CAPUT IV.

DE

AEQUATIONIBUS DIFFERENTIO-DIFFERENTIALIBUS IN QUIBUS ALTERA VARIABILIS UNICAM HABET DIMENSIONEM.

Problema 101.

831.

Sumto elemento ∂x constante, si proponatur aequatio hujus formae $\partial\partial y + P\partial x\partial y + Qy\partial x^2 = 0$, ubi P et Q sint functiones quaecunque ipsius x, eam ad aequationem differentialem primi gradus revocare.

Solutio.

Ponendo $\partial y = p\partial x$ et $\partial p = q\partial x$, aequatio proposita induit hanc formam $q + Pp + Qy = 0$, in qua si methodo ante exposita statuamus $p = uy$ et $q = vy$, obtinebimus hanc inter x, u et v aequationem $v + Pu + Q = 0$, hincque $v = -Pu - Q$. Tum vero fit

$$\partial y = uy\partial x \text{ et } u\partial y + y\partial u = vy\partial x,$$

ita ut sit

$$\frac{\partial y}{y} = u\partial x = \frac{v\partial x - \partial u}{u},$$

ideoque

$$\partial u + uu\partial x + Pu\partial x + Q\partial x = 0,$$

substituto pro v valore.

Qua aequatione resoluta erit $ly = \int u\partial x$. Vel sine his substitutionibus statim in ipsa aequatione proposita ponamus $y = e^{\int u \partial x}$, unde fit

$$\partial y = e^{\int u \partial x} u \partial x, \text{ et } \partial \partial y = e^{\int u \partial x} (\partial u \partial x + u u \partial x^2),$$

et cum facta substitutione quantitas exponentialis $e^{\int u \partial x}$ ex calculo tollatur, obtinebitur praecedens aequatio differentialis primi gradus

$$\partial u + u u \partial x + P u \partial x + Q \partial x = 0,$$

a cujus resolutione integratio aequationis differentio-differentialis propositae pendet.

Corollarium 1.

832. Haec aequatio differentialis primi gradus pluribus modis in alias formas sibi fere similes transmutari potest. Veluti si ponamus $u = M z$, prodit

$$M \partial z + z (\partial M + P M \partial x) + M M z z \partial x + Q \partial x = 0;$$

ubi pro M ejusmodi functionem ipsius x accipere licet, ut terminus ipsa littera z affectus evanescat, quod fit si

$$\partial M + M P \partial x = 0 \text{ seu } M = C e^{-\int P \partial x}.$$

Corollarium 2.

833. Similis forma prodit ponendo $u = \frac{K}{z}$, fit enim

$$-\frac{K \partial z}{z z} + \frac{\partial K}{z} + \frac{K K \partial x}{z z} + \frac{K P \partial x}{z} + Q \partial x = 0 \text{ seu}$$

$$K \partial z - z (\partial K + K P \partial x) - Q z z \partial x - K K \partial x = 0,$$

ubi secundus terminus, sumendo $K = C e^{-\int P \partial x}$, pariter evanescit.

Corollarium 3.

834. Similis transformatio generalius instituitur, ponendo $u = K + M z$, prodit enim

$$\partial K + M \partial z + z \partial M + KK \partial x + 2 KMz \partial x + MMzz \partial x^2 + KP \partial x + MPz \partial x + Q \partial x = 0.$$

quae ordinata praebet

$$M \partial z + z(\partial M + 2 KM \partial x + MP \partial x) + MMzz \partial x^2 + \partial K + KK \partial x + KP \partial x + Q \partial x = 0;$$

unde secundus terminus tollitur sumendo

$$M = C e^{-\int \partial x (2K + P)} \text{ seu } K = \frac{-\partial M - MP \partial x}{2 M \partial x}.$$

Corollarium 4.

835. Adhuc generalius similis forma oritur, si ponatur $u = \frac{K + Mz}{L + Nz}$, unde prodit

$$\partial z(LM - KN) + L \partial K - K \partial L + z(L \partial M - M \partial L + N \partial K - K \partial N) + zz(N \partial M - M \partial N) + (K + Mz)^2 \partial x + P(K + Mz)(L + Nz) \partial x + Q(L + Nz)^2 \partial x = 0,$$

quae reducitur ad hanc formam

$$0 = \partial z(LM - KN) + z \left\{ \begin{matrix} L \partial M - M \partial L + N \partial K - K \partial N \\ + 2 KM \partial x + P(KN + LM) \partial x + 2 LNQ \partial x \end{matrix} \right\} + zz(N \partial M - M \partial N + MM \partial x + MNP \partial x + NNQ \partial x) + L \partial K - K \partial L + KK \partial x + KLP \partial x + LLQ \partial x:$$

ubi pro K, L, M et N functiones ejusmodi ipsius x accipere licet, ut forma prodeat tractatu facillima.

Scholion.

836. Quoniam hujusmodi aequationes differentio-differentiales, in quibus variabilis y unicam habet dimensionem frequentissime occurrere solent, merito geometrae tantopere in resolvenda aequatione

$$\partial u + uu \partial x + P u \partial x + Q \partial x = 0$$

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studium et operam collocarunt, quae etiam forma generaliori ita repraesentari potest

$$\partial z + P z \partial x + R z z \partial x + Q \partial x = 0,$$

cujus quidem casum eximium $\partial z + z z \partial x = a x^n \partial x$ olim Comes *Riccati* in haud spernendum Analyseos incrementum proposuerat. Inter transformationes autem hujus casus praecipue notari meretur positio $x = t^{\frac{2}{n+2}}$, quae dat

$$\partial z + \frac{2}{n+2} z z t^{\frac{-n}{n+2}} \partial t = \frac{2a}{n+2} t^{\frac{n}{n+2}} \partial t;$$

unde ponendo $z = C t^{\frac{n}{n+1}}$, prodit

$$(n+2) C t^{\frac{n}{n+2}} \partial v + C n t^{\frac{-2}{n+2}} v \partial t$$
$$+ 2 C C t^{\frac{n}{n+2}} v v \partial t = 2 a t^{\frac{n}{n+2}} \partial t; \text{ seu}$$
$$(n+2) C \partial v + \frac{n C v \partial t}{t} + 2 C C v v \partial t = 2 a \partial t:$$

ita ut hic nulla potestas indefinita ipsius t occurrat. Si hic porro ponatur $v = \frac{\alpha}{t} + s$, fiet

$$-\frac{(n+2) C \alpha \partial t}{t t} + (n+2) C \partial s + \frac{n C s \partial t}{t} + 2 C C s s \partial t = 2 a \partial t,$$
$$+\frac{n C \alpha \partial t}{t t} \qquad + \frac{4 C C \alpha s \partial t}{t}$$
$$+\frac{2 C C \alpha \alpha \partial t}{t t}$$

ubi si capiatur $\alpha = \frac{-n}{4 C}$, orietur

$$(n+2) C \partial s + 2 C C s s \partial t = 2 a \partial t - \frac{n(n+4) \partial t}{8 t t}$$

quae forma simplicissima videtur.

Theorema.

837. Sumto elemento ∂x constante, si aequationi differentio-differentiali

$$\partial \partial y + P \partial x \partial y + Q y \partial x^2 = 0$$

satisfaciant integralia particularia $y = M$ et $y = N$, ita ut ratio $M : N$ non sit constans, erit ejus integrale completum $y = \alpha M + \beta N$.

Demonstratio.

Cum valores $y = M$ et $y = N$ satisfaciant aequationi propositae, erit

$$\partial \partial M + P \partial x \partial M + Q M \partial x^2 = 0 \text{ et}$$

$$\partial \partial N + P \partial x \partial N + Q N \partial x^2 = 0,$$

unde patet, ponendo $y = \alpha M + \beta N$, aequationi quoque satisfieri, cum fiat

$$\left.\begin{aligned} &+ \alpha (\partial \partial M + P \partial x \partial M + Q M \partial x^2) \\ &+ \beta (\partial \partial N + P \partial x \partial N + Q N \partial x^2) \end{aligned}\right\} = 0.$$

Quoniam vero hoc integrale $y = \alpha M + \beta N$ duas constantes α et β complectitur, quas pro lubitu definire licet, id completum sit necesse est, nisi forte N sit multiplum ipsius M.

Corollarium 1.

838. Ex datis ergo duobus integralibus particularibus hujusmodi aequationis, ejus integrale completum formari potest, siquidem illa duo integralia sint inter se diversa.

Corollarium 2.

839. Cum posito $y = e^{\int u \partial x}$ seu $u = \frac{\partial y}{y \partial x}$, prodeat

$$\partial u + u u \partial x + P u \partial x + Q \partial x = 0.$$

Si huic aequationi satisfaciant valores $u = \frac{\partial M}{M \partial x}$ et $u = \frac{\partial N}{N \partial x}$, eidem quoque satisfaciet valor $u = \frac{\alpha \partial M + \beta \partial N}{(\alpha N + \beta M) \partial x}$.

Corollarium 3.

840. Si ergo aequationis $\partial u + u u \partial x + P u \partial x +$

$Q\,\partial x = 0$ habeantur duo integralia particularia $u = R$ et $u = S$, ob

$$M = e^{\int R\,\partial x} \text{ et } N = e^{\int S\,\partial x},$$

integrale completum erit

$$u = \frac{\alpha\, e^{\int R\,\partial x} R + \beta\, e^{\int S\,\partial x} S}{\alpha\, e^{\int R\,\partial x} + \beta\, e^{\int S\,\partial x}} \text{ sive}$$

$$u = R + \frac{\beta\, e^{\int S\,\partial x}(S - R)}{\alpha\, e^{\int R\,\partial x} + \beta\, e^{\int S\,\partial x}}.$$

Scholion.

841. Maximi momenti est haec observatio, quod in hujusmodi aequationibus ex cognitis binis integralibus particularibus integrale completum assignari possit. Plerumque autem cognito uno integrali particulari, in eo signum radicale inesse solet, ob cujus ambiguitatem duo simul integralia particularia innotescunt. Ita si aequationi

$$\partial u + u u\,\partial x + P u\,\partial x + Q\,\partial x = u$$

satisfaciat valor $u = T + \sqrt{V}$, eidem satisfaciet $u = T - \sqrt{V}$, unde integrale completum erit

$$u = T + \sqrt{V} - \frac{2 B \sqrt{V}}{\alpha\, e^{2\int \partial x \sqrt{V}} + \beta} \text{ seu}$$

$$u = T + \frac{\alpha\, e^{2\int \partial x \sqrt{V}} \sqrt{V} - \beta \sqrt{V}}{\alpha\, e^{2\int \partial x \sqrt{V}} + \beta}.$$

Ac si forte sit $\sqrt{V}$ imaginarium, puta $\sqrt{V} = X\sqrt{-1}$. ob

$$e^{\pm\int \partial x \sqrt{V}} = \cos \int X\,\partial x + \sqrt{-1}.\sin.\int X\,\partial x, \text{ erit}$$

$$u = T + \frac{(\alpha-\beta)\cos.\int X\,\partial x + (\alpha+\beta)\sin.\int X\,\partial x.\sqrt{-1}}{(\alpha+\beta)\cos.\int X\,\partial x + (\alpha-\beta)\sin.\int X\,\partial x.\sqrt{-1}}\, X\sqrt{-1}.$$

seu posito

$$(\alpha - \beta)\sqrt{-1} = \gamma \text{ et } \alpha + \beta = \delta,$$

$$u = T + \frac{\gamma\cos.\int X\,\partial x - \delta\sin.\int X\,\partial x}{\delta\cos.\int X\,\partial x + \gamma\sin.\int X\,\partial x}.X, \text{ vel etiam}$$

$$u = T + X\,\text{tang.}\left(\int X\,\partial x + \zeta\right).$$

Problema 102.

842. Sumto elemento ∂x constante, invenire integrale completum hujus aequationis differentio-differentialis

$$\partial \partial y + A \partial y \partial x + B y \partial x^2 = 0.$$

Solutio.

Posito $y = e^{\int u \partial x}$ prodit haec aequatio

$$\partial u + u u \partial x + A u \partial x + B \partial x = 0, \text{ sive}$$

$$\partial x = \frac{-\partial u}{u u + A u + B},$$

cui satisfit tribuendo ipsi u ejusmodi valorem constantem ut evadat $u u + A u + B = 0$, qui sunt

$$u = -\tfrac{1}{2} A \pm \sqrt{(\tfrac{1}{4} A A - B)}.$$

Hinc ergo cum habeantur bina integralia particularia $y = e^{\int u \partial x}$, posito $\sqrt{(\tfrac{1}{4} A A - B)} = n$, erit integrale completum

$$y = e^{-\frac{1}{2} A x} (\alpha e^{n x} + \beta e^{-n x}),$$

ac si sit n numerus imaginarius, puta $n = m \sqrt{-1}$, erit

$$y = e^{-\frac{1}{2} A x} (\alpha \cos. m x + \beta \sin. m x) = C e^{-\frac{1}{2} A x} \sin. (m x + \gamma).$$

Sin autem sit $n = 0$, prodibit

$$y = e^{-\frac{1}{2} A x} (\alpha + \beta x).$$

Corollarium 1.

843. Ad integrale ergo aequationis propositae inveniendum, resolvi oportet aequationem algebraicam $u u + A u + B = 0$, quae oritur ex proposita

$$\partial \partial y + A \partial y \partial x + B y \partial x^2 = 0,$$

si loco y, ∂y, $\partial \partial y$, scribatur u^0, u^1, u^2, et elementum ∂x rejiciatur, tum enim binae radices illius aequationis dabunt integrale completum.

Corollarium 2.

844. Scilicet si aequationis $uu + Au + B = 0$ factores sint $(u + f)(u + g)$, ob valores $u = -f$ et $u = -g$, integrale completum erit $y = \alpha e^{-fx} + \beta e^{-gx}$. At si sit $g = f$, erit $y = e^{-fx}(\alpha + \beta x)$.

Corollarium 3.

845. Si aequatio $uu + Au + B = 0$ habeat factores imaginarios, quo casu hujusmodi formam habebit

$$uu + 2fu\cos.\zeta + ff = 0, \text{ erit } u = -f\cos.\zeta \pm f\sqrt{-1}.\sin.\zeta,$$

hincque integrale completum erit:

$$y = e^{-fx\cos.\zeta}(\alpha\cos.fx\sin.\zeta + \beta\sin.fx\sin.\zeta), \text{ sive}$$

$$y = C e^{-fx\cos.\zeta}\sin.(fx\sin.\zeta + \gamma).$$

Scholion.

846. Idem integrale completum reperitur methodo consueta ex aequatione $\partial x = \frac{-\partial u}{uu + Au + B}$: posito enim

$$uu + Au + B = (u + f)(u + g), \text{ erit}$$

$$(g - f)\partial x = \frac{\partial u}{u + g} - \frac{\partial u}{u + f}, \text{ et } C e^{(g-f)x} = \frac{u + g}{u + f};$$

unde fit

$$u = \frac{g - Cf e^{(g-f)x}}{C e^{(g-f)x} - 1}, \text{ seu}$$

$$u = \frac{-\alpha f e^{gx} + \beta g e^{fx}}{\alpha e^{gx} - \beta e^{fx}}.$$

Tum vero fit

$$\int u\partial x = -\int \frac{\alpha f e^{(g-f)x} - \beta g}{\alpha e^{(g-f)x} - \beta}\partial x = -\int \frac{u\partial u}{(u + f)(u + g)};$$

ideoque

$$\int u\,\partial x = \frac{f}{g-f}\,l(u+f) - \frac{g}{g-f}\,l(u+g).$$

Hinc

$$y = e^{\int u\,\partial x} = C\,(u+f)^{\frac{f}{g-f}}\,(u+g)^{\frac{-g}{g-f}}.$$

At est

$$u+f = \frac{\beta\,(g-f)\,e^{fx}}{\alpha\,e^{gx} - \beta\,e^{fx}} \text{ et } u+g = \frac{\alpha\,(g-f)\,e^{gx}}{\alpha\,e^{gx} - \beta\,e^{fx}},$$

unde colligitur mutando constantem C

$$y = \frac{C\,e^{\frac{ffx}{g-f}}\,e^{\frac{-ggx}{g-f}}}{(\alpha\,e^{gx} - \beta\,e^{fx})^{\frac{f-g}{g-f}}} = C\,e^{-(f+g)x}\,(\alpha\,e^{gx} - \beta\,e^{fx}),$$

seu $y = \alpha\,e^{-fx} + \beta\,e^{-gx}$, ut ante. Hinc ergo patet, quantum subsidium afferat formatio integralis completi ex binis particularibus.

Problema 103.

847. Sumto elemento ∂x constante, si proponatur haec aequatio differentio-differentialis

$$\partial\partial y + \frac{A\,\partial y\,\partial x}{x} + \frac{B\,y\,\partial x^2}{xx} = 0,$$

ejus integrale completum invenire.

Solutio.

Ponatur $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, ut habeamus

$$q + \frac{Ap}{x} + \frac{By}{xx} = 0, \text{ seu } q = \frac{-Ap}{x} - \frac{By}{xx}.$$

Sit nunc $p = \frac{uy}{x}$, erit $\partial y = \frac{uy\,\partial x}{x}$, et

$$\partial p = \frac{u\,\partial y + y\,\partial u}{x} - \frac{uy\,\partial x}{xx} = -\frac{A\,\partial y}{x} - \frac{B\,y\,\partial x}{xx},$$

unde colligimus

$$\frac{\partial y}{y} = \frac{u\,\partial x}{x} = \frac{u\,\partial x - B\,\partial x - x\,\partial u}{xu + Ax}, \text{ seu}$$

$$x\partial u + B\partial x + uu\partial x + (A-1)u\partial x = 0,$$

hincque $\frac{\partial x}{x} = \frac{-\partial u}{uu+(A-1)u+B}$; cui particulariter satisfit ponendo $uu + (A-1)u + B = 0$.

Sit primo $uu + (A-1)u + B = (u+f)(u+g)$, erit particulariter $ly = -flx$ et $y = x^{-f}$, similique modo $y = x^{-g}$, unde integrale completum erit

$$y = \alpha x^{-f} + \beta x^{-g}.$$

Si sit $g = f$, statuatur $g = f - \omega$ evanescente ω, erit

$$x^{-g} = x^{-f} . x^{\omega} = x^{-f}(1 + \omega lx),$$

ergo hoc casu fit

$$y = x^{-f}(\alpha + \beta lx).$$

Sit denique

$$uu + (A-1)u + B = uu + 2fu\cos.\zeta + ff, \text{ erit}$$

$$u = -f(\cos.\zeta \pm \sqrt{-1}.\sin.\zeta),$$

ergo particulariter

$$y = x^{-f\cos.\zeta}.x^{\pm f\sqrt{-1}.\sin.\zeta} = x^{-f\cos.\zeta}[\cos.(f\sin.\zeta.lx) \pm \sqrt{-1}.\sin.(f\sin.\zeta.lx)],$$

quare integrale completum erit

$$y = C x^{-f\cos.\zeta}\sin.(f\sin.\zeta.lx + \gamma).$$

Corollarium 1.

848. Hujus ergo aequationis

$$\partial\partial y + (f+g+1)\frac{\partial y\partial x}{x} + \frac{fgy\partial x^2}{xx} = 0,$$

integrale completum est

$$y = \alpha x^{-f} + \beta x^{-g}.$$

Hujus autem

$$\partial\partial y + (2f+1)\frac{\partial y\partial x}{x} + \frac{ffy\partial x^2}{xx} = 0,$$

integrale completum est

$$y = x^{-f}(\alpha + \beta\, l\, x).$$

Corollarium 2.

849. At si aequatio proposita hujusmodi formam habuerit

$$\partial\partial y + (1 + 2f\cos.\zeta)\frac{\partial y\,\partial x}{x} + \frac{ffy\,\partial x}{xx} = 0,$$

tum ejus integrale completum erit

$$y = C\,x^{-f\cos.\zeta}\sin.(f\sin.\zeta\, l\, x + \gamma).$$

Scholion.

850. Similem resolutionem quoque admittit haec aequatio differentio-differentialis

$$\partial\partial y - \frac{n\,\partial y\,\partial x}{x} + A\,x^n\,\partial y\,\partial x + B\,x^{2n}\,y\,\partial x^2 = 0.$$

Ponatur enim $\partial y = x^n\,y\,u\,\partial x$, et cum sit

$$\partial\partial y = x^n\,y\,\partial x\,\partial u + n\,x^{n-1}\,yu\,\partial x^2 + x^{2n}\,yu^2\,\partial x^2,$$

erit per y dividendo

$$x^n\,\partial x\,\partial u + n\,x^{n-1}\,u\,\partial x^2 + x^{2n}\,uu\,\partial x^2 - n\,x^{n-1}\,u\,\partial x^2 + A\,x^{2n}\,u\,\partial x^2 + B\,x^{2n}\,\partial x^2 = 0,$$

hinc

$$\partial u + x^n\,u\,u\,\partial x + A\,x^n\,u\,\partial x + B\,x^n\,\partial x = 0,$$

ideoque

$$x^n\,\partial x = \frac{-\partial u}{uu + Au + B},$$

cui particulariter satisfit ponendo $uu + Au + B = 0$, unde u duplicem consequitur valorem constantem quorum alter sit $u = -f$ alter $u = -g$. Quocirca integralia particularia erunt

$$y = e^{\frac{-f\,x^{n+1}}{n+1}} \quad \text{et} \quad y = e^{\frac{-g\,x^{n+1}}{n+1}}.$$

Sit brevitatis gratia $\frac{x^{n+1}}{n+1} = t$, erit integrale completum

$$y = \alpha e^{-ft} + \beta e^{-gt},$$

pro casu scilicet

$$uu + Au + B = (u+f)(u+g).$$

At pro casu

$$uu + Au + B = (u+f)^2, \text{ erit } y = e^{-ft}(\alpha + \beta t).$$

Casu autem quo

$$uu + Au + B = uu + 2fu\cos.\zeta + ff, \text{ erit}$$

$$y = Ce^{-ft\cos.\zeta}\sin.(ft\sin.\zeta + \gamma).$$

Haec integratio adeo ad hanc formam extendi potest, sumendo pro X functionem quamcunque ipsius x

$$\partial\partial y - \frac{\partial x\,\partial y}{X} + AX\,\partial y\,\partial x + BXXy\,\partial x^2 = 0.$$

Posito enim $\partial y = Xuy\,\partial x$ seu $\frac{\partial y}{y} = Xu\,\partial x$, fit

$$X\partial x = \frac{-\partial u}{uu + Au + B},$$

unde posito $\int X\partial x = t$, integrale completum se habebit ut ante. Scilicet

1) si $A = f + g$ et $B = fg$, erit integrale
$$y = \alpha e^{-ft} + \beta e^{-gt},$$
2) si $A = 2f$ et $B = ff$, erit integrale
$$y = e^{-ft}(\alpha + \beta t),$$
3) si $A = 2f\cos.\zeta$ et $B = ff$, erit integrale
$$y = Ce^{-ft\cos.\zeta}\sin.(ft\sin.\zeta + \gamma).$$

Problema 104.

851. Sumto elemento ∂x constante, si P, Q et X denotent functiones quascunque ipsius x, integrationem hujus aequationis differentio-differentialis

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = X\partial x^2,$$

ad aequationem differentialem primi ordinis reducere.

Solutio.

Hic singulari modo procedamus, loco y binas novas incognitas introducendo. Statuatur scilicet $y = uv$, et cum sit

$$\partial y = u\partial v + v\partial u \text{ et } \partial\partial y = u\partial\partial v - 2\partial u\partial v + v\partial\partial u,$$

aequatio nostra induet hanc formam

$$u\partial\partial v + 2\partial u\partial v + v\partial\partial u + Pu\partial x\partial v + Pv\partial x\partial u$$
$$+ Quv\partial x^2 = X\partial x^2.$$

Jam altera v ita determinetur, ut termini ipsa littera u affecti destruantur, quod fit si

$$\partial\partial v + P\partial x\partial v + Qv\partial x^2 = 0,$$

unde per superiora v per x determinetur, quo facto superest haec aequatio

$$2\partial u\partial v + v\partial\partial u + Pv\partial x\partial u = X\partial x^2,$$

unde cum v jam detur per x, quantitas u definiri debet. Ponatur $\partial u = s\partial x$, eritque

$$v\partial s + 2s\partial v + Psv\partial x = X\partial x,$$

quae multiplicata per $v e^{\int P\partial x}$ integrabilis redditur; prodit enim

$$vvs e^{\int P\partial x} = \int e^{\int P\partial x} Xv\partial x, \text{ ideoque}$$

$$y = \frac{e^{-\int P\partial x}}{vv} \int e^{\int P\partial x} Xv\partial x \text{ et}$$

$$u = \int \frac{e^{-\int P\partial x}\partial x}{vv} \int e^{\int P\partial x} Xv\partial x.$$

Quare cum incognita v fuerit determinata aequatione

$$\partial\partial v - P\partial x\partial v + Qv\partial x^2 = 0,$$

integrale aequationis propositae erit

$$y = v\int \frac{e^{-\int P \partial x}\partial x}{vv}\int e^{\int P \partial x} X v \partial x.$$

Corollarium 1.

852. Ut integratio ad aequationem differentialem primi ordinis revocetur, ponatur $v = e^{\int t \partial x}$, et quantitas t definietur per hanc aequationem

$$\partial t + tt\partial x + Pt\partial x + Q\partial x = 0,$$

quo facto integrale quaesitum erit

$$y = e^{\int t \partial x}\int e^{-\int (P+2t)\partial x}\partial x \int e^{\int (P+t)\partial x} X \partial x.$$

Corollarium 2.

853. Cum sit $(P+t)\partial x = -\frac{\partial t}{t} - \frac{Q\partial x}{t}$, erit

$$e^{\int (P+t)\partial x} = \frac{1}{t} e^{-\int \frac{Q\partial y}{t}}, \text{ hincque}$$

$$y = e^{\int t \partial x}\int e^{\int \frac{Q\partial x}{t} - \int t\partial x} t\partial x \int e^{-\int \frac{Q\partial x}{t}} X \frac{\partial x}{t},$$

ubi duplex integratio ad integrale completum perducit.

Scholion 1.

854. Alio modo qui propius ad ante usitatum accedat, eadem integratio institui potest. Ponatur scilicet pro aequatione proposita $\partial y = ty\partial x + v\partial x$, ubi v certam ipsius x functionem designet ex functione X determinandam. Cum igitur sit

$$\partial\partial y = y\partial t\partial x + (ty\partial x + v\partial x) + \partial v\partial x,$$

erit facta substitutione

$$\left.\begin{array}{l} y\partial t\partial x + tty\partial x^2 + Pty\partial x^2 + Qy\partial x^2 \\ + tv\partial x^2 + \partial v\partial x + Pv\partial x^2 + X\partial x^2 \end{array}\right\} = 0,$$

cujus aequationis utraque pars, tum ea quae per y multiplicatur, quam altera ab y libera, seorsim nihilo aequetur, unde has duas aequationes nanciscimur,

$$\partial t + tt\partial x + Pt\partial x + Q\partial x = 0 \text{ et}$$

$$\partial v + tv\partial x + Pv\partial x = X\partial x,$$

ex quarum illa t per x ut ante definiri debet, tum vero erit ex ista

$$e^{\int(P+t)\partial x} v = \int e^{\int(P+t)\partial x} X\partial x.$$

Jam vero ex aequatione assumta $\partial y - ty\partial x = v\partial x$ colligitur

$$e^{-\int t\partial x} y = \int e^{-\int t\partial x} v\partial x,$$

ubi si loco v valor modo inventus substituatur, praecedens integralis forma obtinetur.

Scholion 2.

855. Ex hac operatione sequi videtur, aequationis propositae

$$\partial\partial y + P\partial y\partial x + Qy\partial x^2 = X\partial x^2,$$

integrationem necessario pendere ab integratione hujus

$$\partial\partial v + P\partial v\partial x + Qv\partial x^2 = 0,$$

quandoquidem hac concessa illa exhiberi potest. Minime tamen hinc vicissim colligere licet, si posterioris resolutio vires nostras superet, etiam priorem nullo modo integrari posse, quin potius facile est infinitos casus exhibere, quibus prior integrationem admittat, cum tamen posterior irresolubilis existat. Sit enim $P = 0$ ut $Q = ax$, atque certum est aequationem posteriorem $\partial\partial v + axv\partial x^2 = 0$, nulla adhuc methodo resolvi posse, cum posito $v = e^{\int t\partial x}$, abeat in

$$\partial t + tt\partial x + ax\partial x = 0;$$

neque tamen hinc sequitur aequationem priorem

$$\partial\partial y + axy\partial x^2 = X\partial x^2,$$

semper esse intractabilem. Infiniti enim casus pro X assignari possunt, quibus integratio succedat. Sumta enim pro y functione quacunque ipsius x, reperietur pro X ejusmodi functio, ut aequationi valor pro y assumtus satisfaciat. Veluti posito $y = \frac{\beta x}{\alpha}$, ob $\partial\partial y = 0$ fit $X = \beta x x$, atque aequationi

$$\partial\partial y + \alpha x y \partial x^2 = \beta x x \partial x^2$$

utique satisfacit integrale $y = \frac{\beta x}{\alpha}$. Interim tamen hoc integrale tantum est particulare, ac dubium adhuc relinquitur, an etiam integrale completum exhiberi possit. At posito $y = \frac{\beta x}{\alpha} + z$, pro integrali completo inveniendo prodit $\partial\partial z + \alpha x z \partial x^2 = 0$, quae cum resolutionem respuat, evidens est integrale completum etiam in genere exhiberi non posse, nisi simul altera aequatio integrationem admittat.

Problema 105.

856. Sumto elemento ∂x constante, invenire integrale completum hujus aequationis differentio-differentialis

$$\partial\partial y + A \partial y \partial x + B y \partial x^2 = X \partial x^3$$

denotante X functionem quamcunque ipsius x.

Solutio.

Posito $y = uv$, aequatio proposita in duas sequentes resolvitur

$$\partial\partial v + A \partial v \partial x + B v \partial x^2 = 0 \text{ et}$$

$$v \partial\partial u + 2 \partial v \partial u + A v \partial x \partial u = X \partial x^2.$$

Quodsi ergo ex priori valor ipsius v per x definiatur, integrale completum ex posteriore ita se habebit, ut ob $P = A$ sit

$$y = v \int \frac{e^{-Ax} \partial x}{vv} \int e^{Ax} X v \partial x,$$

ubi cum duplex integratio integrale completum producat, sufficiet pro v integrale particulare prioris aequationis assumsisse, id quod etiam ex solutione generali patebit. Cum igitur pro resolutione prioris aequationis formanda sit haec aequatio quadratica $tt + At + B = 0$, pro ejus indole tres casus evolvi conveniet.

I. Si $tt + At + B = (t + f)(t + g)$, ut sit $A = f + g$ et $B = fg$, erit $v = \alpha e^{-fx} + \beta e^{-gx}$. Sumatur primo tantum integrale particulare $v = e^{-fx}$, et ob $A = f + g$, fiet

$$y = e^{-fx} \int e^{(f-g)x} \partial x \int e^{gx} X \partial x:$$

sit $e^{(f-g)x} \partial x = \partial R$, et $\int e^{gx} X \partial x = S$, ut fiat

$$y = e^{-fx} \int S \partial R = e^{-fx} (RS - \int R \partial S);$$

at est $R = \frac{1}{f-g} e^{(f-g)x}$, unde colligitur

$$y = \frac{1}{f-g} e^{-gx} S - \frac{1}{f-g} e^{-fx} \int e^{fx} X \partial x, \text{ sive}$$

$$(f-g) y = e^{-gx} \int e^{gx} X \partial x - e^{-fx} \int e^{fx} X \partial x,$$

quod idem integrale prodiisset, si altero particulari $v = e^{-gx}$ usi essemus.

In genere autem sumto $v = \alpha e^{-fx} + \beta e^{-gx}$, statuatur ut ante

$$\frac{e^{-(f+g)x} \partial x}{vv} = \partial R, \text{ et } \int e^{(f+g)x} X v \partial x = S,$$

ut sit pariter

$$y = v \int S \partial R = v (RS - \int R \partial S).$$

Fingamus

$$R = \frac{C e^{\lambda x}}{v}, \text{ et ob } \partial v = -\partial x (\alpha f e^{-fx} + \beta g e^{-gx}),$$

erit

$$\partial R = \frac{C e^{\lambda x} \partial x (\alpha \lambda e^{-fx} + \beta \lambda e^{-gx} + \alpha f e^{-fx} + \beta g e^{-gx})}{vv}$$

Sit jam $\lambda = -g$, et $C\alpha(f-g) = 1$, ut ∂R datum adipiscatur valorem, ob $C = \frac{1}{\alpha(f-g)}$, erit

$$R = \frac{e^{-gx}}{\alpha(f-g)v}, \text{ et } R\partial S = \frac{1}{\alpha(f-g)} e^{fx} X \partial x;$$

tum vero

$$S = \alpha \int e^{gx} X \partial x + \beta \int e^{fx} X \partial x,$$

unde conficitur

$$y = v\left(\frac{e^{-gx}}{(f-g)v}\int e^{gx} X\partial x + \frac{\beta e^{-gx}}{\alpha(f-g)v}\int e^{fx} X\partial x - \frac{1}{\alpha(f-g)}\int e^{fx} X\partial x\right),$$

seu prorsus ut ante

$$(f-g)y = e^{-gx}\int e^{gx} X\partial x - e^{-fx}\int e^{fx} X\partial x.$$

II. Si $tt + At + B = (t+f)^2$, seu $A = 2f$ et $B = ff$, erit ex priori aequatione $v = e^{-fx}(\alpha + \beta x)$. Ponatur ut ante

$$\frac{e^{-2fx}\partial x}{vv} = \partial R, \text{ et } \int e^{2fx} X v \partial x = S,$$

ut habeatur $y = v(RS - \int R\partial S)$. Cum ergo sit

$$\partial R = \frac{\partial x}{(\alpha+\beta x)^2}, \text{ erit } R = -\frac{1}{\beta(\alpha+\beta x)} = -\frac{e^{-fx}}{\beta v}, \text{ et}$$

$$S = \alpha\int e^{fx} X\partial x + \beta\int e^{fx} X x\partial x = \int e^{fx} X\partial x(\alpha+\beta x),$$

quare

$$vRS = -\frac{\alpha}{\beta} e^{-fx}\int e^{fx} X\partial x - e^{-fx}\int e^{fx} X x\partial x, \text{ et}$$

$$\int R\partial S = -\frac{1}{\beta}\int e^{fx} X\partial x,$$

unde conficitur

$$y = e^{-fx} x\int e^{fx} X\partial x - e^{-fx}\int e^{fx} X x\partial x;$$

seu cum sit

$$\partial . e^{fx} y = \partial x \int e^{fx} X \partial x,$$

erit succinctius

$$y = e^{-fx} \int \partial x \int e^{fx} X \partial x.$$

III. Si $tt + At + B = tt + 2ft \cos. \zeta + ff$, seu $A = 2f \cos. \zeta$ et $B = ff$, erit

$$v = e^{-fx \cos. \zeta} \sin. (fx \sin. \zeta + \gamma).$$

Ponatur

$$\frac{e^{-2fx \cos. \zeta} \partial x}{vv} = \frac{\partial x}{\sin. (fx \sin. \zeta + \gamma)^2} = \partial R \text{ et}$$

$$e^{2fx \cos. \zeta} X v \partial x = e^{fx \cos. \zeta} X \partial x \sin. (fx \sin. \zeta + \gamma) = \partial S,$$

ut obtineatur $y = vRS - v\int R \partial S$. At est

$$R = -\frac{1}{f \sin. \zeta} \cdot \frac{\cos. (fx \sin. \zeta + \gamma)}{\sin. (fx \sin. \zeta + \gamma)};$$

hincque

$$vRS = -\frac{1}{f \sin. \zeta} e^{-fx \cos. \zeta} \cos. (fx \sin. \zeta + \gamma) \int e^{fx \cos. \zeta} X \partial x \sin. (fx \sin. \zeta + \gamma)$$

et

$$\int R \partial S = -\frac{1}{f \sin. \zeta} \int e^{fx \cos. \zeta} X \partial x \cos. (fx \sin. \zeta + \gamma).$$

Quocirca obtinebitur

$$fy \sin. \zeta = + e^{-fx \cos. \zeta} \sin. (fx \sin. \zeta + \gamma) \int e^{fx \cos. \zeta} X \partial x \cos. (fx \sin. \zeta + \gamma)$$
$$- e^{-fx \cos. \zeta} \cos. (fx \sin. \zeta + \gamma) \int e^{fx \cos. \zeta} X \partial x \sin. (fx \sin. \zeta + \gamma).$$

Corollarium 1.

857. In hoc postremo integrali si ponamus $fx \sin. \zeta = \Phi$, erit

$$f e^{fx \cos. \zeta} y \sin. \zeta =$$
$$(\sin. \gamma \cos. \Phi + \cos. \gamma \sin \Phi) \int e^{fx \cos. \zeta} X \partial x (\cos. \gamma \cos. \Phi - \sin. \gamma \sin. \Phi)$$
$$+ (\sin. \gamma \sin. \Phi - \cos. \gamma \sin. \Phi) \int e^{fx \cos. \zeta} X \partial x (\sin. \gamma \cos. \Phi + \cos. \gamma \sin. \Phi),$$

seu

$$\int e^{\int x \cos.\zeta}\, y \sin.\zeta =$$
$$+\sin.\gamma\cos.\gamma\cos.\Phi\int e^{\int x\cos.\zeta}X\partial x\cos.\Phi-\sin.\gamma^2\cos.\Phi\int e^{\int x\cos.\zeta}X\partial x\sin.\Phi$$
$$+\cos.\gamma^2\sin.\Phi\int e^{\int x\cos.\zeta}X\partial x\cos\Phi-\sin.\gamma\cos.\gamma\sin.\Phi\int e^{\int x\cos.\zeta}X\partial x\sin.\Phi$$
$$+\sin.\gamma^2\sin.\Phi\int e^{\int x\cos.\zeta}X\partial x\cos.\Phi+\sin.\gamma\cos.\gamma\sin.\Phi\int e^{\int x\cos.\zeta}X\partial x\sin.\Phi$$
$$-\sin.\gamma\cos.\gamma\cos.\Phi\int e^{\int x\cos.\zeta}X\partial x\cos.\Phi-\cos.\gamma^2\cos.\Phi\int e^{\int x\cos.\zeta}X\partial x\sin.\Phi$$

unde patet angulum γ prorsus ex calculo excedere, fit enim

$$\int e^{\int x\cos.\zeta}y\sin\zeta=\sin.\Phi\int e^{\int x\cos.\zeta}X\partial x\cos.\Phi-\cos.\Phi\int e^{\int x\cos.\zeta}X\partial x\sin.\Phi.$$

Corollarium 2.

858. Cum igitur loco unius aequationis duas formaverimus integrandas, vidimus sufficere, si alterius integrale saltem particulare fuerit cognitum. In ambobus enim praecedentibus casibus constantes α et β integrale completum praebentes ex calculo sponte evanuerunt, et casu tertio constans γ itidem excessit.

Exemplum.

859. *Sumto elemento ∂x constante, invenire integrale hujus aequationis*

$$\partial\partial y+A\partial y\partial x+By\partial x^2$$
$$=\partial x^2\,[n(n-1)x^{n-2}+nAx^{n-1}+Bx^n].$$

Hoc exemplum ita est comparatum, ut ei manifesto satisfaciat valor $y=x^n$, qui ejus integrale particulare constituit. Ad completum ergo inveniendum, sit $A=f+g$ et $B=fg$; et cum sit

$$X=fg\,x^n+n(f+g)\,x^{n-1}+n(n-1)\,x^{n-2},\ \text{erit}$$
$$\int e^{gx}X\partial x=fe^{gx}x^n+ne^{gx}x^{n-1}+\alpha,\ \text{et}$$
$$\int e^{fx}X\partial x=ge^{fx}x^n+ne^{fx}x^{n-1}+\beta,$$

unde ex forma inventa prodit integrale completum

$$(f-g)y = fx^n + nx^{n-1} + \alpha e^{-gx} - gx^n - nx^{n-1} - \beta e^{-fx},$$

seu $y = x^n + \frac{\alpha}{f-g} e^{-gx} - \frac{\beta}{f-g} e^{-fx}$,

vel mutata constantium forma

$$y = x^n + \alpha e^{-fx} + \beta e^{-gx}.$$

Si sit $g = f$, ponatur $g = f + \omega$, existente $\omega = 0$, et ob $e^{-gx} = e^{-fx} . e^{-\omega x} = e^{-fx}(1 - \omega x)$, unde pro $\alpha + \beta$ et $-\beta\omega$ scribendo α et β, erit

$$y = x^n + e^{-fx}(\alpha + \beta x).$$

Sin autem sit

$$f = a + b\sqrt{-1} \text{ et } g = a - b\sqrt{-1},$$

cum fiat

$$y = x^n + e^{-ax}(\alpha e^{-bx\sqrt{-1}} + \beta e^{bx\sqrt{-1}}) \text{ ob}$$
$$e^{\pm bx\sqrt{-1}} = \cos. bx \pm \sqrt{-1} . \sin. bx,$$

mutata forma constantium habebimus

$$y = x^n + e^{-ax}(\alpha \cos. bx + \beta \sin. bx).$$

Scholion.

860. In genere autem si hujusmodi aequationis

$$\partial\partial y + A\partial y\partial x + By\partial x^2 = X\partial x^2,$$

constet integrale particulare, seu valor ipsi satisfaciens $y = t$, integrale completum facile reperitur ponendo $y = t + z$. Cum enim per hypothesin sit

$$\partial\partial t + A\partial t\partial x + Bt\partial x^2 = X\partial x^2,$$

facta hac substitutione orietur

$$\partial\partial z + A\partial z\partial x + Bz\partial x^2 = 0,$$

unde si $A = f + g$ et $B = fg$, colligitur

$$z = \alpha e^{-fx} + \beta e^{-gx},$$

sicque integrale completum erit

$$y = t + \alpha e^{-fx} + \beta e^{-gx},$$

quemadmodum etiam in exemplo allato invenimus.

Problema 106.

861. Sumto elemento ∂x constante, si proponatur haec aequatio differentio-differentialis

$$\partial\partial y - \frac{n\,\partial y\,\partial x}{x} + A x^n \partial y\,\partial x + B x^{2n} y\,\partial x^2 = X\,\partial x^2,$$

existente X functione quacunque ipsius x, ejus integrale completum investigare.

Solutio.

Resolutio hujus aequationis ut supra ex positione $y = uv$ derivari posset, sed alia methodo hic utentes levi substitutione eam ad formam problematis praecedentis reducamus. Scilicet ponamus $x^n \partial x = \partial t$, ut sit $x^{n+1} = (n+1)t$, qua substitutione functio X abeat in T, functionem quandam ipsius t. Ne autem assumtio elementi ∂x constantis turbet, hanc conditionem tollamus ponendo $\partial y = p\,\partial x$ et $\partial p = q\,\partial x$, habebimusque

$$q - \frac{np}{x} + A x^n p + B x^{2n} y = X.$$

Cum nunc sit $\partial x = \frac{\partial t}{x^n}$, erit $p = \frac{x^n \partial y}{\partial t}$, hincque sumto elemento ∂t constante

$$\partial p = \frac{n x^{n-1}\partial x\,\partial y}{\partial t} + \frac{x^n \partial\partial y}{\partial t} = q\,\partial x = \frac{q\,\partial t}{x^n},$$

ergo

$$q = \frac{n x^{n-1}\partial y}{\partial t} + \frac{x^{2n}\partial\partial y}{\partial t^2},$$

sicque nostra aequatio erit

$$\frac{x^{2n}\partial\partial y}{\partial t^2} = \frac{n x^{n-1}\partial y}{\partial t} - \frac{n x^{n-1}\partial y}{\partial t} + \frac{A x^{2n}\partial y}{\partial t} + B x^{2n} y = X.$$

Sit $X x^{-2n} = \Theta$, quae quantitas, posito $x^{n+1} = (n+1)t$, ut functio ipsius t spectari potest, sicque fiet

$$\partial\partial y + A\,\partial y\,\partial t + B y\,\partial t^2 = \Theta\,\partial t^2,$$

in qua aequatione elementum ∂t constans est assumtum, cujus ergo integrale per superiora datur.

I. Si $A = f + g$ et $B = fg$, erit integrale

$$(f - g)\, y = e^{-gt} \int e^{gt} \Theta \partial t - e^{-ft} \int e^{ft} \Theta \partial t,$$

ubi restitutis valoribus $\partial t = x^n \partial x$ et $\Theta = X x^{-2n}$, retento brevitatis gratia $t = \frac{1}{n+1} x^{n+1}$, valor ipsius y ita per x exprimetur

$$(f - g)\, y = e^{-gt} \int e^{gt} x^{-n} X \partial x - e^{-ft} \int e^{ft} x^{-n} X \partial x.$$

II. Si $A = 2f$ et $B = ff$, erit integrale

$$y = e^{-ft} t \int e^{ft} \Theta \partial t - e^{-ft} \int e^{ft} \Theta t \partial t, \text{ seu}$$

$$y = e^{-ft} \int \partial t \int e^{ft} \Theta \partial t,$$

quod ergo per x ita exprimetur

$$y = e^{-ft} \int x^n \partial x \int e^{ft} x^{-n} X \partial x.$$

III. Si denique $A = 2 f \cos. \zeta$ et $B = ff$, erit integrale $\int e^{ft\cos \zeta} y \sin. \zeta = \sin. \Phi \int e^{ft\cos.\zeta} \Theta \partial t \cos. \Phi - \cos. \Phi \int e^{ft\cos.\zeta} \Theta \partial t \sin. \Phi$, existente $\Phi = ft \sin. \zeta$; seu $\Phi = \frac{f \sin. \zeta}{n+1} x^{n+1}$, ob $t = \frac{1}{n+1} x^{n+1}$. Quare aequationis propositae integrale erit

$$\int e^{ft\cos.\zeta} y \sin. \zeta = \sin. \Phi \int e^{ft\cos.\zeta} x^{-n} X \partial x \cos. \Phi - \cos. \Phi \int e^{ft\cos.\zeta} x^{-n} X \partial x \sin. \Phi$$

Corollarium 1.

862. Si $n = 0$, aequatio proposita abit in eam ipsam, quam problemate praecedente tractavimus, fitque $t = x$, unde etiam integrale eodem redit.

Corollarium 2.

863. Sin autem sit $n = -1$, aequatio nostra fit

$$\partial \partial y + (A + 1) \frac{\partial y \partial x}{x} + \frac{B y \partial x^2}{xx} = X \partial x^2,$$

ubi ergo erit $t = lx$ et $e^{\lambda t} = x^{\lambda}$, tum vero pro casu tertio angulus $\Phi = f \sin. \zeta . lx$.

Scholion.

864. Methodus qua hic usi sumus, hujusmodi aequationes differentiales integrandi, haud satis naturalis videtur, cum ad has quasi solas formas sit adstricta. Quoniam igitur in aequationibus differentialibus primi gradus inventio factorum, quibus eae per se integrabiles reddantur, insignem fructum polliceri videbatur, ejus quoque usum in aequationibus differentialibus secundi gradus ostendere conemur. Hic quidem nihil tam absolutum expectare licet, quod ad omnes omnino aequationum formas pateat, sed quantillum etiam praestare potuerimus, id haud contemnendum Analyseos incrementum spectari debebit. Hac autem methodo eas potissimum aequationes differentiales, in quibus altera variabilis y cum suis differentialibus unam dimensionem nusquam transgreditur, satis commode tractare licet, hincque via perspicietur, quomodo eam magis excoli oporteat.