V\int e^{\int P\partial x}\frac{\partial x}{VV}\int VX\partial x,$$

quo duplici signo integrali gemina constans arbitraria introducitur integrale completum constituens.

Corollarium 1.

870. Inventio ergo multiplicatoris V pendet etiam a resolutione aequationis differentio-differentialis, quae autem proposita simplicior est aestimanda, quod functionem X non involvat, et quantitas V cum suis differentialibus ∂V et $\partial\partial V$ ubique unam dimensionem constituat.

Corollarium 2.

871. Quodsi ergo ponatur $V = e^{\int v\partial x}$, quantitas v determinabitur per hanc aequationem differentialem primi gradus

$$\partial v + vv\partial x - Pv\partial x + Q\partial x - \partial P = 0,$$

cujus si saltem integrale particulare constet, integratio aequationis propositae absolvi poterit.

Corollarium. 3.

872. Dato autem multiplicatore V vicissim ratio aequationis propositae definitur, ut hoc modo integrabilis evadat. Erit enim vel

$$Q = \frac{\partial P}{\partial x} + \frac{P \partial V}{V \partial x} - \frac{\partial \partial V}{V \partial x^2}, \text{ vel}$$

$$\partial P + \frac{P \partial V}{V} = Q \partial x - \frac{\partial \partial V}{V \partial x}, \text{ vel integrando}$$

$$P V = \frac{\partial V}{\partial x} + \int Q V \partial x, \text{ seu } P = \frac{\partial V}{V \partial x} + \frac{\int Q V \partial x}{V}.$$

Exemplum 1.

873. *Definire formam aequationis differentio-differentialis*

$$\partial \partial y + P \partial y \partial x + Q y \partial x^2 = X \partial x^2,$$

ut multiplicata per $e^{\lambda x}$ *integrabilis evadat.*

Posito multiplicatore $V = e^{\int v \partial x} = e^{\lambda x}$, erit $v = \lambda$, et satisfieri oportet huic aequationi

$$\lambda \lambda \partial x - \lambda P \partial x + Q \partial x - \partial P = 0,$$

unde fit $Q = \lambda P - \lambda \lambda + \frac{\partial P}{\partial x}$. Primum ergo hoc evenit si fuerint P et Q constantes, puta $P = A$ et $Q = B$, ac tum λ definiri oportet ex hac aequatione $\lambda \lambda - A \lambda + B = 0$, qui est casus supra tractatus. Praeterea vero qualiscunque functio P fuerit ipsius x, modo sit $Q = \lambda P - \lambda \lambda + \frac{\partial P}{\partial x}$, aequatio in $e^{\lambda x}$ ducta erit integrabilis, integrali existente

$$e^{\lambda x} [\partial y + y \partial x (P - \lambda)] = \partial x \int e^{\lambda x} X \partial x, \text{ seu}$$

$$\partial y + (P - \lambda) y \partial x = e^{-\lambda x} \partial x \int e^{\lambda x} X \partial x,$$

quae ulterius per $e^{\int P \partial x - \lambda x}$ multiplicata et integrata, dat

$$y = e^{-\int P \partial x + \lambda x} \int e^{\int P \partial x - 2 \lambda x} \partial x \int e^{\lambda x} X \partial x.$$

Corollarium,

874. Sit $P = A + \alpha x$ et $Q = B + \beta x$, erit

$$B + \beta x = A\lambda + \alpha\lambda x - \lambda\lambda + \alpha, \text{ ergo}$$
$$B = A\lambda - \lambda\lambda + \alpha \text{ et } \beta = \alpha\lambda,$$

unde ob $\lambda = \frac{\beta}{\alpha}$, coëfficientes A, B, α, β, ita comparatos esse oportet, ut sit

$$B\alpha\alpha = A\alpha\beta - \beta\beta + \alpha^3, \text{ seu } B\alpha\alpha - \beta\beta = \alpha(A\beta + \alpha\alpha).$$

Exemplum 2.

875. *Definire formam aequationis differentio-differentialis*

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = X\partial x^2,$$

ut per $e^{\int v\partial x}$, existente $v = \frac{\lambda}{x} + \mu x^n$, multiplicata fiat integrabilis.

Cum esse debeat

$$\partial v + vv\partial x - Pv\partial x + Q\partial x - \partial P = 0, \text{ erit}$$
$$\left.\begin{array}{l} -\frac{\lambda}{xx} + \mu n x^{n-1} - \frac{\lambda P}{x} - \mu P x^n \\ +\frac{\lambda\lambda}{xx} + 2\lambda\mu x^{n-1} + \mu\mu x^{2n} + Q - \frac{\partial P}{\partial x} \end{array}\right\} = 0,$$

ergo

$$Q = \frac{\lambda(1-\lambda)}{xx} - (2\lambda + n)\mu x^{n-1} - \mu\mu x^{2n} + \frac{\lambda P}{x} + \mu P x^n + \frac{\partial P}{\partial x}.$$

Ponamus $P = \frac{\alpha}{x} + \beta x^n$, erit

$$Q = \frac{1}{xx}(\lambda - \lambda\lambda + \alpha\lambda - \alpha) + x^{n-1}(\beta\lambda + \alpha\mu + \beta n - 2\lambda\mu - n\mu)$$
$$+ x^{2n}(\beta\mu - \mu\mu).$$

Sit $Q = \frac{\gamma}{xx} + \delta x^{n-1} + \varepsilon x^{2n}$, fierique oportet

$$\lambda\lambda - (\alpha + 1)\lambda + \alpha + \gamma = 0,$$
$$\beta(\lambda + n) + \mu(\alpha - 2\lambda - n) = \delta, \text{ et } \mu(\beta - \mu) = \varepsilon,$$

unde non solum pro multiplicatore litterae λ et μ, sed etiam certa relatio inter litteras α, β, γ, δ, ε, definitur.

Veluti si sit $\gamma = 0$ et $\delta = 0$, erit $(\lambda - \alpha)(\lambda - 1) = 0$, unde $\lambda = \alpha$; tum $(\beta - \mu)(\alpha + n) = 0$, ergo $\alpha = \lambda = -n$ et $\mu\mu - \beta\mu + \varepsilon = 0$. Scilicet aequatio

$$\partial\partial y + \partial x \partial y \left(\beta x^n - \frac{n}{x}\right) + \varepsilon x^{2n} y \partial x^2 = X \partial x^2$$

multiplicatorem recipit $e^{\int v \partial x}$, existente $v = -\frac{n}{x} + \mu x^n$, sumto μ ita ut sit $\mu\mu - \mu + \varepsilon = 0$. Erit ergo multiplicator

$$V = \frac{1}{x^n} e^{\frac{\mu}{n+1} x^{n+1}} \quad \text{et} \quad e^{\int P \partial x} = \frac{1}{x^n} e^{\frac{\beta}{n+1} x^{n+1}}.$$

Quare si ponamus $\frac{1}{n+1} x^{n+1} = t$, erit

$$y = x^n e^{-\beta t} \frac{1}{x^n} e^{\mu t} \int e^{\beta t - 2\mu t} x^n \partial x \int \frac{e^{\mu t} X \partial x}{x^n}, \text{ seu}$$

$$y = e^{(\mu - \beta) t} \int e^{(\beta - 2\mu) t} x^n \partial t \int \frac{e^{\mu t} X \partial x}{x^n}.$$

Corollarium 1.

876. Si sumatur $\gamma = 0$ et $\varepsilon = 0$, erit

$\mu = \beta$, $\beta(\alpha - \lambda) = \delta$ et $(\lambda - \alpha)(\lambda - 1) = 0$, hinc

$\lambda = 1$, et $\delta = (\alpha - 1)\beta$, ideoque

$P = \frac{\alpha}{x} + \beta x^n$, $Q = (\alpha - 1)\beta x^{n-1}$,

et aequationis

$$\partial\partial y + \left(\frac{\alpha}{x} + \beta x^n\right) \partial x \partial y + (\alpha - 1) \beta x^{n-1} y \partial x^2 = X \partial x^2,$$

multiplicator $V = e^{\int v \partial x}$, existente $v = \frac{1}{x} + \beta x^n$, ita ut sit

$$V = x e^{\frac{\beta}{n+1} x^{n+1}} \quad \text{et} \quad e^{\int P \partial x} = x^\alpha e^{\frac{\beta}{n+1} x^{n+1}}.$$

Corollarium 2.

877. Hoc ergo casu, posito $\frac{1}{n+1} x^{n+1} = t$, erit integrale

$$y = x^{1-\alpha} \int x^{\alpha-2} e^{-\beta t} \partial x \int e^{\beta t} X x \partial x,$$

quae forma simplicius exhiberi nequit, propterea quod in genere formula $e^{-\beta t} x^{\alpha-2} \partial x$ integrationem non admittit.

Scholion.

878. Cum igitur inventio multiplicatorum, qui hujusmodi aequationem

$$\partial\partial y + P \partial x \partial y + Q y \partial x^2 = X \partial x^2,$$

integrabilem reddunt, resolutionem hujus aequationis postulet

$$\partial\partial V - P \partial V \partial x + V \partial x (Q \partial x - \partial P) = 0,$$

quae in hac forma continetur

$$\partial\partial y + P \partial x \partial y + Q y \partial x^2 = 0,$$

videndum est, quomodo hanc formam etiam per multiplicatores tractari oporteat. Cujus multiplicator si fingatur V functio quaedam ipsius x, iterum ad praecedentem formam

$$\partial\partial V - P \partial V \partial x + V \partial x (Q \partial x - \partial P) = 0,$$

devenitur, atque si hujus multiplicator statuatur $= U$ functioni ipsius x, hic definietur per hanc aequationem

$$\partial\partial U + P \partial U \partial x + Q U \partial x^2 = 0,$$

ita ut sufficiat alteram harum duarum aequationum resolvisse. Ac supra quidem, ubi $y = u v$ posuimus, ad hanc posteriorem aequationem pervenimus: at mirum non est harum duarum aequationum alteram ab altera pendere, cum prior ex posteriori nascatur ponendo $U = e^{-\int P \partial x} V$, posterior vero ex priori ponendo $V = e^{\int P \partial x} U$, uti tentanti facile patebit. Quoniam igitur hoc modo difficultatem, si quae occurrit, tollere non licet, investigandum est, an forte ejusmodi multiplicator, qui utramque variabilem x et y cum suis differentialibus ∂x et ∂y seu $p = \frac{\partial y}{\partial x}$ involvat, negotium conficiat. At vero facile perspicitur exclusis differentialibus hoc fieri non posse; nam si multiplicator esset V functio ipsarum x et y, ex primo termino $\partial\partial y$ nascetur integralis pars $V\partial y$, quae autem differentiata ponendo $\partial V = M\partial x + N\partial y$ involveret in differentiale partem $N\partial y^2$, in aequatione non occurrentem, quae etiam per reliquas integralis partes tolli non posset. Quare rem tentemus ejusmodi multiplicatoribus, qui etiam rationem differentialium $p = \frac{\partial y}{\partial x}$ complectantur; et cum ipsius y cum suis differentialibus ubique sit idem dimensionum numerus, eadem proprietas etiam in multiplicatore insit necesse est; si enim diversae inessent, singulae seorsim negotium essent confecturae.

Problema 109.

879. Sumto elemento ∂x constante, definire conditiones, ut multiplicator hujus formae $Mp + Ny$, existente $p = \frac{\partial y}{\partial x}$ et M et N functionibus ipsius x, integrabilem reddat hanc aequationem

$$\partial\partial y + P\partial x\partial y + Qy\partial x^2 = 0,$$

ubi P et Q sunt functiones ipsius x.

Solutio.

Ob $\partial y = p\partial x$, nostra aequatio est

$$\partial p + Pp\partial x + Qy\partial x = 0,$$

quae per $Mp + Ny$ multiplicata fit

$$\left.\begin{matrix} Mp\partial p + Ny\partial p + MPp\partial y + NPy\partial y + NQyy\partial x \\ + MQy\partial y \end{matrix}\right\} = 0,$$

quam integrabilem esse oportet. Ob terminos differentiali ∂p affectos pars integrabilis erit $\frac{1}{2}Mpp + Nyp$, unde integrale ipsum statuatur $= \frac{1}{2}Mpp + Nyp + S$. Cujus differentiale cum ipsam illam aequationem praebere debeat, habebimus

$$\begin{aligned}\partial S = & MPp\partial y + NPy\partial y + NQyy\partial x \\ & + MQy\partial y \\ & - \tfrac{1}{2}pp\partial M - yp\partial N \\ & - Np\partial y\end{aligned}$$

quam ergo formulam integrabilem esset oportet, quae cum tantum differentialia primi ordinis ∂x et ∂y complectatur, necesse est ut quantitas p ex calculo egrediatur. Posito ergo $\partial M = M'\partial x$ et $\partial N = N'\partial x$, ob $p\partial x = \partial y$, primus terminus continens p ad nihilum redigi debet, ut sit

$$MPp\partial y - \tfrac{1}{2}M'p\partial y - Np\partial y = 0, \text{ seu}$$
$$MP - \tfrac{1}{2}M' - N = 0, \text{ vel } N = MP - \frac{\partial M}{2\partial x}.$$

Tum vero erit

$$\partial S = y\partial y\,(NP + MQ - N') + NQyy\partial x,$$

cujus formulae integrale est

$$S = \tfrac{1}{2}yy\,(NP + MQ - N'), \text{ vel } S = yy\int NQ\partial x,$$

quas duas formas congruere oportet, unde fit

$$NP + MQ - \frac{\partial N}{\partial x} = 2\int NQ\partial x, \text{ seu}$$
$$N\partial P + P\partial N + M\partial Q + Q\partial M - \frac{\partial\partial N}{\partial x} - 2NQ\partial x = 0,$$

quae aequatio cum illa $N = MP - \frac{\partial M}{2\partial x}$ juncta, conditiones quaesitas determinat, proditque tum aequatio integralis

$$\tfrac{1}{2}Mpp + Nyp + \tfrac{1}{2}yy\left(NP + MQ - \frac{\partial N}{\partial x}\right) = C.$$

Corollarium 1.

880. Si functiones P et Q dentur, indeque M et N definiri oporteat, ob $N = MP - \frac{\partial M}{2\partial x}$, erit

$$\partial N = M\partial P + P\partial M - \frac{\partial\partial M}{2\partial x},$$

et functio M definietur per hanc aequationem

$$\frac{\partial^3 M}{2\partial x^2} - \frac{3P\partial\partial M}{2\partial x}$$
$$+ (PP - \frac{5\partial P}{2\partial x} + 2Q)\partial M + M(2P\partial P - \frac{\partial\partial P}{\partial x} - 2PQ\partial x + \partial Q) = 0,$$

quae ob differentialia tertii ordinis parum juvat.

Corollarium 2.

881. Sin autem multiplicator $Mp + Ny$ detur, ipsa aequatio ita definitur, ut sit primo $P = \frac{N}{M} + \frac{\partial M}{2M\partial x}$, unde ex altera, quae est

$$\partial Q + \frac{Q\partial M}{M} - \frac{2NQ\partial x}{M} = \frac{\partial\partial N}{M\partial x} - \frac{\partial . PN}{M},$$

haecque per $M e^{-2\int\frac{N\partial x}{M}}$ multiplicata, integrale dat

$$MQ e^{-2\int\frac{N\partial x}{M}} = \int e^{-2\int\frac{N\partial x}{M}} (\frac{\partial\partial N}{\partial x} - \partial . PN).$$

Corollarium 3.

882. Sit hoc integrale $= Z$ eritque

$$Z = e^{-2\int\frac{N\partial x}{M}} (\frac{\partial N}{\partial x} - PN) + \int e^{-2\int\frac{N\partial x}{M}} (\frac{2N\partial N}{M} - \frac{2PNN\partial x}{M});$$

quod posterius membrum, pro P valore substituto, abit in

$$\int e^{-2\int\frac{N\partial x}{M}} (\frac{2N\partial N}{M} - \frac{2N^3\partial x}{MM} - \frac{NN\partial M}{MM}),$$

cujus integrale est manifesto $e^{-2\int\frac{N\partial x}{M}} \frac{NN}{M}$, ita ut sit

$$Z = e^{-2\int\frac{N\partial x}{M}} (\frac{\partial N}{\partial x} - \frac{N\partial M}{2M\partial x}) + C, \text{ ideoque}$$

$$Q = \frac{C}{M} e^{2\int\frac{N\partial x}{M}} + \frac{\partial N}{M\partial x} - \frac{N\partial M}{2MM\partial x}.$$

Corollarium 4.

883. Proposita ergo hac aequatione

$$\frac{\partial\partial y}{\partial x} + \left(\frac{N}{M} + \frac{\partial M}{2M\partial x}\right)\partial y + y\left(\frac{C\partial x}{M} e^{2\int\frac{N\partial x}{M}} + \frac{\partial N}{M} - \frac{N\partial M}{2MM}\right) = 0,$$

eam per $\frac{M\partial y}{\partial x} + Ny$ multiplicando, integrale fit

$$\frac{M\partial y^2}{2\partial x^2} + \frac{Ny\partial y}{\partial x} + \tfrac{1}{2}yy\left(Ce^{2\int\frac{N\partial x}{M}} + \frac{NN}{M}\right) = \text{Const.}$$

Scholion.

884. Cum ergo pro M et N quascunque functiones ipsius x accipere liceat, innumerabiles hinc nacti sumus aequationum differentio-differentialium formas, quas ope multiplicatoris $\frac{M\partial y}{\partial x} + Ny$ integrare possumus. Forma scilicet generalis, quae hoc multiplicatore integrabilis redditur, est ut vidimus

$$\frac{\partial\partial y}{\partial x} + \frac{\partial y}{2M\partial x}(\partial M + 2N\partial x)$$
$$+ \frac{y}{2MM}\left(2M\partial N - N\partial M + 2CMe^{2\int\frac{N\partial x}{M}}\partial x\right),$$

ipso integrali existente

$$\frac{M\partial y^2}{2\partial x^2} + \frac{Ny\partial y}{\partial x} + \tfrac{1}{2}yy\left(\frac{NN}{M} + Ce^{2\int\frac{N\partial x}{M}}\right),$$

ubi perspicuum est partem exponentialem constanti C affectam utrinque omitti posse, cum ea sola ista proprietate sit praedita. Quodsi partem exponentialem ad algebraicam reducamus ponendo $e^{2\int\frac{N\partial x}{M}} = L$, erit $\frac{2N\partial x}{M} = \frac{\partial L}{L}$ et $N = \frac{M\partial L}{2L\partial x}$, hincque $\partial N = \frac{M\partial\partial L}{2L\partial x} + \frac{\partial L\partial M}{2L\partial x} - \frac{M\partial L^2}{2LL\partial x}$; unde ista forma

$$\frac{\partial\partial y}{\partial x} + \frac{\partial y}{2\partial x}\left(\frac{\partial M}{M} + \frac{\partial L}{L}\right) + \tfrac{1}{2}y\left(\frac{\partial\partial L}{L\partial x} + \frac{\partial L\partial M}{2LM\partial x} - \frac{\partial L^2}{LL\partial x} + \frac{2CL\partial x}{M}\right),$$

quae per $\frac{M\partial y}{\partial x} + \frac{My\partial L}{2L\partial x}$ multiplicata, integrale praebet

$$\frac{M\partial y^2}{2\partial x^2} + \frac{My\partial L\partial y}{2L\partial x^2} + \tfrac{1}{2}yy\left(\frac{M\partial L^2}{4LL\partial x^2} + CL\right).$$

Vel si ponamus $\frac{\partial M}{M} + \frac{\partial L}{L} = \frac{2\partial K}{K}$, ut sit $M = \frac{KK}{L}$, erit nostra aequatio differentio-differentialis

$$\frac{\partial\partial y}{\partial x} + \frac{\partial y}{\partial x} \cdot \frac{\partial K}{K} + \tfrac{1}{2} y \left(\partial \cdot \frac{\partial L}{L\partial x} - \frac{\partial L^2}{2LL\partial x} + \frac{\partial K \partial L}{KL\partial x} + \frac{2CLL\partial x}{KK}\right) = 0,$$

quae per $\frac{KK}{L}\left(\frac{\partial y}{\partial x} + \frac{y\partial L}{2L\partial x}\right)$ multiplicata, dabit integrale

$$\frac{KK}{2L}\left[\frac{\partial y^2}{\partial x^2} + \frac{y\partial L\partial y}{L\partial x^2} + yy\left(\frac{\partial L^2}{4LL\partial x^2} + \frac{CLL}{KK}\right)\right] = \text{Const.}$$

Exemplum 1.

885. Sit $K = x^m (a+x)^n$ et $L = x^\mu (a+x)^\nu$, erit

$$\frac{\partial K}{K\partial x} = \frac{m}{x} + \frac{n}{a+x} = \frac{ma + (m+n)x}{x(a+x)}, \text{ et}$$

$\frac{\partial L}{L\partial x} = \frac{\mu}{x} + \frac{\nu}{a+x}$; unde coëfficiens ipsius $\frac{1}{2} y\partial x$ erit

$$-\frac{\mu}{xx} - \frac{\nu}{(a+x)^2} - \frac{\mu\mu}{2xx} - \frac{\mu\nu}{x(a+x)} - \frac{\nu\nu}{2(a+x)^2} + \frac{m\mu}{xx} + \frac{m\nu + n\mu}{x(a+x)}$$
$$+ \frac{n\nu}{(a+x)^2} + 2Cx^{2\mu - 2m}(a+x)^{2\nu - 2n}, \text{ seu}$$

$$\frac{\mu(2m - \mu - 2)}{2xx} + \frac{m\nu + n\mu - \mu\nu}{x(a+x)} + \frac{\nu(2n - \nu - 2)}{2(a+x)^2}$$
$$+ 2Cx^{2\mu - 2m}(a+x)^{2\nu - 2n},$$

ubi sequentes casus notasse juvabit.

I. Sit $m = \mu + 1$ et $n = \nu$, erit ipsius $\frac{1}{2} y\partial x$ coëfficiens

$$\frac{\mu\mu + 4C}{2xx} + \frac{\nu(\mu+1)}{x(a+x)} + \frac{\nu(\nu-2)}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{\partial\partial y}{\partial x} + \partial y\left(\frac{\mu+1}{x} + \frac{\nu}{a+x}\right)$$
$$+ \tfrac{3}{4} y\partial x\left[\frac{\mu\mu + 4C}{xx} + \frac{2\nu(\mu+1)}{x(a+x)} + \frac{\nu(\nu-2)}{(a+x)^2}\right] = 0$$

multiplicata per

$$x^{\mu+2}(a+x)^\nu\left[\frac{\partial y}{\partial x} + \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right)\frac{y}{2}\right],$$

integrale dat

$$\tfrac{1}{2} x^{\mu+2} (a+x)^{\nu} \left[\frac{\partial y^2}{\partial x^2} + \frac{y \partial y}{\partial x} \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right) + \right.$$
$$\left. \tfrac{1}{4} y y \left(\frac{\mu\mu + 4C}{xx} + \frac{2\mu\nu}{x(a+x)} + \frac{\nu\nu}{(a+x)^2}\right)\right] = \text{Const}$$

II. Sit $m = \mu + \frac{1}{2}$ et $n = \nu + \frac{1}{2}$, erit ipsius $\frac{1}{2} y \partial x$ coëfficiens

$$\frac{\mu(\mu-1)}{2xx} + \frac{2\mu\nu + \mu + \nu + 4C}{2x(a+x)} + \frac{\nu(\nu-1)}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{\partial \partial y}{\partial x} + \partial y \left(\frac{2\mu+1}{2x} + \frac{2\nu+1}{2(a+x)}\right)$$
$$+ \tfrac{1}{4} y \partial x \left(\frac{\mu(\pi-1)}{xx} + \frac{2\mu\nu + \mu + \nu + 4C}{x(a+x)} + \frac{\nu(\nu-1)}{(a+x)^2}\right) = 0$$

multiplicata per

$$x^{\mu+1} (a+x)^{\nu+1} \left[\frac{\partial y}{\partial x} + \tfrac{1}{2} y \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right)\right],$$

integrale dabit

$$\tfrac{1}{2} x^{\mu+1} (a+x)^{\nu+1} \left[\frac{\partial y^2}{\partial x^2} + \frac{y \partial y}{\partial x} \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right) + \right.$$
$$\left. \tfrac{1}{4} y y \left(\frac{\mu\mu}{xx} + \frac{2\mu\nu + 4C}{x(a+x)} + \frac{\nu\nu}{(a+x)^2}\right)\right] = \text{Const.}$$

III. Sit $m = \mu$ et $n = \nu + 1$, erit ipsius $\frac{1}{2} y \partial x$ coëfficie

$$\frac{\mu(\mu-2)}{2xx} + \frac{\mu(\nu+1)}{x(a+x)} + \frac{\nu\nu + 4C}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{\partial \partial y}{\partial x} + \partial y \left(\frac{\mu}{x} + \frac{\nu+1}{a+x}\right)$$
$$+ \tfrac{1}{4} y \partial x \left(\frac{\mu(\mu-2)}{xx} + \frac{2\mu(\nu+1)}{x(a+x)} + \frac{\nu\nu + 4C}{(a+x)^2}\right) = 0,$$

multiplicata per

$$x^{\mu} (a+x)^{\nu+2} \left[\frac{\partial y}{\partial x} + \tfrac{1}{2} y \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right)\right],$$

dabit integrale

$$\tfrac{1}{2} x^{\mu} (a+x)^{\nu+2} \left[\frac{\partial y^2}{\partial x^2} + \frac{y \partial y}{\partial x} \left(\frac{\mu}{x} + \frac{\nu}{a+x}\right) + \right.$$
$$\left. \tfrac{1}{4} y y \left(\frac{\mu\mu}{xx} + \frac{2\mu\nu}{x(a+x)} + \frac{\nu\nu + 4C}{(a+x)^2}\right)\right] = \text{Const.}$$

Corollarium 1.

886. Sit casu primo $\nu = 2$, $C = -\frac{1}{4}\mu\mu$, habebitur hae aequatio

$$\frac{\partial\partial y}{\partial x}+\frac{(\mu+1)a+(\mu+3)x}{x(a+x)}\partial y+\frac{(\mu+1)y\partial x}{x(a+x)}=0,$$

quae per

$$x^{\mu+2}(a+x)^2\left(\frac{\partial y}{\partial x}+\frac{\mu a+(\mu+2)x}{2x(a+x)}y\right)$$

multiplicata, praebet integrale

$$\tfrac{1}{2}x^{\mu+2}(a+x)^2\left[\frac{\partial y^2}{\partial x^2}+\frac{\mu a+(\mu+2)x}{x(a+x)}\cdot\frac{y\partial y}{\partial x}+yy\left(\frac{\mu}{x(a+x)}+\frac{1}{(a+x)^2}\right)\right]=\text{Const.}$$

Corollarium 2.

887. Sit casu tertio $\mu=2$ et $4C=-\nu\nu$, habebitur ista aequatio

$$\frac{\partial\partial y}{\partial x}+\partial y\cdot\frac{2a+(\nu+3)x}{x(a+x)}+\frac{(\nu+1)y\partial x}{x(a+x)}=0,$$

quae multiplicata per $xx(a+x)^{\nu+2}\left[\frac{\partial y}{\partial x}+\frac{1}{2}y\left(\frac{2}{x}+\frac{\nu}{a+x}\right)\right]$, dabit integrale

$$\tfrac{1}{2}xx(a+x)^{\nu+2}\left[\frac{\partial y^2}{\partial x^2}+\frac{y\partial y}{\partial x}\left(\frac{2}{x}+\frac{\nu}{a+x}\right)+yy\left(\frac{1}{xx}+\frac{\nu}{x(a+x)}\right)\right]=\text{Const.}$$

Exemplum 2.

888. Sit $K=x^m(aa+xx)^n$ et $L=x^\mu(aa+xx)^\nu$, erit $\frac{\partial K}{K\partial x}=\frac{m}{x}+\frac{2nx}{aa+xx}$ et $\frac{\partial L}{L\partial x}=\frac{\mu}{x}+\frac{2\nu x}{aa+xx}$, et aequatio differentio-differentialis hanc induet formam

$$\frac{\partial\partial y}{\partial x}+\partial y\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+$$
$$\tfrac{1}{2}y\partial x\left\{\begin{array}{l}\frac{\mu(2m-\mu-2)}{2xx}+\frac{2n\mu+2\nu(m-\mu+1)}{aa+xx}+\frac{2\nu(2n-\nu-2)xx}{(aa+xx)^2}\\ \qquad+\frac{2Cx^{2\mu-2m}}{(aa+xx)^{2n-2\nu}}\end{array}\right\}=0,$$

cujus in $x^{2m-\mu}(aa+xx)^{2n-\nu}\left[\frac{\partial y}{\partial x}+\frac{1}{2}y\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)\right]$ ductae integrale erit

$$\tfrac{1}{2}x^{2m-\mu}(aa+xx)^{2n-\nu}\times\left\{\begin{array}{l}\frac{\partial y^2}{\partial x^2}+\frac{y\partial y}{\partial x}\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)\\ +\frac{1}{4}yy\left[\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)^2+\frac{4Cx^{2\mu-2m}}{(aa+xx)^{2n-2\nu}}\right]\end{array}\right\}=\text{Const.}$$

Evolvamus hic casus, quibus aequatio differentio-differentialis hanc obtinet formam

$$\frac{\partial\partial y}{\partial x}+\partial y\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+\tfrac{1}{2}y\partial x\left(D+\frac{E}{xx}+\frac{F}{aa+xx}+\frac{Gxx}{(aa+xx)^2}\right)=0.$$

I. Sumatur $\mu = m$ et $\nu = n$,
eritque $D = 2C$, $E = \frac{1}{2}m(m-2)$, $F = 2n(m+1)$, et
$G = 2n(n-2)$.

II. Sumatur $\mu = m-1$ et $\nu = n$,
eritque $D = 0$, $E = 2C + \frac{1}{2}(m-1)^2$, $F = 2n(m+1)$, et
$G = 2n(n-2)$.

III. Sumatur $\mu = m-1$ et $2n-2\nu = -1$, seu $\nu = n+\frac{1}{2}$, erit ultimus terminus $\frac{2C(aa+xx)}{xx} = 2C + \frac{2Caa}{xx}$. Ergo
$D = 2C$, $E = 2Caa+\frac{1}{2}(m-1)^2$, $F = 2(mn+n+1)$,
$G = \frac{1}{2}(2n+1)(2n-5)$.

IV. Sumatur $\mu = m$ et $2n-2\nu = 1$, seu $\nu = n-\frac{1}{2}$, erit ultimus terminus $\frac{2C}{aa+xx}$, ideoque
$D = 0$, $E = \frac{1}{2}m(m-2)$, $F = 2C+2mn+2n-1$,
$G = \frac{1}{2}(2n-1)(2n-3)$.

V. Sumatur $\mu = m+1$ et $\nu = n-\frac{1}{2}$, erit ultimus terminus $\frac{2Cxx}{aa+xx} = 2C - \frac{2Caa}{aa+xx}$, ideoque
$D = 2C$, $E = \frac{1}{2}(m+1)(m-2)$, $F = -2Caa+2n(m+1)$,
$G = \frac{1}{2}(2n-1)(2n-3)$.

VI. Sit $\mu = m-1$ et $\nu = n-\frac{1}{2}$, erit ultimus terminus $\frac{2C}{xx(aa+xx)} = \frac{2C}{aaxx} - \frac{2C}{aa(aa+xx)}$, unde fit
$D = 0$, $E = \frac{2C}{aa}+\frac{1}{2}(m-1)^2$, $F = \frac{-2C}{aa}+2mn+2n-2$,
$G = \frac{1}{2}(2n-1)(2n-3)$.

VII. Sit $\mu = m+1$ et $2n-2\nu = 2$, seu $\nu = n-1$, erit terminus ultimus $\frac{2Cxx}{(aa+xx)^2}$, ideoque

$$D = 0,\ E = \tfrac{1}{2}(m+1)(m-3),\ F = 2n(m+1),$$
$$G = 2C + 2(n-1)^2.$$

VIII. Sit $\mu = m+2$ et $\nu = n-1$, erit terminus ultimus $\frac{2Cx^4}{(aa+xx)^2} = 2C - \frac{2Caa}{aa+xx} - \frac{2Caaxx}{(aa+xx)^2}$, hincque

$$D = 2C,\ E = \tfrac{1}{2}(m+2)(m-4),\ F = -2Caa + 2mn + 2n + 2,$$
$$\text{et } G = -2Caa + 2(n-1)^2.$$

IX. Sit $\mu = m$ et $\nu = n-1$, erit terminus ultimus

$$\frac{2C}{(aa+xx)^2} = \frac{2C}{aa(aa+xx)} - \frac{2Cxx}{a^2(aa+xx)^2},$$

hincque

$$D = 0,\ E = \tfrac{1}{2}m(m-2),\ F = \frac{2C}{aa} + 2mn + 2n - 2,$$
$$G = \frac{-2C}{aa} + 2(n-1)^2.$$

X. Sit $\mu = m-1$ et $\nu = n-1$, erit terminus ultimus

$$\frac{2C}{xx(aa+xx)^2} = \frac{2C}{a^4xx} - \frac{4C}{a^4(aa+xx)} + \frac{2Cxx}{a^4(aa+xx)^2},$$

hincque

$$D = 0,\ E = \frac{2C}{a^4} + \tfrac{1}{2}(m-1)^2,\ F = \frac{-4C}{a^4} + 2mn + 2n - 4,$$
$$G = \frac{2C}{a^4} + 2(n-1)^2.$$

Problema 110.

889. Sumto elemento ∂x constante, si K et L denotent functiones quascunque ipsius x, invenire integrale completum hujus aequationis differentio-differentialis

$$\frac{\partial\partial y}{\partial x} + \frac{\partial y}{\partial x}\cdot\frac{\partial K}{K} + \tfrac{1}{2}y\left(\partial\cdot\frac{\partial L}{L\partial x} - \frac{\partial L^2}{2LL\partial x} + \frac{\partial K\partial L}{KL\partial x} + \frac{2CLL\partial x}{KK}\right) = 0.$$

Solutio.

Quoniam haec aequatio integrabilis redditur, si multiplicetur per $\frac{KK}{L}\left(\frac{\partial y}{\partial x} + \frac{y\partial L}{2L\partial x}\right)$, ejus integrale completum, ut supra vidimus, est

$$\frac{KK}{2L}\left[\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)^2+\frac{CLL}{KK}yy\right]=\text{Const.}$$

quam aequationem differentialem primi gradus adhuc integrari oportet; quod cum ob constantem indefinitam maxime sit diffi ile, ea neglecta, primo saltem integrale particulare investigemus. Erit ergo ex aequatione

$$\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)^2+\frac{CLL}{KK}yy=0$$

radicem extrahendo

$$\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}=\frac{Ly}{K}\sqrt{-C}, \text{ seu } \frac{\partial y}{y}+\frac{\partial L}{2L}=\frac{L\partial x}{K}\sqrt{-C},$$

unde fit

$$y\sqrt{L}=ae^{\int\frac{L\partial x}{K}\sqrt{-C}}.$$

Cum igitur aequationi differentio-differentiali propositae satisfaciant hi duo valores:

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{L\partial x}{K}\sqrt{-C}} \text{ et } y=\frac{\beta}{\sqrt{L}}e^{-\int\frac{L\partial x}{K}\sqrt{-C}},$$

bini conjuncti etiam satisfacient, quibus quoniam duae constantes arbitrariae introducuntur, ejus integrale completum erit

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{L\partial x}{K}\sqrt{-C}}+\frac{\beta}{\sqrt{L}}e^{-\int\frac{L\partial x}{K}\sqrt{-C}},$$

quae expressio valet, si $\sqrt{-C}$ fuerit quantitas realis, sin autem sit imaginaria, erit

$$y=\frac{\gamma}{\sqrt{L}}\sin.\left(\int\frac{L\partial x}{K}\sqrt{C}+\zeta\right):$$

sicque habetur integrale completum aequationis differentio-differentialis propositae.

Corollarium 1.

890. Hinc igitur aequationis differentialis primi gradus

$$\left(\frac{\partial y}{\partial x}+\frac{y\partial L}{2L\partial x}\right)^2+\frac{CLLy^2}{KK}=\frac{AL}{KK},$$

quae per se satis est difficilis, integrale assignare valemus, quod est

$$y = \frac{\alpha}{\sqrt{L}} e^{\int \frac{L \partial x}{K} \sqrt{-C}} + \frac{\beta}{\sqrt{L}} e^{-\int \frac{L \partial x}{K} \sqrt{-C}},$$

modo debita relatio constantium α et β respectu constantis A definiatur.

Corollarium 2.

891. Erit autem per $\sqrt{L}$ multiplicando et differentiando

$$\partial y \sqrt{L} + \frac{y \partial L}{2\sqrt{L}} = \frac{\alpha L \partial x}{K} \sqrt{-C} . e^{\int \frac{L \partial x}{K} \sqrt{-C}}$$
$$- \frac{\beta L \partial x}{K} \sqrt{-C} . e^{-\int \frac{L \partial x}{K} \sqrt{-C}}.$$

Hinc

$$\frac{\partial y}{\partial x} + \frac{y \partial L}{2 L \partial x} = \frac{\sqrt{-C} L}{K} (\alpha e^{\int \frac{L \partial x}{K} \sqrt{-C}} - \beta e^{-\int \frac{L \partial x}{K} \sqrt{-C}}),$$

unde fit

$$\frac{A L}{K K} = \frac{-C L}{K K} (\alpha e^{\int \frac{L \partial x}{KK} \sqrt{-C}} - \beta e^{-\int \frac{L \partial x}{K} \sqrt{-C}})^2$$
$$+ \frac{C L}{K K} (\alpha e^{\int \frac{L \partial x}{K} \sqrt{-C}} + \beta e^{-\int \frac{L \partial x}{K} \sqrt{-C}})^2,$$

ideoque $A = 4 C \alpha \beta$ seu $\beta = \frac{A}{4 C \alpha}$.

Scholion 1.

892. Quamvis ergo aequationem propositam ope idonei multiplicatoris integrare licuerit, altera tamen integratio maximis difficultatibus premi videbatur. Interim tamen ope substitutionis aequatio illa differentialis primi gradus tractatu facilis redditur; posito enim $y = \frac{z}{\sqrt{L}}$, ut sit $\partial y \sqrt{L} + \frac{y \partial L}{2\sqrt{L}} = \partial z$, oritur $\left(\frac{\partial z}{\partial x \sqrt{L}}\right)^2 + \frac{C L z z}{K K} = \frac{A L}{K K}$, hinc $\frac{\partial z}{\partial x} = \frac{L}{K} \sqrt{(A - C z z)}$, seu $\frac{\partial z}{\sqrt{(A - C z z)}} = \frac{L \partial x}{K}$, quae integrata dat

$$l[z\sqrt{-C}+\sqrt{(A-Czz)}]=\int\frac{L\,\partial x}{R}\sqrt{-C}+l\,B;$$

unde praecedens integrale eruitur. Caeterum forma nostrae aequationis differentio-differentialis aliquanto commodius exhiberi potest hoc modo: Si P et R sint functiones quaecunque ipsius x, sumaturque elementum ∂x constans, hujus aequationis

$$\partial\partial y-\partial y\left(\frac{\partial P}{P}+\frac{\partial R}{R}\right)-y\left(\partial.\frac{\partial P}{P}-\frac{\partial P\,\partial R}{PR}+\frac{\alpha\alpha RR\,\partial x^2}{PP}\right)=0,$$

bis integratae integrale completum est

$$y=\alpha P\,e^{a\int\frac{R\,\partial x}{P}}+\beta P\,e^{-a\int\frac{R\,\partial x}{P}},$$

siquidem a sit quantitas realis. At si $a=0$, erit

$$y=P\left(\alpha+\beta\int\frac{R\,\partial x}{P}\right).$$

Sin autem sit $aa=-cc$, erit

$$y=\alpha P\sin.\left(\beta+c\int\frac{R\,\partial x}{P}\right).$$

Tum vero illa aequatio integrabilis redditur, si multiplicetur per $\frac{1}{RR\,\partial x^2}\left(\partial y-\frac{y\,\partial P}{P}\right)$, eritque integrale primum

$$\frac{1}{2\,RR\,\partial x^2}\left[\left(\partial y-\frac{y\,\partial P}{P}\right)^2-\frac{a\,a\,RR}{PP}y^2\,\partial x^2\right]=\text{Const.}$$

Hinc patet in illa aequatione differentio-differentiali commode hanc substitutionem adhiberi $y=P\,z$, qua ea transformatur in

$$\partial\partial z+\partial z\left(\frac{\partial P}{P}-\frac{\partial R}{R}\right)-\frac{a\,a\,RR}{PP}z\,\partial x^2=0,$$

quae per $\frac{PP\,\partial z}{RR\,\partial x^2}$ multiplicata, sponte fit integrabilis. Quin etiam posito $\frac{P}{R}=S$, ut habeatur

$$\partial\partial z+\frac{\partial S\,\partial z}{S}-\frac{a\,a\,z\,\partial x^2}{SS}=0,$$

multiplicator $\frac{SS\,\partial z}{\partial x^2}$, statim dat integrale

$$\frac{SS\,\partial z^2}{2\,\partial x^2}-\tfrac{1}{2}a\,a\,z\,z=\text{Const.}$$

Scholion 2.

893. Vicissim ergo ex hac forma simplicissima

$$S S \partial \partial z + S \partial S \partial z - a a z \partial x^2 = 0,$$

quae per ∂z multiplicata integrabilis redditur, formas magis complicatas derivare potuissemus, ponendo $z = \frac{y}{P}$ et $S = \frac{P}{R}$. Quae quamquam in formis generalibus satis perspicua, tamen in exemplis determinatis plerumque haec derivatio nimis est occulta, quam ut menti occurrere possit. Veluti in casibus §. 888. evolutis, si N°. IX. sumamus $m = 2$, et $C = (n-1)^2 a a$, fiet $D = 0$, $E = 0$, $F = 2 n (n+1)$, et $G = 0$, unde habetur haec aequatio

$$\frac{\partial \partial y}{\partial x} + 2 \partial y \left(\frac{1}{x} + \frac{n x}{a a + x x}\right) + \frac{n(n+1) y \partial x}{a a + x x} = 0, \text{ seu}$$

$$\partial \partial y + \frac{2 \partial x \partial y [a a + (n+1) x x]}{x (a a + x x)} + \frac{n(n+1) y \partial x^2}{a a + x x} = 0,$$

quae integrabilis redditur ope multiplicatoris

$$x x (a a + x x)^{n+1} \left(\frac{\partial y}{\partial x} + \frac{y (a a + n x x)}{x (a a + x x)}\right),$$

integrali existente

$$\tfrac{1}{2} x x (a a + x x)^{n+1} \left[\left(\frac{\partial y}{\partial x} + \frac{y (a a + n x x)}{x (a a + x x)}\right)^2 + \frac{(n-1)^2 a a y y}{(a a + x x)^2}\right] = \text{Const.}$$

Pro integrali ergo particulari erit

$$\frac{\partial y}{y} + \frac{\partial x}{x} + \frac{(n-1) x \partial x}{a a + x x} = \pm \frac{(n-1) a \partial x \sqrt{-1}}{a a + x x},$$

unde colligitur

$$x y (a a + x x)^{\frac{n-1}{2}} = \alpha \left(\frac{a + x \sqrt{-1}}{a - x \sqrt{-1}}\right)^{\pm \frac{(n-1)}{2}}.$$

Ergo bina integralia particularia conjuncta dant

$$y = \frac{\alpha}{x} (a - x \sqrt{-1})^{-n+1} + \frac{\beta}{x} (a + x \sqrt{-1})^{-n+1},$$

integrale completum. Hoc autem casu aequatio nostra ad formam simplicissimam reducetur ope substitutionis

$$y = \frac{z}{x}(aa + xx)^{\frac{1-n}{2}},$$

cujus ratio et inventio difficilius perspicitur.

Problema 111.

894. Sumto elemento ∂x constante, investigare conditiones quibus aequatio differentio-differentialis

$$\partial\partial y + P\partial x\partial y + Qy\partial x^2 = 0,$$

integrabilis redditur, ope multiplicatoris hujus formae

$$\frac{y\partial x^2}{L\partial y^2 + My\partial y\partial x + Nyy\partial x^2},$$

denotantibus litteris L, M, N, P, Q functiones ipsius x.

Solutio.

Tribuatur denominatori hujus fractionis talis forma

$$(\partial y + Ry\partial x)(\partial y + Sy\partial x),$$

ac levi attentione adhibita patet, integrale hujusmodi formam esse habiturum

$$V + l\frac{\partial y + Ry\partial x}{\partial y + Sy\partial x} = \text{Const.}$$

cujus ergo differentiale aequationem propositam producere debet. Dat autem differentiatio

$$\partial V + \frac{(S-R)y\partial x\partial\partial y + (R-S)\partial x\partial y^2 + y\partial x\partial y(\partial R - \partial S) + yy\partial x^2(S\partial R - R\partial S)}{(\partial y + Ry\partial x)(\partial y + Sy\partial x)} = 0,$$

quae ad communem denominatorem reducta, abit in

$$\left.\begin{aligned}(S-R)y\partial x\partial\partial y + (R-S)\partial x\partial y^2 + y\partial x\partial y(\partial R - \partial S) + yy\partial x^2(S\partial R - R\partial S)\\ + \partial V\partial y^2 + (R+S)y\partial x\partial y\partial V + RSyy\partial x^2\partial V\end{aligned}\right\} = 0.$$

Statuatur $\partial V = (S - R)\partial x$, ut aequatio per y divisibilis evadat, sicque orietur haec aequatio

$$\left.\begin{aligned}(S-R)\partial\partial y + \partial y(\partial R - \partial S) + y\partial x(S\partial R - R\partial S)\\ + (SS - RR)\partial x\partial y + RS(S-R)y\partial x^2\end{aligned}\right\} = 0,$$

quae ut cum forma proposita conveniat, fieri oportet

$$P = (R + S) + \frac{\partial R - \partial S}{(S - R)\partial x} \text{ et } Q = RS + \frac{S\partial R - R\partial S}{(S - R)\partial x},$$

quos valores si functiones P et Q habuerint, aequatio

$$\partial\partial y + P\partial x\partial y + Qy\partial x^2 = 0$$

per $\frac{(S - R)y\partial x}{(\partial y + Ry\partial x)(\partial y + Sy\partial x)}$ multiplicata, integrale dabit

$$\int (S - R)\partial x + l\frac{\partial y + Ry\partial x}{\partial y + Sy\partial x} = \text{Const.}$$

Si ponamus $S = M + N$ et $R = M - N$, erit

$$P = 2M - \frac{\partial N}{N\partial x} \text{ et } Q = MM - NN + \frac{\partial M}{\partial x} - \frac{M\partial N}{N\partial x}.$$

Corollarium 1.

895. Quaecunque ergo functiones ipsius x loco M et N assumantur, indeque definiantur

$$P = 2M - \frac{\partial N}{N\partial x} \text{ et } Q = MM - NN + \frac{\partial M}{\partial x} - \frac{M\partial N}{N\partial x},$$

hujus aequationis

$$\partial\partial y + P\partial x\partial y + Qy\partial x^2 = 0$$

integrale erit

$$2\int N\partial x + l\frac{\partial y + (M - N)y\partial x}{\partial y + (M + N)y\partial x} = \text{Const.}$$

Corollarium 2.

896. Si ponatur $y = e^{\int z\partial x}$, fiet nostra aequatio differentialis primi gradus

$$\partial z + zz\partial x + Pz\partial x + Q\partial x = 0,$$

cujus propterea integrale erit

$$2\int N\partial x + l\frac{z + (M - N)}{z + (M + N)} = \text{Const.}$$

Corollarium 3.

897. Si velimus, ut sit $P = 0$ et aequatio habeatur hujus-

modi $\partial\partial y + Qy\partial x^2 = 0$, capi debet $2M = \frac{\partial N}{N\partial x}$ eritque $Q = \frac{\partial M}{\partial x} - MM - NN$, ejusque aequatio integralis

$$2\int N\partial x + l\frac{\partial y + (M-N)y\partial x}{\partial y + (M+N)y\partial x} = \text{Const.}$$

Corollarium 4.

898. In genere autem prout constans capiatur vel $+\infty$ vel $-\infty$, obtinebitur integrale particulare vel

$$\partial y + (M+N)y\partial x = 0 \text{ vel}$$
$$\partial y + (M-N)y\partial x = 0,$$

unde erit vel

$$y = \alpha e^{-\int(M+N)\partial x} \text{ vel } y = \beta e^{-\int(M-N)\partial x},$$

ex quibus nostrae aequationis colligitur integrale completum

$$y = e^{-\int M\partial x}\,\alpha\left(e^{-\int N\partial x} + \beta\, e^{\int N\partial x}\right).$$

Exemplum 1.

Sit $M = \alpha$ et $N = \beta$, erit $P = 2\alpha$ et $Q = \alpha\alpha - \beta\beta$, unde hujus aequationis

$$\partial\partial y + 2\alpha\partial x\partial y + (\alpha\alpha - \beta\beta)y\partial x^2 = 0$$

integrale est

$$2\beta x + l\frac{\partial y + (\alpha-\beta)y\partial x}{\partial y + (\alpha+\beta)y\partial x} = \text{Const.}$$

In quantitatibus autem finitis integrale completum

$$y = e^{-\alpha x}\left(A\, e^{-\beta x} + B\, e^{\beta x}\right).$$

Casu autem quo $\beta\beta = -\gamma\gamma$, aequatio

$$\partial\partial y + 2\alpha\partial x\partial y + (\alpha\alpha + \gamma\gamma)y\partial x^2 = 0$$

bis integrata dat

$$y = A\, e^{-\alpha x}\sin.(\gamma x + C).$$

At si $\gamma = 0$, aequationis

$$\partial\partial y + 2\alpha\partial x\partial y + \alpha\alpha y\partial x^2 = 0$$

integrale est

$$y = e^{-\alpha x}(A + Bx).$$

Exemplum 2.

900. Si $M = \frac{\alpha}{x}$ et $N = \beta x^n$, erit $P = \frac{2\alpha - n}{x}$, et

$$Q = \frac{\alpha\alpha}{xx} - \beta\beta x^{2n} - \frac{\alpha}{xx} - \frac{\alpha n}{xx} = \frac{\alpha(\alpha - n - 1)}{xx} - \beta\beta x^{2n}.$$

Ergo hujus aequationis

$$\partial\partial y + \frac{(2\alpha - n)\partial x \partial y}{x} + \frac{\alpha(\alpha - n - 1) y \partial x^2}{xx} - \beta\beta x^{2n} y \partial x^2 = 0$$

integrale primum est

$$\frac{2\beta}{n+1} x^{n+1} + l.\frac{x\partial y + (\alpha - \beta x^{n+1}) y \partial x}{x \partial y + (\alpha + \beta x^{n+1}) y \partial x} = \text{Const.}$$

Integrale autem secundum

$$y = x^{-\alpha}\left(A e^{\frac{-\beta x^{n+1}}{n+1}} + B e^{\frac{\beta x^{n+1}}{n+1}}\right):$$

si $\beta = 0$, erit id

$$y = x^{-\alpha}(A + B x^{n+1}),$$

sin autem $\beta\beta = -\gamma\gamma$, erit

$$y = A x^{-\alpha} \sin.\left(\frac{\gamma}{n+1} x^{n+1} + C\right).$$

Corollarium 1.

901. Sumto $n = 2\alpha$, ut habeatur haec aequatio

$$\partial\partial y - \frac{\alpha(\alpha+1) y \partial x^2}{xx} - \beta\beta x^{4\alpha} y \partial x^2 = 0,$$

erit ejus integrale completum

$$y = x^{-\alpha}\left(A e^{\frac{-\beta}{2\alpha+1} x^{2\alpha+1}} + B e^{\frac{\beta}{2\alpha+1} x^{2\alpha+1}}\right),$$

si sit $\beta = 0$, erit id

$$y = x^{-\alpha}(A + B x^{2\alpha+1}),$$

at si $\beta\beta = -\gamma\gamma$, erit hoc integrale

$$y = A x^{-\alpha} \sin. \left(\frac{\gamma}{2\alpha+1} x^{2\alpha+1} + C\right).$$

Corollarium 2.

902. Ponamus $\alpha = -1$, ut habeamus hanc aequationem $\partial\partial y - \frac{\beta\beta y \partial x^2}{x^4} = 0$, cujus ergo integrale erit

$$y = x \left(A e^{\frac{\beta}{x}} + B e^{-\frac{\beta}{x}}\right),$$

ubi notandum, si sit $\beta\beta = -\gamma\gamma$, fore $y = A x \sin. \left(\frac{\gamma}{x} + C\right)$.

Exemplum 3.

903. Ponatur $N = \frac{A x^m}{\alpha + \beta x^n}$, ut sit

$$\frac{\partial N}{N \partial x} = \frac{m}{x} - \frac{\beta n x^{n-1}}{\alpha + \beta x^n},$$

et sumatur

$$M = \frac{m}{2x} - \frac{\beta n x^{n-1}}{2(\alpha + \beta x^n)},$$

ut fiat $P = 0$ et

$$Q = \frac{-m}{2xx} - \frac{\beta n (n-1) x^{n-2}}{2(\alpha + \beta x^n)} + \frac{\beta\beta n n x^{2n-2}}{2(\alpha + \beta x^n)^2}$$
$$- \frac{mm}{4xx} + \frac{\beta m n x^{n-2}}{2(\alpha + \beta x^n)} - \frac{\beta\beta n n x^{2n-2}}{4(\alpha + \beta x^n)^2}$$
$$- \frac{A A x^{2m}}{(\alpha + \beta x^n)^2}$$

sive

$$Q = \frac{-m(m+2)}{4xx} + \frac{n(m-n+1)\beta x^{n-2}}{2(\alpha + \beta x^n)} + \frac{\beta\beta n n x^{2n-2} - 4 A A x^{2m}}{4(\alpha + \beta x^n)^2},$$

et ob $\int M \partial x = \frac{1}{2} l N$, erit integrale

$$y = \frac{1}{\sqrt{N}} \left(C e^{-\int N \partial x} + D e^{\int N \partial x} \right)$$

hujus aequationis $\partial \partial y + Q y \partial x^2 = 0$. Ut expressio ipsius Q fiat simplicior, hoc fieri potest pluribus modis, dum numerator partis postremae per $\alpha + \beta x^n$ divisibilis redditur.

I. Sit $m = n - 1$ et $AA = \frac{1}{4} \beta \beta n n$, eritque $Q = -\frac{(nn - 1)}{4xx}$, tum vero

$$N = \frac{\frac{1}{2} \beta n x^{n-1}}{\alpha + \beta x^n} \text{ et } \int N \partial x = \tfrac{1}{2} l (\alpha + \beta x^n);$$

unde aequationis $\partial \partial y - \frac{(nn - 1) y \partial x^2}{4xx} = 0$ integrale est

$$y = \frac{1}{\sqrt{x^{n-1}}} (C + D\alpha + D\beta x^n).$$

II. Sit $2m = -2$ seu $m = -1$ et $4AA = \alpha\alpha nn$, erit

$$Q = \frac{1}{4xx} - \frac{nn\beta x^{n-2}}{2(\alpha + \beta x^n)} + \frac{\beta nn x^{n-2} - \alpha nn x^{-2}}{4(\alpha + \beta x^n)}, \text{ seu}$$

$$Q = \frac{1 - nn}{4xx}, \text{ ut ante.}$$

III. Sit $2m = -n - 2$ seu

$$m = \frac{-n-2}{2} \text{ et } 4AA = -\frac{\alpha^3 nn}{\beta},$$

fietque

$$Q = \frac{-(nn - 4)}{16xx} - \frac{3nn\beta x^{n-2}}{4(\alpha + \beta x^n)} + \frac{nn(\beta\beta x^{n-2} - \gamma\beta x^{-2} + \alpha\alpha x^{-n-2})}{4\beta(\alpha + \beta x^n)}$$

seu

$$Q = \frac{4 - nn}{16xx} + \frac{nn(\alpha\alpha x^{-n-2} - \alpha\beta x^{-2} - 2\beta\beta x^{n-2})}{4\beta(\alpha + \beta x^n)},$$

quae expressio abit in

$$Q = \frac{4-nn}{16xx} + \frac{nn}{4\beta}(\alpha x^{-n-2} - 2\beta x^{-2}) = \frac{4-9nn}{16\,xx} + \frac{nn\alpha}{4\beta x^{n+2}}.$$

Quare cum sit

$$N = +\frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}} \cdot \frac{x^{-\frac{n-2}{2}}}{\alpha+\beta x^n}, \text{ erit}$$

$$\int N\,\partial x = \frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}} \int \frac{\partial x}{(\alpha+\beta x^n)x^{\frac{n+2}{2}}},$$

sumatur $n = \frac{2}{3}$, ut fiat $m = -\frac{4}{3}$, $Q = \frac{\alpha}{9\,\beta\,x^{\frac{2}{3}}}$, et

$$N = \frac{\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \cdot \frac{x^{-\frac{4}{3}}}{\alpha+\beta x^{\frac{2}{3}}}, \text{ hinc}$$

$$\int N\,\partial x = \frac{\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \int \frac{\partial x}{(\alpha+\beta\,x^{\frac{2}{3}})\,x^{\frac{4}{3}}};$$

sicque aequationis

$$\partial\partial y + \frac{\alpha}{9\,\beta\,x^{\frac{2}{3}}}\, y\,\partial x^2 = 0$$

integrale erit

$$y = \frac{1}{\sqrt{N}}(C e^{-\int N\partial x} + D e^{\int N\partial x}).$$

Sin autem capiatur $n = -\frac{2}{3}$, ut fiat

$$m = -\frac{2}{3} \text{ et } Q = \frac{\alpha}{9\,\beta\,x^{\frac{4}{3}}}, \text{ erit}$$

$$N = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}} \cdot \frac{x^{-\frac{2}{3}}}{\alpha+\beta x^{-\frac{2}{3}}} \text{ et}$$

$$\int N\partial x = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\int\frac{x^{\frac{1}{3}}\partial x}{\alpha x+\beta x^{\frac{1}{3}}} = \frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\int\frac{\partial x}{\alpha x^{\frac{2}{3}}+\beta},$$

unde aequatio

$$\partial\partial y + \frac{\alpha y\partial x^2}{9\beta x^{\frac{4}{3}}} = 0$$

simili modo integratur.

Scholion 1.

904. Aequationem ergo $\partial\partial y + A x^m y\partial x^2 = 0$ his casibus integrare licuit, $m = 0$, $m = -4$, $m = -\frac{4}{3}$, $m = -\frac{8}{3}$ et $m = -2$, seu $m = -2 \pm \frac{2}{1}$, et $m = -2 \pm \frac{2}{3}$. Quodsi ulterius ponamus $N = \frac{A x^\lambda}{\alpha+\beta x^n+\gamma x^{2n}}$, simili modo integrationem casuum istius aequationis $m = -2 \pm \frac{2}{5}$ impetrabimus, quibus quoque aequatio differentialis primi gradus

$$\partial z + zz\partial x + A x^m \partial x = 0$$

integrationem admittit. Haec autem casuum integrabilium investigatio nimis est operosa, quam ut eam fusius prosequamur, praesertim cum infra methodus occurrat haec omnia commodius evolvendi.

Scholion 2.

905. Ex his colligere licet, quantus fructus ex inventione multiplicatorum, quibus etiam aequationes differentio-differentiales integrabiles redduntur, expectari queat, etiamsi exempla hic tractata tantum leve hujus methodi specimen referant. Aliquas autem sal-

tem multiplicatorum formas hic sum contemplatus, neque ullum est dubium, quin plures aliae formae pari successu in usum vocari queant. In hoc porro capite tantum ejusmodi aequationes differentio-differentiales tractavimus, in quibus altera variabilis y cum suis differentialibus ∂y et $\partial\partial y$ ubique unicam obtinet dimensionem. Verum eadem methodus quoque ad alia hujusmodi aequationum genera extenditur, quae etsi parum adhuc est exculta, tamen usu non carebit sequens applicatio, ubi integratio aliarum aequationum differentialium secundi gradus, quae aliis methodis tractatu difficillimae videntur, ope multiplicatorum docebitur.